Routing Games with Progressive Filling*

Tobias Harks[†] Martin Hoefer[‡] Kevin Schewior[§] Alexander Skopalik[¶]
September 10, 2018

Abstract

Max-min fairness (MMF) is a widely known approach to a fair allocation of bandwidth to each of the users in a network. This allocation can be computed by uniformly raising the bandwidths of all users without violating capacity constraints. We consider an extension of these allocations by raising the bandwidth with arbitrary and not necessarily uniform time-depending velocities (allocation rates). These allocations are used in a game-theoretic context for routing choices, which we formalize in progressive filling games (PFGs).

We present a variety of results for equilibria in PFGs. We show that these games possess pure Nash and strong equilibria. While computation in general is NP-hard, there are polynomial-time algorithms for prominent classes of Max-Min-Fair Games (MMFG), including the case when all users have the same source-destination pair. We characterize prices of anarchy and stability for pure Nash and strong equilibria in PFGs and MMFGs when players have different or the same source-destination pairs. In addition, we show that when a designer can adjust allocation rates, it is possible to design games with optimal strong equilibria. Some initial results on polynomial-time algorithms in this direction are also derived.

1 Introduction

Max-min fairness is a widely used paradigm for bandwidth allocation problems in telecommunication networks, most prominently, it is used as a reference point for designing flow control/congestion control protocols such as TCP (Transport Control Protocol), see [33] for a more detailed discussion. In a max-min fair allocation, the bandwidth of a user cannot be increased without decreasing the bandwidth of another user, who already receives a smaller bandwidth. Max-min fairness also plays an important role in the model of Kelly et al. [20], where congestion control protocols have been interpreted as distributed algorithms at sources and links in order to solve a global optimization problem (cf. [24, 23, 29] for further works in this area). Each user is associated with an increasing, strictly concave bandwidth utility function and the congestion control algorithms aim at maximizing aggregate utility subject to capacity constraints on the links. Mo and Walrand [29] showed

^{*}This research was partially supported by the German Research Foundation (DFG) within the Cluster of Excellence MMCI at Saarland University, the Research Training Group "Methods for Discrete Structures" (GRK 1408) and the Collaborative Research Center "On-The-Fly Computing" (SFB 901). It was also partially supported by the EU within FET project MULTIPLEX (contract no. 317532) and the Marie-Curie grant "Protocol Design" (no. 327546, funded within FP7-PEOPLE-2012-IEF).

[†]Dept. of Quantitative Economics, Maastricht University, t.harks@maastrichtuniversity.nl

[‡]Max-Planck-Institut für Informatik and Saarland University, mhoefer@mpi-inf.mpg.de

[§]Institut für Mathematik, TU Berlin, schewior@math.tu-berlin.de

[¶]Dept. of Computer Science, Paderborn University, skopalik@mail.upb.de

that within the model of Kelly et al., there is a family of utility functions whose global optimum corresponds to a max-min fair bandwidth allocation and they devised a distributed max-min fair congestion control protocol, see also [32] (Section 2.2) and [27]. For further distributed max-min fair congestion control protocols, we refer to [36, 38]. There are several important generalizations of max-min fairness such as weighted max-min fairness [36] and utility max-min fairness [12]. In a weighted max-min fair allocation, the weighted bandwidth of a user cannot be increased without decreasing the weighted bandwidth of another user, who already receives a smaller weighted bandwidth. In a utility max-min fair allocation, each user is associated with an increasing (not necessarily concave) bandwidth utility function and an allocation is utility max-min fair if the utility of a user cannot be increased without decreasing the utility of another user, who already receives a smaller utility. Utility max-min fairness (and also weighted max-min fairness) has been proposed for giving some applications (e.g., real-time applications, or multi-media) a possibly larger bandwidth share than others.

It is well known that (weighted) max-min fair allocations can be easily implemented by simple polynomial time water-filling algorithms that raise the bandwidth of every user at a (weighted) uniform speed and, whenever a link capacity is exhausted, fixes the bandwidth of those users traversing this link [7]. As we will show in this paper, also utility max-min fair allocations can be implemented by simple polynomial time water-filling algorithms that raise the bandwidth of every user at a user-specific speed.

While most works in the area of flow control/congestion control assume that the routes of users are fixed a priori, we study in this paper the flexibility of strategic route choices by users (or players from now on) as a means to obtain high bandwidth. We introduce a general class of strategic games that we term routing games with progressive filling. In such a game, there is a finite set of resources and a strategy of a player corresponds to a subset of resources. Resources have capacities and the utility of every player equals the obtained bandwidth which in turn is defined by a predefined water-filling algorithm. If the allowable subsets of a player correspond to the set of routes connecting the player's source with its terminal, we obtain single-path routing modeling IP (Internet Protocol) routing. Since IP routing is typically updated at a much slower timescale than the flow control, we assume that flow control (modeled in this paper as a water-filling algorithm) converges instantly to a "fair" allocation (max-min fair or generalizations thereof) after each route update. The assumption that flow control converges instantly before route updates are triggered has been made and justified before, see, e.g., Wang et al. [34]. Thus, once a player chooses a new route his bandwidth share is determined by executing the water-filling algorithm. We will impose mild conditions on the class of allowable water-filling algorithms: (i) for every player and every point in time the integral of the rate function is non-negative and the integral of the rate function grows monotonically; (ii) for every player the integral of the rate function tends to infinity as time goes to infinity. While condition (i) is natural, condition (ii) simply ensures that the water filling algorithm terminates and the induced strategic game is well-defined. Note that even though water-filling algorithms are centralized algorithms we demonstrate that they represent a wide range of fairness concepts including max-min fairness, weighted max-min fairness and utility max-min fairness for which distributed and fast converging congestion control protocols are known [28, 29, 36, 38].

We consider existence, computation and quality of equilibria in routing games with progressive filling. In a pure Nash equilibrium (PNE for short), no player obtains strictly higher bandwidth by unilaterally changing his route. If coordinated deviations by players are allowed (for instance by a single player coordinating several sessions or by a set of players connected via peer-to-peer

overlay networks), the Nash equilibrium concept is not sufficient to analyze stable states of a game. For this situation, we adopt the stricter notion of a strong equilibrium (SE for short) proposed by Aumann [4]. In a SE, no coalition (of any size) can change their routes and strictly increase the bandwidth of each of its members (while possibly lowering the bandwidth of players outside the coalition). Every SE is a PNE, but not conversely. Thus, SE constitute a very robust and appealing stability concept for which only a few existence results are known in the literature.

1.1 Our Results

Existence. For progressive filling games we prove that if water-filling algorithms satisfy conditions (i) and (ii), every sequence of profitable deviations of coalitions of players must be finite and, hence, SE always exist. Previously, it was only known that PNE exist if the water-filling algorithm corresponds to the max-min fair allocation [37]. Thus, our results establish for the first time that routing and congestion control admits a PNE (and even SE) for routing games where weighted- and utility max-min fair congestion control protocols are used. We show that our assumptions (i) and (ii) are "minimal" in the sense that if one of them is dropped, there is a corresponding two-player game without PNE.

Complexity. In light of its practical importance, we study routing games with water-filling algorithms inducing the max-min fair allocation. We first focus on the computational complexity of SE and PNE. We give an algorithm that computes a SE for any progressive filling game under max-min fair allocations. Our algorithm iteratively reduces the number of players allowed on a resource. After each such reduction, a packing oracle is invoked that checks whether or not there is a feasible strategy profile that respects the allowed numbers of players on every resource. If the oracle finds a feasible allocation, the algorithm proceeds and, otherwise, we fix strategies for a suitable subset of players. Obviously, the running time of the algorithm crucially relies on the running time of the packing oracle. It is known, however, that if the strategy spaces correspond to, e.g., the set of paths of a single-commodity network, or to bases of a matroid defined on a player-specific subset of resources, the oracle can be implemented in polynomial time, thereby ensuring polynomial-time computation of SE. We complement this result by showing various hardness results of computing SE. In addition, we show a bound on the number of values of the potential function that also represents an upper bound on the number of improvement steps to reach a PNE.

Quality. To measure the quality of an equilibrium, we use the achieved throughput defined as the sum of the player's bandwidths. This performance measure corresponds to utilitarian social welfare and is the standard performance measure in traffic engineering. We use notions of price of stability (PoS) and price of anarchy (PoA), which relate the cost of an equilibrium to the cost of a social optimum. The standard definition of an optimum would refer to a set of route choices such that throughput is maximized for a waterfilling algorithm with given allocation rates. In addition, our bounds continue to hold even with respect to an optimum that is allowed to set arbitrary routes and bandwidths respecting the resource capacities. Computing this general optimum is known in combinatorial optimization as the maximum k-splittable flow problem.

We provide tight bounds for SE and PNE. In general, the PoS and PoA are n, which is tight for both PNE and SE, even in single-commodity PFGs or multi-commodity MMFGs. In single-commodity MMFGs, PoS for PNE and SE is $\left(2 - \frac{1}{n}\right)$, PoA for PNE is n and PoA for SE is 4.

All bounds except the latter are tight. In addition, our algorithm that computes SE for single-commodity MMFGs in polynomial time yields SE that match the PoS bound. In addition, we show some improved bounds on the PoA for PNE in singleton PFGs.

Protocol Design. Using fixed allocation rates, improving upon the $(2-\frac{1}{n})$ -bound is impossible in the worst case. We show, however, that it is possible to show better results when we have slight flexibility in allocation rates. We assume the freedom to "design a protocol" and adjust weights in a weighted MMF waterfilling algorithm towards the topology of the instance. This allows to design a game with an optimal SE that coincides with the maximum k-splittable flow. While computing such an optimum is NP-hard, the result also shows that starting from any α -approximation to the maximum k-splittable flow, we can design weights and a starting state, such that every sequence of unilateral (coalitional) improvement moves leads to a PNE (SE) with the same approximation ratio. We apply this approach in games with 3 players, where we can find in polynomial time a solution that is a 1.5-approximation and represents a PNE for the chosen weights.

1.2 Related Work

Combined routing and congestion control has been studied by several works (cf. [11, 35, 21, 19]). In all these works, the existence of an equilibrium is proved by showing that it corresponds to an optimal solution of an associated convex utility maximization problem. This, however, implies that every user possibly splits the flow among an exponential number of routes which might be critical for some applications. For instance, the standard TCP/IP protocol suite uses single path routing, because splitting the demand comes with several practical complications, e.g., packets arriving out of order, packet jitter due to different path delays etc. This issue has been explicitly addressed by Orda et al. [30].

Another related class of games are *congestion games*, where there is a set of resources, and the pure strategies of players are subsets of this set. Each resource has a delay function depending on the load, i.e., the number of players that select strategies containing the respective resource. These games allow to model network structures, but they fail to incorporate a realistic allocation of network capacities. The reason is that, even though we can define bandwidths allocated on an edge as a function of the number of players using it, the bandwidth of a player would be given by the sum of bandwidth allocated on edges he uses. This problem is addressed by bottleneck congestion games [10] where the bandwidth of one player is rather given by the maximum bandwidth among the edges he uses. It is known that strong equilibria exist for bottleneck congestion games [15]. The complexity of computing PNE and SE in these games was further investigated in [13], where a central result is an algorithm called Dual Greedy that computes SE. On single-commodity network or matroid bottleneck congestion games, it can be implemented to run in polynomial time. Still, for an arbitrary state, the computation of a coalitional improvement step turned out to be NP-hard, even for these classes. The PoA for PNE in bottleneck games can be polynomial in the network size. for social cost being the sum of player delays [10] or maximum player delay [9, 6]. For the latter cost function, the PoA for SE becomes 2 for symmetric games with linear delays [18]. Improved results were obtained for variants, in which players individual costs are exponential or polynomial functions of their delays [17, 8].

