Status Update Control and Analysis under Two-Way Delay

Mohammad Moltafet, Markus Leinonen, Marian Codreanu, and Roy D. Yates

Abstract-We study status updating under two-way delay in a system consisting of a sampler, a sink, and a controller residing at the sink. The controller controls the sampling process by sending request packets to the sampler. Upon receiving a request, the sampler generates a sample and transmits the status update packet to the sink. Transmissions of both request and status update packets encounter random delays. We develop optimal control policies to minimize the average age of information (AoI) using the tools of Markov decision processes in two scenarios. We begin with the system having at most one active request, i.e., a generated request for which the sink has not yet received a status update packet. Then, as the main distinctive feature of this paper, we initiate pipelining-type status updating by studying a system having at most two active requests. Furthermore, we conduct AoI analysis by deriving the average AoI expressions for the Zero-Wait-1, Zero-Wait-2, and Wait-1 policies. According to the Zero-Wait-1 policy, whenever a status update packet is delivered to the sink, a new request packet is inserted into the system. The Zero-Wait-2 policy operates similarly, except that the system holds two active requests. According to the Wait-1 policy, whenever a status update packet is delivered to the sink, a new request is sent after a waiting time which is a function of the current AoI. Numerical results illustrate the performance of each status updating policy under different system parameter

Index Terms- AoI, two-way delay, status updating control, Markov decision process.

I. INTRODUCTION

A key enabler for emerging time-critical applications (e.g., industrial automation or drone control) in wireless sensor networks (WSNs) and Internet of things (IoT) is the timely delivery of status updates of real-world physical processes. The age of information (AoI) is a relatively new destination-centric metric for the information freshness in status update systems [1]–[3]. A status update packet contains a measured value of a monitored process and a time stamp for when the sample was generated. For each source node, the AoI at a destination measures the difference between the current time and the time stamp of the most recent received status update [1], [4].

In this paper, we study status updating under two-way delay in the discrete-time system depicted in Figure 1. A sampler monitors a random process and sends updates through

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a "forward link" service facility to a sink/monitor. A controller (residing at the sink) controls the sampling process by sending request packets to the sampler on a "reverse link" control channel.

The two-way delay refers to the consideration that packets on both forward and reverse links experience random delays. In particular, each request packet requires a random time to be delivered to the sampler. Upon receiving a request, the sampler immediately generates a sample and initiates the transmission of the corresponding status update packet, which requires a random time to arrive at the sink. This model arises in many applications, where the monitor (sink) is responsible for controlling the sampling process of a remote source (e.g., a wireless sensor); accordingly, the monitor sends sampling control messages (i.e., packets) over the wireless links, which arrive at the source after a random delay. Such delayed control messaging is not present in the majority of the existing literature on AoI analysis and control; the few exceptions are elaborated in Section II.

A. Contributions

The delayed control channel opens up a framework to study status updating under *pipelining-type* packet flow, where multiple requests are concurrently being processed in the system. In this setting, we define an *active request* as a request that the controller has created, but for which the sink has not yet received the corresponding status update packet. As the main contribution, we take a step towards improving the timeliness of status updates in pipelining-based status update systems as follows. We initially consider a system with at most one active request, thus operating in a stop-and-wait fashion. Then, initiating pipelining-type status updating, we consider a system with at most two active requests. As this system leads to queued packets when a server is busy, waiting buffers are introduced at both forward and reverse links.

The first part of the paper deals with AoI-optimal status updating *control* in the two scenarios regarding the number of active requests. Assuming that the controller has global knowledge of the system state (i.e., the existence of packets at the waiting buffers and servers and their age values), our objective is to find the optimal times to send request messages, i.e., the optimal control policy, that minimize the average AoI. For each scenario, we formulate a Markov decision process (MDP) problem that we subsequently solve via the relative value iteration (RVI) algorithm.

The second part of the paper focuses on *AoI analysis*. Since, in practice, the global system state is oftentimes not

available at the controller, we analyze the performance of two control policies that operate solely based on the controller's local information: 1) the zero-wait policy with one active request (Zero-Wait-1 policy), and 2) the zero-wait policy with two active requests (Zero-Wait-2 policy). According to the Zero-Wait-1 policy, whenever a status update packet is delivered to the sink, a new request packet is immediately inserted into the system. The Zero-Wait-2 policy operates similarly; except the system always maintains two active requests. We derive the closed-form expression of the average AoI under the Zero-Wait-1 and Zero-Wait-2 policies.

Furthermore, driven by the fact that the zero-wait discipline is not necessarily age-optimal [5], [6], we modify the Zero-Wait-1 policy by introducing a waiting time into the requesting procedure. Under this Wait-1 policy, whenever a status update packet is delivered to the sink, a new request is sent after a waiting time which is a function of the current AoI. We derive the closed-form expression of the average AoI under the Wait-1 policy.

Finally, numerical results are presented to show the performance of each status updating policy under different system parameter values. Moreover, the threshold-type structure of an optimal status updating policy is illustrated.

B. Organization

The paper is organized as follows. Related work is discussed in Section II. The system model and AoI definition are presented in Section III. Optimal control policies to minimize the average AoI are presented in Section IV. Average AoI analysis for Zero-Wait-1, Zero-Wait-2, and Wait-1 policies is provided in Section V. Numerical results are presented in Section VI. Finally, concluding remarks are made in Section VII.

II. RELATED WORK

Analysis of AoI in various status update systems has witnessed great interest. The first analytical results were derived in [4], where the authors characterized the average AoI for M/M/1, D/M/1, and M/D/1 first-come first-served queueing models. In [7], the authors proposed the peak AoI as an alternative metric to evaluate the information freshness. Since then, average AoI and peak AoI have extensively been studied in various queueing models. The average (peak) AoI for various queueing models with exponentially distributed service time and Poisson arrivals has been studied in, e.g., [7]-[12]. Besides exponentially distributed service time and Poisson arrivals, AoI has also been studied under various arrival processes and service time distributions in, e.g., [10], [13]-[20]. The work [21], building on [22], introduced an effective mathematical framework, the stochastic hybrid systems (SHS) technique, to calculate the average AoI in more general status update systems. Furthermore, the work [23] extended the SHS analysis to calculate the moment generating function of the AoI, opening a way to characterize distributional properties of the age. Subsequently, the SHS technique has been used to characterize the average AoI and moment generating functions under various queueing models and packet management policies in, e.g., [24]-[35]. AoI has also been studied in various discrete-time queueing systems [36]-[40].

