Dynamic Random Geometric Graphs*

J. Díaz[1] D. Mitsche[2] X. Pérez[1]

[1]Llenguatges i Sistemes Informàtics, UPC, 08034 Barcelona
[2]Institut für Theoretische Informatik, ETH Zürich, 8092 Zürich
{diaz,xperez}@lsi.upc.edu, dmitsche@inf.ethz.ch

Abstract

In this work we introduce *Dynamic Random Geometric Graphs* as a basic rough model for mobile wireless sensor networks, where communication distances are set to the known threshold for connectivity of static random geometric graphs. We provide precise asymptotic results for the expected length of the connectivity and disconnectivity periods of the network. We believe the formal tools developed in this work could be of use in future studies in more concrete settings. In addition, for static random geometric graphs at the threshold for connectivity, we provide asymptotic expressions on the probability of existence of components according to their sizes.

1 Introduction

Random Geometric graphs (RGG) have been for a long time used as a model of large autonomous networks, as sensor networks, where the network agents are represented by the vertices of the RGG, and the direct connectivity of agents is represented by the edges (see for example [Gil, MKPS, ASSC, HGPC, RMM]). A random geometric graph is a graph resulting from placing a set of n vertices (agents) uniformly at random and independently on the unit torus $[0, 1)^2$, and connecting two vertices if and only if their *distance* is at most the given radius r, the distance depending on the type of metric being used. As we survey in Section 2, quite a lot of work has been done on *static* RGG, whose results can mostly be found in the book by M. D. Penrose [Pen03].

Currently, there has been an increasing interest for MANETs (mobile ad-hoc networks). Quite a bit of work has been done from the empirical point of view. In [Camp] and in [JBAS] the authors propose several models where connections in the network are created and destroyed as the agents move further apart or closer together. Moreover, they do empirical studies on network topology and routing performance.

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The paper [GHSZ] also deals with the problem of maintaining connectivity of mobile agents communicating by radio frequency, but from an orthogonal perspective to the one in the present paper. It describes a *kinetic data structure* to maintain the connected components of the union of unit-radius disks moving in the plane.

In the present paper, we consider the study of the connectivity of RGG, as the agents move randomly around the unit torus $[0,1)^2$. In particular, starting from a given random geometric graph, where the radius r is the known threshold for connectivity on RGG (see Section 2), each vertex chooses independently and uniformly at random an angle $\alpha \in [0, 2\pi)$, and moves a distance s in that direction for a period of m steps. Then a new angle is selected independently for each vertex, and the process repeats. In the literature, this model is denoted as the *Random Direction Mobility* model. Our main result is that we provide precise asymptotic results for the expected number of steps that the graph remains connected once it becomes connected, and the expected number of steps the graph remains disconnected once it becomes disconnected. These expressions will be given as a function of the number of vertices n, the number of steps in the same direction m (which we consider as an arbitrary but fixed natural number) and the step size s. As it will be indicated in Section 3, the proof techniques will be different for different sizes s of basic step. In addition, for the static model of RGG where the radius r is the known threshold for connectivity on RGG, we provide asymptotic bounds on the probability of occurrence of components according to their sizes. All the computations were made using the usual Euclidean distance in the torus, but we believe that similar results can be obtained for any ℓ_p-norm, $1 \leq p \leq \infty$. Moreover, our results also can be extended to any d-dimensional unit torus of bounded dimension.

To the best of our knowledge, the present work is the first work in which the dynamic connectivity of RGG is studied theoretically. In [DPSW] the loosely related problem of the connectivity of the ad-hoc graph produced by w agents moving randomly along the edges of a $n \times n$ grid is studied. In [NTLL] the authors use a similar model to the one used in the present paper to prove that if the vertices are initially distributed uniformly at random, the distribution remains uniform at any time. On the other hand, the number of components in $\mathcal{G}(n,r)$ isomorphic to a fixed graph (and thus the probability of finding components in $\mathcal{G}(n,r)$ of a given size) is extensively studied in [Pen03]. However the range of r covered there does not exceed $\Theta(\sqrt{1/n})$, below the connectivity threshold treated in the present paper. In fact, a percolation argument in [Pen03] shows that with probability $1 - o(1)$ no components other than isolated vertices and the giant one exist at the connectivity threshold, whithout giving accurate bounds on this probability.

2 Static Properties

In this section, we survey some of the known results about static RGG, which will be the starting point to derive our results. Although the threshold for connectivity of RGG has a long and exciting history, due to lack of space, we only refer to [Pen03].

Given a set V of n agents and a positive real $r = r(n)$, each agent is placed at some random position in the unit torus $[0,1)^2$ selected independently and uniformly at random (u.a.r.). We define $\mathcal{G}(n,r)$ as the random graph having V as the vertex set, and with an edge connecting each pair of vertices u and v at distance $d(u,v) \leq r$, where $d(\cdot,\cdot)$ denotes the Euclidean distance in the torus. Unless otherwise stated, all our stated results are asymptotic as $n \to \infty$. As usual, the abbreviation a.a.s. stands for *asymptotically almost surely*, i.e., with probability $1 - o(1)$. We assume hereinafter that $r = o(1)$. Otherwise an easy Balls and Bins argument shows that $\mathcal{G}(n,r)$ is trivially a.a.s. connected.

Let X be the random variable counting the number of isolated vertices in $\mathcal{G}(n,r)$. Then, by multiplying the probability that one vertex is isolated by the number of vertices we obtain

Lemma 1. $\mathbf{E}(X) = n(1-\pi r^2)^{n-1} = ne^{-\pi r^2 n - O(r^4 n)}$.

Define $\mu = ne^{-\pi r^2 n}$. Observe from Lemma 1 that this parameter μ is closely related to $\mathbf{E}X$. In fact, $\mu = o(1)$ iff $\mathbf{E}X = o(1)$, and if $\mu = \Omega(1)$ then $\mathbf{E}X \sim \mu$. Moreover the asymptotic behaviour of μ characterizes the connectivity of $\mathcal{G}(n,r)$. In fact (see [Pen97],[Pen99]),

Theorem 2.

- *If $\mu \to 0$, then a.a.s. $\mathcal{G}(n,r)$ is connected.*
- *If $\mu = \Theta(1)$, then a.a.s. $\mathcal{G}(n,r)$ consists of one giant component of size $> n/2$ and a Poisson number (with parameter μ) of isolated vertices.*
- *If $\mu \to \infty$, then a.a.s. $\mathcal{G}(n,r)$ is disconnected.*

From the definition of μ we deduce that $\mu = \Theta(1)$ iff $r = \sqrt{\frac{\ln n \pm O(1)}{\pi n}}$. Therefore as a weaker consequence we conclude that the property of connectivity of $\mathcal{G}(n,r)$ exhibits a sharp threshold at $r = \sqrt{\frac{\ln n}{\pi n}}$. Theorem 2 also implies that, if $\mu = \Theta(1)$, the components of size 1 (i.e. isolated vertices) are predominant and have the main contribution to the connectivity of $\mathcal{G}(n,r)$. In fact if $\mathcal{C}$ (respectively $\mathcal{D}$) denotes the event that $\mathcal{G}(n,r)$ is connected (respectively disconnected), we have the following

Corollary 3. *Assume that $\mu = \Theta(1)$. Then*

$$\mathbf{Pr}(\mathcal{C}) \sim \mathbf{Pr}(X = 0) \sim e^{-\mu}, \qquad \mathbf{Pr}(\mathcal{D}) \sim \mathbf{Pr}(X > 0) \sim 1 - e^{-\mu}.$$

Observe that we used the fact that, if $\mu = \Theta(1)$, the probability that $\mathcal{G}(n,r)$ has some component of size greater than 1 other than the giant component is $o(1)$. We give more accurate bounds on this probability. Moreover we characterize the probability of having components of any fixed size. Before stating this more precisely we need some definitions.

Given a component C of $\mathcal{G}(n,r)$, we say that C is *embeddable* if it is contained in some square Q with sides parallel to the axes of the torus and length $1 - 2r$. In other words, C is embeddable if it can be mapped into the square $[r, 1-r]^2$ by a translation in the torus. Embeddable components do not wrap around the torus. Througout the paper and often without explicitly mencioning it, we assume in all geometrical descriptions involving an embeddable component C that C is contained in $[r, 1-r]^2$ and regard the torus $[0,1)^2$ as the unit square. Hence terms as "left", "right", "above" and "below" are globally defined. On the other hand, components which are not embeddable must have large size (at least $\Omega(1/r)$). Note that sometimes several non-embeddable components can coexist together. However, there are some non-embeddable components which are so spread around the torus that do not allow any room for other non-embeddable ones. We call these components *solitary*, and by definition we can have at most one solitary component. We cannot disprove the existence of this solitary component, since with probability $1 - o(1)$ there exists one giant component of this nature. For components which are not solitary (i.e., either embeddable or non-embeddable but able to coexist with other non-embeddable ones), we give asymptotic bounds on the probability of their existence according to their size.

Given a fixed integer $i \geq 1$, let Z_i be the number of components in $\mathcal{G}(n,r)$ of size exactly i. For large enough n, we can assume these to be embeddable, since $r = o(1)$. Moreover, for any fixed $\epsilon > 0$, let $Z'_{\epsilon,i}$ be the number of components of size exactly i which have all their vertices at distance at most ϵr from their leftmost one. Finally, $\widetilde{Z}_i$ denotes the number of components of size at least i and which are not solitary. For simplicity, we often denote by $X = Z_1$ the number of isolated vertices and by $\widetilde{X} = \widetilde{Z}_2$ the number of non-solitary components other than isolated vertices. Notice that $Z'_{\epsilon,i} \leq Z_i \leq \widetilde{Z}_i$. However, in the following results we show that asymptotically all the weight in the probability that $\widetilde{Z}_i > 0$ comes from components which also contribute to $Z'_{\epsilon,i}$. This means that the more common components of size at least i are those ones of size exactly i with all their vertices close together.

Lemma 4. *Let $i \geq 2$ be a fixed integer, and $0 < \epsilon < 1/2$ also fixed. Then,*

$$\mathbf{E}Z'_{\epsilon,i} = \Theta(1/\log^{i-1} n)$$

Proof. First observe that with probability 1, for each component C which contributes to $Z'_{\epsilon,i}$, C has a unique leftmost vertex u and the vertex v in C at greatest distance from u is also unique. Hence, we can restrict our attention to this case.

Fix an arbitrary set $V_1 \subset V$ of i vertices, with two distinguished ones u and v. Let $\mathcal{E}$ be the following event: All vertices in V_1 are at distance at most ϵr from u and to the right of u; vertex v is the one in V_1 with greatest distance from u; and the vertices of V_1 form a component of $\mathcal{G}(n,r)$. If $\mathbf{Pr}(\mathcal{E})$ is multiplied by the number of possible choices of u, v and the remaining $i-2$ vertices of V_1, we get

$$\mathbf{E}Z'_{\epsilon,i} = n(n-1)\binom{n-2}{i-2}\mathbf{Pr}(\mathcal{E}). \tag{1}$$

In order to bound the probability of $\mathcal{E}$ we need some definitions. Let $\rho = d(u,v)$ and let $\mathcal{S}$ be the set of all points in the torus $[0,1)^2$ which are at distance at most r

from some vertex in V_1. (Notice that ρ and $\mathcal{S}$ depend on the random position of the vertices in V_1). We first need bounds of the area of $\mathcal{S}$ in terms of ρ. Observe that $\mathcal{S}$ is contained in the circle of radius $r+\rho$ and center u, and then

$$|\mathcal{S}| \leq \pi(r+\rho)^2. \tag{2}$$

Now let $u_l = u$, u_r, u_t and u_b be respectively the leftmost, rightmost, topmost and bottommost vertices in V_1 (some of these vertices possibly equal). Assume w.l.o.g. that the vertical length of V_1 (i.e., the vertical distance between u_t and u_b) is at least $\rho/\sqrt{2}$. Otherwise, the horizontal length of V_1 has this property and we can rotate the descriptions in the argument. The upper halfcircle with center u_t and the lower halfcircle with center u_b are disjoint and are contained in $\mathcal{S}$. If u_r is at greater vertical distance from u_t than from u_b, then consider the rectangle of height $\rho/(2\sqrt{2})$ and width $r - \rho/(2\sqrt{2})$ with one corner on u_r and above and to the right of u_r. Otherwise, consider the same rectangle below and to the right of u_r. This rectangle is also contained in $\mathcal{S}$ and its interior does not intersect the previously described halfcircles. Analogously, we can find another rectangle of height $\rho/(2\sqrt{2})$ and width $r - \rho/(2\sqrt{2})$ to the left of u_l and either above or below u_l with the same properties. Hence,

$$|\mathcal{S}| \geq \pi r^2 + 2\left(\frac{\rho}{2\sqrt{2}}\right)\left(r - \frac{\rho}{2\sqrt{2}}\right). \tag{3}$$

From (2), (3) and the fact that $\rho < r/2$, we can write

$$\pi r^2\left(1 + \frac{1}{6}\frac{\rho}{r}\right) < |\mathcal{S}| < \pi r^2\left(1 + \frac{5}{2}\frac{\rho}{r}\right) < \frac{9\pi}{4}r^2. \tag{4}$$

Now consider the probability P that the $n-i$ vertices not in V_1 lie outside $\mathcal{S}$. Clearly $P = (1-|\mathcal{S}|)^{n-i}$. Moreover, by (4) and using the fact that $e^{-x-x^2} \leq 1-x \leq e^{-x}$ for all $x \in [0, 1/2]$, we obtain

$$e^{-(1+5\rho/(2r))\pi r^2 n - (9\pi r^2/4)^2 n} < P < \frac{e^{-(1+\rho/(6r))\pi r^2 n}}{(1-9\pi r^2/4)^i},$$

and after a few manipulations

$$\left(\frac{\mu}{n}\right)^{1+5\rho/(2r)} e^{-(9\pi r^2/4)^2 n} < P < \left(\frac{\mu}{n}\right)^{1+\rho/(6r)} \frac{1}{(1-9\pi r^2/4)^i}. \tag{5}$$

