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Bounds on the Free Distance of Periodically Time-Varying SC-LDPC Codes

Massimo Battaglioni, Marco Baldi, and Franco Chiaraluce

***Abstract*—Time-invariant spatially coupled low-density parity-check (TI-SC-LDPC) codes can be obtained by unwrapping quasi-cyclic (QC) LDPC codes. This results in a free distance that is lower bounded by the minimum distance of the underlying QC-LDPC codes. By introducing some variability in the syndrome former matrix, time-varying (TV) SC-LDPC codes are obtained, which trade an improved error correction performance for an increased decoding memory requirement and decoding complexity. A family of codes able to combine the advantages of TI-SC-LDPC codes with those of TV-SC-LDPC codes is that of periodically time-varying (PTV) SC-LDPC codes, based on a finite and periodic variation of the syndrome former matrix. In this paper we focus on such codes, and derive new upper bounds on the free distance of PTV-SC-LDPC code ensembles as well as on specific codes. By using these bounds, we show that PTV-SC-LDPC codes can achieve important improvements in the free distance over TI-SC-LDPC codes even using a very small period of variability, which corresponds to a minimal increase in memory and complexity. We also validate the new upper bounds through numerical experiments and assess the error correction performance of the corresponding codes through Monte Carlo simulations.**

***Index Terms*—Convolutional codes, free distance, LDPC codes, spatially coupled codes, time-invariant codes, time-varying codes.**

I. Introduction

The proof that spatially coupled low-density parity-check (SC-LDPC) codes achieve the channel capacity for a large number of channels [1] has led to a renewed interest toward this class of codes, which were first proposed as counterparts of LDPC block codes under the name of LDPC convolutional codes [2]. Many properties of SC-LDPC codes, such as their girth and free distance (that is, the minimum distance in the convolutional domain), are at least as good as those of the underlying block codes (see, for example, [3], [4]).

The decoding complexity and latency of SC-LDPC codes are proportional to the code constraint length which, in turn, is proportional to the product between the block length a (that is, the number of rows of the syndrome former matrices forming the parity-check matrix) and the code memory m_s (for these quantities we use the same nomenclature and definition as in [5]). Therefore, in order to keep reasonably small values of complexity and latency, either a or m_s are usually chosen relatively small.

Time-invariant (TI) SC-LDPC codes [4] can be obtained by *unwrapping* [2] quasi-cyclic LDPC (QC-LDPC) codes and have recently attracted a lot of attention, given the extreme simplicity with which they can be represented (they have unitary period T), and their nice features under different perspectives (error rate, girth, trapping set size) [6]–[8]. It is also shown in [4] that the free distance of a TI-SC-LDPC code is lower bounded by the minimum distance of the associated QC-LDPC code. However, the unitary period of time-invariant SC-LDPC codes imposes some upper bounds on their characteristics, such as girth and free distance, which cannot be improved by just increasing their memory, given a fixed value of the block length, or vice versa.

These shortcomings can be overcome by employing periodically time-varying (PTV) SC-LDPC codes. For instance, for a fixed small block length a, the behavior of time-varying code ensembles can be improved by letting the period T increase with m_s, as done for example in [2], [9], [10]. However, the large number of degrees of freedom afforded by PTV-SC-LDPC codes, especially those with a substantial period, entail a considerable representation cost, scaling as $O(T)$. This necessitates a significant increase in storage requirements and escalates the complexity involved in designing good codes.

Therefore, it is worth investigating the properties of PTV-SC-LDPC codes with period only slightly larger than one. The girth of these codes has already been studied in [5], [11], where it is shown that a slight increase in the period yields a significant increase in an upper bound on their girth and enables the practical design of codes achieving such an upper bound. Moreover, in the same works, the finite length performance of the corresponding codes is assessed, showing that PTV-SC-LDPC codes with period 2 significantly outperform TI-SC-LDPC codes with the same code rate, period, memory and girth. We argue that, among some other factors, such a significant improvement in the finite length performance is also due to a potentially large increase in the free distance of these PTV-SC-LDPC codes with small period with respect to their time-invariant counterparts. In this paper, we address such an issue and introduce new tools for the study of the free distance of PTV-SC-LDPC codes.

A. Related works

Costello proved in [12] that (standard, non-LDPC) PTV convolutional codes have larger free distances than TI convolutional codes. In the same paper, asymptotic lower and upper bounds on the free distance are provided. The free distance of TV-SC-LDPC codes has also been studied in [13] in the asymptotic regime, where it is shown that (for a certain ensemble) it grows linearly with the constraint length. Based

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on a similar approach, existence-type lower bounds on the free distance are given for an ensemble of PTV-SC-LDPC codes and tail-biting codes in [14]. For protograph-based codes, the ensemble average free distance of PTV-SC-LDPC codes is lower bounded in [15] and upper bounded in [16] (again by probabilistic, existence-type bounds). Smarandache and Vontobel derive upper bounds on the free distance of all the codes belonging to any ensemble of TI-SC-LDPC codes in [17] and also provide bounds that are tailored to any specific code in the ensemble.

B. Our contribution

In this paper, we introduce some new upper bounds on the free distance of PTV-SC-LDPC codes with period T, which also highlight some important advantages of these codes over TI-SC-LDPC codes. Moreover, the derived bounds can be extended to general TV-SC-LDPC codes by considering a period $T \to \infty$. We show that, by fixing the period T of a PTV-SC-LDPC code and increasing its memory, the free distance can increase only up to a certain value, given by one of the upper bounds we propose.

We prove that, in any particular ensemble, the upper bound on the free distance of PTV-SC-LDPC codes is lower bounded by the upper bound on the free distance of their associated TI-SC-LDPC codes. Then, we show that for small values of T, if the code parameters are accurately chosen, the free distance of these codes can grow more than linearly with the period. Differently from the approach in [14], [15], we do not tailor the bound to the tail-biting version of the code. Moreover, differently from [16], we do not consider a version of the code with coupling length equal to the period. Rather, we generalize the approach introduced for TI codes in [17] to the periodically time-varying case. This approach has the important benefit that the newly derived bounds can be computed mathematically in a relatively easy way, whereas those in [14], [15] require the computation of the minimum distance of tail-biting codes, that is a computationally heavier task. In fact, the minimum distance computation problem is known to be nondeterministic polynomial time (NP)-hard, as first conjectured in [18] and later confirmed in [19]. So, even though the tail-biting codes are block codes, the lower bounds on the free distance of their convolutional counterparts become tighter and tighter for increasing values of their block length, requiring a significant computational effort for the evaluation of their minimum distance. Still, we remark that our approach can be used together with that in [14], [15], in order to find both a lower and an upper bound on the free distance of a PTV-SC-LDPC code.

C. Paper outline

The paper is organized as follows. In Section II we introduce the notation used throughout the paper and give some background notions. In Section III we derive bounds on the free distance of ensembles of PTV-SC-LDPC codes. In Section IV we provide some numerical examples. In Section V we assess the error rate performance of some PTV-SC-LDPC codes with small period. Conclusions are finally drawn in Section VI.

II. Notation and background

In this section we introduce the notation we use throughout the paper and we provide some background notions.

Given two integers a and b, we denote by $[a,b]$ the set of integers $\{y \,|\, a \leq y \leq b\}$. $\mathbb{F}_2[x]$ is the ring of polynomials with coefficients in the binary Galois field and $\mathbb{F}_2[x]/(x^p+1)$ is the same ring modulo (x^p+1). According to the isomorphism between $p \times p$ circulant matrices and polynomials in $\mathbb{F}_2[x]/(x^p+1)$, any polynomial $q(x)$ univocally corresponds to a $p \times p$ circulant matrix, and the exponents of $q(x)$ represent the positions of the non-zero elements in the first column (or row) of the corresponding circulant matrix. We use bold upper case letters (resp., lower case) to denote matrices (resp., vectors). For a vector $\mathbf{a}$ of length n, the ith entry is indicated as a_i. For an $m \times n$ matrix $\mathbf{A}$, the entry at position (i,j) is indicated as $a_{i,j}$, the ith row as $\mathbf{A}_{i,:}$ and the jth column as $\mathbf{A}_{:,j}$. Given an $m \times n$ matrix $\mathbf{A}$, and a set $K \subset [0, n-1]$, $\mathbf{A}_K$ is the submatrix of $\mathbf{A}$ formed by the columns of $\mathbf{A}$ with indexes in K.