A fundamental drawback of bottleneck congestion games is that the bandwidth allocated to a player on a specific edge is *solely* a function of the number of players on it. If one of the players does not exhaust his allocated bandwidth share (e.g., because he has a smaller bottleneck on another

edge) the remaining bandwidth remains unused. In max-min fair allocations [16], this leftover is fairly distributed among players who can make use of it.

Yang et al. [37] introduced so-called MAXBAR-games which correspond to progressive filling games using max-min fair allocations. They show that these games possess PNE and that the price of anarchy for PNE is n in these games, where n is the number of players. It is also shown that iterative computation of unilateral improvement steps converge in polynomial time to a PNE if the number of players is constant.

Amaldi et al. [2] considered a centralized approach to computing routes maximizing the aggregated bandwidth subject to max-min fair allocations. They show hardness results (for multicommodity networks) and devise an exact algorithm using column generation. Kleinberg et al. [25] devise approximation algorithms (and hardness results) for the same problem using an approximate version of max-min fairness.

In terms of combinatorial optimization, the problem of computing a strategy with maximum aggregated bandwidth (without fairness constraints) is related to the maximum k-splittable flow problem [5]. In contrast to the ordinary maximum flow problem, the number of paths flow is sent along is bounded by k, for each commodity. Positive results were especially found for the single-commodity case. For k=2 and k=3, a $\frac{3}{2}$ -approximation was given and this result was generalized to a 2-approximation for arbitrary fixed k. It turned out that, asymptotically, any approximation with a factor of smaller than $\frac{6}{5}$ is NP-hard to obtain. Furthermore, for k=2, $\frac{3}{2}$ is exactly the inapproximability bound [26].

2 Progressive Filling Game

A progressive filling game is a tuple $(N, R, (c_i)_{r \in R}, (S_i)_{i \in N}, (v_i)_{i \in N}, (u_i)_{i \in N})$, where $N = \{1, \ldots, n\}$ is the set of players, $R = \{1, \ldots, m\}$ is the set of resources, $c_r \in \mathbb{R}_+$ is the capacity of resource r for each $r \in R$. The allocation rate is defined as $v_i : \mathbb{R}_+ \to \mathbb{R}_+$ and is assumed to be (Riemann) integrable. The aggregated rate (or bandwidth) of player i at time t' is defined as $V_{i}(t') = \int_{0}^{t'} v_{i}(t) dt$. We assume that for all $i \in N$, $V_{i} \geq 0$, $V_{i}(t)$ is monotonically non-decreasing in t, and $\lim_{t\to\infty} V_i(t) = \infty$. We denote by $S_i \subseteq \mathcal{P}(R)$ the set of strategies of player i, for each $i \in N$, and $S = S_1 \times \cdots \times S_n$ are the set of states. Note that this definition is kept very general and can be restricted to model more specific objects, e.g. networks. An allocation in state $S \in \mathcal{S}$ is a vector $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ of feasible bandwidths, i.e., $\sum_{i \in N: r \in S_i} a_i \leq c_r$, for each $r \in R$. The *i*-th component of a is called the bandwidth or capacity of player i (in a). Given S, we create an allocation the following way. Each of the players starts off with a bandwidth $b_i = 0$. We raise their bandwidths with the velocity $v_i(t)$ at time step $t \in \mathbb{R}$ until a further increase would lead to non-feasible capacities (i.e., one of the resources is saturated). At this point, we fix the bandwidths of all the corresponding (saturated) players and continue with the other ones. See Algorithm 1 for a formal description. For given S, we denote by $t_i(S)$ the finishing time, i.e., the time when player i's bandwidth is fixed. Thus, the payoff for player i is given by $u_i(S) = V_i(t_i(S)) = a_i$. We can easily extend our model to allow for player-specific payoff functions of the form $u_i(S) = U_i(V_i(t_i(S))) = U_i(a_i)$, where U_i is a differentiable and strictly increasing bandwidth utility function. As long as U_i is strictly increasing (yielding a monotone payoff transformation), an allocation is a PNE (SE) in the new game iff it is one in the original game. We now state a useful observation linking the outcome of Algorithm 1 with different fairness concepts.

Proposition 1. Let U_i , $i \in N$ be a set of nonnegative, differentiable and strictly increasing bandwidth

Algorithm 1 Progressive Filling(PF)

```
Parameters: A progressive filling game \mathcal{G} = (N, R, (c_i)_{r \in R}, (\mathcal{S})_{i \in N}, (v_i)_{i \in N})
Input: A state S = (S_1, \ldots, S_n) \in \mathcal{S}.
Output: The bandwidth b_i for each player i \in N.
   1: b_i \leftarrow 0, for all i \in N; N' \leftarrow N
   2: N_r \leftarrow \{i \in N \mid r \in S_i\}; c'_r \leftarrow c_r, for all r \in R
   3: while N' \neq \emptyset do
                t^{\star} \leftarrow \min\{t' \mid \exists r \in R\}
                                              with \sum_{i \in N_r \cap N'} \int_0^{t'} v_i\left(t\right) dt = c'_r and N_r \cap N' \neq \emptyset
               \begin{array}{c} \text{choose } r^{\star} \text{ with } \sum_{i \in N_{r^{\star}} \cap N'} \int_{0}^{t^{\star}} v_{i}\left(t\right) dt = c'_{r^{\star}} \\ \text{ and } N_{r^{\star}} \cap N' \neq \emptyset \\ \text{for each } i \in N_{r^{\star}} \cap N' \text{ do} \\ b_{i} \leftarrow \int_{0}^{t^{\star}} v_{i}\left(t\right) dt \\ N' \leftarrow N' \setminus \{i\} \end{array}
   5:
   6:
   7:
   8:
                        for each r \in S_i do
   9:
                               c'_r \leftarrow c'_r - b_i
 10:
                        end for
 11:
 12:
                end for
 13: end while
 14: return (b_1, \ldots, b_n)
```

utility functions and let w_i , $i \in N$ be a set of nonnegative weights. For given progressive filling game and state S, the following holds:

- 1. If for all $i \in N$: $v_i(t) = 1$ and $u_i(S) = V_i(t_i(S))$, Algorithm 1 computes a max-min fair bandwidth allocation under S.
- 2. If for all $i \in N : v_i(t) = w_i$ and $u_i(S) = V_i(t_i(S))$, Algorithm 1 computes a weighted max-min fair bandwidth allocation under S.
- 3. If for all $i \in N$: $v_i(t) = \frac{d}{dt}(U_i^{-1}(t))$ and $u_i(S) = U_i(V_i(t_i(S)))$, Algorithm 1 computes a utility max-min fair bandwidth allocation under S.

Proof. As (1) and (2) are known in the literature (cf. [7]) we only prove (3). In order to obtain a utility max-min fair allocation, we need to ensure that while raising rates, the bandwidth utilities must be equally distributed. Thus, starting with t = 0 we set $U_i(V_i(t) = t$ for all $i \in N$. This is equivalent to $U_i^{-1}(t) = V_i(t)$ using that U_i is strictly increasing and thus invertible. Differentiating both sides leads to $v_i(t) = \frac{d}{dt} \left(U_i^{-1}(t) \right)$ as claimed. Since U_i is strictly increasing, its inverse is also strictly increasing (and also nonnegative), hence, $v_i(t)$ satisfies all assumptions needed. Now it follows by standard arguments (cf. [12]) that the resulting allocation is utility max-min fair.

3 Existence of Equilibria

We first study game-theoretic properties of a progressive filling game. We show that SE exist and moreover every sequence of improving deviations of coalitions converges to a SE.

Theorem 2. Every progressive filling game has a SE and every sequence of improving deviations of coalitions converges to a SE.

Proof. Let \mathcal{G} be a PFG and S a state in this game. For a player i, recall that we denote by $t_i(S)$ the finishing time, i.e., the point in time when his bandwidth is fixed by Algorithm 1 on S. Likewise, we denote by $\tilde{t}_r(S)$ the point in time when resource r gets saturated. In the remainder of the proof we crucially exploit the monotone relationship between the obtained bandwidth and the finishing time of every player. By the monotonicity of the V_i 's, if a player strictly improves his obtained bandwidth by using an alternative strategy, then the new finishing time must strictly increase.

For a state S, we define a lexicographical potential function $\phi: \mathcal{S} \to \mathbb{R}^n_+$ as the vector of finishing times sorted in non-decreasing order, i.e., $\phi(S) = (t_{i_1}(S), \dots, t_{i_n}(S))$ with $\{i_1, \dots, i_n\} = N$ and $t_{i_j}(S) \leq t_{i_{j+1}}(S)$.

The next lemma shows that in a state S an improving move of a coalition C to a state T implies that $\phi(S) \prec \phi(T)$ where \prec denotes the lexicographic ordering of vectors. Thus, a \prec -maximal state must be a SE. This implies the existence of the potential function and thereby the theorem. \square

Lemma 3. Let $C \subseteq N$ be a coalition which has an improving move from $S = (S_1, ..., S_n)$ to $T = (T_1, ..., T_n)$ where $S, T \in \mathcal{S}$. Then we have

- (a) $t_i(T) \ge t_i(S)$, for all $i \in N$ with $t_i(S) \le \min_{j \in C} t_j(S)$, and
- (b) $t_i(T) > \min_{j \in C} t_j(S)$, for all $i \in N$ with $t_i(S) > \min_{j \in C} t_j(S)$.

Proof. For some player i, note that we have $\tilde{t}_r(S) > t^*$ for all $r \in T_i$ if and only if $t_i(S) > t^*$ for some $t^* \in \mathbb{R}$. Hence, it suffices to show that

- (a') $\tilde{t}_r(T) \geq \tilde{t}_r(S)$, for all $r \in R$ with $\tilde{t}_r(S) \leq \min_{j \in C} t_j(S)$, and
- (b') $\tilde{t}_r(T) > \min_{j \in C} t_j(S)$, for all $r \in R$ with $\tilde{t}_r(S) > \min_{j \in C} t_j(S)$.

For all $r \in T_i$ for some $i \in C$, the claim directly follows because we have $t_i(T) > t_i(S) \ge \min_{i \in C} t_i(S)$ and thus $\tilde{t}_r(T) > \min_{i \in C} t_i(S)$. So let $r \in R$ such that r is not used in T by any player from C.

In S, no resource which is used by a player from C has been saturated before $\min_{i \in C} t_i(S)$. Consequently, the bandwidth allocated to a player i is identical at time $\min_{i \in C} t_i(S)$ in S and T for all $i \in N \setminus C$. Since in T resource r is used by exactly the same players from $N \setminus C$ as in S and by no player from C, the residual capacity of r at a time $t \leq \min_{i \in C} t_i(S)$ is in T at least as high as in S.

This last result immediately implies (a'). For (b'), let $\tilde{t}_r(S) > \min_{j \in C} t_j(S)$. This means that the residual capacity at time $\min_{i \in C} t_i(S)$ is above zero in S and hence also in T. By the continuity of the indefinite integrals of the allocation rate functions, we obtain $\tilde{t}_r(T) > \min_{j \in C} t_j(S)$.

Note that the above result applies to PFGs in full generality, that is, only requiring that the functions V are non-negative, non-decreasing, and tend to infinity for t going to infinity. We now show that the assumptions underlying this result cannot be relaxed. Clearly, relaxing non-negativity or relaxing the unboundedness of V makes not much sense. Negative aggregated rates have no physical meaning, and for a bounded V there exists a game with large enough capacities for which Algorithm 1 does not terminate. More interestingly, suppose we have an allocation rate

function for which the aggregated bandwidth V(t) is non-monotonic. Note that this extension still allows to use Algorithm 1 to calculate the allocation via progressive filling. We show that for any such function, Theorem 2 does not hold anymore. This is even true if we restrict to two-player games with symmetric strategy spaces.