Event $\mathcal{E}$ can also be described as follows: There is some nonnegative real $\rho \leq \epsilon r$ such that v is placed at distance ρ from u and to the right of u; all the remaining vertices in V_1 are deployed inside the halfcircle of center u and radius ρ; and the $n-i$ vertices not in V_1 lie outside $\mathcal{S}$. Hence, $\mathbf{Pr}(\mathcal{E})$ can be upper (lower) bounded by integrating with respect to ρ the probability density function of $d(u,v)$ times the probability that the remaining $i-2$ selected vertices lie inside the right halfcircle of center u and radius ρ times the upper (lower) bound on P we obtained in (5):

$$\Theta(1)\, I(5/2) \leq \mathbf{Pr}(\mathcal{E}) \leq \Theta(1)\, I(1/6), \tag{6}$$

where

$$
\begin{aligned}
I(\beta) &= \int_0^{\epsilon r} \pi\rho \left(\frac{\pi}{2}\rho^2\right)^{i-2} \frac{1}{n^{1+\beta\rho/r}}\, d\rho. \\
&= \frac{2}{n}\left(\frac{\pi}{2}r^2\right)^{i-1} \int_0^{\epsilon} x^{2i-3} n^{-\beta x}\, dx
\end{aligned} \tag{7}
$$

Since i is fixed, for $\beta = 5/2$ or $\beta = 1/6$,

$$
\begin{aligned}
I(\beta) &= \Theta\left(\frac{\log^{i-1} n}{n^i}\right) \int_0^{\epsilon} x^{2i-3} n^{-\beta x}\, dx \\
&= \Theta\left(\frac{\log^{i-1} n}{n^i}\right) \frac{(2i-3)!}{(\beta \log n)^{2i-2}} \\
&= \Theta\left(\frac{1}{n^i \log^{i-1} n}\right),
\end{aligned} \tag{8}
$$

and the statement follows from (1), (6) and (8). □

Lemma 5. *Let $i \geq 2$ be a fixed integer. Let $\epsilon > 0$ be also fixed. Then*

$$\mathbf{Pr}(\widetilde{Z}_i - Z'_{\epsilon,i} > 0) = O(1/\log^i n)$$

Proof. We assume throughout this proof that $\epsilon \leq 10^{-18}$, and prove the claim for this case. The case $\epsilon > 10^{-18}$ follows from the fact that $(\widetilde{Z}_i - Z'_{\epsilon,i}) \leq (\widetilde{Z}_i - Z'_{10^{-18},i})$.

Consider all components in $\mathcal{G}(n, r)$ which are not solitary. Remove from these components the ones of size at most i and diameter at most ϵr, and denote by Y the number of remaining components. By construction $\widetilde{Z}_i - Z'_{\epsilon,i} \leq Y$, and therefore it is sufficient to prove that $\mathbf{Pr}(Y > 0) = O(1/\log^i n)$. The components counted by Y are classified into several types according to their size and diameter. We deal with each type separately.

Part 1. Consider all the components in $\mathcal{G}(n, r)$ which have diameter at most ϵr and size between $i + 1$ and $\log n/37$. Call them components of type 1, and let Y_1 denote their number.

For each ℓ, $i + 1 \leq \ell \leq \log n/37$, let E_ℓ be the expected number of components of type 1 and size ℓ. We observe that these components have all of their vertices at distance at most ϵr from the leftmost one. Therefore, we can apply the same argument we used for bounding $\mathbf{E}Z'_{\epsilon,i}$ in the proof of Lemma 4. Note that (1), (6) and (7) are also valid for sizes not fixed but depending on n. Thus we obtain

$$E_\ell \leq O(1) n(n-1)\binom{n-2}{\ell-2} I(1/6),$$

where $I(1/6)$ is defined in (7). We use the fact that $\binom{n-2}{\ell-2} \leq (\frac{ne}{\ell-2})^{\ell-2}$ and get

$$E_\ell = O(1) \log n \left(\frac{e}{2}\frac{\log n}{\ell - 2}\right)^{\ell-2} \int_0^{\epsilon} x^{2\ell-3} n^{-x/6}\, dx \tag{9}$$

The expression $x^{2\ell-3}n^{-x/6}$ can be maximised for $x \in \mathbb{R}^+$ by elementary techniques, and we deduce that

$$x^{2\ell-3}n^{-x/6} \leq \left(\frac{2\ell-3}{(e/6)\log n}\right)^{2\ell-3}.$$

Then we can bound the integral in (9) and get

$$\begin{aligned} E_\ell &= O(1)\log n\left(\frac{e}{2}\frac{\log n}{\ell-2}\right)^{\ell-2}\epsilon\left(\frac{2\ell-3}{(e/6)\log n}\right)^{2\ell-3} \\ &= O(1)\left(\frac{36}{2e}\frac{(2\ell-3)^2}{(\ell-2)\log n}\right)^{\ell-2}\ell \end{aligned}$$

Note that for $\ell \leq \log n/37$ the expression $\left(\frac{36}{2e}\frac{(2\ell-3)^2}{(\ell-2)\log n}\right)^{\ell-2}\ell$ is decreasing with ℓ. Hence we can write

$$E_\ell = O\left(\frac{1}{\log^{i+1} n}\right), \qquad \forall \ell\ :\ i+3 \leq \ell \leq \frac{1}{37}\log n.$$

Moreover the bounds $E_{i+1} = O(1/\log^i n)$ and $E_{i+2} = O(1/\log^{i+1} n)$ are obtained from Lemma 4, and hence

$$\mathbf{E}Y_1 = \sum_{\ell=i+1}^{\frac{1}{37}\log n} E_\ell = O\left(\frac{1}{\log^i n}\right) + O\left(\frac{1}{\log^{i+1} n}\right) + \frac{\log n}{37}O\left(\frac{1}{\log^{i+1} n}\right) = O\left(\frac{1}{\log^i n}\right),$$

and then $\mathbf{Pr}(Y_1 > 0) \leq \mathbf{E}Y_1 = O(1/\log^i n)$.

Part 2. Consider all the components in $\mathcal{G}(n,r)$ which have diameter at most ϵr and size greater than $\log n/37$. Call them components of type 2, and let Y_2 denote their number.

We tessellate the torus with square cells of side $y = \lfloor(\epsilon r)^{-1}\rfloor^{-1}$ ($y \geq \epsilon r$ but also $y \sim \epsilon r$). We define a box to be a square of side $2y$ consisting of the union of 4 cells of the tessellation. Consider the set of all possible boxes. Note that any component of type 2 must be fully contained in some box.

Let us fix a box b. Let W be the number of vertices which are deployed inside b. Clearly W has a binomial distribution with mean $\mathbf{E}W = (2y)^2 n \sim (2\epsilon)^2 \log n/\pi$. By setting $\delta = \frac{\log n}{37\mathbf{E}W} - 1$ and applying Chernoff inequality to W, we have

$$\mathbf{Pr}(W > \frac{1}{37}\log n) = \mathbf{Pr}(W > (1+\delta)\mathbf{E}W) \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^{\mathbf{E}W} = n^{-\frac{(\log(1+\delta)-\frac{\delta}{1+\delta})}{37}}.$$

Note that $\delta \sim \frac{\pi}{148\epsilon^2} - 1 > e^{79}$, and therefore

$$\mathbf{Pr}(W > \frac{1}{37}\log n) < n^{-2.1}.$$

Then taking a union bound over the set of all $\Theta(r^{-1}) = \Theta(n/\log n)$ boxes, the probability that there is some box with more than $\frac{1}{37}\log n$ vertices is $O(1/(n^{1.1}\log n))$. Then since each component of type 2 is contained in some box, we have

$$\mathbf{Pr}(Y_2 > 0) = O(1/(n^{1.1}\log n)).$$

Part 3. Consider all the components in $\mathcal{G}(n,r)$ which are embeddable and have diameter at least ϵr. Call them components of type 3, and let Y_3 denote their number.

We tessellate the torus into square cells of side αr, for some $\alpha = \alpha(\epsilon) > 0$ fixed but small enough. Let C be a component of type 3. Let $\mathcal{S} = \mathcal{S}_C$ be the set of all points in the torus $[0,1)^2$ which are at distance at most r from some vertex in C. Remove from $\mathcal{S}$ the vertices of C and the edges (represented by straight segments) and denote by $\mathcal{S}'$ the outer connected topologic component of the remaining set. By construction, $\mathcal{S}'$ must contain no vertex.

Now let $u_l = u$, u_r, u_t and u_b be respectively the leftmost, rightmost, topmost and bottommost vertices in C (some of these vertices possibly equal). Assume w.l.o.g. that the vertical length of C (i.e., the vertical distance between u_t and u_b) is at least $\epsilon r/\sqrt{2}$. Otherwise, the horizontal length of C has this property and we can rotate the descriptions in the argument. The upper halfcircle with center u_t and the lower halfcircle with center u_b are disjoint and are contained in $\mathcal{S}'$. If u_r is at greater vertical distance from u_t than from u_b, then consider the rectangle of height $\epsilon r/(2\sqrt{2})$ and width $r - \epsilon r/(2\sqrt{2})$ with one corner on u_r and above and to the right of u_r. Otherwise, consider the same rectangle below and to the right of u_r. This rectangle is also contained in $\mathcal{S}'$ and its interior does not intersect the previously described halfcircles. Analogously, we can find another rectangle of height $\epsilon r/(2\sqrt{2})$ and width $r - \epsilon r/(2\sqrt{2})$ to the left of u_l and either above or below u_l with the same properties. Hence, taking into account that $\epsilon \leq 10^{-18}$, we have

$$|\mathcal{S}'| \geq \pi r^2 + 2\left(\frac{\epsilon r}{2\sqrt{2}}\right)\left(r - \frac{\epsilon r}{2\sqrt{2}}\right) > \left(1 + \frac{\epsilon}{5}\right)\pi r^2. \tag{10}$$

Let $\mathcal{S}^*$ be the union of all the cells in the tessellation which are fully contained in $\mathcal{S}'$. We loose a bit of area compared to $\mathcal{S}'$. However, if α was chosen small enough, we can guarantee that $\mathcal{S}^*$ is topologically connected and has area $|\mathcal{S}^*| \geq (1+\epsilon/6)\pi r^2$. This α can be chosen to be the same for all components of type 3.

Hence, we showed that the event $(Y_3 > 0)$ implies that some connected union of cells $\mathcal{S}^*$ of area $|\mathcal{S}^*| \geq (1+\epsilon/6)\pi r^2$ contains no vertices. By removing some cells from $\mathcal{S}^*$, we can assume that $(1+\epsilon/6)\pi r^2 \leq |\mathcal{S}^*| < (1+\epsilon/6)\pi r^2 + \alpha^2 r^2$. Let $\mathcal{S}^*$ be any union of cells with these properties. (Note that there are $\Theta(1/r^2) = \Theta(n/\log n)$ many possible choices for $\mathcal{S}^*$.) The probability that $\mathcal{S}^*$ contains no vertices is

$$(1 - |\mathcal{S}^*|)^n \leq e^{-(1+\epsilon/6)\pi r^2 n} = \left(\frac{\mu}{n}\right)^{1+\epsilon/6}.$$

Therefore, we can take the union bound over all the $\Theta(n/\log n)$ possible $\mathcal{S}^*$, and obtain an upper bound of the probability that there is some component of the type 3:

$$\mathbf{Pr}(Y_3 > 0) \leq \Theta\left(\frac{n}{\log n}\right)\left(\frac{\mu}{n}\right)^{1+\epsilon/6} = \Theta\left(\frac{1}{n^{\epsilon/6}\log n}\right).$$

Part 4. Consider all the components in $\mathcal{G}(n,r)$ which are not embeddable but not solitary either. Call them components of type 4, and let Y_4 denote their number.

We tessellate the torus $[0,1)^2$ into $\Theta(n/\ln n)$ small square cells of side length αr, where $\alpha > 0$ is a sufficiently small positive constant.

Let C be a component of type 4. Let $\mathcal{S} = \mathcal{S}_C$ be the set of all points in the torus $[0,1)^2$ which are at distance at most r from some vertex in C. Remove from $\mathcal{S}$ the vertices of C and the edges (represented by straight segments) and denote by $\mathcal{S}'$ the remaining set. By construction, $\mathcal{S}'$ must contain no vertex.

Suppose there is a horizontal or a vertical band of width $2r$ in $[0,1)^2$ which does not intersect the component C (assume w.l.o.g. that it is the topmost horizontal band consisting of all points with the y-coordinate in $[1-2r,1)$). Let us divide the torus into vertical bands of width $2r$. All of them must contain at least one vertex of C, since otherwise C would be embeddable. Select any 9 consecutive vertical bands and pick one vertex of C with maximal y-coordinate in each one. For each one of these 9 vertices, we select the left upper quartercircle centered at the vertex if the vertex is closer to the right side of the band or the right upper quartercircle otherwise. These nine quartercircles we chose are disjoint and must contain no vertices by construction. Moreover, they belong to the same connected component of the set $\mathcal{S}'$, which we denote by $\mathcal{S}''$, and which has area $|\mathcal{S}''| \geq (9/4)\pi r^2$. Let $\mathcal{S}^*$ be the union of all the cells in the tessellation of the torus which are completely contained in $\mathcal{S}''$. We loose a bit of area compared to $\mathcal{S}''$. However, as usual, by choosing α small enough we can guarantee that $\mathcal{S}^*$ is connected and it has an area $|\mathcal{S}^*| \geq (11/5)\pi r^2$. Note that this α can be chosen to be the same for all components C of this kind.