Besides the age analysis, numerous works have focused on devising AoI-optimal status updating control procedures. Indeed, the optimization of, e.g., the sampling times of the sensors, scheduling, and resource management plays a critical role in the performance of status update systems. The works [6], [41]–[56] investigated the optimal sampling problem in status update systems, whereas the works [5], [57]–[63] studied the optimal sampling problem with energy harvesting sources. A comprehensive literature review of recent works in AoI can be found in [3].

An efficient status update system optimally balances between too infrequent updates and excessively frequent updates that induce queueing delays. To this end, an intuitive approach is to send new updates as the system becomes empty. However, the work [5] established a new notion for optimal status update control by showing that a lower average AoI value may be achieved by purposely asking the sensor to wait before taking and transmitting a fresh sample. The work [6] generalized the idea in [5] and studied optimal sampling with respect to different age penalty functions, where a source communicates over a Markov channel (imposing correlated packet transmission times) subject to a sampling rate constraint. The work [48] proposed a new mutual-information-based metric to quantify information freshness and subsequently studied a sampling problem, where a Markov source sends updates through a queue to the sink. The authors proved that the optimal sampling policy is a threshold policy and found the optimal threshold. The work [49] further extended the study of the sampling problem in [6] by considering more general non-linear functions of AoI and investigating both continuoustime and discrete-time systems. The work [57] considered an energy harvesting sensor and derived the optimal threshold, in terms of the current energy state and estimated age, to trigger a new sample to minimize the average AoI. Therein too, the threshold policy conforms to the idea of intentionally waiting prior to sampling, as in [5].

Extending beyond mere sampling, status update control has also been studied in more general network optimization frameworks. The authors of [50], [64] considered a multisource status update system in which the different sources send their updates through a shared channel causing random delays. They studied joint sampling and transmission scheduling for optimizing the information freshness. The authors of [51] considered two source nodes generating heterogeneous traffic with different power supplies and studied the peak-age-optimal status update scheduling. In [58], the authors considered an energy harvesting source sending status updates over a noisy channel and derived the optimal status updating policy to minimize the average AoI. In [52], the authors studied transmission scheduling of status updates over an error-prone communication channel via automatic repeat request (ARQ) protocols to minimize the average AoI subject to the constraint on the average number of transmissions. The work [53] studied the problem of optimal scheduling and radio resource allocation in a WSN to minimize the average total transmit power subject to an average AoI constraint. The work in [54] studied the AoI-optimal sampling and scheduling in a relay-based multi-source status update system under an average number of

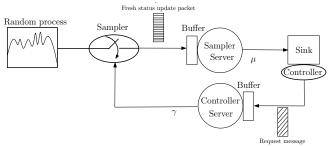


Fig. 1: The considered status update system with two-way delay.

transmissions constraint. None of the aforementioned works, however, considers a delayed control channel.

The most related works to our paper are [55], [56], which both incorporate a delayed feedback/ACK channel in a status update system. The work in [55] considered a system consisting of a sensor, a controller, a forward sensor-to-controller channel, and a backward controller-to-sensor channel. A transmission in both the forward and backward channel experiences a random delay. A sensor can generate a new packet only after receiving a confirmation message of the previous packet from the controller, thus conforming to a *stop-and-wait* ARQ mechanism. The authors introduced waiting times (in line with [5], [6]) at both the sensor and controller ends and derived optimal status updating policies to minimize the average AoI or to perform remote estimation of a Wiener process. The work [56] considered a system where status updates are transmitted through an unreliable channel and upon delivery of a packet, a feedback message is sent to the sampler, which experiences a random delay. Similarly to [55], stop-and-wait packet flow is assumed. They derived an optimal sampling policy that minimizes a long-term average age penalty under a sampling rate constraint.

In contrast to [55], [56], our paper relaxes the restriction to stop-and-wait mechanisms. As conjectured in [55, Remark 2] and [56, Footnote 2], our paper shows that, instead of working in a stop-and-wait fashion, requesting for status updates *anticipatively* is, in general, beneficial. Indeed, the consideration of pipelining (in the form of two active requests) in a status update system under two-way delay is the most distinctive feature of our paper. Also, differently to [55], we consider a strictly obedient sampler ("sensor" in [55]) so that once it receives a request message, it is not allowed to wait prior to sampling. Overall, our analysis is quite different and is conducted for a discrete-time system, as opposed to the continuous-time systems in [55], [56].

III. SYSTEM MODEL AND AOI DEFINITION

A. System Model

We consider a status update system consisting of one sampler and one sink, as depicted in Fig. 1. We assume slotted communication with normalized slots $t \in \{0,1,\dots\}$; slot t indicates the time interval [t,t+1). The sampler monitors a random process and the sink is interested in timely information about the status of the process.

1) Request Messaging: We consider that the sampler can be commanded to take a sample of the process at will.

More precisely, we consider a *controller* at the sink who sends *request messages* (i.e., request packets) to the sampler; once the sampler receives the request, it takes a sample. We consider that each request packet requires a random time to be successfully delivered to the sampler. We model this as a *controller server* residing at the link from the controller to the sampler. We assume that the controller server serves request packets according to a geometrically distributed service time with mean $1/\gamma$. Moreover, the controller server is associated with a *controller buffer* that stores arriving request packets whenever the server is busy.