Theorem 4. Let v be such that $V: \mathbb{R}_+ \to \mathbb{R}_+, t' \mapsto \int_0^{t'} v(t) dt$ satisfies $V \geq 0$ and $\lim_{t \to \infty} V(t) = \infty$. If V(t) is not monotone, there is a two-player PFG \mathcal{G}_v with symmetric strategy spaces that does not have a PNE and only uses v and one constant function as allocation rate functions.

Proof. Let v be an allocation rate function such that the aggregated rate function V is not monotone. By the continuity and non-negativity of V, there is $t_1 > 0$ such that for every $\epsilon > 0$, there is $t_2 = t_2(\epsilon) \in (t_1, t_1 + \epsilon)$ with $V(t_1) > V(t_2)$ (see [14, Lemma 3.1]). Thus, we can choose t_2 satisfying $t_2 < t_1 + \epsilon$ for any $\epsilon > 0$ to be specified later. Since v is Riemann integrable and thus on the interval $[0, t_2]$ bounded, its indefinite integral V has a Lipschitz constant $\rho > 0$ on $[0, t_2]$.

We now describe the game \mathcal{G}_v with two players $\{1,2\}$. We set $R=\{r_1,r_2,r_3\}$ with $c_{r_1}=c_{r_2}=(\rho+1)t_1+V(t_1)$ and $c_3=(\rho+1)t_2+V(t_2)$. Furthermore, the sets of strategies are $\mathcal{S}_1=\mathcal{S}_2=\{\{r_1,r_3\},\{r_2,r_3\}\}$. As allocation rate functions, we use $v_1\equiv v$ and $v_2\equiv \rho+1$. We claim that, whenever both players share one of the resources r_1 or r_2 , the shared resource is saturated at time t_1 and player 2 gets bandwidth $(\rho+1)t_1$ while player 1 gets bandwidth $V(t_1)$. To see this, we use the Lipschitz inequality $\frac{V(t)-V(t_1)}{t_1-t}<(\rho+1)$ for all $t\in[0,t_1)$ implying $(\rho+1)t+V(t)<(\rho+1)t_1+V(t_1)$ for all $t\in[0,t_1)$. On the other hand, whenever player 2 is alone on either r_1 or r_2 , resource r_3 is saturated at time t_2 using again $\frac{V(t)-V(t_2)}{t_2-t}<(\rho+1)$ for all $t\in[0,t_2)$. By choosing $t_2< t_1+V(t_1)/(\rho+1)$ (hence $t_2=t_2(\epsilon)$ with $\epsilon=V(t_1)/(\rho+1)$) we get $(\rho+1)t_2<(\rho+1)t_1+V(t_1)$ and, thus, none of the resources r_1 or r_2 gets saturated before t_2 . Consequently, player 2 gets bandwidth $(\rho+1)t_2>(\rho+1)t_1$ while player 1 gets bandwidth $V(t_2)< V(t_1)$. Hence, there is no PNE.

4 Max-Min-Fair Progressive Filling Games

A special case of progressive filling games arises if all players raise their bandwidth uniformly, i.e., $v_i(t) = 1$ for all $i \in N$. This leads to allocations that are max-min fair. We call such a game max-min-fair progressive filling game or MMFG. More formally, let $S \in \mathcal{S}$ be a state and $\mathcal{A} = \{a \mid a \text{ is an allocation in } S\}$, then the unique \leq -maximal a^* in \mathcal{A} is the max-min fair allocation. In the following, we will study the computational complexity and efficiency of SE and PE in MMFGs.

4.1 Computing Equilibria

Similar to [13], we use a dual greedy algorithm [31] to compute strong equilibria. Our dual greedy algorithm is allowed to query a strategy packing oracle that solves the strategy packing problem which is the following: The input is given by a set R of $m \in \mathbb{N}$ resources, n sets of strategies $S_i \in \mathcal{P}(R)$, for all $i \in \{1, \ldots, n\}$, along with upper bounds $u_r \in \{0, \ldots, n\}$, for each $r \in R$. The output is a state $(S_1, \ldots, S_n) \in S_1 \times \cdots \times S_n$ satisfying the upper bounds, i.e., $|\{i \in N \mid r \in S_i\}| \leq u_r$, for all $r \in R$, if it exists. Otherwise the output is the information that no such state exists.

The dual greedy algorithm initially allows an upper bound of $u_r = n$ players on each resource r and every resource and every player is initially considered *free*. The algorithm starts with an arbitrary state S of strategies for players. It iteratively decrements one of the bounds u_r on a free resource providing minimum bandwidth if each resource was used by u_r players. After each

Algorithm 2 Dual Greedy Algorithm

```
Let \mathfrak{O} denote the strategy packing oracle.
Input: A MMFG \mathcal{G} = (N, R, (c_i)_{r \in R}, (\mathcal{S})_{i \in N})
Output: A SE in \mathcal{G}.
  1: b_i \leftarrow 0, for all i \in N; N' \leftarrow N
  2: u_r \leftarrow n, c_r' \leftarrow c_r, for all r \in R
  3: while N' \neq \emptyset do
             (S_i')_{i \in N'} \leftarrow \mathfrak{O}\left(R, (S_i)_{i \in N'}, (u_r)_{r \in R}\right)
             choose r^* \in \arg\min_{r \in E: u_r > 0} \frac{c'_r}{u_r}
  5:
             u_{r^{\star}} \leftarrow u_{r^{\star}} - 1
  6:
             if \mathfrak{O}\left(R, (\mathcal{S}_i)_{i \in N'}, (u_r)_{r \in R}\right) = \emptyset then
  7:
                   u_{r^{\star}} \leftarrow u_{r^{\star}} + 1
b \leftarrow \frac{c'_{r^{\star}}}{u_{r^{\star}}}
  8:
  9:
                    for each i \in N' with r^* \in S'_i do
 10:
                          S_i \leftarrow S_i' \\ N' \leftarrow N' \setminus \{i\}
11:
12:
                          for each r \in S_i do
13:
                                 u_r \leftarrow u_r - 1
14:
                                 c'_r \leftarrow c'_r - b
 15:
 16:
                    end for
17:
 18:
             end if
 19: end while
20: return S
```

decrement, it checks the existence of a strategy profile respecting the new upper bounds on the number of players using it by invoking the strategy packing oracle. When a decrease produces infeasible bounds, i.e., when there is no state of the game respecting the new bounds, it reverts the last decrease. Now we know that in the profile that was returned by the oracle, exactly u_r players are using r and it is infeasible to further reduce u_r . Thus, the algorithm turns r into a fixed resource, and also fixes the u_r players as well as their strategies. In addition, it decreases every resource capacity by the amount given to the u_r fixed players in their strategies. Then it continues with the remaining players, resources, and residual capacities. For a formal statement of the algorithm see Algorithm 2.

Theorem 5. The dual greedy algorithm computes a SE.

Proof. The main idea of the proof is similar to [13], i.e., the iterative assignment of Dual Greedy yields a lexicographically maximal vector of bandwidths. Consider on each resource the residual capacity not yet assigned to fixed players. We can assume that this residual capacity is offered in equal shares to the remaining free players. Thus, the share of each free player only depends on the number of free players using it. Hence, as long as no players are fixed, the game can be seen equivalently as a bottleneck congestion game. In addition, once a resource and players are fixed, then the bandwidth of a fixed player is smaller than the equal share of residual capacity on every free resource he uses. This allows to inductively show correctness of the algorithm.

More formally, fix a run of the dual greedy algorithm on the input instance given in the formal description and denote the output by $S = (S_1, \ldots, S_n)$. Furthermore, by b_i , for a player $i \in N$, we denote his bandwidth calculated just before his strategy was fixed. We start off with proving the following useful lemma.

Lemma 6. Consider the t-th run of the main loop in Algorithm 2 where t > 1. If $u_r > 0$, the value of $\frac{c'_r}{u_r}$ is not smaller than the value in the (t-1)-th run of the main loop.

Proof (Lemma). Observe that, in one run of the main loop, the fraction $\frac{c'_r}{u_r}$ for some $r \in R$ can only be changed for the following two reasons.

Case 1: The resource r is chosen in line 5 and the oracle does not evaluate to \emptyset in line 7. Then u_r is decremented, i.e., the above fraction is increased.

Case 2: A resource r^* (not necessarily $r \neq r^*$) is chosen in line 5, the oracle evaluates to \emptyset in line 7 and r occurs in k different strategies S'_i obtained from the oracle in line 4 where $1 \leq k \leq u_{r^*} \leq n$. According to the calculations from line 13 to line 16, the new value of the above fraction is

$$\frac{c_r' - k \cdot \frac{c_{r^*}'}{u_{r^*}}}{u_r - k} = \frac{u_{r^*} \cdot c_r' - k \cdot c_{r^*}'}{u_{r^*} \cdot u_r - k \cdot u_{r^*}}$$

where we let $k < u_r$ since, otherwise, the new u_r is 0. Further, we have

$$\frac{u_{r^{\star}} \cdot c_r' - k \cdot c_{r^{\star}}'}{u_{r^{\star}} \cdot u_r - k \cdot u_{r^{\star}}} \ge \frac{c_r'}{u_r}$$

which is equivalent to

$$u_{r^*} \cdot c'_r - k \cdot c'_{r^*} \ge u_{r^*} \cdot c'_r - k \cdot u_{r^*} \cdot \frac{c'_r}{u_r}$$

and because of the choice of r^* gives us $c'_{r^*}/u_{r^*} \leq c'_r/u_r$, which implies the claim.

Now let N_k be the set of players whose strategies are fixed as a consequence of the oracle's k-th evaluation to \emptyset . We show by induction on k that none of the players from $N_1 \uplus \cdots \uplus N_k$ will be part of a coalition performing an improving move, for all k. Note that this proves the theorem because we have $N = N_1 \uplus \cdots \uplus N_l$ for some $l \in \mathbb{N}$.

The base case of k=0 follows trivially. Now assume that the statement holds for some k < l. To see that this implies the statement for k+1, observe that the strategies S_i and bandwidths b_i of the players $i \in N_1 \uplus \cdots \uplus N_k$ are already fixed. Now suppose there is a coalition $C \subseteq N$ with $C \cap N_{k+1} \neq \emptyset$ profitably deviating from S to $T = (S'_C, S_{-C})$. We consider the state of the variables at line 9 after the oracle's k+1-th evaluation to \emptyset .

Since N_1, \ldots, N_k are not participating in the improvement step, Lemma 6 implies that $N_{k+1} \in \operatorname{argmin}_{j \in C} b_j(S)$. Thus, for $i^* \in N_{k+1}$, Lemma 3 can be used to obtain that

- $b_i(T) \geq b_{i^*}(S)$, for all $i \in N$ with $b_i(S) = b_{i^*}(S)$, and
- $b_i(T) > b_{i^*}(S)$, for all $i \in N$ with $b_i(S) > b_{i^*}(S)$.

Again by Lemma 6, this means that we have $l_r(T) \leq u_r$, for all $r \in R$. In particular, we even have $l_{r^*}(T) < u_{r^*}$ as $C \cap N_{k+1} \neq \emptyset$ and the players from C strictly improve. Such a state T may, however, not exist by the evaluation of the oracle to \emptyset .

Dual Greedy can be implemented in polynomial time given an efficient strategy packing oracle. Hence, the problem of computing SE in MMFGs is polynomial-time reducible to the strategy packing problem. There are several non-trivial cases in which the strategy packing problem is polynomial-time solvable, e.g., for single-commodity networks [13]. Thus, we obtain the following result.

Corollary 7. SE can be computed in polynomial time for single-commodity network MMFGs.

In contrast, the strategy packing problem turns out to be NP-hard even if we generalize to symmetric (non-network) strategy spaces.