Given a set S, $|S|$ represents its cardinality. Transposition is denoted with $^\top$. We define $\mathcal{W}(\cdot)$ as a function returning the Hamming weight of its argument, if the argument is a vector, or the number of non-zero coefficients, if the argument is a polynomial. If the argument is a polynomial matrix (or vector), $\mathcal{W}(\cdot)$ returns a matrix (or vector) containing the weights of its polynomial entries. If the argument is a binary matrix (or vector), the function returns its Hamming weight. The operator $\mathrm{perm}(\cdot)$ returns the permanent of its argument, that is,

$$\mathrm{perm}(\mathbf{A}) \triangleq \sum_{\sigma \in S_n} \prod_{j=0}^{n-1} a_{j,\sigma(j)},$$

where $\mathbf{A}$ is an $n \times n$ matrix and S_n is the symmetric group. The $\min^*$ operator, introduced in [20], returns the smallest positive entry of a list if the list contains only positive entries, otherwise it returns $+\infty$.

A. SC-LDPC codes

Let us briefly recall time-invariant, periodically time-varying and time-varying codes. Time-varying SC-LDPC codes with asymptotic rate $R_\infty = \frac{a-c}{a}$ and minimum distance d_{free} are characterized by a parity-check matrix $\mathbf{H}$ such that

$$\mathbf{H}^\top = \begin{bmatrix} \ddots & & \ddots & & & & & \\ \mathbf{H}_0^\top(0) & \cdots & \mathbf{H}_{m_s}^\top(m_s) & \mathbf{0} & & & & \\ & & & \ddots & & & & \\ & & & \mathbf{0} & \mathbf{H}_0^\top(t) & \cdots & \mathbf{H}_{m_s}^\top(t+m_s) & \mathbf{0} \\ & & & & \ddots & & \ddots & \end{bmatrix} \quad (1)$$

where each block $\mathbf{H}_i(t)$, $i \in [0, m_s]$, is a binary matrix with size $c \times a$, namely,

$$\mathbf{H}_i(t) = \begin{bmatrix} (h_i)_{0,0}(t) & \dots & (h_i)_{0,a-1}(t) \\ \vdots & \ddots & \vdots \\ (h_i)_{c-1,0}(t) & \dots & (h_i)_{c-1,a-1}(t) \end{bmatrix}, \quad (2)$$

where a is defined as the *block length* of the code.

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The parity-check matrix $\mathbf{H}$ is said to be (d_v, d_c)-regular if all its rows have Hamming weight d_c and all its columns have Hamming weight d_v. If only one of these conditions is verified, the code is said to be row-regular or column-regular, respectively. Otherwise, when none of the conditions is satisfied, the code is called irregular. We also define

$$\mathbf{H_s}(t) \triangleq \left[\mathbf{H}_0^\top(t)|\mathbf{H}_1^\top(t)|\dots|\mathbf{H}_{m_s}^\top(t)\right] \quad (3)$$

as the t-th syndrome former matrix. The variable m_s is the syndrome former memory order (that will be denoted as *memory* in the following, for brevity) and $\nu_s = (m_s+1)a$ is the syndrome former constraint length. If $\mathbf{H}_i(t) = \mathbf{H}_i(t+T)$ for a finite value of T, the corresponding code is said to be *periodically time-varying* with *period* T. When $T = 1$, we say that the code is *time-invariant*. Time-invariant codes are characterized by a fixed $\mathbf{H_s}$, that is, the matrix in (3) does not depend on the variable t.

The convolutional counterpart of the minimum distance for block codes is called *free distance* (denoted as d_{free}), and is defined as the minimum Hamming weight of a non-zero code sequence. We also define the girth g as the length of the shortest cycle(s) in the Tanner graph of a code.

The syndrome former matrix of a TI-SC-LDPC code can also be defined through a symbolic representation exploiting polynomials in $\mathbb{F}_2[D]$. According to such a representation, the code is described by a $c \times a$ *symbolic matrix* having polynomial entries, that is

$$\mathbf{H}^{\text{TI}}(D) \triangleq \begin{bmatrix} h_{0,0}(D) & \dots & h_{0,a-1}(D) \\ \vdots & \ddots & \vdots \\ h_{c-1,0}(D) & \dots & h_{c-1,a-1}(D) \end{bmatrix}, \quad (4)$$

where each $h_{i,j}(D)$, $i \in [0, c-1]$, $j \in [0, a-1]$, is a polynomial in $\mathbb{F}_2[D]$. If $\mathbf{H}^{\text{TI}}(D)$ contains only (non-zero) monomial entries, the code is said to be a *fully-connected monomial code*. The code representation based on $\mathbf{H_s}$ can be converted into that based on $\mathbf{H}^{\text{TI}}(D)$ by setting

$$h_{i,j}(D) = \sum_{m=0}^{m_s} (h_m)_{i,j}\, D^m, \quad (5)$$

where $(h_m)_{i,j}$ is the (i,j) entry of $\mathbf{H}_m$. Next we provide a toy example of conversion of $\mathbf{H_s}$ into $\mathbf{H}^{\text{TI}}(D)$.

Example 1. *Let us consider a code with $T = 1$, $a = 3$ and $c = 2$, characterized by*

$$\mathbf{H}_\mathbf{s}^\top = \begin{array}{cl} \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] & \begin{array}{l} \mathbf{H}_0 \\ \\ \mathbf{H}_1. \\ \\ \mathbf{H}_2 \\ \end{array} \end{array}$$

Then, according to (5)*, we obtain*

$$\mathbf{H}^{\text{TI}}(D) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & D & D^2 \end{bmatrix},$$

where entries of the same color in $\mathbf{H_s}$ and $\mathbf{H}(D)^{\text{TI}}$ correspond to each other.

Let us define the *base matrix* of the code as $\mathbf{B} \triangleq \mathcal{W}(\mathbf{H}(D))$. We also define an *ensemble of codes* $\mathcal{E}(\mathbf{B})$ as the collection of all codes characterized by the same $\mathbf{B}$. In the time-varying case, if $\mathcal{W}(\mathbf{H}_i(D)) = \mathcal{W}(\mathbf{H}_j(D))$, $\forall i \neq j$, without loss of generality we assume that $\mathbf{B} = \mathcal{W}(\mathbf{H}_0(D))$.

B. Tail Biting SC-LDPC Codes

Practically speaking, we need to terminate SC-LDPC codes at some point. By starting from the following section of the semi-infinite parity-check matrix (1)

$$\mathbf{H}_{[0,L]}^\top = \begin{bmatrix} \mathbf{H}_0^\top(0) & \cdots & \mathbf{H}_{m_s}^\top(m_s) & & \\ & \ddots & & \ddots & & \ddots \\ & & & \mathbf{H}_0^\top(L) & \cdots & \mathbf{H}_{m_s}^\top(L+m_s) \end{bmatrix} \quad (6)$$

we obtain a SC-LDPC code terminated in *tail-biting* fashion (or simply, tail-biting SC-LDPC code) with coupling length $L > m_s$, by wrapping back the last $m_s c$ columns of (6) after L time instants. Matrix (7) is thus obtained.

The inverse of this procedure is called *unwrapping* and was initially proposed in [2]. If the tail-biting termination is applied to a time-invariant code, (7) is the parity-check matrix of a QC-LDPC code with block length $a(L+1)$, as initially defined in [21]. The more common circulants (of size L) block form of the parity-check matrix of this QC-LDPC (introduced in [22]) can be obtained by a simple reordering of the rows and columns of (7).

It has been proven in [15, Theorem 4] that the free distance of the unterminated version of a SC-LDPC code $\mathcal{C}$ is lower bounded by the minimum distance of its tail-biting version, as long as the coupling length is larger than the memory. Therefore, in order to practically find this lower bound on d_{free}, one can terminate a SC-LDPC code in tail-biting fashion (obtaining a full-fledged QC-LDPC code) and compute its minimum distance. Clearly, there is a trade-off between the tightness of the bound and the complexity of the computation of the minimum distance. Practically speaking, for values the minimum distance d_{min} in the order of 10^2 or larger, the exact computation of the lower bound is unfeasible, since the best known solvers of the Codeword Finding Problem (CFP) have a computational complexity which is exponential in the weight of the searched codeword(s) [23]. The use of tools for the *approximation* of the minimum distance is thus enforced. In this paper, we take advantage of the approach proposed in [24]. These aspects will be further discussed in Section V.