- 2) Status Updating: If the sampler receives a request packet at the end of slot t-1, it generates a sample at the beginning of slot t. The samples are sent to the sink as status update packets, each containing the measured value of the monitored process and a time stamp representing the time when the sample was generated. We consider that each status update packet requires a random time to arrive at the sink. We model this as a server residing at the link from the sampler to the sink, henceforth called the sampler server. We assume that the sampler server serves the packets according to a geometrically distributed service time with mean $1/\mu$. The sampler server is associated with a sampler buffer that stores an arriving status update packet if the server is busy.
- 3) System Capacity: We will study two scenarios regarding the system capacity, i.e., the maximum number of packets (either request or status update packets) concurrently in the system. We define an active request as a request that the controller has created, but for which the sink has not yet received the corresponding status update packet. In the first scenario, there is at most one active request concurrently in the system. This corresponds to the typical stop-andwait transmission policy, where a new request packet can be inserted only after the preceding requested status update packet is delivered to the sink. To give insights to the queueing dynamics, the system holds at most one status update packet at a time while the buffers are always empty. In the second scenario, we consider a system with at most two active requests. This takes a step toward *pipelining*, where the system is simultaneously processing multiple packets. This leads to more intricate queueing dynamics: for example, the system may have two request or two status update packets so that one packet is waiting in a buffer.

B. AoI Definition

The AoI at the sink is defined as the time elapsed since the last successfully received status update packet was generated. Formally, let t_i denote the time slot at which the ith request message is generated. This request message completes service at the controller server at time slot $t_i'-1$ and is delivered to the sampler server at time slot t_i' , which is also the time slot that the ith status update packet is both generated at the sampler and arrives at the sampler server. Finally, to close the round trip passage of update i, let \bar{t}_i denote the time slot at which the ith status update packet arrives at the sink. At time slot τ , the index of the most recently received status update packet is given by

$$N(\tau) = \max\{i : \bar{t}_i \le \tau\},\tag{1}$$

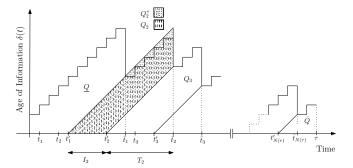


Fig. 2: An example of the evolution of the AoI.

and the time stamp of the most recently received status update packet is $U^{\tau}=t'_{N(\tau)}$. The (discrete) AoI at the sink is defined as the random process $\delta^t=t-U^t$. The evolution of the AoI is given as

$$\delta^t = \begin{cases} t - t'_{N(t)}, & \text{if } t = \bar{t}_{N(t)} \\ \delta^{t-1} + 1, & \text{otherwise.} \end{cases}$$
 (2)

A sample path of the AoI is exemplified in Fig. 2, which illustrates the staircase pattern of the AoI evolution.

To evaluate the AoI, we use the most commonly used metric, the average AoI [3], [65], which is defined in the following. Let $(0,\tau)$ denote an observation interval. Accordingly, the time average AoI at the sink, denoted as Δ_{τ} , is defined as

$$\Delta_{\tau} = \frac{1}{\tau} \sum_{t=1}^{\tau} \delta^{t}. \tag{3}$$

The summation in (3) is equal to the area under δ^t , which can be expressed as a sum of disjoint areas determined by polygons Q, $\{Q_i\}_{i=2}^{N(\tau)}$, and \bar{Q} , as illustrated in Fig. 2. Following the definition of $N(\tau)$ in (1), Δ_{τ} can be calculated as

$$\Delta_{\tau} = \frac{1}{\tau} \left(\underline{Q} + \sum_{i=2}^{N(\tau)} Q_i + \overline{Q} \right)$$

$$= \frac{\underline{Q} + \overline{Q}}{\tau} + \frac{N(\tau) - 1}{\tau} \frac{1}{N(\tau) - 1} \sum_{i=2}^{N(\tau)} Q_i. \tag{4}$$

In our analysis, we use a transition between the discrete AoI process defined in (2) and the AoI process, which is equivalent to (2) except that the age increases continually during a slot. To this end, let Q_i^+ denote a trapezoid associated with Q_i so that Q_i is inscribed in Q_i^+ , as illustrated in Fig. 2 for i=2. The trapezoids Q_i^+ and Q_i^+ are defined similarly. Thus, we can rewrite Δ_{τ} in (4) as

$$\Delta_{\tau} = \frac{Q^{+} + \bar{Q}^{+}}{\tau} + \frac{N(\tau) - 1}{\tau} \frac{1}{N(\tau) - 1} \sum_{i=2}^{N(\tau)} Q_{i}^{+} - \frac{1}{2}, \tag{5}$$

because by considering Q^+ , $\{Q_i^+\}_{i\geq 1}$, and \bar{Q}^+ , a triangle of area 1/2 has been added in each slot to the area under δ^t , and thus, we need to subtract the average area of these triangles, which is 1/2.

Let $T_i = \bar{t}_i - t_i'$ denote the system time of status update packet i, defined as the time duration between the generation time at the sampler and the reception time at the sink. Let $I_i = t_i' - t_{i-1}'$ denote the interarrival time between status update packets i-1 and i at the sampler server. Since Q_i^+ can

be calculated by subtracting the area of the isosceles triangle with sides T_i from the area of the isosceles triangle with sides $I_i + T_i$ (see Fig. 2),

$$Q_i^+ = \frac{1}{2}I_i^2 + I_i T_i. (6)$$

The (time) average AoI, denoted by Δ , is defined as $\Delta = \lim_{\tau \to \infty} \Delta_{\tau}$. With the assumption that $\{(I_i, T_i)\}_{i \geq 1}$ is a stationary ergodic random process, the limit $\lambda = \lim_{\tau \to \infty} (N(\tau) - 1)/\tau$ exists and is given as $\lambda = 1/\mathbb{E}[I_i]$ and the sample average $\frac{1}{N(\tau)-1} \sum_{i=2}^{N(\tau)} Q_i^+$ in (5) converges to the stochastic average $\mathbb{E}[Q_i^+]$. Since the term $(\underline{Q}^+ + \bar{Q}^+)/\tau$ in (5) goes to zero as $\tau \to \infty$, it then follows from (5) and (6) that the average AoI is

$$\Delta = \frac{1}{\mathbb{E}[I_i]} \mathbb{E}\left[\frac{1}{2}I_i^2 + I_i T_i\right] - \frac{1}{2}.\tag{7}$$

Going forward, we will use the average AoI (7) to compare controller policies.