Theorem 8. The strategy packing problem for symmetric strategies is NP-hard.

Proof. We reduce from the strongly NP-hard set packing problem. Given an instance of the set packing problem $\mathcal{I} = (\mathcal{U}, \mathcal{S}, k)$. From \mathcal{I} , we construct the following strategy packing instance \mathcal{J} . As resource set, we choose \mathcal{U} and, for each upper bound, we choose $u_r = 1$. Furthermore, we set $\mathcal{S}_1 = \cdots = \mathcal{S}_k = \mathcal{S}$ for the strategy sets. It is easy to see that there exists a set packing in \mathcal{I} if and only if there exists a strategy packing in \mathcal{J} . This is because each family of subsets $\mathcal{S}' \subseteq \mathcal{S}$ gives a state in \mathcal{J} and vice versa. Obviously, \mathcal{S}' has mutually disjoint elements if and only if the corresponding state is satisfies the upper bounds.

This result permits computation of SE polynomial time by other algorithms than Dual Greedy, but, in fact, computation of SE and strategy packing are *mutually* polynomial-time reducible, even for symmetric games.

Theorem 9. The computation of a SE in symmetric MMFGs is NP-hard.

Proof. We reduce the strategy packing problem to the computation of SE in symmetric MMFGs. Let $\mathcal{I} = \left(R, (\mathcal{S}_i)_{i \in \{1, \dots, n\}}, (u_r)_{r \in R}\right)$ be an instance of the symmetric strategy packing problem, i.e., we have $\mathcal{S}_1 = \dots = \mathcal{S}_k$. We create a symmetric MMFG $\mathcal{G}_{\mathcal{I}}$ the following way. As resources, we define

$$R' = R \cup \{r_1, \dots, r_n, r'_1, \dots, r'_n, r^*\}.$$

The set of strategies for each of the n+1 players is defined by

$$S_i' = \{S_1 \cup \{r_j, r_j'\} \mid 1 \le j \le n \land S_1 \in S_1\} \cup \{\{r_1, \dots, r_n, r^*\}, R \cup \{r^*\}\}.$$

Finally, we set

$$c_r = \begin{cases} u_r + 1, & \text{if } r \in R \\ 2 - \varepsilon, & \text{if } r = r_i \text{ for some } i \\ 1, & \text{if } r = r'_i \text{ for some } i \text{ or } r = r^* \end{cases}$$

as the capacity for each resource $r \in R$ where we choose $\varepsilon < \frac{1}{(n+1)^{n+1}}$. This defines a unique MMFG $\mathcal{G}_{\mathcal{I}}$ with bandwidth functions $(b_i)_{i \in \{1,\dots,n+1\}}$. The model is illustrated in Figure 1. Furthermore, this is obviously a polynomial time reduction (assuming ε is chosen accordingly).

We will now show that, from a SE S in $\mathcal{G}_{\mathcal{I}}$ such that each player gets at least a bandwidth of 1 in S, we can construct a strategy packing in \mathcal{I} . Conversely, we will show that the existence of any other SE already certifies that no such strategy packing exists.

The easier direction is the following. If in a state $S = (S_1, \ldots, S_n)$ in $\mathcal{G}_{\mathcal{I}}$, we have $b_i(S) \geq 1$, for all $i \in \{1, \ldots, n+1\}$, at most for one player i^* we can have $r^* \in S_{i^*}$ because $c_{r^*} = 1$. Furthermore,

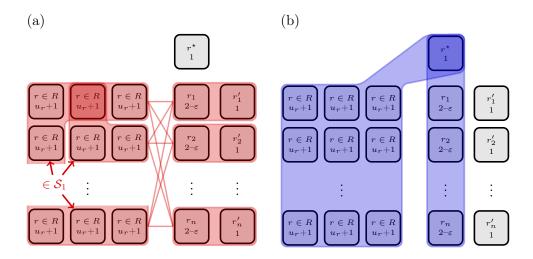


Figure 1: Illustration of the strategies in the proof of Theorem 9. The strategies built from the strategies in the strategy packing instance where a line between two sets indicates that the union is in \mathcal{S}' (a) as well as the two strategies independent of the strategies in strategy packing instance (b) are shown. Note that the resources in (a) and (b) are identical.

as either r^* or some $r'_i \in R'$ occur in every strategy and the latter resources only allow a total bandwidth of n, there must exist at least one such player. Thus, there exists a unique player i^* .

Moreover, we must have $S_{i^*} = R \cup \{r^*\}$ since, otherwise, $r_1, \ldots, r_n \in S_{i^*}$ would hold, i.e., all the other players could get at most bandwidth $\frac{2-\varepsilon}{2}$. Consequently, at most u_r players from $N \setminus \{i^*\}$ may use a certain resource $r \in R$. As, for each $i \in N \setminus \{i^*\}$, it must hold that $S_i \setminus \{r_1, \ldots, r_n, r'_1, \ldots, r'_n\} \in S_1 \subseteq R$, $(S_i \setminus \{r_1, \ldots, r_n, r'_1, \ldots, r'_n\})_{i \in N \setminus \{i^*\}}$ is therefore a strategy packing in \mathcal{I} .

For the other direction, we first introduce a lemma that, informally speaking, says that ε is small enough.

Lemma 10. Let \mathcal{G} be a MMFG such that the capacities u_r are integral, for each $r \in R$. Then, for each $S \in \mathcal{S}$, there is a $\delta \in \mathbb{N}$ with $\delta \leq n^n$ such that, for each player $i \in N$, the bandwidth $b_i(S)$ is $\frac{1}{\delta}$ -integral.

Proof (Lemma). Let $S \in \mathcal{S}$ and fix a run of Algorithm 1 on S. Define N_1, \ldots, N_k to be the partition of N where N_i is the set of players which are fixed in the i-th run of the main loop. By induction on i, we will now show that, for each $i \in \{1, \ldots, k\}$, there is a $\delta_i \in \mathbb{Q}$ with $\delta_i \leq n^i$ such that all bandwidths of players in $\biguplus_{j \leq i} N_j$ as well as all the values of c' are $\frac{1}{\delta_i}$ -integral at the end of the i-th run of the main loop. This already implies the claim since $k \leq n$.

For i=0, there is nothing to be shown. Now let δ_i as above. Since the values of c' are $\frac{1}{\delta_i}$ -integral, the bandwidth calculated in line 4 and assigned in line 7 in the i+1-th run of the main loop is $\frac{1}{|N_{i+1}|\cdot\delta_i}$ -integral. The same holds for the values of c' changed in line 10. So we set $\delta_{i+1}:=|N_{i+1}|\cdot\delta_i$. By $|N_{i+1}| \leq n$, we have $\delta_i \leq n^{i+1}$.

Now let $S = (S_1, ..., S_n)$ be a SE in $\mathcal{G}_{\mathcal{I}}$ such that there exists a player i with $b_i(S) < 1$ and consider three different cases.

- Case 1: There exists no player j such that $r^* \in S_j$. By Lemma 10, we know that $b_i(S) < 1 \varepsilon$. So it is profitable for this player to unilaterally deviate to the strategy $\{r_1, \ldots, r_n, r^*\}$ yielding a bandwidth of at least 1ε for him. Hence, S is no SE, in contradiction to our assumption.
- Case 2: There exists a player j such that $S_j = \{r_1, \ldots, r'_n, r^*\}$ but no player k exists with $S_k = R \cup \{r^*\}$. This means that all players get a bandwidth of less than 1 in S (because either r^* or a resource r'_i occurs in each strategy). Therefore, they would all profitably deviate to a state T with $b_i(T) = 1$, for all players i. We now show that, however, such a state would exist in $\mathcal{G}_{\mathcal{I}}$ if there was a strategy packing in \mathcal{I} . This immediately implies that there is no strategy packing in \mathcal{I} .

Let (S'_1, \ldots, S'_n) be the strategy packing in \mathcal{I} . In T, player i uses the strategy $S' \cup \{r_i, r'_i\}$, for $i \in \{1, \ldots, n\}$. Further, player n + 1 uses the strategy $R \cup \{r^*\}$. It can easily be verified that each player gets bandwidth 1 in this state.

Case 3: There exists a player j such that $S_i = R \cup \{r^*\}$. We again distinguish two cases.

Case a: First, consider the case where another player k exists with $k \neq j$ and $r^* \in S_k$. This means that both players j and k get a bandwidth of less than 1. Furthermore, each other player must also get a bandwidth of less than 1 since, otherwise (i.e., if there exists a player l getting at least bandwidth 1), player j could unilaterally and profitably deviate the following way. Player j imitates player l on l and moreover chooses resources l l such that l is not used in l giving him a bandwidth at least as large as l l With the same argumentation as is Case 4.1, we can hence infer that there is no strategy packing in l.

Case b: Now let player j be the unique player with $r^* \in S_j$ and further let $S_j = R \cup \{r^*\}$. We show again that $b_j(S) < 1$ and apply the same argumentation as in Case 4.1 (the preconditions of Lemma 10 are fulfilled since only resources with integral capacities are saturated). If i = j, we are finished. So let $i \neq j$ and suppose that, in S, each resource $r \in R$ is used by at most $u_r + 1$ players. Since S is a SE, we know that, in this case, each r'_i is used by at most one player. Hence, Algorithm 1 calculates a bandwidth of 1 for each player; contradiction. Thus, there is a resource $r \in R$ used by more than $u_r + 1$ players.

If all r'_i are used by one player each, there is a resource in R that is the first one saturated in Algorithm 1 (by the existence of r), which implies the claim. So let $i^* \in \{1, \ldots, n\}$ such that r'_{i^*} is a resource not used in S and suppose $b_i(S) < b_j(S)$. Then, player i could, however, replace the resources from $\{r_1, \ldots, r_n, r'_1, \ldots, r'_n\}$ he currently uses by $\{r_{i^*}, r'_{i^*}\}$, resulting in a bandwidth at least as large as $b_j(S)$.

4.2 Efficiency of Equilibria

In this section we investigate the quality of SE in terms of social welfare, i.e., the sum of allocated bandwidth. In a game \mathcal{G} , let S^* with allocation a be the state in \mathcal{S} that maximizes $\sum_{i\in N} a_i$. Further, let $\mathcal{S}^{SE}\subseteq \mathcal{S}^{NE}\subseteq \mathcal{S}$ denote the set of SE and NE, respectively. We denote $\mathrm{SW}_{\mathcal{G}}(S) = \sum_{i\in N} b_i(S)$. Then, the price of stability and price of anarchy, PoS and PoA, are defined

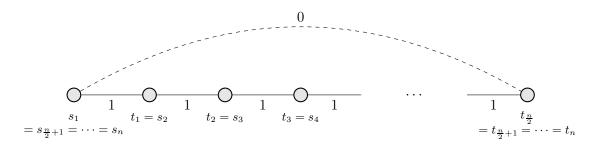


Figure 2: Illustration of the network of the game \mathcal{G}_n in the proof of Theorem 11.

as $\inf_{S \in \mathcal{S}^{NE}} \frac{\operatorname{SW}_{\mathcal{G}}(S^{\star})}{\operatorname{SW}_{\mathcal{G}}(S)} = \inf_{S \in \mathcal{S}^{NE}} \frac{\sum_{i \in N} a_i}{\sum_{i \in N} b_i(S)}$ and $\sup_{S \in \mathcal{S}^{NE}} \frac{\operatorname{SW}_{\mathcal{G}}(S^{\star})}{\operatorname{SW}_{\mathcal{G}}(S)}$, respectively. For the strong price of stability and anarchy, SPoS and SPoA, \mathcal{S}^{SE} is considered instead of \mathcal{S}^{NE} . Furthermore, the same measures can be applied to classes of games where they are simply the supremum of all individual measures.