Suppose otherwise that all horizontal and vertical bands of width $2r$ in $[0,1)^2$ contain at least one vertex of C. Since C is not solitary it must be possible that it coexists with some other non-embeddable component C'. Then all vertical bands or all horizontal bands of width $2r$ must also contain some vertex of C' (assume w.l.o.g. the vertical bands do). Let us divide the torus into vertical bands of width $2r$. We can find a simple path Π with vertices in C' which passes through 11 consecutive bands. For each one of the 9 internal bands, pick the uppermost vertex of C in the band below Π (in the torus sense). As before each one of these vertices contributes with a disjoint quartercircle which must be empty of vertices, and by the same argument we obtain a connected union of cells of the tessellation, which we denote by $\mathcal{S}^*$, with area $|\mathcal{S}^*| \geq (11/5)\pi r^2$ and containing no vertices.

Hence, we showed that the event $(Y_4 > 0)$ implies that some connected union of cells $\mathcal{S}^*$ of area $|\mathcal{S}^*| \geq (11/5)\pi r^2$ contains no vertices. By repeating the same argument we used for components of type 3 but replacing $(1+\epsilon/6)\pi r^2$ by $(11/5)\pi r^2$, we get

$$\mathbf{Pr}(Y_4 > 0) = \Theta\left(\frac{1}{n^{6/5}\log n}\right).$$

□

Lemma 6. *Let $i \geq 2$ be a fixed integer. Let $0 < \epsilon < 1/2$ be fixed.*

$$\mathbf{E}[Z'_{\epsilon,i}]_2 = O(1/\log^{2i-2} n)$$

Proof. As in the proof of Lemma 4, we assume that each component C which contributes to $Z'_{\epsilon,i}$ has a unique leftmost vertex u, and the vertex v in C at greatest distance from u is also unique. In fact, this happens with probability 1.

Choose any two disjoint sets of vertices of size i each, namely V_1 and V_2, with four distinguished vertices $u_1, v_1 \in V_1$ and $u_2, v_2 \in V_2$. Let $\mathcal{E}$ be the event that the following conditions hold for both $j = 1$ and $j = 2$: All vertices in V_j are at distance at most ϵr from u_j and to the right of u_j; vertex v_j is the one in V_j with greatest distance from u_j; and the vertices of V_j form a component of $\mathcal{G}(n, r)$. If $\mathbf{Pr}(\mathcal{E})$ is multiplied by the number of possible choices of u_j, v_j and the remaining vertices of V_j, we get

$$\mathbf{E}[Z'_{\epsilon,i}]_2 = O(n^{2i})\mathbf{Pr}(\mathcal{E}). \tag{11}$$

In order to bound the probability of $\mathcal{E}$ we need some definitions. For each $j \in \{1, 2\}$, let $\rho_j = d(u_j, v_j)$ and let $\mathcal{S}_j$ be the set of all the points in the torus $[0, 1)^2$ which are at distance at most r from some vertex in V_j. (Obviously ρ_j and $\mathcal{S}_j$ depend on the random position of the vertices in V_j.) Also define $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$.

Let $\mathcal{F}$ be the event that $d(u_1, u_2) > 3r$. This holds with probability $1 - O(r^2)$. In order to bound $\mathbf{Pr}(\mathcal{E} \mid \mathcal{F})$, we apply a similar approach to the one in the proof of Lemma 4. In fact, observe that if $\mathcal{F}$ holds then $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$. Therefore in view of (4) we can write

$$\pi r^2(2 + (\rho_1 + \rho_2)/(6r)) < |\mathcal{S}| < \frac{18\pi}{4} r^2, \tag{12}$$

and using the same elementary techniques that gave us (5) we get

$$(1 - |\mathcal{S}|)^{n-2i} < \left(\frac{\mu}{n}\right)^{2+(\rho_1+\rho_2)/(6r)} \frac{1}{(1 - 18\pi r^2/4)^{2i}}. \tag{13}$$

Now observe that $\mathcal{E}$ can also be described as follows: For each $j \in \{1, 2\}$ there is some nonnegative real $\rho_j \le \epsilon r$ such that v_j is placed at distance ρ_j from u_j and to the right of u_j; all the remaining vertices in V_j are deployed inside the halfcircle of center u_j and radius ρ_j; and the $n - i$ vertices not in V_j lie outside $\mathcal{S}_j$. In fact, rather than this last condition we only require for our bound that all vertices in $V \setminus (V_1 \cup V_2)$ are placed outside $\mathcal{S}$. Clearly, this has probability $(1 - |\mathcal{S}|)^{n-2i}$. Then, from (13) and following an analogous argument to the one that leads to (6), we obtain the bound

$$\begin{aligned}\mathbf{Pr}(\mathcal{E} \mid \mathcal{F}) &\le \Theta(1) \int_0^{\epsilon r}\int_0^{\epsilon r} \pi\rho_1 \left(\frac{\pi}{2}\rho_1^2\right)^{i-2} \pi\rho_2 \left(\frac{\pi}{2}\rho_2^2\right)^{i-2} \frac{1}{n^{2+(\rho_1+\rho_2)/(6r)}}\, d\rho_1 d\rho_2 \\ &= \Theta(1)\, I(1/6)^2,\end{aligned}$$

where $I(1/6)$ is defined in (7). Thus from (8) we conclude

$$\mathbf{Pr}(\mathcal{E} \wedge \mathcal{F}) \le \Theta(1)\, P(\mathcal{F})\, I(1/6)^2 = O\left(\frac{1}{n^{2i}\log^{2i-2} n}\right). \tag{14}$$

Otherwise, suppose that $\mathcal{F}$ does not hold (i.e., $d(u_1, u_2) \le 3r$). Observe that $\mathcal{E}$ implies that $d(u_1, u_2) > r$, since u_1 and u_2 must belong to different components.

Hence the circles with centers on u_1 and u_2 and radius r have an intersection of area less than $(\pi/2)r^2$. These two circles are contained in $\mathcal{S}$ and then we can write $|\mathcal{S}| \geq (3/2)\pi r^2$. Note that $\mathcal{E}$ implies that all vertices in $V \setminus (V_1 \cup V_2)$ are placed outside $\mathcal{S}$ and that for each $j \in \{1,2\}$ all the vertices in $V_j \setminus \{u_j\}$ are at distance at most ϵr and to the right of u_j. This gives us the following rough bound

$$\mathbf{Pr}(\mathcal{E} \mid \overline{\mathcal{F}}) \leq \left(\frac{\pi}{2}(\epsilon r)^2\right)^{2i-2} \left(1 - \frac{3\pi}{2}r^2\right)^{n-2i} = O(1)\left(\frac{\log n}{n}\right)^{2i-2}\left(\frac{\mu}{n}\right)^{3/2}.$$

Multiplying this by $\mathbf{Pr}(\overline{\mathcal{F}}) = O(r^2) = O(\log n/n)$ we obtain

$$\mathbf{Pr}(\mathcal{E} \wedge \overline{\mathcal{F}}) = O\left(\frac{\log^{2i-1} n}{n^{2i+1/2}}\right), \tag{15}$$

which is negligible compared to (14). The statement follows from (11), (14) and (15). $\square$

Theorem 7. *Let $i \geq 2$ be a fixed integer. Let $0 < \epsilon < 1/2$ be fixed. Then*

$$\mathbf{Pr}(\widetilde{Z}_i > 0) \sim \mathbf{Pr}(Z_i > 0) \sim \mathbf{Pr}(Z'_{\epsilon,i} > 0) = \Theta\left(\frac{1}{\log^{i-1} n}\right)$$

Proof. From Corollary 1.12 in [Bol], we have

$$\mathbf{E}Z'_{\epsilon,i} - \frac{1}{2}\mathbf{E}[Z'_{\epsilon,i}]_2 \leq \mathbf{Pr}(Z'_{\epsilon,i} > 0) \leq \mathbf{E}Z'_{\epsilon,i},$$

and therefore by Lemmata 4 and 6 we obtain

$$\mathbf{Pr}(Z'_{\epsilon,i} > 0) = \Theta(1/\log^{i-1} n).$$

Combining this and Lemma 5, yields the statement. $\square$

3 Dynamic Properties

We define the dynamic model as follows. Given a positive real $s = s(n)$ and an arbitrary but fixed natural number $m \geq 1$, we consider the following random process $(\Omega_k)_{k\in\mathbb{Z}} = (\Omega_k(n,s,m))_{k\in\mathbb{Z}}$: At step $k = 0$, n agents are scattered independently and u.a.r. over the torus $[0,1)^2$, as in the static model. Moreover each agent chooses u.a.r. an angle $\alpha \in [0, 2\pi)$, and moves in the direction of α, travelling distance s at each step. Angles change every m steps for all agents. More formally, for each agent v and for each interval $[k, k+m]$ with $k \in \mathbb{Z}$ divisible by m, an angle in $[0, 2\pi)$ is chosen independently and u.a.r., and this angle determines the direction of v between steps k and $k+m$. Note that we are also considering negative steps, which is interpreted as if the agents were already moving around the torus ever before step $k = 0$. Furthermore, given a positive $r = r(n) \in \mathbb{R}$ such that $r = o(1)$, a random graph process can be derived from $(\Omega_k)_{k\in\mathbb{Z}}$. For any $k \in \mathbb{Z}$, the vertex set of $\mathcal{G}_k(n,r,s,m)$ is the set of agents, and we join by an edge all pairs of agents which are at Euclidean distance at most r at Ω_k. From now on we always use the term vertex instead of agent to stress that we have the corresponding graph in mind. We derive asymptotic results on $(\mathcal{G}_k(n, r, s, m))_{k\in\mathbb{Z}}$ as $n \to \infty$.

We use the following lemma proven in [NTLL].

Lemma 8. *At any fixed step $k \in \mathbb{Z}$, the vertices are distributed over the torus $[0, 1)^2$ independently and u.a.r. Consequently for any $k \in \mathbb{Z}$, $\mathcal{G}_k(n, r, s, m)$ has the same distribution as $\mathcal{G}(n, r)$.*

In the remaining of the section, we focus our attention around the threshold of connectivity and we assume that $\mu = \Theta(1)$, or equivalently $r = \sqrt{\frac{\ln n \pm O(1)}{\pi n}}$. Most of the arguments in the proofs, require the analysis of $\mathcal{G}_k(n, r, s, m)$ at two consecutive steps k and $k + 1$. There is a convenient way of describing the events involving these two steps: We assign to each vertex v a triple $\phi(v) = (x, y, z) \in [0, 1)^3$ where (x, y) is the position of v at step k and $\alpha = 2\pi z$ is the angle with respect to the horizontal axis of the direction v is travelling. We note that the behaviour of v at steps k and $k+1$ is uniquely determined by $\phi(v)$. From Lemma 8, we observe that for each vertex v, $\phi(v)$ is selected uniformly at random in $[0, 1)^3$ and independently from the other vertices. Often for sake of simplicity, $\phi(v)$ will be written simply as v. Moreover when there is just one step k involved v could also denote the position of v in the torus at step k. The meaning will always be clear from the context.

We define a few useful sets which are used in our argument. Given a vertex v, we define Γ_v to be the set of triples $(x, y, z) \in [0, 1)^3$ such that (x, y) is at distance at most r (in the torus) from the position of v at step k. Similarly, we define Γ'_v as the set of triples $(x, y, z) \in [0, 1)^3$ such that $(x+s\cos(2\pi z), y+s\sin(2\pi z))$ is at distance at most r (in the torus) from the position of v at step $k+1$. For any two vertices u and v, define $d_k(u, v)$ to be the distance (in the torus) between the positions of u and v at step k. Note that $\phi(u) \in \Gamma_v$ iff $d_k(u, v) \le r$, and $\phi(u) \in \Gamma'_v$ iff $d_{k+1}(u, v) \le r$. Moreover, the probability of each of these events is exactly $|\Gamma_v| = |\Gamma'_v| = \pi r^2$. Furthermore, we observe that vertex v is isolated at step k (resp. $k + 1$) iff no other vertex is mapped by ϕ into Γ_v (resp. Γ'_v).