IV. OPTIMAL CONTROL FOR AVERAGE AOI MINIMIZATION

In this section, we study and design optimal control policies for the considered status updating system. More precisely, our goal is to determine the optimal times for the controller to send request messages to minimize the average AoI at the sink. We split the control design into two cases. In Section IV-A, we consider the system with at most one active request, hereinafter referred to as *1-Packet system*. In Section IV-B, we consider the system with at most two active requests, hereinafter referred to as *2-Packet system*.

In this section, we use a common assumption (see e.g., [52], [66]–[68]) that the AoI is upper-bounded by a sufficiently large value $\bar{\Delta}$ so that $\delta^t \in \{1,2,\ldots,\bar{\Delta}\}$. This enables tractable design of optimal control policies by making the state spaces finite, while acknowledging that when the status information becomes excessively stale by reaching $\bar{\Delta}$, the time-critical end application would not be affected if counting further.

A. 1-Packet System

In this section, we formulate the optimal status update control problem as a Markov decision process (MDP) problem and propose a relative value iteration (RVI) algorithm to find an optimal control policy.

1) State: Let $s^t \in \mathcal{S}$ denote the state of the system in slot t, where \mathcal{S} is the state space. We define the system state as $s^t = (\delta^t, e_{\mathrm{s}}^t, E_{\mathrm{s}}^t, \Delta_{\mathrm{s}}^t)$ with the following four elements: 1) δ^t is the AoI at the sink, 2) $e_{\mathrm{s}}^t \in \{0,1\}$ is an occupancy indicator for the controller server; $e_{\mathrm{s}}^t = 1$ indicates that there is a packet at the controller server, and $e_{\mathrm{s}}^t = 0$ otherwise, 3) $E_{\mathrm{s}}^t \in \{0,1\}$ is an occupancy indicator for the sampler server, 4) Δ_{s}^t is the age of the status update packet at the sampler server.

¹Such ergodicity assumptions are common in the AoI literature [4], [65]. For the controller policies considered in this work, explicit verification can be nontrivial but not especially interesting.

²The same convention is used for all binary occupancy indicators throughout the paper.

- 2) Action: The action taken in slot t is denoted as $a^t \in \mathcal{A} = \{0,1\}$, where $a^t = 1$ indicates that the controller generates a request packet at the beginning of slot t, and $a^t = 0$ otherwise.
- 3) Policy: A policy π is a (possibly randomized) mapping from the state space $\mathcal S$ to action space $\mathcal A$. It determines an action a^t taken in the current state s^t . Note that according to the 1-Packet system constraint, the set of admissible policies must satisfy that if there is an active request in the system, i.e., either $e^t_s=1$ or $E^t_s=1$, then $a^t=0$.
- 4) Problem Formulation: Given an initial state $s^0 \in \mathcal{S}$, the long-term average AoI, obtained by following a policy π , is defined as

$$J_{\pi}(s^0) = \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[\delta^t \mid s^0\right], \tag{8}$$

where the expectation is with respect to the action taken in each slot and the randomness in the service times, i.e., required time to serve each packet in the controller and sampler servers. The main objective is to find the optimal policy π^* that provides the minimum long-term average AoI in the 1-Packet system, i.e.,

(**PI**)
$$\pi^*(s^0) = \underset{\pi \in \Pi_1 \text{ body}}{\arg \min} J_{\pi}(s^0),$$
 (9)

where $\Pi_{1\text{-Packet}}$ is the set of all admissible policies subject to the 1-Packet system constraint. Next, we recast problem (**PI**) as an MDP problem and solve it (i.e., find the optimal policy) using the RVI algorithm.

5) MDP Modeling of Problem (PI): The MDP of the 1-Packet system is defined by the tuple $(\mathcal{S},\mathcal{A},\Pr(s^{t+1}|s^t,a^t),C(s^t,a^t))$, with $C(s^t,a^t)$ denoting the (immediate) cost of taking action a^t in state s^t . Specifically, we define $C(s^t,a^t)=\delta^{t+1}$, the AoI at slot t+1. The state transition probability $\Pr(s'|s,a)=\Pr(s^{t+1}|s^t,a^t)$ gives the probability of moving from the current state $s=s^t$ to the next state $s'=s^{t+1}$ under an action $a=a^t$.

To facilitate a compact description of $\Pr(s'|s,a)$, we employ the shorthand notations $\bar{\gamma} \triangleq 1 - \gamma$ and $\bar{\mu} \triangleq 1 - \mu$, as well as the operator

$$[\xi]^{\bar{\Delta}} \triangleq \min\{\xi + 1, \bar{\Delta}\}. \tag{10}$$

We also adopt the convention that \star indicates an irrelevant age value when the sampler server is idle. Then, considering the system dynamics, the transition probability from state $s=(\delta,e_{\rm s},E_{\rm s},\Delta_{\rm s})$ to the next state $s'=(\delta',e'_{\rm s},E'_{\rm s},\Delta'_{\rm s})$ under an action a, is characterized as:

$$\Pr(s'|s = (\delta, 0, 0, \star), a)$$

$$= \begin{cases} 1, & a = 0, \ s' = ([\delta]^{\bar{\Delta}}, 0, 0, \star); \\ \gamma, & a = 1, \ s' = ([\delta]^{\bar{\Delta}}, 0, 1, 0); \\ \bar{\gamma}, & a = 1, \ s' = ([\delta]^{\bar{\Delta}}, 1, 0, \star); \\ 0, & \text{otherwise.} \end{cases}$$
(11a)

$$\Pr(s'|s = (\delta, 1, 0, \star), a)$$

$$= \begin{cases} \gamma, & a = 0, \ s' = ([\delta]^{\bar{\Delta}}, 0, 1, 0); \\ \bar{\gamma}, & a = 0, \ s' = ([\delta]^{\bar{\Delta}}, 1, 0, \star); \\ 0, & \text{otherwise.} \end{cases}$$
(11b)

$$\Pr(s'|s = (\delta, 0, 1, \Delta_{s}), a)$$

$$= \begin{cases} \mu, & a = 0, \ s' = ([\Delta_{s}]^{\bar{\Delta}}, 0, 0, \star); \\ \bar{\mu}, & a = 0, \ s' = ([\delta]^{\bar{\Delta}}, 0, 1, [\Delta_{s}]^{\bar{\Delta}}); \\ 0, & \text{otherwise.} \end{cases}$$
(11c)