The maximum capacity allocation problem (MCAP) is given by the problem of computing an allocation a' which maximizes $\sum_{i \in N} a'_i$. Note that we have $\sum_{i \in N} a'_i \geq \sum_{i \in N} a_i$ and that this inequality may even be strict since a' is not necessarily computed by progressive filling.

In general, one cannot hope to find SE with good social welfare. There are network MMFGs in which even the best PNE is a factor of $\Omega(n)$ worse than the optimum. This matches the upper bound of O(n) on the PoA for network MMFGs shown in [37].

Theorem 11. The PoS and SPoS in multi-commodity network MMFGs are $\Omega(n)$.

Proof. For a given $n \in \mathbb{N}$, we construct a network MMFG \mathcal{G}_n with n players and PoS of more than $\frac{n}{4}$. We assume w.l.o.g. that $2 \mid n$.

The network underlying \mathcal{G}_n consists of $\frac{n}{2}$ consecutive edges each of which has capacity 1 and connects the source and sink nodes s_i, t_i of one respective player i. The source and sink nodes of the other $\frac{n}{2}$ players are the first and last vertex of this path. Additionally, there is one edge with the capacity 0 between these two vertices. This network is illustrated in Figure 2.

At first, note that there are two strategies for each of the players. A player can either choose the path through the 1-edges or the path which has the 0-edge in it. The latter path will, however, not be taken in a PNE since avoiding the 0-edge always results in a bandwidth strictly larger than 0. Thus, in the unique PNE S, each 1-edge is congested with $\frac{n}{2} + 1$ players, resulting in a social welfare of $SW_{\mathcal{G}_n}(S) = n \cdot \frac{1}{\frac{n}{2}+1} = \frac{2n}{n+2}$.

If, however, the players $i \in \left\{\frac{n}{2}+1,\ldots,n\right\}$ altruistically take the direct 0-capacity path (s_i,t_i) instead, all the other players get a bandwidth of 1 by sticking to their paths from S. Consequently, a lower bound on the PoS is $\frac{n}{2}$ $\frac{n}{2}$ $\frac{n}{2}$ $\frac{n+2}{2}$ $\frac{n+2}{4}$ $\frac{n}{4}$.

In contrast, when all players have the same strategy set, the best SE achieves a good approximation, and such a good SE is found by Dual Greedy (for single-commodity networks even in polynomial time).

Theorem 12. The PoS and SPoS in symmetric MMFGs are $2 - \frac{1}{n}$, and this bound is tight. The Dual Greedy computes an SE achieving this quarantee.

Proof. For the upper bound, we use an idea from [5] and define the uniform MCAP as the restriction of the MCAP to uniform bandwidth values, i.e., we additionally require that the found allocation is a vector (a, \ldots, a) for some $a \in \mathbb{R}$. It is easy to see that the smallest bandwidth in the state S_{DG} computed by Dual Greedy solves the uniform MCAP. That is $\min_{i \in N} b_i(S_{DG}) = v$ where $n \cdot v$ is the optimal value of the uniform MCAP.

Lemma 13. Let $S = (S_1, ..., S_n)$ be a solution of Dual Greedy on \mathcal{G} and $n \cdot v$ be the optimal value of the uniform MCAP, for $n \in \mathbb{N}$ and $v \in \mathbb{R}$. Then, we have $v = \min_{i \in N} b_i(S)$.

Proof (Lemma). We show the lemma in two steps:

 $v \leq \min_{i \in N} b_i(S)$: By Lemma 6, $\min_{i \in N} b_i(S)$ is exactly the bandwidth allocated to a player after the oracle has evaluated to \emptyset for the first time. Again by Lemma 6, such an evaluation to \emptyset means that there is no state $S' \in \mathcal{S}$ with $b_j(S') > \min_{i \in N} b_i(S)$, for all $i \in N$. This implies the claim since, otherwise, such a state S' is given by the optimal solution of the uniform MCAP.

 $v \geq \min_{i \in N} b_i(S)$: Suppose $v < \min_{i \in N} b_i(S)$. Then we construct a feasible solution of the uniform MCAP with a larger value. We choose S as state and $\min_{i \in N} b_i(S)$ as bandwidth for each player (which is feasible since we only possibly lower the feasible bandwidth $b_i(S)$, for all i). This may, however, not happen as the solution to the uniform MCAP has the value $n \cdot v$ by assumption.

Thus, the upper bound follows from the next lemma.

Lemma 14. An optimum to the uniform MCAP is a $\left(2-\frac{1}{n}\right)$ -approximation for the MCAP.

Proof (Lemma). Let $n \cdot v$ be the optimal value of the uniform MCAP. Consider an arbitrary feasible solution attained by the state $S = (S_1, \ldots, S_n)$ and the respective allocation (a_1, \ldots, a_n) . Define $\alpha_i \in \mathbb{R}_+$ such that $a_i = \alpha_i \cdot v$, for all $i \in N$.

Now suppose that $\sum_{i\in N} \alpha_i > 2n-1$. Then we can construct a new state $S' = (S'_1, \ldots, S'_n)$ with a corresponding bandwidth v' > v for each player. In S', we use at most $\lceil \alpha_i \rceil - 1$ copies of the strategy S_i , for all i, and no other strategy. Furthermore, we set $v' := \min_{i \in N} \frac{\alpha_i}{\lceil \alpha_i \rceil - 1} \cdot v$. Three properties of this solution remain to be shown:

- 1. There are at least n (not necessarily different) strategies constructed for S' above. Using that $\lceil \alpha_i \rceil 1 \ge \alpha_i 1$ holds for all i, we get: $\sum_{i \in N} (\lceil \alpha_i \rceil 1) \ge \sum_{i \in N} (\alpha_i 1) = (\sum_{i \in N} \alpha_i) n > n 1$. As $\sum_{i \in N} (\lceil \alpha_i \rceil 1)$ must be integer, it follows that $\sum_{i \in N} (\lceil \alpha_i \rceil 1) \ge n$.
- 2. The constructed bandwidths are feasible. We only use the strategies from S for which (a_1, \ldots, a_n) is an allocation. So it suffices to see that for all i, in the constructed solution, the total capacity on S_i is at most as high as a_i : $(\lceil \alpha_i \rceil 1) \cdot v' = (\lceil \alpha_i \rceil 1) \cdot \min_{j \in N} \frac{\alpha_j}{\lceil \alpha_j \rceil 1} \cdot v \le (\lceil \alpha_i \rceil 1) \cdot \frac{\alpha_i}{\lceil \alpha_i \rceil 1} \cdot v = a_i$.
- 3. It indeed holds that v' > v: by $\lceil \alpha_i \rceil 1 < \alpha_i$, we have $\frac{\alpha_i}{\lceil \alpha_i \rceil 1} > 1$ for all i, i.e., we also have $\min_{i \in N} \frac{\alpha_i}{\lceil \alpha_i \rceil 1} > 1$ and hence v' > v.

Thus, we have constructed a new solution of the uniform MCAP with a higher value than $n \cdot v$. So the initial solution cannot be maximal, i.e., we obtain a contradiction. So we must have $\sum_{i \in N} \alpha_i \leq 2n-1$, which implies

$$\frac{\sum_{i \in N} a_i}{n \cdot v} = \frac{\sum_{i \in N} \alpha_i \cdot v}{n \cdot v} = \frac{\sum_{i \in N} \alpha_i}{n} \leq \frac{2n-1}{n}.$$

For the lower bound consider for given $n \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$ a single-commodity network MMFG $\mathcal{G}_{n,\varepsilon}$. From the source to the sink node, there are n-1 parallel edges each of which has capacity $1-\varepsilon$. Moreover, there is one single edge with capacity n. In the optimal state, every edge is used by one player each, i.e., we obtain a social welfare of $2n-1-(n-1)\cdot\varepsilon$. In a NE, however, every player uses the edge with capacity n because the bandwidth for each player is at least $1>1-\varepsilon$ on this edge. Thus, the social welfare is exactly n in this state. Therefore, it holds that

$$\operatorname{PoS}(\mathcal{G}_{n,\varepsilon}) = \frac{2n-1-(n-1)\cdot\varepsilon}{n} = 2-\frac{1}{n}-\frac{n-1}{n}\cdot\varepsilon$$
.

In symmetric games even the worst SE is still a 4-approximation. For n=2, we can tighten the bound on the SPoA to the lower bound of the SPoS of $\frac{3}{2}$.

Theorem 15. The SPoA for symmetric MMFGs is at most $4 - \frac{6}{n+1}$.

Proof. Let \mathcal{G} be a symmetric MMFG and let S be a SE in this game. Then in S each player must get at least a bandwidth of $\frac{1}{2} \max_{i \in N} b_i(S)$, as otherwise this player could profitably imitate a player in $\operatorname{argmax}_{i \in N} b_i(S)$ by choosing the same strategy. Thus, we can lower bound the social welfare by

$$SW_{\mathcal{G}}(S) = \sum_{i \in N} b_i(S) \ge \left(\frac{n-1}{2} + 1\right) \cdot \max_{i \in N} b_i(S). \tag{1}$$

State S_{DG} computed by Dual Greedy in \mathcal{G} is such that $\min_{i \in N} b_i(S_{DG}) = v$ where $n \cdot v$ is the optimal value of the uniform MCAP. Consequently, for any other SE S, we must have $\max_{i \in N} b_i(S) \geq v$, because otherwise all the players could profitably switch to their strategies in S_{DG} . Using Equation 1, this means $SW_{\mathcal{G}}(S) \geq \frac{n+1}{2} \cdot v$, and hence we obtain

$$\frac{\max_{S' \in \mathcal{S}} \operatorname{SW}_{\mathcal{G}}\left(S'\right)}{\operatorname{SW}_{\mathcal{G}}\left(S\right)} \leq \frac{2n}{n+1} \cdot \frac{\max_{S' \in \mathcal{S}} \operatorname{SW}_{\mathcal{G}}\left(S'\right)}{n \cdot v} \\ \leq \frac{2n}{n+1} \cdot \frac{2n-1}{n} = \frac{4n-2}{n+1} .$$

Theorem 16. The SPoA for symmetric MMFGs with 2 players is $\frac{3}{2}$ and this bound is tight.

Proof. Let \mathcal{G} be a symmetric MMFG with n=2 and let S be a SE in this game. Further, let S' be an arbitrary (optimal) state. W.l.o.g., we may assume that $b_1(S) \leq b_2(S)$ and $b_1(S') \leq b_2(S')$.

Note that $b_1(S) \ge b_1(S')$ or $b_2(S) \ge b_2(S')$ must hold. Otherwise switching from S to their strategies in S' would be profitable for both players. Thus, the following case distinction is complete.

Case 1: We have $b_1(S) \ge b_1(S')$. We can also derive an upper bound on $b_2(S')$. If $b_2(S') > 2 \cdot b_1(S)$, player 1 could profitably deviate to S'_2 in S. So we must have $b_2(S') \le 2 \cdot b_1(S)$. Thus,

$$\frac{\mathrm{SW}_{\mathcal{G}}\left(S^{\prime}\right)}{\mathrm{SW}_{\mathcal{G}}\left(S\right)} = \frac{b_{1}\left(S^{\prime}\right) + b_{2}\left(S^{\prime}\right)}{b_{1}\left(S\right) + b_{2}\left(S\right)} \leq \frac{3 \cdot b_{1}\left(S^{\prime}\right)}{2 \cdot b_{1}\left(S\right)} \leq \frac{3}{2} \ .$$

Case 2: We have $b_2(S) \ge b_2(S')$. We find an upper bound on $b_2(S)$. Since $b_1(S) < \frac{1}{2} \cdot b_2(S)$ would mean that player 1 could profitably imitate player 2 in S, it holds that $b_1(S) \ge \frac{1}{2} \cdot b_2(S)$. This implies

$$\frac{\mathrm{SW}_{\mathcal{G}}\left(S'\right)}{\mathrm{SW}_{\mathcal{G}}\left(S\right)} = \frac{b_{1}\left(S'\right) + b_{2}\left(S'\right)}{b_{1}\left(S\right) + b_{2}\left(S\right)} \le \frac{2 \cdot b_{2}\left(S'\right)}{\frac{3}{2} \cdot b_{2}\left(S\right)} \le \frac{4}{3} < \frac{3}{2} \ .$$

The lower bound immediately follows from Theorem 12.