We need the following

Lemma 9. *Assume $\mu = \Theta(1)$. There exists a constant $\epsilon > 0$ such that the following statements are true (for large enough n): For any two vertices u and v (including the case $u = v$),*

1. *if $d_k(u, v) > r$ then $|\Gamma_u \cap \Gamma_v| \le \frac{\pi}{2}r^2$.*
2. *if $s < r/7$ and $d_k(u, v) > r - 2s$ then $|(\Gamma_u \cup \Gamma'_u) \cap (\Gamma_v \cup \Gamma'_v)| \le (1 - \epsilon)\pi r^2$.*
3. *if $s \ge r/7$ and $s = O(r)$ then $|\Gamma_u \cap \Gamma'_v| \le (1 - \epsilon)\pi r^2$.*
4. *if $s = \omega(r)$ then $|\Gamma_u \cap \Gamma'_v| = O(r^3 \frac{s+1}{s}) = o(r^2)$.*

Proof.
Statement 1:

Let O and Q be respectively the positions of u and v at step k. Assume w.l.o.g. that the segment $\overline{OQ}$ is vertical and that O is above Q. Then, let $\mathcal{S}$ be the set of points above O and at distance at most r from O, and $\mathcal{S}' = \mathcal{S} \times [0,1) \subset [0,1)^3$. Clearly, $|\mathcal{S}'| = \pi r^2/2$, $\mathcal{S}' \subset \Gamma_u$ and $\mathcal{S}' \cap \Gamma_v = \emptyset$, and the statement follows.
Statement 2:

Let R and T be respectively the positions of u and v at step $k+1$. The distance between R and T is greater than $3r/7$, since $d_{k+t}(u,v) \geq d_k(u,v) - 2s > r - 4s$. Let $\mathcal{S}_u$ (respectively $\mathcal{S}_v$) be the set of points at distance at most $8r/7$ from R (respectively T) Note that $\mathcal{S}_u$ and $\mathcal{S}_v$ are two circles of radius $8r/7$ with centers at distance greater than $3r/7$. Trivial computations show that the area of $\mathcal{S}_u \cap \mathcal{S}_v$ is at most $(1-\epsilon)\pi r^2$ for some $\epsilon > 0$. We define $\mathcal{S}'_u = \mathcal{S}_u \times [0,1)$ and $\mathcal{S}'_v = \mathcal{S}_v \times [0,1)$. Clearly, $\mathcal{S}'_u \supset \Gamma_u \cup \Gamma'_u$ and $\mathcal{S}'_v \supset \Gamma_v \cup \Gamma'_v$. Hence $|(\Gamma_u \cup \Gamma'_u) \cap (\Gamma_v \cup \Gamma'_v)| \leq |\mathcal{S}'_u \cap \mathcal{S}'_v| = |\mathcal{S}_u \cap \mathcal{S}_v| \leq (1-\epsilon)\pi r^2$.
Statement 3:

Let O be the position of u at time k and T the position of v at time $k+1$. Let w be any other vertex different from u and v. Observe that $|\Gamma_u \setminus \Gamma'_v|$ is the probability that $d_k(u,w) \leq r$ but $d_{k+1}(v,w) > r$. Let Q be the position of w at time k. Suppose that Q is at distance at most r from O (i.e., $d_k(u,w) \leq r$) but greater than $13r/14$ from T. (This happens with probability at least $(1 - 13^2/14^2)\pi r^2$.) Let α be the angle of $\overrightarrow{TQ}$ with respect to the horizontal axis. If w moves between steps k and $k+1$ towards a direction in $[\alpha - \pi/3, \alpha + \pi/3]$, then it increases the distance with respect to T at least $s/2 \geq r/14$ and then $d_{k+1}(v,w) > r/14 + 13r/14 = r$. This range of directions has probability $1/3$. Summarising, we proved that $|\Gamma_u \setminus \Gamma'_v| \geq (1 - 13^2/14^2)\pi r^2/3$, and the statement follows.
Statement 4:

Given a vertex $w \in V$ different from u and v, observe that $|\Gamma_u \cap \Gamma'_v|$ is the probability that $d_k(u,w) \leq r$ and also $d_{k+1}(v,w) \leq r$. Suppose first that $s < 1/2$. We claim that no matter which position has w at step k the probability that $d_{k+1}(u,w) \leq r$ is at most $(2+\epsilon)r/s$ for some $\epsilon > 0$. In fact, let O be the position of w at step k and T the position of v at step $k+1$. We assume $O \neq T$ (the case $O = T$ is trivial) and let α be the angle of $\overrightarrow{OT}$ with respect to the horizontal axis. If w moves between steps k and $k+1$ towards a direction not in $[\alpha - \arcsin(r/s), \alpha + \arcsin(r/s)]$ then $d_{k+1}(w,v) > r$. Hence, $|\Gamma_u \cap \Gamma'_v|$ is at most πr^2 times $(2+\epsilon)r/s$, which satisfies the statement. The case $s \geq 1/2$ is a bit more delicate, since w may loop many times around the torus while moving between steps k and $k+1$. In fact, as we move along the circumference of radius s centered on O we cross the axes of the torus $\Theta(1+s)$ times. This gives the extra factor $(1+s)$ in the statement, which is negligible when $s = o(1)$ but grows large when $s = \omega(1)$. □

For each vertex v we define $\Delta_v := \Gamma'_v \setminus \Gamma_v$ and $\Delta'_v := \Gamma_v \setminus \Gamma'_v$. Given any two vertices u and v, the probability that $d_k(u,v) \leq r$ and $d_{k+1}(u,v) > r$ (i.e., they are joined by an edge at step k but not at step $k+1$) is exactly $|\Delta'_u| = |\Delta'_v|$. This probability does not depend of the particular vertices and of k and will be denoted by q.

Lemma 10. *The probability that two vertices u and v are at distance at most r at step k but more than r at step $k+1$ is*

$$q \sim \begin{cases} \frac{4}{\pi} sr & \text{if } s = o(r), \\ \Theta(r^2) & \text{if } s = \Theta(r), \\ \pi r^2 & \text{if } s = \omega(r). \end{cases}$$

Proof. Let O be the position of u at step k and R the position of u at step $k+1$, at distance s from O.

Case $s = o(r)$.

Let Q be the position of v at step k. Then Q must lie in the circle of radius r centered on O, but outside the circle of radius $r - s$ and center R. Let θ be the the angle $\angle ROQ$ and ρ the distance between R and Q. We have that ρ must be greater than $r - s$, since otherwise u and v would share an edge at step $k+1$ by the triangular inequality. The maximum value that ρ can take is, by the cosine theorem,

$$\sqrt{r^2 + s^2 - 2rs\cos\theta}.$$

Given ρ, let α be the angle determined from the range of all possible directions that v can move. Again by the cosine theorem,

$$\alpha = 2\arccos\left(\frac{r^2 - s^2 - \rho^2}{2s\rho}\right).$$

Then, the probability that v is at distance at most r from u and moves in the right direction so that the distance becomes greater than r is

$$\begin{aligned} q &= \int_0^{2\pi} \int_{r-s}^{\sqrt{r^2+s^2-2rs\cos\theta}} \frac{\alpha}{2\pi} \rho d\rho d\theta \\ &= \int_0^{2\pi} \int_{r-s}^{\sqrt{r^2+s^2-2rs\cos\theta}} \frac{1}{\pi} \arccos\left(\frac{r^2 - s^2 - \rho^2}{2s\rho}\right) \rho d\rho d\theta \\ &= 2\int_0^{\pi} \frac{1}{2\pi}\left((r^2 + s^2 - 2rs\cos\theta)\arccos\frac{r\cos\theta - s}{\sqrt{r^2 + s^2 - 2rs\cos\theta}} - rs\sin\theta - \theta r^2\right) d\theta \end{aligned}$$

By looking at the Taylor series with respect to s/r of the expression inside the integral divided by r^2, we get

$$q = \int_0^{\pi} r^2 \left(-\frac{2\theta\cos\theta}{\pi}\frac{s}{r} + O\left(\left(\frac{s}{r}\right)^2\right)\right) d\theta = \frac{4}{\pi} sr\left(1 + O\left(\frac{s}{r}\right)\right). \tag{16}$$

Case $s = \Theta(r)$.

From (16), we first observe that if $s \leq \epsilon r$ for some small enough constant $\epsilon > 0$, then $q = \Theta(r^2)$. Assume then in the following that $s \geq \epsilon r$. Consider the circle of radius r and center O. Take the chord which is perpendicular to the segment $\overline{OR}$ and at distance r from R. This chord divides the circle into two regions. One of them has the property that all the points inside are at distance at least r from R and moreover it has area at least $\epsilon\sqrt{2\epsilon - \epsilon^2}r^2$.

Suppose that at step k some vertex v is located inside that region. (This happens with probability at least $\epsilon\sqrt{2\epsilon - \epsilon^2}r^2$.) Let Q be its position and T the new position at step $k+1$. Let us consider the circle centered on R and passing through Q. We observe that, with probability at least $1/2$, the new position T of v will be further away from R than Q is, since it is sufficient that v chooses an angle in the outer side of the tangent at Q. Therefore, the probability in the statement of the result is at least $\frac{1}{2}\epsilon\sqrt{2\epsilon - \epsilon^2}r^2$.

Case $s = \omega(r)$.

From Lemma 9(4), we obtain

$$q = |\Delta'_u| = |\Gamma_u \setminus \Gamma'_u| = |\Gamma_u| - |\Gamma_u \cap \Gamma'_u| = \pi r^2 - o(r^2).$$

□

We also need the following

Lemma 11. *Suppose that n points are selected independently and uniformly at random from $[0,1)^2$ (or from $[0,1)^3$). For some fixed integer $i \geq 2$, let $\mathcal{S}_1, \ldots, \mathcal{S}_i$ $(i \geq 2)$ be disjoint measurable subsets of $[0,1)^2$ (or $[0,1)^3$) with area (or volume) $s_1, \ldots, s_i$ respectively.*

1. *The probability that no selected point belongs to $\mathcal{S}_1$ and at least one lies in $\mathcal{S}_2$ is*

$$\sim (1 - s_1)^n (1 - e^{-s_2 n/(1-s_1)}),$$

where asymptotics are with respect to the number of points.

2. *If moreover $s_j = o(1)$, for all $1 \leq j \leq i$, then the probability that no selected point belongs to $\mathcal{S}_1$ and at least one lies in $\mathcal{S}_j$ for all $2 \leq j \leq i$ is*

$$\sim (1 - s_1)^n \prod_{j=2}^{i} (1 - e^{-s_j n}),$$

Proof. The probability in the first statement is

$$P_1 = (1 - s_1)^n - (1 - s_1 - s_2)^n = (1 - s_1)^n \left(1 - \left(1 - \frac{s_2}{1 - s_1}\right)^n\right).$$

Trivial asymptotic analysis yields the statement.

The probability in the second statement is by inclusion-exclusion

$$
\begin{aligned}
P_2 &= \sum_{c_j \in \{0,1\},\, 2 \le j \le i} (-1)^{\sum_{j=2}^{i} c_j} \left(1 - \left(s_1 + \sum_{j=2}^{i} c_j s_j\right)\right)^n \\
&= \sum_{c_j \in \{0,1\},\, 2 \le j \le i-1} (-1)^{\sum_{j=2}^{i-1} c_j} \left(1 - \left(s_1 + \sum_{j=2}^{i-1} c_j s_j\right)\right)^n \left(1 - \left(1 - \frac{s_i}{1 - \sum_{j=2}^{i-1} c_j s_j}\right)^n\right) \\
&\sim \left(1 - e^{-s_i n}\right) \sum_{c_j \in \{0,1\},\, 2 \le j \le i-1} (-1)^{\sum_{j=2}^{i-1} c_j} \left(1 - \left(s_1 + \sum_{j=2}^{i-1} c_j s_j\right)\right)^n
\end{aligned}
$$

and the argument follows by induction. □

We are now in good position to study the changes experienced by the isolated vertices between two consecutive steps k and $k+1$. Extending the notation in Section 2, we denote by $X_k = (Z_1)_k$ the number of isolated vertices of $\mathcal{G}(n,r)$ at step k. Also, for any two consecutive steps k and $k+1$, we define the following random variables: B_k is the number of vertices of $\mathcal{G}_k(n,r,s,m)$ which are isolated at step $k+1$ but not at step k; D_k is the number of vertices of $\mathcal{G}_k(n,r,s,m)$ which are isolated at step k but not at step $k+1$; S_k is the number of vertices of $\mathcal{G}_k(n,r,s,m)$ which are isolated at both steps k and $k+1$. We often denote them just X, B, D and S for simplicity whenever k and $k+1$ are understood. Note that B and D have the same distribution, since any birth of an isolated vertex corresponds to a death of an isolated vertex in the time-reversed process and vice versa.

To state the following result, we need one more definition: We say that two events $\mathcal{E} = \mathcal{E}(n)$ and $\mathcal{F} = \mathcal{F}(n)$ are asymptotically independent if $\mathbf{Pr}(\mathcal{E} \wedge \mathcal{F}) \sim \mathbf{Pr}(\mathcal{E})\mathbf{Pr}(\mathcal{F})$.

Proposition 12. *Assume $\mu = \Theta(1)$. Then for any two consecutive steps,*

$$\mathbf{E}B = \mathbf{E}D \sim \mu(1 - e^{-qn}), \quad \mathbf{E}S \sim \mu e^{-qn}.$$

Moreover for any fixed $j_1, j_2, j_3 \in \mathbb{N}$ we have that

1. *If $s = o(1/rn)$ then the events $(B > 0)$, $(D > 0)$ and $(S = j_3)$ are asymptotically mutually independent and*

$$\mathbf{Pr}(B > 0) \sim \mathbf{E}B, \quad \mathbf{Pr}(D > 0) \sim \mathbf{E}D, \quad \mathbf{Pr}(S = j_3) \sim e^{-\mathbf{E}S} \frac{(\mathbf{E}S)^{j_3}}{j_3!}.$$

2. *If $s = \Theta(1/rn)$ then*

$$\mathbf{Pr}((B = j_1) \wedge (D = j_2) \wedge (S = j_3)) \sim e^{-\mathbf{E}B} \frac{(\mathbf{E}B)^{j_1}}{j_1!} e^{-\mathbf{E}D} \frac{(\mathbf{E}D)^{j_2}}{j_2!} e^{-\mathbf{E}S} \frac{(\mathbf{E}S)^{j_3}}{j_3!}.$$

3. *If $s = \omega(1/rn)$ then the events $(B = j_1) \wedge (D = j_2)$ and $(S > 0)$ are asymptotically mutually independent and*

$$\mathbf{Pr}((B = j_1) \wedge (D = j_2)) \sim e^{-\mathbf{E}B} \frac{(\mathbf{E}B)^{j_1}}{j_1!} e^{-\mathbf{E}D} \frac{(\mathbf{E}D)^{j_2}}{j_2!}, \quad \mathbf{Pr}(S > 0) \sim \mathbf{E}S.$$

Proof. Given a vertex v, let B_v the indicator function of the event that v is not isolated at step k but isolated at step $k+1$. D_v and S_v are defined analogously so that

$$B = \sum_{v \in V} B_v, \quad D = \sum_{v \in V} D_v, \quad S = \sum_{v \in V} S_v.$$

Observe that for the birth of an isolated vertex v between step k and $k+1$ we must have $\phi(w) \notin \Gamma'_v$, for any vertex $w \neq v$, but there must exist some vertex $w \neq v$ with $\phi(w) \in \Gamma_v$. Thus from Lemmata 10 and 11 we have

$$\mathbf{Pr}(B_v = 1) \sim (1 - r^2\pi)^{n-1}(1 - e^{-q(n-1)/(1-r^2\pi)}) \sim e^{-r^2\pi n - O(\ln^2 n/n)}(1 - e^{-qn})$$

Then,

$$\mathbf{E}B = \sum_{v \in V} \mathbf{Pr}(B_v = 1) \sim n e^{-r^2\pi n}(1 - e^{-qn}) = \mu(1 - e^{-qn})$$

Since B and D have the same distribution, we also have $\mathbf{E}D = \mathbf{E}B$.