III. Bounds on the Free Distance of PTV-SC-LDPC Codes

In order to find bounds on the free distance of PTV-SC-LDPC codes, we need the following results that links their representation to that in (4) used for TI-SC-LDPC codes.

Remark 1. *The T syndrome former matrices representing a PTV-SC-LDPC code with memory m_s, each of size $(m_s+1)c \times a$, can be included into a single larger syndrome former matrix, representing a TI-SC-LDPC code, i.e.,*

$$\mathbf{H}_\mathbf{s}^\top \triangleq \begin{bmatrix} \mathbf{H}_0^\top(0) & \cdots & \mathbf{H}_{m_s}^\top(m_s) & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{H}_0^\top(T-1) & \cdots & \mathbf{H}_{m_s}^\top(T+m_s-1) \end{bmatrix}, \quad (8)$$

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$$\tilde{\mathbf{H}}^{\top}_{[0,L]} = \begin{bmatrix} \mathbf{H}_0^{\top}(0) & \mathbf{H}_1^{\top}(1) & \dots & \mathbf{H}_{m_s}^{\top}(m_s) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_0^{\top}(1) & \dots & \dots & \mathbf{H}_{m_s}^{\top}(m_s+1) & \dots & \vdots \\ & & \ddots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{H}_{m_s}^{\top}(L+1) & \mathbf{0} & & \mathbf{H}_0^{\top}(L-m_s+1) & \dots & \dots & \mathbf{H}_{m_s-1}^{\top}(L) \\ \mathbf{H}_{m_s-1}^{\top}(L+1) & \mathbf{H}_{m_s}^{\top}(L+2) & \mathbf{0} & & \ddots & \ddots & \vdots \\ \vdots & & & & & \mathbf{H}_0^{\top}(L-1) & \mathbf{H}_1^{\top}(L) \\ \mathbf{H}_1^{\top}(L+1) & \dots & \mathbf{H}_{m_s}^{\top}(L+m_s) & \mathbf{0} & & & \mathbf{H}_0^{\top}(L) \end{bmatrix} \tag{7}$$

with all zeros outside the main block diagonal. Such a TI-SC-LDPC code, having block length aT and cT parity-symbols, can also be represented through a $Tc \times Ta$ symbolic matrix in the form (4)*, having the structure that will be discussed in Lemma 1. By definition, the memory of the resulting TI-SC-LDPC code, denoted as $\hat{m_s}$, is given by the number of rows in* (8)*, divided by the number of parity-symbols (Tc), minus* 1*, i.e.,*

$$\hat{m_s} = \left\lceil \frac{c(m_s+1)+c(T-1)}{Tc} \right\rceil - 1 = \left\lceil \frac{m_s}{T} \right\rceil .$$

This implies that (8) *can also be written as*

$$\mathbf{H_s} \triangleq \left[\hat{\mathbf{H}}_0^{\top} | \hat{\mathbf{H}}_1^{\top} | \dots | \hat{\mathbf{H}}^{\top}_{\lceil \frac{m_s}{T} \rceil} \right], \tag{9}$$

where

$$\hat{\mathbf{H}}_0 = \begin{bmatrix} \mathbf{H}_0(0) & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \mathbf{H}_1(0) & \mathbf{H}_0(1) & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{H}_{T-1}(0) & \mathbf{H}_{T-2}(1) & \dots & \dots & \mathbf{H}_0(T-1) \end{bmatrix}, \tag{10}$$

$$\hat{\mathbf{H}}_i = \begin{bmatrix} \mathbf{H}_{iT}(0) & \mathbf{H}_{iT-1}(1) & \cdots & \mathbf{H}_{(i-1)T+1}(T-1) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}_{(i+1)T-1}(0) & \mathbf{H}_{(i+1)T-2}(1) & \cdots & \mathbf{H}_{iT}(T-1) \end{bmatrix}, \tag{11}$$

$$\hat{\mathbf{H}}_{\lceil \frac{m_s}{T} \rceil} = \begin{bmatrix} \mathbf{H}_{\lceil \frac{m_s}{T} \rceil T}(0) & \mathbf{H}_{\lceil \frac{m_s}{T} \rceil T-1}(1) & \cdots & \cdots & \mathbf{H}_{(\lceil \frac{m_s}{T} \rceil -1)T+1}(T-1) \\ \mathbf{0} & \mathbf{H}_{\lceil \frac{m_s}{T} \rceil T}(1) & \cdots & \cdots & \mathbf{H}_{(\lceil \frac{m_s}{T} \rceil -1)T+2}(T-1) \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{H}_{\lceil \frac{m_s}{T} \rceil T}(T-2) & \mathbf{H}_{\lceil \frac{m_s}{T} \rceil T-1}(T-1) \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{H}_{\lceil \frac{m_s}{T} \rceil T}(T-1) \end{bmatrix}. \tag{12}$$

with $i \in [1, \lceil \frac{m_s}{T} \rceil - 1]$. We remark that $\mathbf{H}_j(k) = \mathbf{0}$, $\forall j > m_s, \forall k$.

Lemma 1. *A PTV-SC-LDPC code with period T, block length a, c parity symbols per time instant and constraint length $(m_s+1)a$ can be represented through a $Tc \times Ta$ symbolic matrix of the type in* (4)*, having the following form*

$$\mathbf{H}^{\mathrm{PTV}}(D,T) = \begin{bmatrix} h_{0,0}(D) & \cdots & h_{0,T(a-1)}(D) \\ \vdots & \ddots & \vdots \\ h_{T(c-1),0}(D) & \cdots & h_{T(c-1),T(a-1)}(D) \end{bmatrix}, \tag{13}$$

where the following necessary and sufficient conditions hold

1) *for $i \in [(k-1)c, kc-1]$ and $j \in [ka, Ta-1]$, $\forall k \in [1, T-1]$, $h_{i,j}(D)$ cannot contain constant terms,*
2) *for $i \in [k_{\text{row}}c, (k_{\text{row}}+1)c-1]$, and $j \in [k_{\text{col}}a, (k_{\text{col}}+1)a-1]$, $\forall k_{\text{row}}, k_{\text{col}} \in [0, T-1]$, the largest possible degree of $h_{i,j}(D)$ is $\lfloor \frac{m_s - k_{\text{row}} + k_{\text{col}}}{T} \rfloor \leq \lceil \frac{m_s}{T} \rceil$,*
3) *if $m_s < T-1$, for $i \in [(m_s+k)c, Tc-1]$, $j \in [0, ka-1]$, $\forall k \in [1, T-m_s+1]$, then $h_{i,j}(D) = 0$.*

Proof: By employing (5) to convert (8) in symbolic form, we obtain a typical symbolic matrix of a TI-SC-LDPC code in the form (13). However, the particular form of $\mathbf{H_s}$ imposes some constraints on the entries of $\mathbf{H}^{\mathrm{PTV}}(D,T)$, as shown next.