Based on the MDP construction above, problem (PI) is equivalently cast as an MDP problem

$$\pi^*(s^0) = \underset{\pi \in \Pi_{1\text{-Packet}}}{\min} \left\{ \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[C(s^t, a^t) \mid s^0\right] \right\}. \tag{12}$$

6) Optimal Policy: According to [69, Proposition 4.2.1], for a given initial state s^0 , if we can find a scalar \bar{V} and values V_s^* for all $s \in \mathcal{S}$ that satisfy

$$\bar{V} + V_s^* = \min_{a \in \mathcal{A}} \left\{ C(s, a) + \sum_{s' \in \mathcal{S}} \Pr(s'|s, a) V_{s'}^* \right\}, \quad (13)$$

then \overline{V} is the optimal average cost for problem (PI), $J_{\pi^*}(s^0)$, and the actions $a \in \mathcal{A}$ that attain the minimum in (13) constitute an optimal policy, $\pi^*(s^0)$.

Next, we define a condition under which the Bellman's equation in (13) has a solution and the optimal average cost is independent of the initial state s^0 [69, Proposition 4.2.3].

Definition 1 (Weak accessibility condition). The weak accessibility condition holds for an MDP problem if the states can be divided into two subsets S_c and S_t such that: I) for any two states s and s' in subset S_c , there exists a stationary policy such that the probability of moving from state s to state s' after some slots is non-zero, and II) all states in S_t are transient under every stationary policy [69, Definition 4.2.2].

In the following lemma, we prove that the weak accessibility condition holds for the MDP problem defined in (12).

Lemma 1. For any $(\mu, \gamma) \in \{(\theta_1, \theta_2) \mid 0 < \theta_1 \le 1, \ 0 < \theta_2 \le 1, \ (\theta_1, \theta_2) \ne (1, 1)\}$, the weak accessibility condition holds for the MDP problem (12).

Proof. If all deterministic policies in $\Pi_{1\text{-Packet}}$ are unichain, the weak accessibility condition holds [69, Proposition 4.2.5]. A policy is unichain if in the Markov chain induced by the policy, there exists a state that can be reached from any other states with a non-zero probability [70, Exercise 4.3]. In the MDP problem (12), under any deterministic policy: I) for $\gamma \neq 1$ and $0 < \mu \le 1$, the state $(\bar{\Delta}, 1, 0, \star)$ is accessible from any other states and II) for $\gamma = 1$ and $0 < \mu < 1$, the state $(\bar{\Delta}, 0, 1, \bar{\Delta})$ is accessible from any other states. The justification for case I is that when there is a packet at the controller server, the probability of the event of having unsuccessful transmissions in $\bar{\Delta}$ consecutive slots is non-zero. A similar argument can be stated for case II. Thus, every deterministic policy is unichain and consequently, the weak accessibility condition holds.

Given that the weak accessibility condition holds, we can find an optimal deterministic policy π^* and the optimal average cost J^* , which are independent of the initial state s^0 , by solving the Bellman's equation in (13) [69, Proposition 4.2.6]. We solve it using the RVI algorithm [69, Section 4.3], which turns the Bellman's equation into an iterative procedure. The

Algorithm 1: RVI algorithm

- 1 **Input:** 1) State transition probabilities Pr(s'|s, a), 2) Stopping criterion threshold ϵ ;
- **2 Initialization:** 1) Set $V_s^* = 0$, $\forall s \in \mathcal{S}$, 2) determine an arbitrary reference state $s_{\text{ref}} \in \mathcal{S}$, and 3) set $\phi > \epsilon$;

$$\begin{array}{lll} \textbf{3} & \textbf{while} & \phi > \epsilon & \textbf{do} \\ \textbf{4} & \textbf{for} & s \in \mathcal{S} & \textbf{do} \\ \textbf{5} & & \pi_s^* \leftarrow \\ & \arg \min_{a \in \mathcal{A}} \left\{ C(s,a) + \sum_{s' \in \mathcal{S}} \Pr(s'|s,a) V_{s'}^* \right\}; \\ \textbf{6} & & V_s \leftarrow \\ & \left\{ C(s,\pi_s^*) + \sum_{s' \in \mathcal{S}} \Pr(s'|s,\pi_s^*) V_{s'}^* \right\} - V_{s_{\text{ref}}}^*; \\ \textbf{7} & \textbf{end} \\ \textbf{8} & \phi \leftarrow \max_{s \in \mathcal{S}} \{ |V_s - V_s^*| \}; \\ \textbf{9} & & V_s^* \leftarrow V_s, \, \forall s \in \mathcal{S}; \end{array}$$

10 end

11 **Output:** 1) Optimal policy π^* , 2) optimal objective function value $J_{\pi^*} = V_{s_{\rm ref}}^*$.

steps of the RVI algorithm are summarized in Algorithm 1. In each iteration, the optimal deterministic action in state s, π_s^* , and the scalar V_s^* for each state $s \in \mathcal{S}$ are updated as follows:

$$\begin{split} & \pi_s^* \leftarrow \arg\min_{a \in \mathcal{A}} \left\{ C(s, a) + \sum_{s' \in \mathcal{S}} \Pr(s'|s, a) V_{s'}^* \right\}, \\ & V_s^* \leftarrow \left\{ C(s, \pi_s^*) + \sum_{s' \in \mathcal{S}} \Pr(s'|s, \pi_s^*) V_{s'}^* \right\} - V_{s_{\text{ref}}}^*, \end{split}$$

where $s_{\rm ref} \in \mathcal{S}$ is an arbitrary chosen reference state. Once the iterative process has converged, the algorithm provides an optimal policy π^* and the optimal value of the average AoI, obtained as $J_{\pi^*} = V_{s_{\rm ref}}^*$. The following theorem shows that the RVI algorithm presented in Algorithm 1 converges to an optimal deterministic policy.