Similarly, if a vertex v is isolated at time k and at time $k+1$ as well, for all $w \neq v$, we must have $w \notin (\Gamma_v \cup \Gamma'_v)$, and thus

$$\mathbf{Pr}(S_v = 1) \sim (1 - (r^2\pi + q))^{n-1} \sim e^{-(r^2\pi+q)n - O(\ln^2 n/n)}$$

Then,

$$\mathbf{E}S = \sum_{v \in V} \mathbf{Pr}(S_v = 1) \sim n e^{-(r^2\pi+q)n} = \mu e^{-qn}$$

For any fixed naturals j_1, j_2, j_3 with sum $j = j_1 + j_2 + j_3$, we want to compute the joint factorial moments

$$\mathbf{E}([B]_{j_1}[D]_{j_2}[S]_{j_3}). \tag{17}$$

This amounts to multiplying $[n]_j$ (i.e., the number of ordered choices of j vertices $v_1, \ldots, v_j$) by $\mathbf{Pr}(\mathcal{E}_{j_1,j_2,j_3})$, where

$$\mathcal{E}_{j_1,j_2,j_3} = (\bigwedge_{i_1=1}^{j_1} B_{v_{i_1}} = 1) \wedge (\bigwedge_{i_2=j_1+1}^{j_1+j_2} D_{v_{i_2}} = 1) \wedge (\bigwedge_{i_3=j_1+j_2+1}^{j} S_{v_{i_3}} = 1) \tag{18}$$

is the event that for some particular j vertices, the first j_1 become isolated, the next j_2 stop being isolated and the last j_3 stay isolated. Obviously, $\mathbf{Pr}(\mathcal{E}_{j_1,j_2,j_3})$ does not depend on the particular choice of vertices.

Recall that each vertex v is assigned independently and uniformly at random a point $\phi(v) = (x, y, z) \in [0, 1)^3$, where (x, y) is the position of v in the torus at step k

and $\alpha = 2\pi z$ is the angle of its trajectory between steps k and $k+1$. Let us denote by $V_j = (v_1, \dots, v_j)$ the ordered set of vertices of our choice.

The event $\mathcal{E}_{j_1,j_2,j_3}$ can also be described as follows in terms of ϕ: none of the vertices in $V \setminus V_j$ is mapped by ϕ into the set

$$\Gamma = \bigcup_{i_1=1}^{j_1} \Gamma'_{v_{i_1}} \cup \bigcup_{i_2=j_1+1}^{j_1+j_2} \Gamma_{v_{i_2}} \cup \bigcup_{i_3=j_1+j_2+1}^{j} (\Gamma_{v_{i_3}} \cup \Gamma'_{v_{i_3}}),$$

and each of the sets in the following collection

$$\Sigma = \{\Delta'_{v_1}, \dots, \Delta'_{v_{j_1}}, \Delta_{v_{j_1+1}}, \dots, \Delta_{v_{j_1+j_2}}\}$$

must contain at least one $\phi(v)$ for some $v \in V$.

Case $s = \Theta(1/(rn))$.

Let $\mathcal{F}$ be the event that all pairs of vertices in V_j are at distance greater than $2r+4s$ at step k. This event has probability $1 - O(r^2)$. We observe that if $\mathcal{F}$ holds, then for any pair u and v of the j selected vertices, $\Gamma_u \cap \Gamma_v = \emptyset$, $\Gamma'_u \cap \Gamma'_v = \emptyset$ and $\Gamma_u \cap \Gamma'_v = \emptyset$. Therefore the probability of $\mathcal{E}_{j_1,j_2,j_3}$ is easy to compute, since $|\Gamma| = j\pi r^2 + j_3 q$ and the sets in Σ are disjoint. Therefore, from Lemmata 10 and 11

$$\mathbf{Pr}(\mathcal{E}_{j_1,j_2,j_3} \mid \mathcal{F}) \sim (1 - j\pi r^2 - j_3 q)^n (1 - e^{-qn})^{j_1+j_2} \sim \left(\frac{\mu}{n}\right)^j (1 - e^{-qn})^{j_1+j_2} e^{-j_3 qn}. \quad (19)$$

We claim that this is the main contribution to $\mathbf{Pr}(\mathcal{E}_{j_1,j_2,j_3})$. In fact if $\mathcal{F}$ does not hold (i.e., some of the vertices in V_j are at distance at most $2r+4s$), then $\mathbf{Pr}(\mathcal{E}_{j_1,j_2,j_3} \mid \bar{\mathcal{F}})$ is larger than (19), but this is balanced out by the fact that $\mathbf{Pr}(\bar{\mathcal{F}})$ is small. We say a vertex $v_i \in V_j$ is *restricted* if it has some other vertex $v_{i'} \in V_j$ with greater index ($i' > i$) at distance at most $2r + 4s$ at step k. Suppose that p of the vertices in V_j are restricted. This happens with probability $O(r^{2p})$. Notice that the distance between any u and v in V_j at step k cannot be smaller than $r - 2s$, since otherwise they would be joined by an edge at both steps k and $k+1$, which is not compatible with $\mathcal{E}_{j_1,j_2,j_3}$. Then we deduce $|\Gamma| \geq (j-p)\pi r^2 + \epsilon\pi r^2$, since each unrestricted vertex in V_j contributes in at least πr^2 to $|\Gamma|$ and the first restricted one gives the term $\epsilon\pi r^2$, by Lemma 9(2). Therefore, this has probability $O(1/n^{j-p+\epsilon})$. Summarizing, the weight in $\mathbf{Pr}(\mathcal{E}_{j_1,j_2,j_3})$ coming from situations with p restricted vertices is $O(r^{2p}/n^{j-p+\epsilon}) = O(\ln^p n/n^{j+\epsilon})$, and is thus negligible. (Notice that the conditions involving the sets in Σ play no important role here, since they hold with constant probability.) Therefore,

$$\mathbf{E}([B]_{j_1}[D]_{j_2}[S]_{j_3}) = [n]_j \mathbf{Pr}(\mathcal{E}_{j_1,j_2,j_3}) \sim (\mathbf{E}B)^{j_1} (\mathbf{E}D)^{j_2} (\mathbf{E}S)^{j_3} \quad (20)$$

The statement follows from Theorem 1.23 in [Bol].

Case $s = o(1/(rn))$.

In this case we must prove a weaker statement, since (20) does not hold when s is extremely small. In fact, we compute $\mathbf{E}([B]_{j_1}[D]_{j_2}[S]_{j_3})$ but we restrict to $j_1, j_2 \in$

$\{0, 1, 2\}$. Following the same notation as for the case $s = \Theta(1/(rn))$ and by an analogous argument we obtain

$$\mathbf{Pr}(\mathcal{E}_{j_1,j_2,j_3} \mid \mathcal{F}) \sim \left(\frac{\mu}{n}\right)^j (1 - e^{-qn})^{j_1+j_2} e^{-j_3 qn} \sim \left(\frac{\mu}{n}\right)^j (qn)^{j_1+j_2} \tag{21}$$

The case that $\mathcal{F}$ does not hold is slightly more delicate, since the sets in Σ need not be disjoint and this may affect the probability. Suppose for instance that u and v in V_j are at distance at most $2r + 4s$ at step k, and that the events required by $\mathcal{E}_{j_1,j_2,j_3}$ which involve u and v are $B_u = 1$ and $D_v = 1$. (The other possible situations are analyzed analogously). The probability that some vertex in V is mapped into the intersection $\Delta'_u \cap \Delta_v$ is bounded by the expected number of vertices falling there. This is at most n times the probability that a vertex $w \in V$ is mapped by ϕ into $\Delta'_u \cap \Delta_v$ which is at most the probability that u falls into Δ'_u and v into Δ_w conditional upon $d_k(u, v) \leq 2r + 4s$. This probability is $O(nq^2/r^2) = O(n^2q^2/\ln n)$, which is smaller than the probability that both Δ'_u and Δ_v contain at least one vertex, for Δ'_u and Δ_v being disjoint sets. Unfortunately, if $j_1 \geq 2$ or $j_2 \geq 2$ there is another situation which must be taken into account. Imagine that for $u, v \in V_j$ we have $\phi(u) \in \Delta_v$ (or $\phi(u) \in \Delta'_v$), which also implies $\phi(v) \in \Delta_u$ (or $\phi(v) \in \Delta'_u$). This event has probability $O(q/n^{1+\epsilon})$ for some $\epsilon > 0$, which might be larger (when s is really small) than the probability that both Δ'_u and Δ_v contain at least one vertex conditional upon $\mathcal{F}$. Summarizing, we obtain

$$\begin{aligned}
&\mathbf{E}([B]_{j_1}[D]_{j_2}[S]_{j_3}) = [n]_j \mathbf{Pr}(\mathcal{E}_{j_1,j_2,j_3}) \sim (\mathbf{E}B)^{j_1}(\mathbf{E}D)^{j_2}(\mathbf{E}S)^{j_3}, \quad \text{if } j_1 < 2, j_2 < 2 \\
&\mathbf{E}([B]_2[D]_{j_2}[S]_{j_3}) = o(\mathbf{E}(B[D]_{j_2}[S]_{j_3})) \\
&\mathbf{E}([B]_{j_1}[D]_2[S]_{j_3}) = o(\mathbf{E}([B]_{j_1}D[S]_{j_3}))
\end{aligned} \tag{22}$$

The results for the joint probability of these events follow by using upper and lower bounds given in [Bol], Section 1.4, applied to several variables.

Case $s = \omega(1/(rn))$ but also $s = O(r)$.

In this case we compute $\mathbf{E}([B]_{j_1}[D]_{j_2}[S]_{j_3})$ but we restrict to $j_3 \in \{0, 1, 2\}$. Following the same notation as for the case $s = \Theta(1/(rn))$ and by an analogous argument we obtain

$$\mathbf{Pr}(\mathcal{E}_{j_1,j_2,j_3} \mid \mathcal{F}) \sim \left(\frac{\mu}{n}\right)^j (1 - e^{-qn})^{j_1+j_2} e^{-j_3 qn} \sim \left(\frac{\mu}{n}\right)^j e^{-j_3 qn} \tag{23}$$

We claim that this is the main contribution to $\mathbf{Pr}(\mathcal{E}_{j_1,j_2,j_3})$ if $j_3 \leq 1$. In fact, suppose that $\mathcal{F}$ does not hold and that p of the vertices in V_j are restricted. Since $j_3 \leq 1$, then the only possible event required by $\mathcal{E}_{j_1,j_2,j_3}$ which contributes to S is (S_{v_j}), which involves the last vertex of our j-tuple. This vertex cannot be restricted by definition. Then we deduce $|\Gamma| \geq (j - p)\pi r^2 + j_3 q + \epsilon\pi r^2$, since the unrestricted vertices in V_j contribute in $(j-p)\pi r^2 + j_3 q$ to $|\Gamma|$ and the first restricted one gives the term $\epsilon\pi r^2$, by Lemma 9(2,3). Therefore, the probability of $\mathcal{E}_{j_1,j_2,j_3}$ in this situation is $O(e^{-j_3 qn}/n^{j-p+\epsilon})$, which combined with the probability $O(r^{2p})$ that p vertices are restricted has negligible weight compared to (23). Unfortunately, if $j_3 = 2$ and we have p restricted vertices in V_j, we can just assure that $|\Gamma| \geq (j-p)\pi r^2 + q + \epsilon\pi r^2$, which may affect the unconditional probability of $\mathcal{E}_{j_1,j_2,j_3}$. We obtain

$$\begin{aligned} \mathbf{E}([B]_{j_1}[D]_{j_2}[S]_{j_3}) &= [n]_j \mathbf{Pr}(\mathcal{E}_{j_1,j_2,j_3}) \sim (\mathbf{E}B)^{j_1}(\mathbf{E}D)^{j_2}(\mathbf{E}S)^{j_3}, \quad \text{if } j_3 < 2 \\ \mathbf{E}([B]_{j_1}[D]_{j_2}[S]_2) &= o(\mathbf{E}([B]_{j_1}[D]_{j_2}S)) \end{aligned} \tag{24}$$

As before, the results for the joint probability of these events follow by using upper and lower bounds given in [Bol], Section 1.4, applied to several variables.

Case $s = \omega(r)$.

We compute again $\mathbf{E}([B]_{j_1}[D]_{j_2}[S]_{j_3})$ restricted to $j_3 \in \{0,1,2\}$. Let $\mathcal{F}'$ be the event that all pairs of vertices in V_j are at distance greater than $2r$ at step k and also at step $k+1$. This event has probability $1 - O(r^2)$. We observe that if $\mathcal{F}'$ holds, then for any pair u and v of the j selected vertices, $\Gamma_u \cap \Gamma_v = \emptyset$, $\Gamma'_u \cap \Gamma'_v = \emptyset$ and $\Gamma_u \cap \Gamma'_v = \emptyset$. Therefore the probability of $\mathcal{E}_{j_1,j_2,j_3}$ is easy to compute, since $|\Gamma| = j\pi r^2 + j_3 q$ and the sets in Σ are disjoint. Therefore, from Lemmata 10 and 11

$$\mathbf{Pr}(\mathcal{E}_{j_1,j_2,j_3} \mid \mathcal{F}') \sim \left(\frac{\mu}{n}\right)^j (1 - e^{-qn})^{j_1+j_2} e^{-j_3 qn} \sim \left(\frac{\mu}{n}\right)^j e^{-j_3 qn} \tag{25}$$

The remaining of the argument is analogous to the previous case but changing $\mathcal{F}$ to $\mathcal{F}'$ and using Lemma 9(4).