1) From (10) we notice that, by construction, all the entries $(\hat{h}_0)_{i,j}$, where $i \in [(k-1)c, kc-1]$ and $j \in [ka, Ta-1]$, $\forall k \in [1, T-1]$, are equal to 0 (corresponding to the all-zero portion on the top right part of $\mathbf{H_s}^{\top}$). Substituting into (5), we readily obtain that, for the same set of indices i, j, k, $h_{i,j}(D)$ cannot contain constant terms, but only entries with degree strictly larger than 0.
2) Let us consider the 0-th time instant, i.e., columns of $\mathbf{H_s}^{\top}$ with indexes in $[0, a-1]$. Let us also consider $m \in [0, \lceil \frac{m_s}{T} \rceil]$; then, the largest value of m for which $\mathbf{H}_{zT+k_{\text{row}}}(0)$, $k_{\text{row}} \in [0, T-1]$, is not necessarily an-all zero matrix is m^* such that $m^* T + k_{\text{row}} \leq m_s$, i.e., $m^* \leq \lfloor \frac{m_s - k_{\text{row}}}{T} \rfloor$. Thus, when substituting into (5), the largest value of m in the sum is m^* which, in its turn, cannot be larger than $\lfloor \frac{m_s - k_{\text{row}}}{T} \rfloor$. This implies that the degree of $h_{i,j}(D)$ is upper bounded by $\lfloor \frac{m_s - k_{\text{row}}}{T} \rfloor$, with $i \in [kc, (k+1)c-1]$, $\forall k \in [0, T-1]$, and $j \in [0, a-1]$. We can extend the same reasoning to the other time instants, taking into account that the k_{col}-th syndrome former matrix of the PTV-SC-LDPC code is shifted $k_{\text{col}}c$ rows down with respect to the 0-th one. Therefore, $m^* \leq \lfloor \frac{m_s - k_{\text{row}} + k_{\text{col}}}{T} \rfloor$. We notice that $k_{\text{col}} - k_{\text{row}} \leq T-1$, and thus $m^* \leq \lfloor \frac{m_s - k_{\text{row}} + k_{\text{col}}}{T} \rfloor \leq \lceil \frac{m_s}{T} \rceil$, coherently with the fact that $\mathbf{H}^{\mathrm{PTV}}(D,T)$ cannot have entries with degree larger than the memory of the code.
3) If $m_s < T-1$, all the entries $(\hat{h}_0)_{i,j}$, where $i \in [(m_s+k)c, Tc-1]$ and $j \in [0, ka-1]$, $\forall k \in [1, T-m_s+1]$, are equal to 0, and so are all the entries below them in $\mathbf{H_s}^{\top}$. By substituting again into (5), we readily obtain that, for the same set of indices i, j, k, $h_{i,j}(D) = 0$. This characterizes the zero portion on the bottom left of $\hat{\mathbf{H}}_0$, existing only when $m_s < T-1$.

If any of the above conditions is not met, then (13) does not correspond to a syndrome former matrix as in (8), mak-

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ing those three conditions necessary. Furthermore, conditions 2) and 3) enforce the existence of a diagonal-like pattern containing T $(m_s+1)c \times a$ transposed syndrome former matrices, with an all-zero region beyond its bottom-left border. Instead, condition 1) determines the existence of an all-zero region beyond the top-right border of such a diagonal pattern, respectively. Therefore, these conditions are also sufficient to describe any PTV-SC-LDPC code with period T, block length a, and c parity symbols per time instant. ■

By considering a symbolic matrix of the type (13), we can derive an upper bound on the ensemble and code free distance as follows. Notice that, given a code $\mathcal{C}$ described by $\mathbf{H}^{\mathrm{PTV}}(D,T)$ in (13), the following Theorem 1 gives an upper bound which holds for all the codes belonging to the same ensemble as $\mathcal{C}$, since the permanent operator is only applied on (portions of) the base matrix $\mathcal{W}(\mathbf{H}^{\mathrm{PTV}}(D,T))$ and not on the symbolic matrix itself. The subsequent Theorem 2, instead, provides a tighter bound that is specific for a code $\mathcal{C}$, since the permanent operator is applied to (portions of) the symbolic matrix.

Theorem 1. *The free distance d_{free} of any PTV-SC-LDPC code with period T, block length a and c parity symbols per time instant, belonging to the ensemble $\mathcal{E}(\mathcal{W}(\mathbf{H}^{\mathrm{PTV}}(D,T)))$, is bounded as follows*

$$d_{\mathrm{free}} \leq \min^*_{\substack{L\subseteq[0,Ta-1]\\|L|=Tc+1}} \sum_{i\in L} \mathrm{perm}(\mathcal{W}(\mathbf{H}^{\mathrm{PTV}}_{L\setminus i}(D,T))) \triangleq d_{\mathcal{E},T}. \tag{14}$$

Proof: Due to Lemma 1, we can represent an ensemble of PTV-SC-LDPC codes with a $Tc\times Ta$ base matrix. Therefore, the theorem follows from the same arguments as in [17, Theorem 8], applied to such a larger matrix. ■

Theorem 2. *The free distance of any PTV-SC-LDPC code with period T, block length a and c parity symbols per time instant described by $\mathbf{H}^{\mathrm{PTV}}(D,T)$ is bounded as follows*

$$d_{\mathrm{free}} \leq \min^*_{\substack{L\subseteq[0,Ta-1]\\|L|=Tc+1}} \sum_{i\in L} \mathcal{W}(\mathrm{perm}(\mathbf{H}^{\mathrm{PTV}}_{L\setminus i}(D,T))) \triangleq d_{\mathcal{C},T}. \tag{15}$$

Proof: Due to Lemma 1, we can represent a PTV-SC-LDPC code with a $Tc\times Ta$ symbolic matrix. Therefore, the theorem follows from the same arguments as in [17, Theorem 7], applied to such a larger matrix. ■

We remark that, for both Theorems 1 and 2, the $\min^*$ operator is computed over all possible subsets of $[0,Ta-1]$ of size $Tc+1$. Also notice that, due to the definition of the $\min^*$ operator, the trivial bound $d_{\mathrm{free}}<\infty$ may be found, depending on the code symbolic matrix. Also note that, for any $\mathcal{C}\in\mathcal{E}$, $d_{\mathcal{C},T}\leq d_{\mathcal{E},T}$. Hence, $d_{\mathcal{E},T}$ provides an ultimate bound on the free distance achievable by a given family of PTV-SC-LDPC codes. This concept is formalized in the following statement.

Corollary 1. *The free distance d_{free} of any PTV-SC-LDPC code with period T, block length a and c parity symbols per time instant, belonging to the ensemble $\mathcal{E}(\mathcal{W}(\mathbf{H}^{\mathrm{PTV}}(D,T)))$, is upper bounded by $d_{\mathcal{E},T}$, independent of its memory m_s.*

Proof: As shown in Theorem 1, $d_{\mathcal{E},T}$ is computed by calculating

$$\mathrm{perm}(\mathcal{W}(\mathbf{H}^{\mathrm{PTV}}_{L\setminus i}(D,T))),$$

for each element of all the possible subsets of $[0,Ta-1]$ of size $Tc+1$. We notice that the permanent operation is applied on portions of the base matrix, which does not contain any information on the memory of the code. Therefore, the upper bound given by (14) does not depend on m_s, and the free distance of any code in the ensemble is upper bounded by $d_{\mathcal{E},T}$, independent of its memory. ■

Therefore, Corollary 1 states that, given a certain ensemble, we can design PTV-LDPC codes with as large memory as possible, but upon reaching $d_{\mathcal{E},T}$, the free distance will stop improving. It is therefore interesting to study lower and upper bounds on $d_{\mathcal{E},T}$ for PTV-SC-LDPC codes, which is done through the following further results.

Lemma 2 ([25, Equation (10)])**.** *The permanent of a square matrix $\mathbf{A}$ in $\mathbb{F}_2^{n\times n}$ with row sums $r_i=\mathcal{W}(\mathbf{A}_{i,:})$, $i\in[0,n-1]$ is upper bounded by*

$$\prod_{i=0}^{n-1}(r_i!)^{\frac{1}{r_i}}.$$

Lemma 3 ([26, Theorem 2])**.** *The permanent of a square matrix $\mathbf{A}$ in $\mathbb{F}_2^{n\times n}$ (admitting non-zero permanent) with row sums $r_i=\mathcal{W}(\mathbf{A}_{i,:})\geq t$, $i\in[0,n-1]$ is lower bounded by $t!$.*

Lemma 4. *Given $\mathbf{A}\in\mathbb{F}_2^{n\times(n+1)}$ with column sums $c_i=\mathcal{W}(\mathbf{A}_{:,i})=c$, $\forall i\in[0,n]$, it holds that*

$$\mathrm{perm}\left(\mathbf{A}_{[0,n]\setminus j}\right)=0$$

for either at most $n-c$ values of j, or all $j\in[0,n]$.