Theorem 1. The RVI algorithm presented in Algorithm 1 converges to an optimal deterministic policy and returns the optimal value of the average AoI.

Proof. According to [69, Page 209], it is sufficient to show that the Markov chain induced by every deterministic policy is unichain and aperiodic. In the proof of Lemma 1, we showed that any deterministic policy induces a unichain Markov chain. According to [71, Exercise 4.1], since the recurrent states $(\bar{\Delta},1,0,\star)$ (for $\gamma\neq 1$ and $0<\mu\leq 1$) and $(\bar{\Delta},0,1,\bar{\Delta})$ (for $\gamma=1$ and $0<\mu<1$) have self-transitions, the Markov chain induced by every deterministic policy is aperiodic.

B. 2-Packet System

Here, we extend the MDP construction of the 1-Packet system (described in Section IV-A) for the 2-Packet system and subsequently derive the optimal status update control policy.

For the 2-Packet system, the action space and the cost function are the same as for the 1-Packet system. However,

the system state needs to be redefined to account for the two active requests.

1) State: Let $\underline{s}^t \in \underline{S}$ denote the system state in slot t, where \underline{S} is the state space. We define the system state as $\underline{s}^t = (\delta^t, e_{\mathrm{b}}^t, e_{\mathrm{s}}^t, E_{\mathrm{b}}^t, E_{\mathrm{s}}^t, \Delta_{\mathrm{b}}^t, \Delta_{\mathrm{s}}^t)$ with the following seven elements: 1) δ^t is the AoI at the sink, 2) $e_{\mathrm{b}}^t \in \{0,1\}$ is the occupancy indicator for the controller buffer, 3) $e_{\mathrm{b}}^t \in \{0,1\}$ is the occupancy indicator for the controller server, 4) $E_{\mathrm{b}}^t \in \{0,1\}$ is the occupancy indicator for the sampler buffer, 5) $E_{\mathrm{s}}^t \in \{0,1\}$ is the occupancy indicator for the sampler server, 6) Δ_{b}^t is the age of the status update packet at the sampler buffer, and 7) Δ_{s}^t is the age of the status update packet at the sampler server.

Remark 1. Having the four occupancy indicators $e_{\rm s}^t$, $e_{\rm s}^t$, $E_{\rm b}^t$, and $E_{\rm s}^t$ in the state implies that the controller (i.e., the decision-maker) has global knowledge on the system's occupancy. This consideration may not always be practical, and thus, the derived control policy establishes a performance bound for policies operating under local/imperfect knowledge.

2) Problem Formulation: Given an initial state \underline{s}^0 , the average AoI, obtained by following a policy π , is defined as

$$\underline{J}_{\pi}(\underline{s}^{0}) = \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{\pi} \left[\delta^{t} \mid \underline{s}^{0} \right]. \tag{14}$$

Note that according to the 2-Packet system constraint, the set of admissible policies must satisfy that if there are two active requests in the system, i.e., $(e_{\rm b}^t,e_{\rm s}^t)=(1,1)$ or $(e_{\rm s}^t,E_{\rm s}^t)=(1,1)$ or $(E_{\rm b}^t,E_{\rm s}^t)=(1,1)$, then $a^t=0$. The problem of finding the optimal policy π^* that provides the minimum long-term average AoI in the 2-Packet system is formulated as

$$(\mathbf{PII}) \ \pi^*(\underline{s}^0) = \underset{\pi \in \Pi_{2,\text{Packel}}}{\arg\min} \ \underline{J}_{\pi}(\underline{s}^0), \tag{15}$$

where $\Pi_{2\text{-Packet}}$ is the set of all admissible policies subject to the 2-Packet system constraint.

Next, we recast problem (**PII**) as an MDP problem and solve it using the RVI algorithm.

3) MDP Modeling for Problem (PII): The MDP of the 2-Packet system is defined by the tuple $(\underline{\mathcal{S}}, \mathcal{A}, \Pr(\underline{s}^{t+1}|\underline{s}^t, a^t), C(\underline{s}^t, a^t))$. The (immediate) cost function is defined as $C(\underline{s}^t, a^t) = \delta^{t+1}$ (similarly as for problem (PI)). The transition probability $\Pr(\underline{s}'|\underline{s}, a)$ of moving from state $\underline{s} = (\delta, e_b, e_s, E_b, E_s, \Delta_b, \Delta_s)$ to the next state $\underline{s}' = (\delta', e_b', e_s', E_b', E_s', \Delta_b', \Delta_s')$ under an action a is characterized as:

$$\Pr(\underline{s}'|\underline{s} = (\delta, 0, 0, 0, 0, \star, \star), a)$$

$$= \begin{cases} 1, & a = 0, \ \underline{s}' = ([\delta]^{\bar{\Delta}}, 0, 0, 0, 0, \star, \star); \\ \gamma, & a = 1, \ \underline{s}' = ([\delta]^{\bar{\Delta}}, 0, 0, 0, 1, \star, 0); \\ \bar{\gamma}, & a = 1, \ \underline{s}' = ([\delta]^{\bar{\Delta}}, 0, 1, 0, 0, \star, \star); \\ 0, & \text{otherwise.} \end{cases}$$
(16a)

$$\Pr(\underline{s}'|\underline{s} = (\delta, 1, 1, 0, 0, \star, \star), a)$$

$$= \begin{cases} \gamma, & a = 0, \ \underline{s'} = ([\delta]^{\bar{\Delta}}, 0, 1, 0, 1, \star, 0); \\ \bar{\gamma}, & a = 0, \ \underline{s'} = ([\delta]^{\bar{\Delta}}, 1, 1, 0, 0, \star, \star); \\ 0, & \text{otherwise.} \end{cases}$$
(16b)