$\square$

Taking into account that $X_k = D_k + S_k$ and $X_{k+1} = S_k + B_k$, from Proposition 12 we can completely characterize the number of isolated vertices at two consecutive steps in the case $s = \Theta(1/(rn))$. For the other ranges of s, the result is weaker but still sufficient for our further usage. We remark that if $s = o(1/(rn)$ then creations and destructions of isolated vertices are rare, but a Poisson number of isolated vertices is present at both consecutive steps. Otherwise if $s = \omega(1/(rn)$ then the isolated vertices which are present at both consecutive steps are rare since, but a Poisson number of them are created and also a Poisson number destroyed.

Now in order to characterize the connectivity of $\mathcal{G}_k(n,r,s,m)$, we need to bound the probability that components other than isolated vertices and the giant one appear at some step. We know by Theorem 2 that a.a.s. this does not occur at one single step k. However during long periods of time this event could affect the connectivity and must be considered.

Extending the notation in Section 2, given a step k let $\widetilde{X}_k = (\widetilde{Z}_2)_k$ be the number of non-solitary components other than isolated vertices occurring at step k. We show that they have a negligible effect compared to isolated vertices in the dynamic evolution of connectivity.

Lemma 13. *Assume that $\mu = \Theta(1)$ and $s = o(1/(rn))$. Then,*

- $\mathbf{Pr}(\widetilde{X}_k > 0 \wedge \widetilde{X}_{k+1} = 0) = \mathbf{Pr}(\widetilde{X}_k = 0 \wedge \widetilde{X}_{k+1} > 0) = o(srn)$,
- $\mathbf{Pr}(\widetilde{X}_k > 0 \wedge B_k > 0) = \mathbf{Pr}(\widetilde{X}_k > 0 \wedge D_k > 0) = o(srn)$.

Proof. We will only prove that $\mathbf{Pr}(\widetilde{X}_k > 0 \wedge \widetilde{X}_{k+1} = 0) = o(rsn)$ and $\mathbf{Pr}(\widetilde{X}_k > 0 \wedge D_k > 0) = o(rsn)$. The other cases follow analogously.

For any such component C of size at least 2 we denote by $diam(C)$ the diameter of C, i.e., the largest Euclidean distance between any two vertices in C and by $|C|$ the size of C. We distinguish a few cases depending on diameter and/or size of C and on whether C is embeddable. For each Case i we treat in this proof, let $\widetilde{X}_k^i$ be the number of components of the particular type studied in this case at step k. Then it is enough showing for each Case i that $\mathbf{Pr}(\widetilde{X}_k^i > 0 \wedge \widetilde{X}_{k+1} = 0) = o(rsn)$ and $\mathbf{Pr}(\widetilde{X}_k^i > 0 \wedge D_k > 0) = o(rsn)$.

Case 1: (C is embeddable and moreover $diam(C) > \sqrt{2}(4+\epsilon)r$ for some small $\epsilon > 0$:)

Consider a component C of size ℓ, $2 \leq \ell \leq n$, at step k, with all these properties above. We tessellate $[0,1)^2$ into $\Theta(n/\ln n)$ small square cells of side length δr, where $\delta = \delta(\epsilon)$ is a sufficiently small positive constant. We can assume w.l.o.g. that C is contained in $Q = [r, 1-r]^2$ (otherwise apply a translation to the coordinates of the torus). In this section of the proof, all distances will be measured with respect to the unit square rather than the torus. Let us consider the set of points in $[0,1)^2$ at distance at most r from some vertex of C. Let us remove from this region the vertices of C and the segments joining each pair of vertices of C at distance at most r, and denote by $\mathcal{S}$ the outer connected component of the remaining. By construction, $\mathcal{S}$ must not contain any vertex at step k. We shall give a lower bound of the area of $\mathcal{S}$. Let the vertical (or horizontal) length of C be the maximal vertical (or horizontal) distance between all the pairs of vertices in C. Since $diam(C) > \sqrt{2}(4+\epsilon)r$, the vertical or horizontal length of C must be larger than $(4+\epsilon)r$. Assume w.l.o.g. the vertical length of C has this property. Let u and v be vertices in C with respectively lowest and highest y-coordinate at step k. Let us denote their coordinates by (u_x, u_y) and (v_x, v_y). Note that the lower halfcircle of radius r centered at u, and the upper halfcircle of radius r centered at v must be contained in $\mathcal{S}$. Consider also the leftmost (rightmost, respectively) vertex l_u (r_u, respectively) in C whose y-coordinate is between u_y and $u_y + 2r$ (possibly this vertex is equal to u). If the y-coordinate of l_u is above $u_y + r$, then consider the left lower quartercircle of radius r centered at l_u, otherwise consider the left upper quartercircle of radius r centered at l_u. In any case, the corresponding quartercircle must be contained in $\mathcal{S}$. Similarly, if the y-coordinate of r_u is above $u_y + r$, then the right lower quartercircle of radius r centered at r_u must be contained in $\mathcal{S}$, otherwise the right upper quartercircle must be. By the same argument, we get that for the leftmost (rightmost, respectively) vertex l_v (r_v, respectively) in C whose y-coordinate is between v_y and $v_y - 2r$ one corresponding left (right, respectively) quartercircle of radius r contained in $\mathcal{S}$. Note that by construction all these four quartercircles are disjoint.

Moreover, since $d_k(u,v) \geq (4+\epsilon)r$, an additional area around C with y-coordinate between $u_y + 2r$ and $v_y - 2r$ of size at least ϵr^2 is contained in $\mathcal{S}$. Altogether, the area of $\mathcal{S}$ is at least $(2 + \epsilon/\pi)r^2\pi$.

Let $\mathcal{S}_1$ be the union of all the cells in the tessellation of the torus which are completely contained in $\mathcal{S}$. By choosing δ small enough we can guarantee that $\mathcal{S}_1$ is connected and it has an area $|\mathcal{S}_1| \geq (2+\delta')\pi r^2$ for some small $\delta' > 0$. Note that this

δ can be chosen to be the same for all components C of this kind.

So far, we have shown that $(\widetilde{X}_k^1 > 0)$ implies that some connected set of area $(2+\delta')r^2\pi$ and which is a union of cells of the tesselation contains no vertices at step k. There are $O(n/\ln n)$ of such sets. Now let $\mathcal{S}_1$ be any of these sets, and let v_1 and v_2 be any two vertices in V. Let $\mathcal{E}_{\mathcal{S}_1,v_1,v_2}$ be the event that $d_k(v_1,v_2) \le r$ and $d_{k+1}(v_1,v_2) > r$ or vice versa (i.e., an edge between v_1 and v_2 appears or disappears between steps k and $k+1$), and moreover $\mathcal{S}_1$ contains no vertices at step k. Then

$$\mathbf{Pr}(\mathcal{E}_{\mathcal{S}_1,v_1,v_2}) \le (1-(2+\delta')\pi r^2)^{n-1} q$$

Each of the events $(\widetilde{X}_k^1 > 0 \wedge \widetilde{X}_{k+1} = 0)$ and $(\widetilde{X}_k^1 > 0 \wedge D_k > 0)$ implies $\mathcal{E}_{\mathcal{S}_1,v_1,v_2}$ for some connected union of cells $\mathcal{S}_1$ and vertices v_1 and v_2. Therefore taking a union bound over all $\mathcal{S}_1$, v_1 and v_2, we deduce that their probabilities are at most

$$O(n/\ln n)n^2\mathbf{Pr}(\mathcal{E}_{\mathcal{S}_1,v_1,v_2}) = O(n/\ln n)n^2 n^{-2-\delta'} q = o(qn),$$

which is $o(srn)$ in view of Lemma 10.

Case 2: $(4+\epsilon)r < diam(C) \le \sqrt{2}(4+\epsilon)r$:

Let u and v be vertices in C such that $d_k(u,v) = diam(C)$. Then we can assume w.l.o.g. that u and v have the same x-coordinate and that u is below v (otherwise we rotate the geometrical descriptions in the argument), and apply the same argument as in the previous case.

Case 3: $diam(C) \le (4+\epsilon)r$ *and* $|C| \ge 17\ln n$:

As before, we tessellate $[0,1)^2$ with square cells of side length δr for some small $\delta > 0$. Consider a component C of diameter at most $(4+\epsilon)r$ which exists at step k, and assume w.l.o.g. that none of its edges intersects any of the sides of $[0,1)^2$. Let v_1, v_2, v_3 and v_4 be respectively the vertices of C with highest y-coordinate, lowest y-coordinate, highest x-coordinate and lowest x-coordinate. Let $\mathcal{S}_3$ be the minimal rectangle made as a union of cells of the tessellation and containing v_1, v_2, v_3 and v_4. By construction, this rectangle has area $|\mathcal{S}_3| \le (4+\epsilon+2\delta)^2 r^2 \le 17r^2$ if ϵ and δ small enough, and moreover it contains C.

Let now $\mathcal{S}_3$ be any rectangular union of cells with asymptotically area $17r^2$, and u and v any two vertices in V. If W is the number of vertices from $V \setminus \{u,v\}$ which are placed inside $\mathcal{S}_3$ at step k, then $\mathbf{E}W \sim (n-2)17r^2 \sim (17/\pi)\ln n$. Then the probability that $d_k(u,v) \le r$ and $d_{k+1}(u,v) > r$ or vice versa, and moreover $\mathcal{S}_3$ contains at least $17\ln n > 3\mathbf{E}W$ vertices from $V \setminus \{u,v\}$ is (by Chernoff bound) at most

$$\mathbf{Pr}(W \ge 3\mathbf{E}W \wedge (u \in \Delta_v \vee u \in \Delta'_v)) < e^{-4\mathbf{E}W/4} 2q = o(q/n^3)$$

Taking a union bound of this probability as we did in Case 1 over the possible $O(n/\ln n)$ different sets $\mathcal{S}_3$ and $n(n-1)$ choices for vertices u and v, completes the argument for this case.

Case 4: $\epsilon r \le diam(C) \le (4+\epsilon)r$*:*

We tesselate the torus into square boxes of side $(8+2\epsilon)r$. Let $\mathcal{B}$ be the set containing all these boxes and the new boxes resulting from displacing each of those a distance of $(4+\epsilon)r$ to the right, left, up and down. We observe that any component of the studied type must be completely contained in one of the boxes of $\mathcal{B}$. We also consider a finer tesselation of the torus into square cells of side δr, for a small enough $\delta = \delta(\epsilon) > 0$. We refer to the elements of $\mathcal{B}$ as boxes and to the squares of the finer tesselation as cells.

Let us fix a box $b \in \mathcal{B}$. Suppose that at step k we have some component C contained in b such that $\epsilon r \leq diam(C) \leq (4+\epsilon)r$. Then, arguing as in Case 1, we conclude that there is a set $\mathcal{S}$ which is the connected union of cells, has area $(1+\epsilon')\pi r^2$ for some $\epsilon' > 0$, is empty of vertices at step k, and moreover intersects b. Note that the number of unions of cells with these properties is $\Theta(1)$. Therefore, taking a union bound we obtain that the probability that b contains a component of the studied type at step k is at most $\Theta(1)e^{-(1+\epsilon')\pi r^2 n} = \Theta(n^{-1-\epsilon'})$.

An arrangement of vertices in b is a number ℓ, a choice of ℓ vertices, and an assigment of a position inside b for each of the ℓ vertices. Suppose that b contains less than $17 \ln n$ vertices. Now, conditional to any particular arrangement of vertices in b with $\ell < 17 \ln n$, the probability that some edge involving a vertex inside b changes between steps k and $k+1$ is at most $17qn \ln n$. Observe that this bound is independent of the particular arrangement.

Hence, conditional upon b containing less than $17 \ln n$ vertices, the probability that b contains some component of the studied type at step k and moreover some edge involving a vertex inside b changes between steps k and $k+1$ is at most $\Theta(n^{-1-\epsilon'} qn \ln n)$.

By an argument analogous to Case 3 (Chernoff bound), we prove that the situation in which b contains at least $17 \ln n$ vertices at step k has negligible weight and therefore the probability that b contains some component of the studied type at step k and moreover some edge involving a vertex inside b changes between steps k and $k+1$ is at most $\Theta(n^{-1-\epsilon'} qn \ln n)$.

Taking union bound over all $\Theta(n/\ln n)$ boxes in $\mathcal{B}$ we prove that the probability that some box contains a component of the studied type and moreover some edge involving a vertex in the box changes is at most $\Theta(n^{-\epsilon'} qn) = o(\mathrm{qn})$. This event contains the event that some component of the studied type exists at step k and no medium size components exist at step $k+1$, and hence the first statement is proved.

The same argument can be used to bound $\mathbf{Pr}(\widetilde{X}_k^4 > 0 \wedge D_k > 0)$.

Case 5: (C is not embeddable)

We tessellate $[0,1)^2$ into $\Theta(n/\ln n)$ small square cells of side length δr, where $\delta = \delta(\epsilon)$ is a sufficiently small positive constant.

Let us consider the set of points in $[0,1)^2$ at distance at most r from some vertex of C. Let us remove from this region the vertices of C and the segments joining each pair of vertices of C at distance at most r, and call $\mathcal{S}'$ to the remaining set.