Proof: We define $\mathbf{A}^*=\mathbf{A}_{[0,n]\setminus j^*}$, for a value $j^*\in[0,n]$. Let us assume that $\mathrm{perm}\left(\mathbf{A}^*\right)\neq 0$. Since $\mathbf{A}$ is binary, the permanent of $\mathbf{A}^*$ is positive, and this implies that there exists at least an element σ^* of the symmetric group S_n such that $\prod_{i=0}^{n-1}a^*_{i,\sigma^*(i)-1}>0$. In other words, $\mathbf{A}^*$ can be written as $\mathbf{A}^*=\mathbf{P}^*+\mathbf{E}^*$, where $\mathbf{P}^*$ is a permutation matrix of size n and $\mathbf{E}^*\in\mathbb{F}_2^{n\times n}$ is such that $e_{k,l}=0$ if $p_{k,l}=1$ and each column contains $c-1$ ones. We now notice that considering $\hat{j}\neq j^*$ corresponds to replacing a column of $\mathbf{A}^*$ (in particular, the one coinciding with the $\hat{j}$th column of $\mathbf{A}$) with $\mathbf{A}_{:,j^*}$. Then, since by hypothesis $\mathbf{A}_{:,j^*}$ contains c ones, there are exactly c values of j $(j_0,j_1,\dots,j_{c-1}\neq j^*)$ such that

$$|\mathbf{P}^*_{:,j_m}\cap\mathbf{A}_{:,j^*}|=1,\quad\forall m\in[0,c-1]. \tag{16}$$

This implies that $\mathrm{perm}\left(\mathbf{A}_{[0,n]\setminus j_m}\right)>0$, $\forall m\in[0,c-1]$, since the permanent of any permutation matrix is unitary. Therefore, there are at least $c+1$ values of j (the above c ones and j^*) having non-zero permanent, since (16) is a sufficient, but not necessary, condition. In other words, there are at most $n+1-c-1=n-c$ values of j such that

$$\mathrm{perm}\left(\mathbf{A}_{[0,n]\setminus j}\right)=0.$$

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This proves the first part of the lemma.

Removing the initial hypothesis that $\mathrm{perm}\left(\mathbf{A}_{[0,n]\setminus j^*}\right) \neq 0$, for some $j^* \in [0,n]$ leaves the possibility that

$$\mathrm{perm}\left(\mathbf{A}_{[0,n]\setminus j}\right) = 0$$

for all $j \in [0,n]$, which is the second part of the lemma. ■

We now prove an upper bound on the ensemble free distance, which holds for both regular and irregular PTV-SC-LDPC codes.

Theorem 3. *Let $\mathcal{C}$ be a PTV-SC-LDPC code with period T drawn from the ensemble $\mathcal{E}$, described by a base matrix $\mathbf{B} \in \mathbb{F}_2^{Tc\times Ta}$ (and associated vector $\mathbf{b}$), with block length a and c parity symbols per time instant. Then, if $d_{\mathcal{E},T}$ is finite,*

$$d_{\mathcal{C},T} \leq d_{\mathcal{E},T} \leq (Tc+1) \max_{\substack{S\subseteq[0,Ta-1]\\|S|=Tc}} \prod_{j\in S} b_j!^{\frac{1}{b_j}}. \tag{17}$$

Proof: According to (14), $d_{\mathcal{E},T}$ is computed as the sum of $Tc+1$ permanents of $Tc \times Tc$ matrices. Therefore, if $d_{\mathcal{E},T}$ is finite,

$$d_{\mathcal{C},T} \leq d_{\mathcal{E},T} \leq (Tc+1)\lambda,$$

where λ is the largest value that the permanent of any valid $Tc \times Tc$ matrix can assume. In order to compute λ, we resort to Lemma 2, which gives an upper bound on the permanent of a square matrix with known row sums. Since it is convenient to operate with column sums, we can work on the transpose of the considered matrix, and the lemma holds, as well. In our case, by definition, the column sums of $\mathcal{W}(\mathbf{H}^{\mathrm{PTV}}(D,T))$ are given by the elements of the vector $\mathbf{b}$. Therefore, λ is obtained as the largest permanent over all subsets S of $[0, Ta-1]$, having size Tc, that is,

$$\max_{\substack{S\subseteq[0,Ta-1]\\|S|=Tc}} \prod_{j\in S} b_j!^{\frac{1}{b_j}}.$$

■

The above considerations become simpler when column-regular codes are considered, as shown below.

Corollary 2. *Let $\mathcal{C}$ be a d_v column-regular PTV-SC-LDPC code with period T drawn from the ensemble $\mathcal{E}$, described by the base matrix $\mathbf{B} \in \mathbb{F}_2^{Tc\times Ta}$ (and the associated vector $\mathbf{b}$), with block length a and c parity symbols per time instant. Then, if $d_{\mathcal{E},T}$ is finite,*

$$d_{\mathcal{C},T} \leq d_{\mathcal{E},T} \leq (Tc+1) d_v!^{\frac{Tc}{d_v}}. \tag{18}$$

Proof: Eq. (18) follows from the following equality, holding when all the columns of $\mathbf{B}$ have the same Hamming weight d_v

$$\max_{\substack{S\subseteq[0,Ta-1]\\|S|=Tc}} \prod_{j\in \mathcal{S}} b_j!^{\frac{1}{b_j}} = \prod_{j=0}^{Tc-1} b_j!^{\frac{1}{b_j}} = \left(d_v!^{\frac{1}{d_v}}\right)^{Tc}.$$

■

Starting from the previous general analysis, we devote special attention to the family of fully-connected monomial codes [4], [17], [27], for which some interesting results can be derived. In order to perform a fair comparison between the upper bounds of TI- and PTV-SC-LDPC codes and to simplify the analysis, we make the following assumption.

Assumption 1. *From now on we consider PTV-SC-LDPC codes with period T drawn from the ensemble $\mathcal{E}(\mathbf{B})$ such that $\mathcal{W}(\mathbf{H}_i(D)) = \mathcal{W}(\mathbf{H}_j(D))$, $\forall i \neq j \in [0, T-1]$, and $\mathbf{B} = \mathcal{W}(\mathbf{H}_0(D))$.*

Notice that this is a pessimistic assumption, since it reduces the degrees of freedom on the design of PTV-SC-LDPC codes.

We now focus on the family of fully-connected monomial codes, for which the following result holds.

Corollary 3. *Let $\mathcal{C}$ and $\mathcal{C}'$ be a TI-SC-LDPC fully monomial code and a PTV-SC-LDPC fully monomial code with period T, respectively, with syndrome former matrices drawn from the same ensemble $\mathcal{E}(\mathbf{B})$ with the same block length a, and c parity symbols per time instant. Then, if $d_{\mathcal{E},T}$ is finite,*

$$d_{\mathcal{E},1} \leq d_{\mathcal{E},T} \leq (Tc+1)c!^T. \tag{19}$$

Proof: The rightmost part of the inequality follows from the fact that, for fully connected monomial codes, $d_v = c$, which is substituted into (18).

In order to prove the leftmost part of the inequality, we need to prove that $d_{\mathcal{E},T} \geq (c+1)! = d_{\mathcal{E},1}$. Owing to the fact that $d_v = c$, $\forall i$, we can apply Lemma 3 (working on the transpose of the considered matrix, since column sums are known, rather then row sums) to each $Tc \times Tc$ submatrix of $\mathcal{W}\left(\mathbf{H}^{\mathrm{PTV}}(D,T)\right)$, thus obtaining that their permanents are lower bounded by $c!$. We now consider the $Tc \times Tc+1$ submatrices $\mathcal{W}\left(\mathbf{H}_L^{\mathrm{PTV}}(D,T)\right)$, $L \subseteq [0, Ta-1], |L| = Tc+1$, needed to compute (14). Due to Lemma 4, at least $c+1$ permanents of any $Tc \times Tc$ submatrix of $\mathcal{W}\left(\mathbf{H}_L^{\mathrm{PTV}}(D,T)\right)$ are non-zero, leading to

$$d_{\mathcal{E},T} \geq (c+1)c! = (c+1)! = d_{\mathcal{E},1}.$$

■

We stress the fact that $d_{\mathcal{E},T}$ provides a bound on the ensemble free distance. Therefore, once a suitable ensemble has been chosen, in order to adhere as much as possible to the proposed upper bounds, the entries of $\mathbf{H}^{\mathrm{PTV}}$ should be chosen in such a way that also the upper bound on the code free distance, $d_{\mathcal{C},T}$, computed as in Theorem 2, is maximized. It is important to stress that the exact achievable improvement depends on the considered ensemble. In the next sections, we provide some numerical examples which permit us to quantitatively assess the improvement in the free distance achievable by PTV-SC-LDPC codes over TI-SC-LDPC codes.