In addition, we show a lower bound of $\Omega(n/k)$ on the k-SPoA for k-SE, where only deviations of coalitions of size at most k are considered.

Theorem 17. The k-SPoA for single-commodity network MMFGs is in $\Omega\left(\frac{n}{k}\right)$.

Proof. We construct a family of single-commodity networks MMFG $\mathcal{G}_{n,k}$ with SPoA $\frac{n}{k}$. As we are showing an asymptotical lower bound, we may assume w.l.o.g. that $k \mid n$.

The game $\mathcal{G}_{n,k}$ consists of k gadgets $G_{n,i}$ for $i \in \{1, \ldots, k\}$ where gadget $G_{n,i} = (V_{n,i}, E_{n,i}, c_{n,i})$ is the following network. For the vertices and edges, we set

$$V_{n,i} = \{u_i, v_{i,1}, \dots, v_{i,n}, w_{i,1}, \dots, w_{i,n}, u_{i+1}\},$$

$$E_{n,i} = \{u_i\} \times \{v_{i,1}, \dots, v_{i,n}\} \cup \{(v_{i,j}, w_{i,j}) \mid 1 \le j \le n\}$$

$$\cup \{(w_{i,j}, v_{i,j+1}) \mid 1 \le j \le n-1\} \cup \{w_{i,1}, \dots, w_{i,n}\} \times \{u_{i+1}\}$$

and, further, we let $c_{n,i}(e) = 1$, for all $e \in E$.

By arranging the $G_{i,n}$ in a row, we obtain the network underlying $\mathcal{G}_{n,k}$. More specifically, this network is

$$(V_{n,1} \cup \cdots \cup V_{n,k}, E_{n,1} \cup \cdots \cup E_{n,k} \cup E_{n,k}^{\star}, c_{n,1} \cup \cdots \cup c_{n,k} \cup c_{n,k}^{\star})$$

with the source and sink nodes u_1 and u_{k+1} , respectively, and where

$$E_{n,k}^{\star} = \left\{ (u_1, v_{i,1}) \mid 2 \le i \le n \right\} \cup \left\{ (w_{n,1}, u_{n+1}) \mid 1 \le i \le n-1 \right\}.$$

Moreover, $c_{n,k}^{\star}$ is again constantly 1 on $E_{n,k}^{\star}$. This network is illustrated in Figure 3.

Since all the edges have capacity 1, the optimal social welfare is n. In particular, in an optimal state player i chooses the path $(u_1, v_{1,i}, w_{1,i}, u_2, v_{2,i}, \ldots, w_{n,i}, u_{n+1})$, which can easily be verified. Thus, the optimal social welfare is n.

We now describe a state S^* (also shown in Figure 3) that will be shown to be a k-SE and attain a social welfare of k, implying the claim. We partition the player set N into N_1, \ldots, N_k with each player $j \in N_i$ choosing the path $(u_1, v_{i,1}, w_{i,1}, v_{i,2}, w_{i,2}, \ldots, v_{i,n}, w_{i,n}, u_{i+1})$, for all $i \in \{1, \ldots, k\}$. Then, obviously, the paths of the players from different sets N_i are pairwise arc-disjoint. Hence, the social welfare in S^* is indeed k.

It remains to be shown that S^* is indeed a k-SE. Towards a contradiction, suppose that there is a coalition $C \subseteq N$ with $|C| \le k$ and a state S' such that each player in C strictly improves when moving from S^* to $T = \left(S'_C, S^*_{-C}\right)$. We distinguish two cases and will use implicitly that in S^* there is no edge with more than $\frac{n}{k}$ players on it.

Case 1: There is an $i \in \{1, ..., k\}$ such that $N_i \cap C = \emptyset$. W.l.o.g., i is the minimum i with this property. Furthermore, we let p be the number of such sets, i.e. $p = |\{j \mid N_i \cap C = \emptyset\}|$.

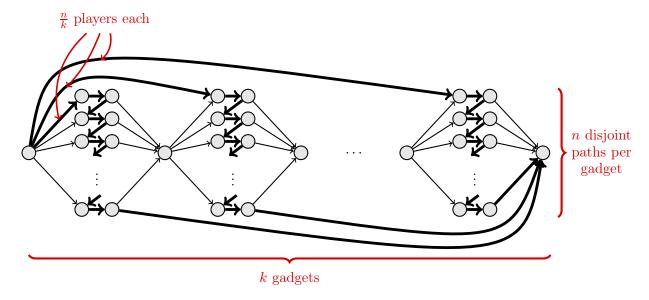


Figure 3: Illustration of the network of the game $\mathcal{G}_{n,k}$ with the SE S^* in the proof of Theorem 17.

First note that, if a player from C passes through a gadget $G_{n,j}$ with $N_j \cap C = \emptyset$, the global bottleneck edge will be used by at least k+1 players and thus by at least one player from C. So each player from C must use at least one edge from $E_C = \{(w_{j,n}, u_{k+1}) \mid 1 \leq j < i\}$ $\cup \{(u_1, v_{j,1}) \mid i \leq j \land N_j \cap C \neq \emptyset\}$.

Now note that there are $n - p \cdot \frac{k}{n} - |C|$ players from $N \setminus C$ on E_C and we have $|E_C| = k - p$. Thus, in T, there is a global bottleneck edge that has at least

$$\frac{n - p \cdot \frac{n}{k}}{k - p} = \frac{n}{k}$$

players and, among them, one player from C on it. This player does not strictly improve.

Case 2: For each $i \in \{1, ..., k\}$, it holds that $|N_i \cap C| = 1$. Then, on each path from u_1 to u_{k+1} there is an edge used by exactly $\frac{n}{k} - 1$ players from $N \setminus C$. Thus, adding the players from C to them produces a global bottleneck edge with at least $\frac{n}{k}$ players on it. Consequently, the players from C on the global bottleneck edge cannot strictly improve.

5 General Progressive Filling Games

5.1 Complexity and Convergence

The lexicographical potential function for PFGs implies that the length of each coalitional improvement sequence is finite. By $\Phi_{\rm ord}$, we denote the set of ordered values of ϕ , i.e., $\Phi_{\rm ord} := {\rm img} \, ({\rm ord} \circ \phi)$ where ord is the function which orders a vector, say ascendingly. The cardinality of the above set provides an upper bound on the length of improvement sequences.

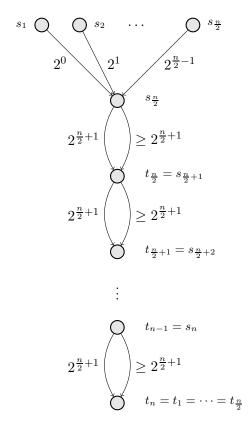


Figure 4: Illustration of the network in the proof of Theorem 18.

For a MMFG with n players and m resources, Yang et al. [37] provide an upper bound of $(mn)^n$ on the number of improvement steps to reach a PNE. In the following, we show that it is not possible to get this result by just bounding $|\Phi_{\text{ord}}|$.

Theorem 18. There is a family of network MMFGs \mathcal{G}_n with $m \in \Theta(n)$ and respective potential function Φ_{ord} such that $|\Phi_{\mathrm{ord}}|$ is in $2^{\Omega(n^2)} = \omega((n^2)^n)$.

Proof. Since we are proving an asymptotical lower bound, we may assume w.l.o.g. that $2 \mid n$. We now describe the multigraph underlying \mathcal{G}_n .

For each player $i \in \left\{\frac{n}{2}+1,\ldots,n\right\}$, we have a gadget in this multigraph. This gadget consists of two parallel edges, both connecting the source and the sink nodes $(s_i \text{ and } t_i, \text{ respectively})$ of the particular player. One of these edges (referred to as the *left* edge) has a capacity of $2^{\frac{n}{2}+1}$ whereas the other one also has at least this capacity. The gadgets are arranged in a row such that $s_i = t_{i+1}$ for $i \in \left\{\frac{n}{2}+1,\ldots,n-1\right\}$. All the other players $i \in \left\{1,\ldots,\frac{n}{2}\right\}$ have one disjoint source node s_i each and t_n as sink node. Moreover, there is one edge connecting s_i and $s_{\frac{n}{2}}$ with capacity 2^{i-1} . This results in a network as shown in Figure 4. Obviously, the number of edges is in $\Theta(n)$.

Note that, independently of the path a player $i \in \{1, \dots, \frac{n}{2}\}$ chooses, he is always assigned the respective bandwidth 2^{i-1} in the max-min fair allocation. This is because the residual capacity of an edge from the gadgets is larger than each of the bandwidth of players from $\{1, \dots, \frac{n}{2}\}$, even if all these players use this edge.

Consequently, the players $\{1, \dots, \frac{n}{2}\}$ are capable of choosing any natural number between $2^{\frac{n}{2}} + 1$

and $2^{\frac{n}{2}+1}$ for the residual capacity of each of the $\frac{n}{2}$ left edges in the different gadgets. More specifically, let $x_{\frac{n}{2}}x_{\frac{n}{2}-1}\dots x_1$ be the binary representation of a natural number x such that $2^{\frac{n}{2}+1}-x$ is from that interval. To obtain the desired residual capacity on a left edge in a given gadget, player i simply chooses this edge in his path if and only if we have $x_i=1$. This has indeed the desired effect since $2^{\frac{n}{2}+1}-\sum_{i\in N: x_{i=1}}2^{i-1}=2^{\frac{n}{2}+1}-x$. We now give a lower bound on the number of different ordered allocation vectors. Since we want to derive a lower bound, it suffices to show the claim for allocations where the residual capacity of the left edge in the i-th gadget (i.e. the one of player $\frac{n}{2}+i$) is between $2^{\frac{n}{2}}+1+(i-1)\cdot\left\lfloor(2^{\frac{n}{2}}-1)/n\right\rfloor$ and $2^{\frac{n}{2}}+1+i\cdot\left\lfloor(2^{\frac{n}{2}}-1)/n\right\rfloor$ and player i chooses this edge. In these allocations, the bandwidth of player i occurs in the ordered allocation vector before the one of player i+1, for all $i\in\left\{\frac{n}{2}+1,\dots,n-1\right\}$. Consequently, the claim is implied by the following bound on the number of ordered allocations

$$\left\lfloor \frac{2^{\frac{n}{2}} - 1}{n} \right\rfloor^{\frac{n}{2}} = \left(\frac{2^{\Omega(n)}}{2^{\mathcal{O}(\log n)}} \right)^{\Omega(n)} = 2^{\Omega(n) \cdot \Omega(n)} = 2^{\Omega(n^2)}.$$

We now provide an *upper* bound on the number of ordered values of the potential, even for general progressive filling games. For $m = \Theta(n)$, this yields an upper bound of $2^{O(n^2)}$.

Theorem 19. For arbitrary PFGs with the potential function ϕ , it holds that $|\Phi_{\text{ord}}| \leq 2^{n^2} \cdot m^n$.

Proof. Let \mathcal{G} be a PFG with potential function ϕ . We claim that the number of different vectors up to the k-th position (for $k \leq n$) in $|\Phi_{\text{ord}}|$ is at most $2^{k \cdot n} \cdot m^k$. It is shown via induction on k.