Suppose there is a horizontal or a vertical band of width $2r$ in $[0,1)^2$ which does not intersect the component at step k (assume w.l.o.g. that it is the topmost horizontal band consisting of all points with the y-coordinate in $[1-2r, 1)$). Let us divide the torus into vertical bands of width $2r$. All of them must contain at least one vertex of C, since otherwise C would be embeddable. Select any 9 consecutive vertical bands and pick one vertex of C with maximal y-coordinate in each one. For each one of these 9 vertices, we select the left upper quartercircle centered at the vertex if the vertex is closer to the right side of the band or the right upper quartercircle otherwise. These nine quartercircles we chose are disjoint and must contain no vertices by construction. Moreover, they belong to the same connected component of the set $\mathcal{S}'$, which we denote by $\mathcal{S}$. Let $\mathcal{S}_1$ be the union of all the cells in the tessellation of the torus which are completely contained in $\mathcal{S}$. As usual, by choosing δ small enough we can guarantee that $\mathcal{S}_1$ is connected and it has an area $|\mathcal{S}_1| \geq (9\pi/4)r^2$ for some small $\delta' > 0$. Note that this δ can be chosen to be the same for all components C of this kind.

Suppose otherwise that all horizontal and vertical bands of width $2r$ in $[0,1)^2$ contain at least one vertex of C at step k. Since C is not solitary it must be possible that it coexists with some other non-embeddable component C' at step k. Then all vertical bands or all horizontal bands of width $2r$ must also contain some vertex of C' (assume w.l.o.g. the vertical bands do). Let us divide the torus into vertical bands of width $2r$. We can find a simple path Π with vertices in C' which passes through 11 consecutive bands. For each one of the 9 internal bands, pick the uppermost vertex of C in the band below Π (in the torus sense). As before each one of these vertices contributes with a disjoint quartercircle which must be empty of vertices, and by the same argument we obtain a connected union of cells of the tessellation, which we denote by $\mathcal{S}_1$, with area $|\mathcal{S}_1| \geq (9\pi/4)r^2$ and containing no vertices. The rest of the argument follows exactly as in Case 1.

Case 6: $diam(C) \leq \frac{1}{n\sqrt{\log n}}$ and $|C| \leq 17\log n$:

In this case we use a very simple calculation to get an upper bound for $\mathbf{Pr}(\widetilde{X}_k^6 > 0 \wedge \widetilde{X}_{k+1} = 0)$. The argument is the following: for any fixed ℓ-tuple of vertices, the probability that it forms a component C of size ℓ at step k is at most $((diam(C))^2\pi)^{\ell-1}$, since the position of the first vertex can be chosen arbitrarily, but all other $\ell-1$ vertices must be placed in a ball of radius at most $diam(C)$ centered at the first vertex. Let $\mathcal{S}$ denote the indicator event that the fixed ℓ-tuple of vertices forms a component of size ℓ at step k. Furthermore, let v_1 and v_2 be two vertices in V, and at least one of them in C (assume w.l.o.g. v_1). For a fixed C denote by $\mathcal{E}^C_{\mathcal{S},v_1,v_2}$ the event that $d_k(v_1,v_2) \leq r$ and $d_{k+1}(v_1,v_2) > r$ or vice versa, and moreover the indicator event $\mathcal{S}$ is true. Denote by $\mathcal{E}^\ell_{\mathcal{S},v_1,v_2}$ the event that for one of the $\binom{n}{\ell} \leq \frac{n^\ell}{\ell!}$ choices of ℓ vertices the event $\mathcal{E}^C_{\mathcal{S},v_1,v_2}$ is true. Note that $\mathbf{Pr}(\widetilde{X}_k^6 > 0 \wedge \widetilde{X}_{k+1} = 0)$ implies the event $\mathcal{E}^\ell_{\mathcal{S},v_1,v_2}$ for some ℓ. Thus we obtain that

$$\begin{aligned}
&\mathbf{Pr}(\mathcal{E}^\ell_{\mathcal{S},v_1,v_2}) \\
&\sim ((\tfrac{1}{n\sqrt{\log n}})^2\pi)^{\ell-1}\tfrac{n^\ell}{\ell!}\Theta(qn\ell),
\end{aligned}$$

which is after summing over all possible values of ℓ, $2 \leq \ell \leq 17\log n$, still $o(rsn)$ and therefore $\mathbf{Pr}(\widetilde{X}_k^6 > 0 \wedge \widetilde{X}_{k+1} = 0) = o(rsn)$. The same argument (even simpler, since the $\Theta(qn\ell)$ term can be replaced by a $\Theta(qn)$ term) can be used to show that $\mathbf{Pr}(\widetilde{X}_k^6 > 0 \wedge D_k = 0)$ is also $o(qsn)$, as desired.

Case 7: $\frac{1}{n\sqrt{\log n}} \leq diam(C) \leq \frac{\sqrt{\pi}}{\sqrt{n\log n}}$ and $|C| \leq 0.01\log n$:

We fix an ℓ-tuple C of vertices, $2 \leq \ell \leq 0.01 \log n$, which forms a component of size ℓ of diameter exactly d. Let u and v be two vertices of C such that $d_k(u,v) = diam(C) =: d$, and assume w.l.o.g. as in Case 1 that u and v have the same x-coordinate, and u is the bottommost vertex, and v the topmost vertex of C. Note that for such a component to exist the halfcircle of radius r centered at u and below u as well as the halfcircle of radius r centered at v and above u must not contain any vertex at step k. Let us denote this area by S_1. Moreover, as in Case 1, an additional area of total size rd to the right (left) of the rightmost (leftmost) vertex of C must not contain any vertex at step k either, which we call $\mathcal{S}_2$. Denote by $\mathcal{S}$ the union of $\mathcal{S}_1$ and $\mathcal{S}_2$. Note that $|\mathcal{S}| \geq r^2\pi + rd$. Let v_1 and v_2 be two vertices in V, and at least one of them in C (assume w.l.o.g. v_1). For a fixed C denote by $\mathcal{E}^C_{\mathcal{S},v_1,v_2}$ the event that $d_k(v_1,v_2) \leq r$ and $d_{k+1}(v_1,v_2) > r$ or vice versa, and moreover $\mathcal{S}$ contains no vertices at step k. Note that the event $(\widetilde{X}^7_k > 0 \wedge \widetilde{X}_{k+1} = 0)$ implies for some l and C the event $\mathcal{E}^C_{\mathcal{S},v_1,v_2}$. Denote by $\mathcal{E}^\ell_{\mathcal{S},v_1,v_2}$ the union of all events event $\mathcal{E}^C_{\mathcal{S},v_1,v_2}$ for all C with $|C| = \ell$ and observe that $\mathbf{Pr}(\mathcal{E}^\ell_{\mathcal{S},v_1,v_2}) \leq \frac{n^\ell}{\ell!}\mathbf{Pr}(\mathcal{E}^C_{\mathcal{S},v_1,v_2})$.

$$
\begin{aligned}
&\mathbf{Pr}(\mathcal{E}^\ell_{\mathcal{S},v_1,v_2})\\
&\sim \tfrac{n^\ell}{\ell!}\Theta(qn\ell)\tfrac{\mu}{n}\pi^{\ell-1}\int_{x=\frac{1}{n\sqrt{\log n}}}^{\frac{\sqrt{\pi}}{\sqrt{n\log n}}}(x^2)^{\ell-1}e^{-rxn}dx\\
&\sim \tfrac{n^\ell}{(\ell-1)!}\Theta(qn)\tfrac{\mu}{n}\pi^{\ell-1}\left(-Exp_{2-2\ell}(rxn)x^{2\ell-1}\right)\Big|_{x=\frac{1}{n\sqrt{\log n}}}^{x=\frac{\sqrt{\pi}}{\sqrt{n\log n}}},
\end{aligned}
$$

where $Exp_n(x)$ is the exponential integrating function defined as $Exp_n(x) := \int_{t=1}^{\infty}\frac{e^{-xt}dt}{t^n}$. Plugging in the asymptotic expansion $Exp_n(x) = \frac{e^{-x}}{x}(1-\frac{n}{x}+\frac{n(n+1)}{x^2}+\ldots)$ we see that the value of the integral is at most $\frac{(2\ell-2)!(\sqrt{\pi})^{2\ell-1}}{(n\log n)^{\ell-1/2}}$ (the value is obtained by plugging in the upper bound for x). Thus we obtain

$$
\begin{aligned}
&\mathbf{Pr}(\mathcal{E}^\ell_{\mathcal{S},v_1,v_2})\\
&\sim \tfrac{n^\ell}{(\ell-1)!}\Theta(qn)\tfrac{\mu}{n}\pi^{\ell-1}\tfrac{(2\ell-2)!(\sqrt{\pi})^{2\ell-1}}{(n\log n)^{\ell-1/2}}\\
&\leq \Theta(qn)\tfrac{\mu}{\sqrt{n}}\tfrac{(1.5\pi^2\ell)^\ell}{(\log n)^{\ell-1/2}},
\end{aligned}
$$

and after taking a union bound over all sizes ℓ, $2 \leq \ell \leq 0.01\log n$, we obtain that $\mathbf{Pr}(\widetilde{X}^7_k > 0 \wedge \widetilde{X}_{k+1} = 0) = o(rsn)$, as desired. As before, the same argument can be used to show that $\mathbf{Pr}(\widetilde{X}^7_k > 0 \wedge D^k > 0) = o(rsn)$.

Case 8: $\frac{\sqrt{\pi}}{\sqrt{n\log n}} \leq diam(C) \leq \epsilon r$ and $|C| \leq 0.01\log n$ ($\epsilon \ll 0.01$ being a sufficiently small constant):
We again compute the same integral as in Case 7, but in this case the value at the lower bound of the integral is the dominant one. Using the same notation as in that case we obtain

$$
\begin{aligned}
&\mathbf{Pr}(\mathcal{E}^\ell_{\mathcal{S},v_1,v_2})\\
&\sim \tfrac{n^\ell}{\ell!}\Theta(qn\ell)\pi^{\ell-1}\tfrac{\mu}{n}\tfrac{(2\ell-2)!(\sqrt{\pi})^{2\ell-1}}{(n\log n)^{\ell-1/2}},
\end{aligned}
$$

and after taking a union bound over all sizes ℓ we obtain that $\mathbf{Pr}(\widetilde{X}_k^8 > 0 \wedge \widetilde{X}_{k+1} = 0) = o(rsn)$.

Case 9: diam$(C) \leq \epsilon r$ *and* $0.01 \log n \leq |C| \leq 17 \log n$*:*
For ϵ sufficiently small, we obtain by the same calculations as in Case 6

$$\begin{aligned}&\mathbf{Pr}(\mathcal{E}^{\ell}_{\mathcal{S},v_1,v_2}) \\ &\sim (\epsilon\sqrt{\tfrac{\log n}{\pi n}})^2\pi)^{\ell-1}\tfrac{n^{\ell}}{\ell!}\Theta(qn\ell),\end{aligned}$$

and after taking a union bound over all possible values of ℓ we get that $\mathbf{Pr}(\widetilde{X}_k^9 > 0 \wedge \widetilde{X}_{k+1} = 0) = o(rsn)$.

□

Now we can characterise the connectivity of $\mathcal{G}_k(n, r, s, m)$ at two consecutive steps. We denote by $\mathcal{C}_k$ the event that $\mathcal{G}_k(n, r, s, m)$ is connected at step k, and by $\mathcal{D}_k = \overline{\mathcal{C}_k}$ the event that $\mathcal{G}_k(n, r, s, m)$ is disconnected at step k.

Corollary 14. *Assume that* $\mu = \Theta(1)$*. Then,*

$$\begin{aligned}\mathbf{Pr}(\mathcal{C}_k \wedge \mathcal{D}_{k+1}) &\sim e^{-\mu}(1 - e^{-\mathbf{E}B}), & \mathbf{Pr}(\mathcal{D}_k \wedge \mathcal{C}_{k+1}) &\sim e^{-\mu}(1 - e^{-\mathbf{E}B}) \\ \mathbf{Pr}(\mathcal{C}_k \wedge \mathcal{C}_{k+1}) &\sim e^{-\mu}e^{-\mathbf{E}B}, & \mathbf{Pr}(\mathcal{D}_k \wedge \mathcal{D}_{k+1}) &\sim 1 - 2e^{-\mu} + e^{-\mu}e^{-\mathbf{E}B}\end{aligned}$$

Proof. First observe that $X_k = S_k + D_k$ and $X_{k+1} = S_k + B_k$. Therefore we have

$$\mathbf{Pr}(X_k = 0 \wedge X_{k+1} > 0) = \mathbf{Pr}(S_k = 0 \wedge D_k = 0 \wedge B_k > 0),$$

and by Proposition 12 we get

$$\mathbf{Pr}(X_k = 0 \wedge X_{k+1} > 0) \sim e^{-\mathbf{E}S - \mathbf{E}D}(1 - e^{-\mathbf{E}B}) \sim e^{-\mu}(1 - e^{-\mathbf{E}B}). \tag{26}$$

We want to connect this probability with $\mathbf{Pr}(\mathcal{C}_k \wedge \mathcal{D}_{k+1})$. In fact, by partitioning $(X_k = 0 \wedge X_{k+1} > 0)$ and $(\mathcal{C}_k \wedge \mathcal{D}_{k+1})$ into disjoint events, we obtain

$$\begin{aligned}\mathbf{Pr}(X_k = 0 \wedge X_{k+1} > 0) &= \mathbf{Pr}(\mathcal{C}_k \wedge X_{k+1} > 0) + \mathbf{Pr}(\mathcal{D}_k \wedge X_k = 0 \wedge X_{k+1} > 0), \\ \mathbf{Pr}(\mathcal{C}_k \wedge \mathcal{D}_{k+1}) &= \mathbf{Pr}(\mathcal{C}_k \wedge X_{k+1} > 0) + \mathbf{Pr}(\mathcal{C}_k \wedge \mathcal{D}_{k+1} \wedge X_{k+1} = 0),\end{aligned}$$

and thus we can write

$$\mathbf{Pr}(\mathcal{C}_k \wedge \mathcal{D}_{k+1}) = \mathbf{Pr}(X_k = 0 \wedge X_{k+1} > 0) + P_1 - P_2, \tag{27}$$

where $P_1 = \mathbf{Pr}(\mathcal{C}_k \wedge \mathcal{D}_{k+1} \wedge X_{k+1} = 0)$ and $P_2 = \mathbf{Pr}(\mathcal{D}_k \wedge X_k = 0 \wedge X_{k+1} > 0)$.