IV. Numerical results

In order to provide some practical evidence of the new bounds and of their tightness, in this section we numerically compute the upper bound $d_{\mathcal{E},T}$, and the upper and lower

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bounds on its value, as explained in the previous sections. In the next section, instead, we assess the error rate performance of some randomly generated codes in the ensemble described next.

Let us assume that all the syndrome former matrices of the PTV-SC-LDPC codes are randomly drawn from the ensemble of $(3,4)$-regular fully-connected monomial codes, defined by

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{20}$$

By applying (14) with $T=1$, as first shown in [27], we obtain that the free distance of any TI-SC-LDPC code in this ensemble is upper bounded by 24. We have considered 10 000 randomly generated PTV-SC-LDPC codes with the same base matrix, corresponding to a few values of $T>1$.[1] We remark that, when represented as TI-SC-LDPC codes, the generated codes can belong to different ensembles. In Figs. 1 and 2 we show with dots the upper bound $d_{\mathcal{E},T}$ on the free distance of these codes, computed by using (14), for $T=2$ and $T=3$, respectively. We observe that the free distance of these PTV-SC-LDPC codes is upper bounded by values that are significantly larger than 24, and that the difference increases as T increases. The dashed lines in Figs. 1 and 2 represent the lower (red) and upper (black and blue) bounds given by (19). We notice that the potential increase in the bound on the free distance passing from $T=1$ to $T=2$ is more than linear, since it passes from 24 to 120 (largest value found empirically). Similarly, passing from $T=1$ to $T=3$, the upper bound on the free distance goes from 24 to 312 (largest value found empirically). We notice that, since the value of m_s does not influence $d_{\mathcal{E},T}$ (see Corollary 1), increasing the memory alone does not help in achieving higher upper bounds on the code free distance.

We remark that this substantial increase in the upper bound on the ensemble free distance does not necessarily imply that any PTV-SC-LDPC code has a larger free distance than the TI-SC-LDPC codes in the same ensemble $\mathcal{E}$. Therefore, in order to provide a tighter upper bound on the free distance of these codes, we have repeated the experiment by computing $\Delta = \frac{d_{\mathcal{E},T}}{d_{\mathcal{C},T}}$, for 10 000 codes with $T=2$ and $T=3$ such that $m_s \leq 300$ (the value of the memory has been chosen as an example, others can be easily considered), thus comparing (14) and (15). The parameter Δ is quite relevant, since it measures the impact of the entries of the code symbolic matrix on the upper bound on the free distance. On the one hand, if for the same ensemble many high values of Δ occur, they may denote that $d_{\mathcal{E},T}$ is a loose bound, probably due to a relatively small value of the memory m_s. Instead, sporadic high values of Δ may correspond to bad choices of the symbolic matrix entries.

We remark that, in order to compute Δ, we need to specify the value of the memory because, differently from $d_{\mathcal{E},T}$, for which the permanent is computed on the base matrix, $d_{\mathcal{C},T}$ must be computed on the symbolic matrix of the code, as follows from Theorem 2.

[1]The code ID in Figs. 1 and 2 stops slightly before 10 000 because the randomly generated codes for which the trivial bound $d_{\text{free}} \leq \infty$ was obtained were discarded. This is also true for Fig. 3.

The normalized frequency of the obtained values of Δ is shown in Fig. 3 for $T=2$ and $T=3$.[2]

It is apparent from Fig. 3a that, when $T=2$, despite the moderate value of the memory ($\hat{m}_s = \frac{m_s}{T} = 150$), there exist many codes for which $d_{\mathcal{C},2}$ is equal to the corresponding value of $d_{\mathcal{E},2}$ (precisely, this happens in 58.5% of the tested cases). Moreover, $\Delta \leq 1.5$ in 99.2% of the tested cases. The mean value of Δ is 1.04 and its variance is 0.012. This clearly highlights that codes for which the bounds on the code and the ensemble free distance are close can be designed. The relatively small mean and variance values observed also suggest that it should be possible (although not theoretically guaranteed) to find codes with relatively large values of the free distance, after a careful design phase. Similar conclusions can be drawn for the case of $T=3$, shown in Fig. 3b. However, in the latter case the values of Δ are slightly more dispersed. In fact, we notice that $\Delta = 1$ in 27.5% of the tested cases. We argue that this is due to the fact that, when $T=3$, we have $\hat{m}_s = \frac{m_s}{T} = 100$, i.e., the memory of the corresponding TI-SC-LDPC code is smaller. This implies that more terms in the computation of (15) are likely to be canceled in the computation of the permanent of the matrices $\mathbf{H}^{\mathrm{PTV}}_{L\setminus i}(D,3)$. Still, the largest found value of Δ does not change (it is 3.5 in both cases) and $\Delta \leq 1.5$ in 97.5% of the tested cases. Moreover, the mean value of Δ is 1.08 and its variance is 0.02. Smaller values of the constraint on m_s yield larger values of Δ. The converse holds for larger values of the constraint.

V. Error rate performance

In order to investigate the performance of the studied codes when used in a real error correction setting, let us assess the bit error rate (BER) performance of some of the TI- and PTV-SC-LDPC fully monomial codes with period $T=2$ and $T=3$ considered in Section IV. For such a purpose, we resort to Monte Carlo simulations of binary phase shift keying (BPSK) modulated transmissions over the Additive White Gaussian Noise (AWGN) channel, using the Log-Likelihood Ratio Sum-Product Algorithm (LLR-SPA) decoder [28] running 100 iterations.

The first code we consider ($\mathcal{C}_1$) is a TI-SC-LDPC code with $a=d_c=4$, $c=d_v=3$, $m_s=150$ and $g=6$ (and, obviously, $T=1$), described by the following symbolic matrix

$$\mathbf{H}^{\mathrm{TI}}_{\mathcal{C}_1}(D) = \begin{bmatrix} 1 & D^{144} & D^{5} & D^{106} \\ D^{41} & D^{73} & 1 & D^{4} \\ D^{150} & 1 & D^{128} & 1 \end{bmatrix}. \tag{21}$$

The syndrome former constraint length of the code is $\nu_s = (150+1)4 = 604$. The matrix $\mathbf{H}^{\mathrm{TI}}_{\mathcal{C}_1}(D)$ in (21) has been randomly picked from its ensemble (performing rejection sampling only to avoid codes having $g=4$) and, according to Theorem 2, the free distance of the corresponding TI-SC-LDPC code is upper bounded by 24. We have estimated the lower bound given in [15, Theorem 4] by using the tool in

[2]For the sake of reproducibility, we specify that the width of the bins of the histograms has been chosen equal to 0.01.

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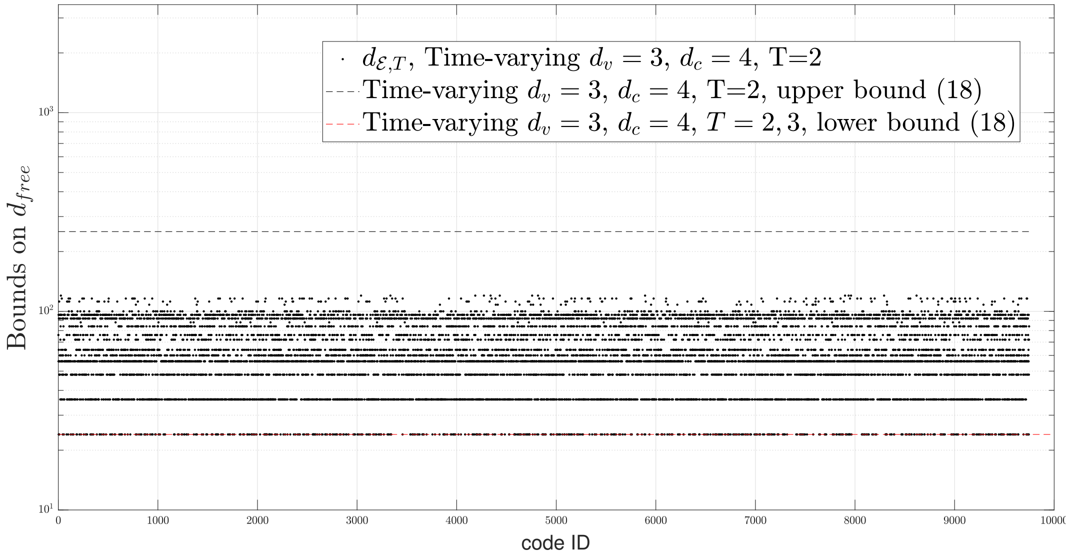

Fig. 1. Bounds on the ensemble free distance of $(3, 4)$-regular fully-connected monomial TI- and PTV-SC-LDPC codes with $T = 2$.