$$\Pr(\underline{s}'|\underline{s} = (\delta, 0, 1, 0, 0, \star, \star), a)$$

$$\begin{cases}
\gamma, & a = 0, \ \underline{s'} = ([\delta]^{\bar{\Delta}}, 0, 0, 0, 1, \star, 0); \\
\bar{\gamma}, & a = 0, \ \underline{s'} = ([\delta]^{\bar{\Delta}}, 0, 1, 0, 0, \star, \star); \\
\gamma, & a = 1, \ \underline{s'} = ([\delta]^{\bar{\Delta}}, 0, 1, 0, 1, \star, 0); \\
\bar{\gamma}, & a = 1, \ \underline{s'} = ([\delta]^{\bar{\Delta}}, 1, 1, 0, 0, \star, \star); \\
0, & \text{otherwise.}
\end{cases} (16c)$$

$$\Pr(\underline{s}'|\underline{s} = (\delta, 0, 1, 0, 1, \star, \Delta_{s}), a)$$

$$= \begin{cases} \bar{\gamma}\bar{\mu}, & a = 0, \ \underline{s'} = ([\delta]^{\bar{\Delta}}, 0, 1, 0, 1, \star, [\Delta_{\mathrm{s}}]^{\bar{\Delta}}); \\ \gamma\bar{\mu}, & a = 0, \ \underline{s'} = ([\delta]^{\bar{\Delta}}, 0, 0, 1, 1, 0, [\Delta_{\mathrm{s}}]^{\bar{\Delta}}); \\ \bar{\gamma}\mu, & a = 0, \ \underline{s'} = ([\Delta_{\mathrm{s}}]^{\bar{\Delta}}, 0, 1, 0, 0, \star, \star); \\ \gamma\mu, & a = 0, \ \underline{s'} = ([\Delta_{\mathrm{s}}]^{\bar{\Delta}}, 0, 0, 0, 1, \star, 0); \\ 0, & \text{otherwise.} \end{cases}$$
(16d)

$$\begin{split} &\Pr(\underline{s}'|\underline{s} = (\delta, 0, 0, 1, 1, \Delta_{\mathrm{b}}, \Delta_{\mathrm{s}}), a) \\ &= \begin{cases} \mu, & a = 0, \ \underline{s}' = ([\Delta_{\mathrm{s}}]^{\bar{\Delta}}, 0, 0, 0, 1, \star, [\Delta_{\mathrm{b}}]^{\bar{\Delta}}); \\ \bar{\mu}, & a = 0, \ \underline{s}' = ([\delta]^{\bar{\Delta}}, 0, 0, 1, 1, [\Delta_{\mathrm{b}}]^{\bar{\Delta}}, [\Delta_{\mathrm{s}}]^{\bar{\Delta}}); \\ 0, & \text{otherwise}. \end{cases} \end{split}$$

$$\Pr(s'|s = (\delta, 0, 0, 0, 1, \star, \Delta_s), a)$$

$$\begin{aligned}
\bar{\mu}, & a = 0, \ \underline{s'} = ([\delta]^{\bar{\Delta}}, 0, 0, 0, 1, \star, [\Delta_{s}]^{\bar{\Delta}}); \\
\mu, & a = 0, \ \underline{s'} = ([\Delta_{s}]^{\bar{\Delta}}, 0, 0, 0, 0, \star, \star); \\
\bar{\gamma}\bar{\mu}, & a = 1, \ \underline{s'} = ([\delta]^{\bar{\Delta}}, 0, 1, 0, 1, \star, [\Delta_{s}]^{\bar{\Delta}}); \\
\bar{\gamma}\mu, & a = 1, \ \underline{s'} = ([\Delta_{s}]^{\bar{\Delta}}, 0, 1, 0, 0, \star, \star); \\
\gamma\bar{\mu}, & a = 1, \ \underline{s'} = ([\delta]^{\bar{\Delta}}, 0, 0, 1, 1, 0, [\Delta_{s}]^{\bar{\Delta}}); \\
\gamma\mu, & a = 1, \ \underline{s'} = ([\delta]^{\bar{\Delta}}, 0, 0, 0, 1, \star, 0); \\
0, & \text{otherwise.}
\end{aligned}$$
(16f)

Based on the above definitions, the MDP formulation of problem (PII) is formulated as

$$\pi^*(\underline{s}^0) = \underset{\pi \in \Pi_{2, \text{Packet}}}{\min} \left\{ \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[C(\underline{s}^t, a^t) \mid \underline{s}^0 \right] \right\}. \quad (17)$$

4) Optimal Policy: First, similar to the proof of Lemma 1 for the 1-Packet system, one can show that the weak accessibility condition holds for the MDP problem of the 2-Packet system in (17) for any $(\mu,\gamma) \in \{(\theta_1,\theta_2) \mid 0 < \theta_1 \leq 1, \ 0 < \theta_2 \leq 1, \ (\theta_1,\theta_2) \neq (1,1)\}$. Here, the recurrent states are: I) state $(\bar{\Delta},1,1,0,0,\star,\star)$ for $\gamma \neq 1$ and $0 < \mu \leq 1$, and II) state $(\bar{\Delta},0,0,1,1,\bar{\Delta},\bar{\Delta})$ for $\gamma=1$ and $0 < \mu < 1$.

By the weak accessibility condition, the optimal average cost \underline{J}_{π^*} and optimal policy π^* are derived by solving the following equation for each $\underline{s} \in \underline{\mathcal{S}}$ (cf. (13)):

$$\underline{J}_{\pi^*} + V_{\underline{s}}^* = \min_{a \in \mathcal{A}} \left\{ C(\underline{s}, a) + \sum_{\underline{s}' \in \underline{\mathcal{S}}} \Pr(\underline{s}' | \underline{s}, a) V_{\underline{s}'}^* \right\}. \quad (18)$$

The RVI algorithm presented in Algorithm 1 is used to iteratively solve for \underline{J}_{π^*} and V_s^* in (18).