Now suppose that $s = o(1/(rn))$. In that case, $\mathbf{Pr}(X_k = 0 \wedge X_{k+1} > 0) = \Theta(srn)$ (see (26) and Proposition 12). Also observe that $\mathcal{D} \wedge (X = 0)$ implies that $\widetilde{X} > 0$. In fact, we must have at least two components of size greater than 1, so at least one of these must be non-solitary. Then, we have that $P_1 \leq \mathbf{Pr}(\widetilde{X}_k = 0 \wedge \widetilde{X}_{k+1} > 0)$ and $P_2 \leq \mathbf{Pr}(\widetilde{X}_k > 0 \wedge B_k > 0)$, and from Lemma 13 we get

$$P_1, P_2 = o\big(\mathbf{Pr}(X_k = 0 \wedge X_{k+1} > 0)\big). \tag{28}$$

Otherwise if $s = \Omega(1/(rn))$, then $\mathbf{Pr}(X_k = 0 \wedge X_{k+1} > 0) = \Theta(1)$. In this case, we simply use the fact that $P_1 \leq \mathbf{Pr}(\widetilde{X}_{k+1} > 0) = o(1)$ and $P_2 \leq \mathbf{Pr}(\widetilde{X}_k > 0) = o(1)$ (see Theorem 7 and Lemma 8), and deduce that (28) also holds.

Finally, the asymptotic expression of $\mathbf{Pr}(\mathcal{C}_k \wedge \mathcal{D}_{k+1})$ is obtained from (26), (27) and (28). Moreover, by considering the time-reversed process, we deduce that $\mathbf{Pr}(\mathcal{D}_k \wedge \mathcal{C}_{k+1}) = \mathbf{Pr}(\mathcal{C}_k \wedge \mathcal{D}_{k+1})$. The remaing probabilities in the statement are computed from Corollary 3 and Lemma 8, and using the fact that

$$\begin{aligned}\mathbf{Pr}(\mathcal{C}_k \wedge \mathcal{C}_{k+1}) &= \mathbf{Pr}(\mathcal{C}_k) - \mathbf{Pr}(\mathcal{C}_k \wedge \mathcal{D}_{k+1}),\\ \mathbf{Pr}(\mathcal{D}_k \wedge \mathcal{D}_{k+1}) &= \mathbf{Pr}(\mathcal{D}_k) - \mathbf{Pr}(\mathcal{D}_k \wedge \mathcal{C}_{k+1}).\end{aligned}$$

□

Let $\mathcal{A}$ be an event in the static model (i.e. $\mathcal{A} \subseteq \mathcal{G}(n,r)$). We denote by $\mathcal{A}_k$ the event that $\mathcal{A}$ holds at step k. In the $\mathcal{G}_k(n,r,s,m)$ model, we define $L_k(\mathcal{A})$ to be the number of consecutive steps that $\mathcal{A}$ holds starting at step k (possibly 0 if A_k does not hold). Note that the distribution of $L_k(\mathcal{A})$ does not depend on k, and we will sometimes omit the k when it is understood or not relevant.

Lemma 15. *Consider any event $\mathcal{A}$ in the static model. If we have that $\mathbf{E}(L(\mathcal{A})) < +\infty$ (but possibly $\mathbf{E}(L(\mathcal{A})) \to +\infty$ as $n \to +\infty$), then conditional upon $\mathcal{A}_k$ but not A_{k-1} we have*

$$\mathbf{E}(L_k(\mathcal{A}) \mid \overline{\mathcal{A}_{k-1}} \wedge \mathcal{A}_k) = \frac{\mathbf{Pr}(\mathcal{A})}{\mathbf{Pr}(\overline{\mathcal{A}_{k-1}} \wedge \mathcal{A}_k)},$$

which does not depend on k.

Proof. We have that

$$L_{k-1} + 1[\overline{\mathcal{A}_{k-1}}]L_k = 1[\mathcal{A}_{k-1}] + L_k$$

and taking expectations and using the hypothesis that $\mathbf{E}(L(\mathcal{A})) < +\infty$ we get

$$\mathbf{E}(1[\overline{\mathcal{A}_{k-1}}]L_k(\mathcal{A})) = \mathbf{Pr}(\mathcal{A}), \quad \forall k.$$

Using the fact that

$$\mathbf{E}(L_k(\mathcal{A}) \mid \overline{\mathcal{A}_{k-1}} \wedge \mathcal{A}_k) = \frac{\mathbf{E}(1[\overline{\mathcal{A}_{k-1}} \wedge \mathcal{A}_k]L_k(\mathcal{A}))}{\mathbf{Pr}(\overline{\mathcal{A}_{k-1}} \wedge \mathcal{A}_k)} = \frac{\mathbf{E}(1[\overline{\mathcal{A}_{k-1}}]L_k(\mathcal{A}))}{\mathbf{Pr}(\overline{\mathcal{A}_{k-1}} \wedge \mathcal{A}_k)},$$

the result follows. □

To prove that $\mathbf{E}(L(\mathcal{C})) < +\infty$ and $\mathbf{E}(L(\mathcal{D})) < +\infty$ we need the following technical lemma.

Lemma 16. *Let $\beta \in \mathbb{N}$ the smallest number such that $(\beta - 3)ms \geq 10$. Then, for any vertex v at an arbitrary point in $[0,1)^2$ at any time k, the probability that v is in a fixed square cell $c_v \subseteq [0,1)^2$ of side length $r/2$ at time $(k'-k) + \beta m$, which we call its* destination cell *(where k' is the first time after k at which a new angle is chosen), is at least $p := p(n) > 0$, where p is a function depending only on n.*

Proof. Fix an arbitrary vertex v and a fixed destination cell c_v of side length $r/2$. We prove the statement only for the case when $k' = k$, since otherwise we simply wait for the first $k' - k \leq m - 1$ steps until a new angle is chosen by all vertices and our analysis starts with k' (the condition $(\beta - 3)ms \geq 10$ takes care of this). Now fix $\beta \in \mathbb{N}$ such that β is the smallest number with the property $(\beta - 3)ms \geq 10$ (the constant 10 is quite arbitrary and can be made smaller). For any time t denote by $d_t^{(v)}$ the Euclidean distance from the start position of vertex v to the closest point of c_v at time t. To prove the desired lower bound on the probability of reaching c_v we prove a lower bound on the probability of a strategy that is sufficient to reach c_v at time $k + \beta m$: if $d_t^{(v)} \geq 2ms$ (where t is an arbitrary time $\in [k, k + \beta m]$ at which a new angle is chosen) we choose an angle $\alpha \in [a - \pi/4, a + \pi/4]$, where a is the angle that corresponds to the line going through v and its closest point in c_v at time t. If an angle inside this interval is chosen, by the theorem of cosines,

$$d_{t+m}^{(v)} \leq \sqrt{(d_t^{(v)})^2 + (ms)^2 - \sqrt{2} d_t^{(v)} ms} \leq d_t^{(v)} - \frac{1}{2} ms.$$

If always an angle out of the interval of size $\pi/2$ (centered at the angle of the line through v and its closest point in c_v at that time) is chosen, then by the choice of β as in the statement of the lemma we are guaranteed that at the latest at time $k + (\beta - 2)m$ we have $d_{k+(\beta-2)m}^{(v)} < 2ms$. If $d_t^{(v)} < 2ms$ for some time t then we distinguish two cases: if $t < (k - 2)m$ (and if t is such that a new angle is chosen at time t) we choose an angle such that at time $t + m$ we still have $d_{t+m}^{(v)} < 2ms$. Note that in such a case the interval of angles that can be chosen from has size at least π. Otherwise, if we are at time $k + (\beta - 2)m$ then we have to take into account that after $2m$ more steps we have to end up in c_v. Therefore the last two times the angle has to be chosen out of an interval of size $\Theta(r)$, such that at time $t + km$ vertex v ends up (possibly after a suitable detour) exactly in c_v. Since angles are chosen uniformly at random, the probability of choosing an angle according to the strategy is at least

$$(1/4)^{(\beta-2)} (\Theta(r))^2 = (1/4)^{\Theta(1)/ms} (\Theta(r))^2 =: p(n) > 0,$$

for some function $p(n)$, since r (and possibly also s) only depends on n. □

The next lemma allows us to apply Lemma 15.

Lemma 17.

$$\mathbf{E}(L(\mathcal{C})) < +\infty, \quad \mathbf{E}(L(\mathcal{D})) < +\infty.$$

Proof. Since all vertices choose their angles independently from each other, the probability that all vertices v end up after βm steps in their destination cell c_v, is by Lemma 16, at least p^n. Since we allow negative times as well, we can w.l.o.g. apply Lemma 16 with $k' = 0$ and $-m + 1 \leq k \leq 0$ and a $\beta \in \mathbb{N}$ satisfying the conditions of that lemma. If we choose for all vertices the same destination cell c, and all vertices are after βm steps in this destination cell c, then, by construction, at time βm all these vertices form a clique, and hence the graph is connected. Thus we have

$$\begin{aligned}&\mathbf{Pr}(\mathcal{D}_0 \wedge \mathcal{D}_{\beta m} \wedge \mathcal{D}_{2\beta m} \wedge \ldots \wedge \mathcal{D}_{d\beta m})\\ &\leq (1-p^n)^d =: e^{-f(n)d}\end{aligned}$$

for some function $f(n) > 0$ which depends only on n (and also on s, β, but they also depend on n only). Therefore

$$\begin{aligned}&\mathbf{E}(L(\mathcal{D})) = \sum_{d=0}^{\infty} d\, \mathbf{Pr}(\mathcal{D}_0 \wedge \ldots \wedge D_d)\\ &\leq \sum_{d=0}^{\infty} d\, \mathbf{Pr}(\mathcal{D}_0 \wedge \mathcal{D}_{\beta m} \wedge D_{2\beta m} \wedge \ldots \mathcal{D}_{\lfloor \frac{d}{\beta m} \rfloor \beta m})\\ &\leq \sum_{d=0}^{\infty} d\, e^{-f'(n)d} < +\infty,\end{aligned}$$

since f' depends only on n (but not on d). Note however, that it is still possible that $\mathbf{E}(L(\mathcal{D}))$ goes to ∞ as n goes to ∞.
The same argument can be used to show that $\mathbf{E}(L(\mathcal{C})) < +\infty$. In this case we can for example choose for all but one vertex the same destination cell c, but for one special vertex i we choose its destination cell c_i such that any two points in c_i and c are at Euclidean distance strictly greater than r. If after βm steps all vertices end up in their destination cells (which happens with positive probability) then i will be an isolated vertex at that time, and thus the graph is not connected. Since the bounds from Lemma 16 hold for an arbitrary start position and any fixed destination cell, the same bounds also apply in this case.

□

We are now ready to prove our main theorem which characterizes the expected number of steps the graph remains (dis)connected once it becomes (dis)connected.

Theorem 18.

$$\mathbf{E}(L_k(\mathcal{C}) \mid \mathcal{D}_{k-1} \wedge \mathcal{C}_k) \sim \frac{1}{(1-e^{-\mathbf{E}B})} = \begin{cases} \frac{\pi}{4srn} & \textit{if } srn = o(1),\\ \frac{1}{(1-e^{-4srn/\pi})} & \textit{if } srn = \Theta(1),\\ 1 & \textit{if } srn = \omega(1),\end{cases}$$

$$\mathbf{E}(L_k(\mathcal{D}) \mid \mathcal{C}_{k-1} \wedge \mathcal{D}_k) \sim \frac{e^{\mu}-1}{(1-e^{-\mathbf{E}B})} = \begin{cases} \frac{\pi(e^{\mu}-1)}{4srn} & \textit{if } srn = o(1),\\ \frac{e^{\mu}-1}{(1-e^{-4srn/\pi})} & \textit{if } srn = \Theta(1),\\ e^{\mu}-1 & \textit{if } srn = \omega(1).\end{cases}$$

Proof. Since by Lemma 17, $\mathbf{E}(L_k(\mathcal{C})) < +\infty$, $\mathbf{E}(L_k(\mathcal{D})) < +\infty$, we can apply the formula of Lemma 15 and the results follow by Corollary 14. □

4 Conclusions.

In this extended abstract, we have introduced the dynamic random geometric graph and studied the expected length of the connectivity and disconnectivity periods, considering different step sizes s and different lengths m during which the angle remains invariant, always considering the static connectivity threshold $r = r_c$. We believe that a similar analysis can be performed for other values of r as well.

The *random direction mobility* model simulates the behavior of a swarm of mobile agents as sensors or robots, which move randomly to monitor an unknown territory or to search in it. There exist other models such as the *way-point model*, where each agent chooses randomly a fixed way-point (from a set of pre-determined way-points) and moves there, and when it arrives it chooses another and moves there, and so on [Camp]. A possible line of future research is to do a study similar to the one developed in this paper for this way-point model. We believe that the techniques developed in this paper will prove very useful to carry out that study.

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