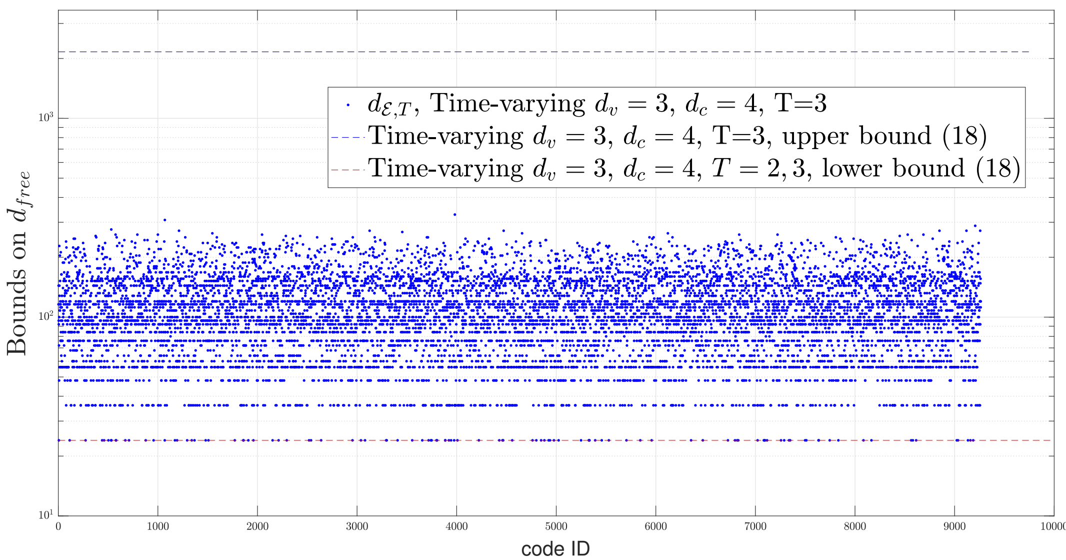

Fig. 2. Bounds on the ensemble free distance of $(3, 4)$-regular fully-connected monomial TI- and PTV-SC-LDPC codes with $T = 3$.

[29], based on the method described in [24], and obtained that $22 \leq d_{\text{free}} \leq 24$.

The second code we consider ($\mathcal{C}_2$) is a PTV-SC-LDPC fully monomial code with $T = 2$, $a = d_c = 4$, $c = d_v = 3$, $m_s = 142$ and $g = 6$. In this case, the syndrome former constraint length is $\nu_s = 572$.[3] is slightly smaller than that of $\mathcal{C}_1$. We remark that, this way, the performance of these PTV codes is evaluated pessimistically, since larger memories (combined with a good design) can yield better performances.

[3]Notice that the value of the memory $m_s = 150$ is provided as an upper bound to our search algorithm, and therefore the memory of all $\mathcal{C}_i$'s, with $1 < i \leq 5$,

It is described by the following symbolic matrix

$$\mathbf{H}^{\mathrm{PTV}}_{\mathcal{C}_2}(D) = \begin{bmatrix} D^{61} & 0 & 1 & 0 & 0 & D^{6} & 0 & D^{59} \\ D^{32} & 0 & D^{12} & 0 & 0 & 0 & D^{52} & D \\ 0 & D^{44} & 0 & D^{29} & D^{2} & D^{60} & 0 & 0 \\ 0 & 1 & 0 & D^{59} & 1 & 0 & 1 & 0 \\ 0 & D^{27} & 0 & 1 & D^{30} & 1 & 0 & 0 \\ 1 & 0 & D^{15} & 0 & 0 & 0 & D^{71} & D^{3} \end{bmatrix}, \tag{22}$$

which was extracted from its ensemble since Theorem 2 gives $d_{\mathcal{C},T} = 116$ as the upper bound on the code free distance. For $m_s \leq 150$, codes with larger values of $d_{\mathcal{C},T}$ have not been found. Also in this case, we have combined our method with that in [15], operating on the QC version of $\mathcal{C}_2$ of block length $1\,440$. However, in this case the tool

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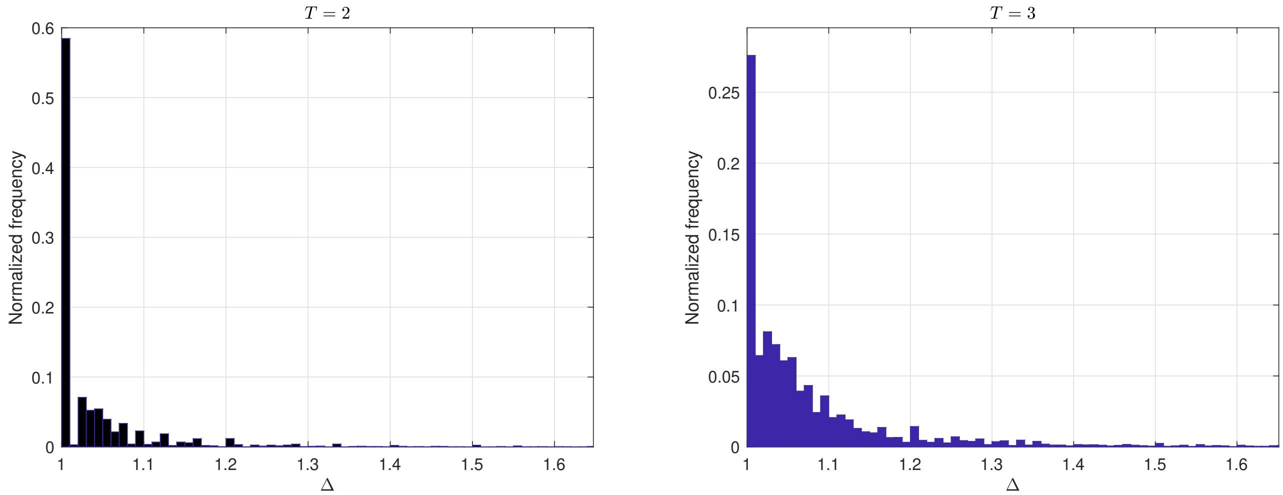

(a) (b)

Fig. 3. Normalized frequency of the ratio between $d_{\mathcal{E},T}$ and $d_{\mathcal{C},T}$, for 10 000 PTV-SC-LDPC codes with (a) $T = 2$ and (b) $T = 3$.

in [29] found no codewords of weight smaller than 116, hinting that the free distance is close to the upper bound. In fact, in this case we were unable to estimate the lower bound in [15], due to the larger block length of the code and the resulting increase in the computational complexity of the minimum distance estimate. The upper bound on the block code minimum distance provided by [17] states that, for the QC version of $\mathcal{C}_2$, $d_{\min} \leq 108$. We also consider two other PTV-SC-LDPC codes with $T = 2$ and m_s comparable to that of $\mathcal{C}_2$ (and the same values of g, a, c, d_v and d_c), but characterized by smaller values of $d_{\mathcal{C},T}$. They are described by the following symbolic matrices:

$$\mathbf{H}_{\mathcal{C}_3}^{\mathrm{PTV}}(D) = \begin{bmatrix} 1 & D^{61} & 1 & D^{29} & D^9 & 0 & D^{47} & 0 \\ D^{61} & D^{72} & 0 & 0 & D^{45} & D^{11} & D & D \\ D^{32} & 1 & 0 & D^{53} & 0 & 0 & D^{28} & D^{62} \\ 0 & 0 & 0 & 0 & 0 & D^{50} & 0 & D^{58} \\ 0 & 0 & D^{64} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & D^{50} & 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \tag{23}$$

$$\mathbf{H}_{\mathcal{C}_4}^{\mathrm{PTV}}(D) = \begin{bmatrix} 0 & 0 & 0 & D^3 & 0 & 0 & D^{71} & D^{71} \\ 0 & D^{51} & D^5 & 1 & D^{21} & D^8 & 0 & D^{38} \\ 0 & D^{60} & 0 & 0 & 0 & D^3 & 0 & 0 \\ D^{43} & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ D^7 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & D^{50} & D^{38} & D^7 & 0 & D^{68} & 1 \end{bmatrix}, \tag{24}$$

and have $d_{\min} \leq 48$ and $d_{\min} \leq 56$, respectively. Their syndrome former constraint length is 580 and 576, respectively, hence slightly larger than that of $\mathcal{C}_2$.