For k=0, the claim is clear as there is only the vector of dimension 0. So let $n \geq k > 0$ and assume there are at most $2^{(k-1)\cdot n} \cdot m^{(k-1)}$ different vectors up to the position k-1 in Φ_{ord} . We now fix the first k-1 positions of a vector in Φ_{ord} and bound the number of entries at the k-th position. Note that one can calculate the next finishing time given the resource which is saturated and the subset of players on that resource. Since there are $2^n \cdot m$ such combinations, the claim follows. \square

Theorem 9 shows that computing SE is NP-hard in MMFGs. For general PFGs with constant allocation rates (i.e., weighted MMF allocations), the same result holds even for single-commodity network games with two players. Hence, extending Dual Greedy to compute SE in polynomial time for this case is impossible.

Theorem 20. Let $v_1 \neq v_2$ be two constant allocation rate functions and consider the class of single-commodity network PFGs with two players and v_1, v_2 as allocation rate functions. In this class, the computation of SE is NP-hard.

Proof. We reduce from the 2-directed-arc-disjoint-paths problem (2DADP). Let \mathcal{I} be an instance of this problem. W.l.o.g., we can assume that there are paths in D from s_1 to t_1 and from s_2 to t_2 and, further, that $v_1 \equiv 1$ and $v_2 \equiv \lambda$ where $\lambda < 1$. Furthermore, we choose an $\varepsilon \in (0, 1 - \lambda)$.

We construct the network underlying the single-commodity network PFG $\mathcal{G}_{\mathcal{I}}$ by keeping D, adding the source and sink nodes s and t, respectively, and the four edges (s, s_1) , (s, s_2) , (t_1, t) and (t_2, t) . The capacities are

$$c(s, s_1) = c(t_1, t) = 1 + \lambda \text{ and } c(s, s_2) = c(t_2, t) = \lambda + \varepsilon.$$

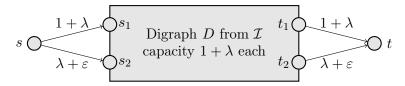


Figure 5: Illustration of the network of the game $\mathcal{G}_{\mathcal{I}}$ in the proof of Theorem 20.

The capacities of all edges occurring in D are set to $1 + \lambda$. This construction is illustrated in Figure 5.

We first show that each SE S with two arc-disjoint paths from s to t certifies that \mathcal{I} is solvable. To see this, note that player 1 will always choose a path of the form (s, s_1, \ldots, t_1, t) because, even if he has to share an edge with player 2, he gets bandwidth $1 > \lambda + \varepsilon$. Hence, the path of player 2 must indeed connect s_2 and t_2 .

We also show that each SE S without this property certifies that no arc-disjoint paths from s_1 to t_1 and from s_2 to t_2 exist. In S, both players share a common edge, i.e., player 1 gets a bandwidth of 1 and player 2 a bandwidth of λ . Thus, if there were two arc-disjoint paths (s, s_1, \ldots, t_1, t) and (s, s_2, \ldots, t_2, t) , both players could profitably change to these paths and get bandwidths of $1 + \lambda$ and $\lambda + \varepsilon$.

Let us instead consider PNE, which may be easier to compute than SE. Similar to a result from [37] for MMFGs, we first show that one can efficiently compute a unilateral improvement step for a given player in a PFG with constant allocation rate functions (if it exists). Using Theorem 19, computation of PNE can be done efficiently for a constant number of players.

Lemma 21. In PFGs with constant allocation rate functions, an improving move of any player i in any state S can be computed in polynomial time if it exists.

Proof. The bandwidth of player i in the state $(\{r\}, S_{-i})$ can by computed in polynomial time by Algorithm 1. Further, for a given strategy S'_i , the bandwidth of player 1 only depends on the resource which gets saturated first, i.e., $b_i(S'_i, S_{-i}) = \min_{r \in S'_i} b_i(\{r\}, S_{-i})$, which can easily be verified on Algorithm 1. Thus, it suffices to calculate $\min_{r \in S'_i} b_i(\{r\}, S_{-i})$ for all possible alternative strategies S'_i to decide whether there is an improvement step from S for player i.

If the strategies are given explicitly as input, this value can be explicitly computed for each of the strategies. If strategies are given implicitly in the form of a network, we can use, e.g., Dijkstra's algorithm to find a path P^* with the maximum $\min_{r \in P^*} b_i(\{r\}, S_{-i})$.

Corollary 22. A PNE can be computed in polynomial time in PFGs with constant allocation rate functions and a constant number of players.

5.2 Quality of Equilibria

In this section we prove results on PoA and PoS for NE in PFGs. In general, Theorem 11 in the previous section yields a lower bound on the PoA in MMFGs of $\Omega(n)$. In fact, n is also the correct upper bound on the PoA, for every PFG.

Theorem 23. The PoA in PFGs is at most n.

Proof. Consider a PFG \mathcal{G} with state set $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$. Then choose

$$S_j \in \underset{S \in \mathcal{S}_1 \cup \dots \cup \mathcal{S}_n}{\operatorname{argmax}} \min_{r \in S} c_r$$

as a strategy with the maximum bottleneck resource, where j is a player with $S_j \in \mathcal{S}_j$. Now consider an arbitrary NE S' and distinguish two cases.

Case 1: There is a saturated resource among the resources of S_j in S'. By the choice of S_j ,

$$SW_{\mathcal{G}}(S') \geq \min_{r \in S_i} c_r \geq \frac{1}{n} SW_{\mathcal{G}}(S^*)$$
,

for any other (optimal) state S^* .

Case 2: There is no saturated resource among the resources of S_j in S' but adding a bandwidth of $\delta > 0$ on S_j would saturate a resource. If player j already uses a resource from S_j in S'_j , then S' is no NE because using S_j as strategy instead would increase the bandwidth of player j by δ .

So let $S'_j \cap S_j = \emptyset$. Then player j must get a bandwidth of at least δ in S' since it would be profitable to use S_j as strategy instead. Consequently, we get

$$\operatorname{SW}_{\mathcal{G}}\left(S'\right) \ge \left(\min_{r \in S_{j}} c_{r} - \delta\right) + \delta = \min_{r \in S_{j}} c_{r} \ge \frac{1}{n} \operatorname{SW}_{\mathcal{G}}\left(S^{\star}\right)$$

for any other (optimal) state S^* .

An improved result can be obtained for singleton games. We have already seen a lower bound of $2 - \frac{1}{n}$ on the PoA in Theorem 12, as the tightness construction is a symmetric singleton MMFG. We now prove that this lower bound is tight, even in general singleton PFGs.

Theorem 24. The PoA in singleton PFGs is $2 - \frac{1}{n}$ and this bound is tight.

Proof. For a state $S \in \mathcal{S}$, we denote the resources used in S by $R_S = \{r \in R \mid l_r(S) \neq 0\}$. As the limits of the indefinite integrals of the allocation rate functions for $x \to \infty$ are also ∞ , we get that

$$SW_{\mathcal{G}}(S) = \sum_{r \in R_S} c_r.$$

Further, let S be a NE in \mathcal{G} and S^* an arbitrary other (optimal) state. Since no player unilaterally deviates from his strategy in S to a resource $r \notin R_S$, we must have

$$c_r \leq \min_{i \in N} b_i(S) \leq \frac{\operatorname{SW}_{\mathcal{G}}(S)}{n} ,$$
 (2)

for all $r \notin R_S$. In particular, this holds for each resource $r \in R_{S^*} \setminus R_S$. Distinguish two cases:

Case 1: We have $R_S \cap R_{S^*} = \emptyset$. Then, by the previous observation, it follows that

$$\frac{\operatorname{SW}_{\mathcal{G}}\left(S^{\star}\right)}{\operatorname{SW}_{\mathcal{G}}\left(S\right)} = \frac{\sum_{r \in R_{S'} \setminus R_{S}} c_{r}}{\sum_{r \in R_{S}} c_{r}} \leq \frac{\operatorname{SW}_{\mathcal{G}}\left(S\right)}{\operatorname{SW}_{\mathcal{G}}\left(S\right)} = 1.$$

Case 2: We have $R_S \cap R_{S^*} \neq \emptyset$. Then, again by the previous observation, it follows that

$$\frac{\operatorname{SW}_{\mathcal{G}}(S^{\star})}{\operatorname{SW}_{\mathcal{G}}(S)} \leq \frac{\sum_{r \in R_{S}} c_{r} + \sum_{r \in R_{S^{\star}} \setminus R_{S}} c_{r}}{\sum_{r \in R_{S}} c_{r}}$$

$$\leq \frac{\left(1 + \frac{n-1}{n}\right) \cdot \operatorname{SW}_{\mathcal{G}}(S)}{\operatorname{SW}_{\mathcal{G}}(S)}$$

$$= 2 - \frac{1}{n} .$$

Finally, we show a lower bound of n on the PoS if we leave the singleton case. For multicommodity network MMFGs we proved a bound of $\Omega(n)$ in Theorem 11 in the previous section. We show that this lower bound can even be established in single-commodity network PFGs. The reason for this is that a player with a fast-growing bandwidth may make his decision (nearly) unaffected of the decisions of all the other player decisions and hence possibly blocks all strategies for other players. This argument even applies if we only allow constant allocation rate functions.

Theorem 25. The PoS in single-commodity network PFGs with constant allocation rate functions is at least n.

Proof. For each $\varepsilon \in (0,1]$, we construct a family of single-commodity PFGs $\mathcal{G}_{n,\varepsilon}$ with n players and PoS at least $\frac{n}{1+2\varepsilon}$. For such a game, we employ a gadget $G_{n,i}$ from the proof of Theorem 17 as underlying network. We omit the i in the indices and call u_i and u_{i+1} simply s and t, respectively. Further, we adapt the capacities of the edges in the following way. We set

$$c_e = \begin{cases} 1 + \varepsilon, & \text{if } e = (s, v_1) \text{ or } e = (w_n, t) \text{ or } e \text{ is not incident to } s \text{ or } t \\ 1, & \text{else} \end{cases}$$

for all $e \in E$. To obtain a PFG from this network, we equip player 1 with an allocation rate function which is constantly 1 and all the other players from $\{1, \ldots, n\}$ with functions which are constantly $\frac{\varepsilon}{n}$.

Consider a state S with social welfare n. Such a state evolves if player i chooses the path (s, v_i, w_i, t) , for all i. Player 1, however, has an incentive to use the path $(s, v_1, w_1, v_2, \ldots, v_n, w_n, t)$ instead of any other path with capacity 1 – even if player 1 had to share a $(1 + \varepsilon)$ -edge with all the other players he would get a bandwidth larger than 1. More specifically, if this edge is the first one saturated by Algorithm 1 (which is the case when player 1 chooses the considered path), the finishing time of this edge is larger than 1.

Now let S' be the NE with the highest social welfare. By the previous considerations, player 1 uses the path $(s, v_1, w_1, v_2, \dots, v_n, w_n, t)$ and obviously gets at most a bandwidth of $1 + \varepsilon$ in S'. Since we chose $\varepsilon \leq 1$, all the other players get at most a bandwidth of $\frac{2\varepsilon}{n}$. This bound is tight if $\varepsilon = 1$ and all these players only share an edge with player 1.

We can now compare the social welfare of the states S and S' (both illustrated in Figure 6) and obtain that $PoS(\mathcal{G}_n) \geq \frac{n}{1+2\varepsilon}$. Hence, we get $sup\{PoS(\mathcal{G}_{n,\varepsilon}) \mid n \in \mathbb{N} \land \varepsilon \in \mathbb{R} \land \varepsilon > 0\} \geq n$, which implies the claim.