In order to further investigate the effect of the period of the performance, we consider a code denoted as $\mathcal{C}_5$, that is a PTV-SC-LDPC fully monomial code with $T = 3$, $a = 4$, $c = 3$, $m_s = 138$ and $g = 6$, from which $\nu_s = 556$. It is described by the following symbolic matrix:

$$\mathbf{H}_{\mathcal{C}_5}^{\mathrm{PTV}}(D) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & D^2 & D & 0 & D^{27} & 0 & 0 & 0 \\ 0 & D^{16} & 0 & D^{20} & 0 & 0 & 0 & 0 & 0 & 0 & D^{15} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & D & 0 & 0 & D^2 & D^{26} \\ 0 & 0 & D^9 & 0 & 0 & 0 & 0 & 0 & 0 & D^{45} & 0 & D \\ D & 0 & 0 & 0 & 1 & D^{21} & 0 & 0 & 0 & 0 & 0 & D^{23} \\ 0 & 0 & 0 & D^{14} & D^{42} & 0 & D^{10} & 0 & D^{18} & 0 & 0 & 0 \\ 1 & D^5 & 0 & 1 & D^{23} & 0 & 0 & D^8 & 0 & 0 & D^{46} & 0 \\ 0 & 0 & D^{29} & 0 & 0 & 0 & D^{39} & D^8 & 1 & 1 & 0 & 0 \\ D^{34} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & D^{21} & 0 & 0 \end{bmatrix}, \tag{25}$$

for which Theorem 2 gives $d_{\mathcal{C},T} = 214$. No codewords of weight smaller than 214 were found either in this case by the tool in [29] for the QC version of $\mathcal{C}_5$ of block length 1 440. On the other hand, according to [17], its minimum distance is upper bounded by 200.

Finally, in order to investigate the effect of the memory on the performance, we consider the PTV-LDPC fully monomial code with $T = 2$, $a = d_c = 4$, $c = d_v = 3$, and $g = 6$, described by:

$$\mathbf{H}_{\mathcal{C}_6}^{\mathrm{PTV}}(D) = \begin{bmatrix} 0 & D^{45} & D^{33} & D^{214} & 0 & 0 & D & 0 \\ 1 & 1 & D^{78} & D^{44} & 0 & 0 & 0 & 0 \\ 0 & D^{83} & 1 & 0 & D^{53} & 0 & D^{272} & 0 \\ D^{169} & 0 & 0 & 0 & 1 & D^{152} & 0 & D^{45} \\ 0 & 0 & 0 & 0 & D^{41} & 1 & D^{206} & 1 \\ D^{32} & 0 & 0 & 1 & 0 & D^{40} & 0 & D^{104} \end{bmatrix}, \tag{26}$$

characterized by $m_s = 544$, and for which $d_{\text{free}} \leq 114$. The syndrome former constraint length of this code is 2180. Table I summarizes the parameters of all the considered codes.

TABLE I
Parameters of the considered PTV-SC-LDPC codes, all with $a = d_c = 4$, $c = d_v = 3$, and $g = 6$.

Code	T	m_s	ν_s	$d_{\mathcal{C},T}$
$\mathcal{C}_1$	1	150	604	24
$\mathcal{C}_2$	2	142	572	116
$\mathcal{C}_3$	2	144	580	48
$\mathcal{C}_4$	2	143	576	56
$\mathcal{C}_5$	3	138	556	214
$\mathcal{C}_6$	2	544	2180	114

The results of Monte Carlo simulations are shown in Fig. 4. We notice that the PTV codes with period $T = 2$ and $T = 3$ significantly outperform the time invariant one. For example, at $\mathrm{BER} = 10^{-4}$, $\mathcal{C}_2$ shows a 2.22 dB gain with respect to $\mathcal{C}_1$; the gain becomes even larger for $\mathcal{C}_5$, which shows a 2.47 dB gain with respect to the time invariant code $\mathcal{C}_1$. We also notice in Fig. 4 that the performance of $\mathcal{C}_3$ and $\mathcal{C}_4$ follows the trend of the upper bounds on the free distance. Even though this is not to be intended as a general rule, it hints that the maximization

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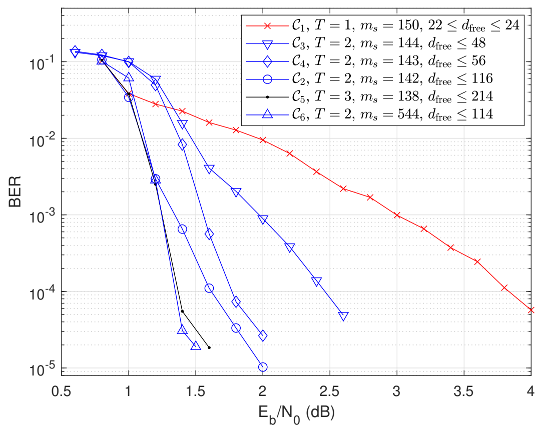

Fig. 4. Bit error rate vs signal-to-noise ratio, for codes with different period and memory.

of the proposed upper bounds can be considered as a broad design method. We remark that, despite the fact that $\mathcal{C}_3$ and $\mathcal{C}_4$ perform worse than $\mathcal{C}_2$, they still achieve a noticeable gain with respect to $\mathcal{C}_1$. We note that all the previously mentioned codes have the same rate ($R = \frac{1}{4}$) and comparable memory and syndrome former constraint length. Instead, regarding the code with larger memory, namely $\mathcal{C}_6$, we note that it exhibits some gain with respect to all the other codes, which however is achieved at the cost of more than tripling the syndrome former constraint length. This hints that, even if we further increase the memory, without increasing the period, the corresponding codes will not exhibit a significant gain.

For the code $\mathcal{C}_1$, we were able to collect a sufficiently large number of error patterns (more than 1000) causing decoding failures, corresponding to codewords. 90% of these error patterns are low-weight codewords of weight 24, i.e., the largest free distance achievable by any TI-SC-LDPC code in the considered ensemble. The remaining 10% are codewords of weight larger than or equal to 44. On the other hand, there are no codewords of $\mathcal{C}_2$ and $\mathcal{C}_5$[4] in the decoding error patterns, which is a clear hint that their free distance is relatively large (as discussed in the concluding paragraph of Section II) and probably close to the upper bounds on the code free distance. Despite the fact that other harmful objects, such as various types of trapping sets, are often the main responsible ones of decoding failures, especially in the error-floor region (i.e., for relatively large values of the signal-to-noise ratio $\frac{E_b}{N_0}$), it was expected (and is evident also from Fig. 4) that such a potential increase in the free distance of these codes would play a critical role on their error rate performance.

VI. Conclusion

We have studied the free distance of PTV-SC-LDPC codes, introducing some new bounds and showing that they can theoretically achieve much better distance properties than their time-invariant counterparts, even for very small periods. Numerical results show that the increase in the free distance is not only theoretical and has a dramatic impact on the error rate performance of PTV-SC-LDPC codes, which turns out to be significantly better than that of TI-SC-LDPC codes having the same parameters, even for very small periods.

As a suggestion for future work, supported by the results in Fig. 4, we foresee that the obtained upper bounds may be used to design specific PTV-SC-LDPC codes with small period having good free distance and error correction performance. A similar approach was used to obtain good QC-LDPC block codes in [30], [31], where the maximization of the upper bound on the ensemble free distance is used as a design criterion. Owing to the connection between QC-LDPC and SC-LDPC codes, such a design method could be easily extended.

[4]As mentioned above, some codewords of the QC counterparts of $\mathcal{C}_2$ and $\mathcal{C}_6$ were found by means of the tool [29], but their weight is larger than the upper bound on the code free distance and, therefore, they do not help in the estimation of the latter.

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