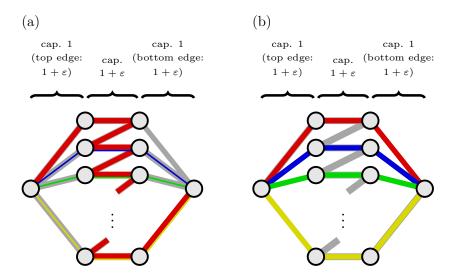


Figure 6: Illustration of the network of the game $\mathcal{G}_{n,\varepsilon}$ from the proof of Theorem 25 with (a) the best NE S' and (b) the optimal state S in terms of social welfare.

5.3 Changing the Allocation Rate Functions

Dual Greedy computes a SE that is a $2 - \frac{1}{n}$ -approximation and this bound is tight. To stabilize better solutions, in this section we take a "protocol design" approach. We assume the waterfilling algorithm can determine a set of constant allocation rate functions for each instance. Interestingly, for any given collection of players, resources, capacities and strategy sets, one can give constant allocation rate functions such that the resulting PFG has an SE with social welfare as high as the optimal value of the MCAP.

Theorem 26. Let \mathcal{G} be a PFG with player set N and v^* be the optimal value of the MCAP. There are constant allocation rate functions $(v_i')_{i\in N}$ such that the maximal social welfare in \mathcal{G} with allocation rate functions replaced by $(v_i')_{i\in N}$ is v^* and the SPoS in this game is 1.

Proof. Let the state $S = (S_1, \ldots, S_n)$ along with the allocation $a = (a_1, \ldots, a_n)$ be an optimal solution of the MCAP. We use allocation rate function $v_i' \equiv a_i$ for each player $i \in N$. We call the corresponding PFG \mathcal{G}' . If we run the progressive filling algorithm in S with v', all finishing times are exactly 1 and the allocation is exactly a.

We show that S is a SE in \mathcal{G} . Towards this, suppose that there is a coalition profitably deviating from S to T. Then, by Lemma 3, the finishing times and thus bandwidths of all players in $N \setminus C$ remain identical in T whereas the players from C strictly improve. Consequently, we have constructed a solution of the MCAP on \mathcal{M} with a higher social welfare – a contradiction.

Not surprisingly, this approach is intractable, as the MCAP is NP-hard to approximate to within a factor $\frac{3}{2} - \varepsilon$, even for arbitrary fixed rates.

Theorem 27. For 2 players, it is NP-hard to approximate the MCAP with a factor of smaller than $\frac{3}{2}$. This also holds for the MCAP with arbitrary fixed rates.

Proof. We use the reduction from the proof of Theorem 20 for $\lambda = 1$ and $\varepsilon = 0$. Since for two players, the allocation rate functions do not affect the social welfare in a given state, our argumentation works completely without allocation rate functions, even for the MCAP with fixed rates.

Each state with optimal social welfare 2 certifies that there are no arc-disjoint paths (v_1, \ldots, v_k) and (w_1, \ldots, w_l) from s_1 to t_1 and from s_2 to t_2 , respectively, since, otherwise, the state $((s_1, v_1, \ldots, v_k, t_1), (s_2, w_1, \ldots, w_l, t_2))$ would attain a social welfare of 3.

Conversely, each state with a social welfare higher than 2 must use two arc-disjoint paths in D. Further, these paths must obviously connect s_1 to t_1 and s_2 to t_2 . Consequently, such a state certifies that the instance of 2DADP is solvable.

This implies that the approximation guarantee of Dual Greedy is optimal for n=2, even without requiring the output to be a SE. The idea behind the previous theorem extends also to approximate solutions of the MCAP. For the MCAP on single-commodity networks, a better $\frac{3}{2}$ -approximation exists for n=3 [5] and can be obtained as follows: Run the maximum capacity augmenting path algorithm [1] on the given network for two iterations and decompose [1] the obtained flow into three paths (plus a circulation). We use this approach to calculate an equilibrium state that is a better approximation than the one calculated by Dual Greedy. By Theorem 12, this is not possible if the allocation rate functions are fixed, even for uniform ones. Adjusting allocation rate functions subject to the instance, however, allows to beat Dual Greedy, at least for n=3 and PNE.

Theorem 28. In single-commodity networks with 3 players, there exist constant allocation rate functions and a PNE that is a $\frac{3}{2}$ -approximation to the MCAP. The allocation rate functions and the PNE can be computed in polynomial time.

Proof. Let $S = (S_1, S_2, S_3)$ and the allocation $a = (a_1, a_2, a_3)$ represent the a $\frac{3}{2}$ -approximation of the MCAP. As allocation rate function for player i, for all $i \in N$, we use the function $v_i \equiv a_i$. Note that, if we run Algorithm 1 on S, the finishing time is 1, for each of the players, and a is exactly the computed allocation. We now invoke best-response dynamics starting from S and iteratively compute and apply unilateral player deviations. By Corollary 19, this procedure can be implemented in polynomial time. We call the resulting state S^* . Using Lemma 3, we know that the finishing times of the players never sink below 1 during that procedure. Consequently, S^* is at most a $\frac{3}{2}$ -approximation to the MCAP.

Indeed, we can start with an arbitrary approximate solution of the MCAP, set the allocation rates such that finishing times are all 1, and then every unilateral (coalitional) improvement dynamics will lead to a PNE (SE) that only improves social welfare. Exploring this idea is a very interesting avenue for future work.

Acknowledgement

We thank Berthold Vöcking for helpful comments regarding the model underlying this paper.

References

- [1] R. Ahuja, T. Magnanti, and J. Orlin. Network Flows: Theory, Algorithms, and Applications. Prentice Hall, 1993.
- [2] E. Amaldi, A. Capone, S. Coniglio, and L. Gianoli. Network optimization problems subject to max-min fair flow allocation. Submitted.

- [3] E. Amaldi, S. Coniglio, L. Gianoli, and C. U. Ileri. On single-path network routing subject to max-min fair flow allocation. Submitted.
- [4] R. Aumann. Acceptable points in general cooperative n-person games. In *Contributions to the Theory of Games IV*, volume 40 of *Annals of Mathematics Study*, pages 287–324, 1959.
- [5] G. Baier, E. Köhler, and M. Skutella. The k-splittable flow problem. Algorithmica, 42(3–4):231–248, 2005.
- [6] R. Banner and A. Orda. Bottleneck routing games in communication networks. *IEEE J. Sel. Areas Commun.*, 25(6):1173–1179, 2007.
- [7] D. Bertsekas and R. Gallager. Data networks (2nd ed.). Prentice Hall, 1992.
- [8] C. Busch, R. Kannan and A. Samman. Bottleneck Routing Games on Grids. *In Proc. GAMENETS*, pages 294–307, 2011.
- [9] C. Busch and M. Magdon-Ismail. Atomic routing games on maximum congestion. *Theoret. Comput. Sci.*, 410(36):3337–3975, 2009.
- [10] R. Cole, Y. Dodis, and T. Roughgarden. Bottleneck links, variable demand, and the tragedy of the commons. *Networks*, 60(3):194–203, 2012.
- [11] H. Han, S. Shakkottai, C.V. Hollot, R. Srikant, D. Towsley. Multi-Path TCP: A Joint Congestion Control and Routing Scheme to Exploit Path Diversity in the Internet. *IEEE/ACM Trans. Networking*, 1(1): 22–33, 2006
- [12] T. Harks and T. Poschwatta. Congestion Control in Utility Fair Networks. *Computer Networks*, 52(15): 2947–2960, 2008
- [13] T. Harks, M. Hoefer, M. Klimm, and A. Skopalik. Computing pure Nash and strong equilibria in bottleneck congestion games. In *Proc. 18th ESA*, vol. 2, pages 29–38, 2010.
- [14] T. Harks and M. Klimm. On the existence of pure Nash equilibria in weighted congestion games. *Math. Oper. Res.*, 37(3):419–436, 2012.
- [15] T. Harks, M. Klimm, and R. Möhring. Strong Nash equilibria in games with the lexicographical improvement property. *Int. J. Game Theory*, 42(2): 461–482, 2013.
- [16] J. Jaffe. Bottleneck flow control. IEEE Trans. Commun., 29(7):954–962, 1981.
- [17] R.Kannan and C. Busch. Bottleneck Congestion Games with Logarithmic Price of Anarchy. In *Proc. 3rd SAGT*, pages 222–233, 2010.
- [18] B. de Keijzer, G. Schäfer, and O. Telelis. On the inefficiency on equilibria in linear bottleneck congestion games. In *Proc. 3rd SAGT*, pages 335–346, 2010.
- [19] F. P. Kelly and T. Voice Stability of end-to-end algorithms for joint routing and rate control Comp. Comm. Rev., 35(2), 5–12, 2005.
- [20] F. P. Kelly, A. K. Maulloo, and D. K. H. Tan. Rate Control in Communication Networks: Shadow Prices, Proportional Fairness, and Stability. *J. Oper. Res. Soc.*, 49:237–52, 1998.

- [21] P. Key, L. Massoulié, D. Towsley. Path selection and multipath congestion control. *Proc. 26th INFOCOM*, pages 143–151, 2007.
- [22] S. Low. A duality model of TCP flow controls. In *Proc. ITC Specialist Seminar on IP Traffic Measurement, Modeling and Management*, 2000.
- [23] S. Low, F. Paganini, and J. C. Doyle. Internet Congestion Control. *IEEE Control Systems Magazine*, 22, pp. 28–43, 2002.
- [24] S. H. Low and D. E. Lapsley. Optimization Flow Control I. *IEEE/ACM Trans. Netw.*, 7(6):861–874, 1999.
- [25] J. Kleinberg, Y. Rabani, and É. Tardos. Fairness in routing and load balancing. *J. Comput. Syst. Sci.*, 63(1):2–20, 2001.
- [26] R. Koch and I. Spenke. Complexity and approximability of k-splittable flows. *Theoret. Comput. Sci.*, 369(1–3):338–347, 2006.
- [27] L. Mamatas, T. Harks, and V. Tsaoussidis. Approaches to Congestion Control in Packet Networks. J. Internet Engineering, 1(1): 22–33, 2007
- [28] K. Miller and T. Harks. Utility max-min fair congestion control with time-varying delays. In *Proc. 27th INFOCOM*, pages 331–335, 2008.
- [29] J. Mo and J. Walrand. Fair end-to-end window-based congestion control. *IEEE/ACM Trans.* Netw., 8(5), 556 567, 2000.
- [30] Orda, A., R. Rom, N. Shimkin. Competitive routing in multi-user communication networks. *IEEE/ACM Trans. Netw.*: 1, pp. 510–521, 1993.
- [31] A. Schrijver. Combinatorial Optimization: Polyhedra and Efficiency. Springer Verlag, 2003.
- [32] R. Srikant. The Mathematics of Internet Congestion Control. Birkhaeuser, 2003.
- [33] L. Tan, L. Dong, C. Yuan, and M. Zukerman. Fairness Comparison of FAST TCP and TCP Reno. Comput. Commun., 30(6):1375–1382, 2007.
- [34] J. Wang, L. Li, S. H. Low, and J. C. Doyle Cross-Layer Optimization in TCP/IP networks *IEEE/ACM Trans. Netw.*, 13(3):582–568, 2005
- [35] W.-H. Wang, M. Palaniswami, S.H. Low. Optimal flow control and routing in multi-path networks. *Perform. Eval.*: 52: 119–132, 2003.
- [36] B. Wydrowski and M. Zukerman. MaxNet: A congestion control architecture for MaxMin fairness. *IEEE Commun. Lett.*, 6, 512–514, 2002.
- [37] D. Yang, G. Xue, X. Fang, S. Misra, and J. Zhang. Routing in max-min fair networks: A game theoretic approach. In *Proc.* 18th ICNP, pp. 1–10, 2010.
- [38] Y. Zhang, S.-R. Kang, and D. Loguinov. Delay-independent stability and performance of distributed congestion control. *IEEE/ACM Trans. Netw.*, 15(4): 838–851, 2007.