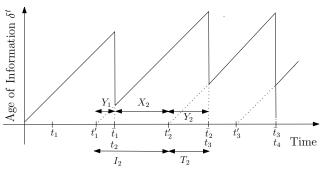


Fig. 3: AoI as a function of time under the Zero-Wait-1 policy.

V. AVERAGE AOI ANALYSIS

In this section, we conduct average AoI analysis in the considered status updating system and derive the average AoI expressions for the following three policies: 1) a zero-wait policy for the 1-Packet system, hereinafter referred to as the *Zero-Wait-1 policy*, 2) a zero-wait policy for the 2-Packet system, hereinafter referred to as the *Zero-Wait-2 policy* (Section V-B), and 3) a waiting-based policy for the 1-Packet system, hereinafter referred to as the *Wait-1 policy* (Section V-C).

A. Zero-Wait-1: AoI Analysis

(16e)

In this section, we derive the average AoI under the Zero-Wait-1 policy. Since the Zero-Wait-1 policy conforms to the *zero-wait* serving discipline (also known as *work-conserving* or *just-in-time* policy [6]), the controller sends a new request message immediately after receiving a status update packet. That is, if a status update packet i-1 is received by the end of slot t-1, the controller generates a request message i at the beginning of slot t, which enters an empty system. AoI evolution under the Zero-Wait-1 policy is illustrated in Fig. 3.

To characterize the average AoI as given by (7), we need to calculate $\mathbb{E}[I_i]$, $\mathbb{E}[I_i^2]$, and $\mathbb{E}[I_iT_i]$. As preliminaries, let X_i be the random variable representing the required time to serve the ith request packet by the controller server. Similarly, let Y_i be the random variable representing the required time to serve the ith status update packet by the sampler server. Because the service times of all request messages are stochastically identical and the service times of all status update packets are stochastically identical, $X_i = ^{\rm st} X$ and $Y_i = ^{\rm st} Y$ for all $i = \{1, 2, \ldots\}$, and $X \sim {\rm Geo}(\gamma)$ and $Y \sim {\rm Geo}(\mu)$. Now we delve into calculations of $\mathbb{E}[I_i]$, $\mathbb{E}[I_i^2]$, and $\mathbb{E}[I_iT_i]$.

1) Derivation of $\mathbb{E}[I_i]$ and $\mathbb{E}[I_i^2]$: As it can be seen in Fig. 3, the interarrival time between status update packets i-1 and i at the sampler server is

$$I_i = Y_{i-1} + X_i. (19)$$

The mean and second moment of the interarrival time, $\mathbb{E}[I_i]$ and $\mathbb{E}[I_i^2]$, are

$$\mathbb{E}[I_i] = \mathbb{E}[Y_{i-1}] + \mathbb{E}[X_i] = \frac{1}{\mu} + \frac{1}{\gamma},$$
 (20)

$$\mathbb{E}[I_i^2] = \mathbb{E}[(Y_{i-1} + X_i)^2]$$

$$= \mathbb{E}[Y_{i-1}^2] + \mathbb{E}[X_i^2] + 2\mathbb{E}[Y_{i-1}]\mathbb{E}[X_i]$$
 (21)

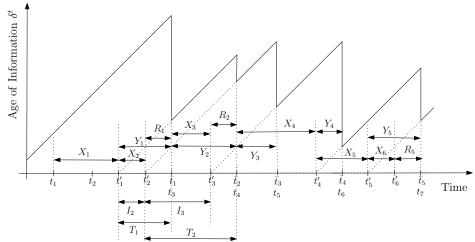


Fig. 4: AoI as a function of time under the Zero-Wait-2 policy.

$$= \frac{2-\mu}{\mu^2} + \frac{2-\gamma}{\gamma^2} + \frac{2}{\mu\gamma},\tag{22}$$

where (21) follows from independence of X_i and Y_{i-1} .

2) Derivation of $\mathbb{E}[I_iT_i]$: As it can be seen in Fig. 3, the system time of packet i, T_i , equals Y_i , the service time of update packet i. Thus,

$$\mathbb{E}[I_i T_i] = \mathbb{E}[(Y_{i-1} + X_i) Y_i]$$

$$\stackrel{(a)}{=} \mathbb{E}[Y_i] \mathbb{E}[Y_{i-1} + X_i] = \frac{1}{\mu} \left(\frac{1}{\mu} + \frac{1}{\gamma}\right), \quad (23)$$

where (a) comes from the independence of Y_i and X_i, Y_{i-1} . Substituting (20), (22), and (23) into (7) yields Theorem 2.

Theorem 2. The average AoI under the Zero-Wait-1 policy is

$$\Delta^{\text{Zero-Wait-1}} = \frac{2}{\mu} + \frac{\mu}{\gamma(\mu + \gamma)} - 1. \tag{24}$$

B. Zero-Wait-2: AoI Analysis

To evaluate the AoI of the Zero-Wait-2 system using (7), we need to track the event B_i that the sampler server is busy (serving update packet i-1) when request message i arrives at the sampler and update packet i is generated. With \overline{B}_i denoting the complement of B_i , we start with the arrival of update packet i-1 at the sampler server and we use the partition $\{B_{i-1}, \overline{B}_{i-1}\}$ to describe what can follow:

- If \overline{B}_{i-1} , then update i-1 immediately goes into service at the sampler server, starting service time Y_{i-1} . Request message i simultaneously starts service time X_i at the controller server.
 - If $X_i \geq Y_{i-1}$, then event \overline{B}_i occurs since update i arrives at the sampler server after update i-1 departs. Thus, update packet i arrives at an idle sampler server and immediately starts service time Y_i . In this case, $I_i = X_i$ and $T_i = Y_i$.
 - If $B_i = \{X_i < Y_{i-1}\}$ occurs, then update i waits for the residual service time R_{i-1} of update i-1 before starting its own service time Y_i . In this case, $I_i = X_i$ and $T_i = R_{i-1} + Y_i$.
- If B_{i-1} , update i-1 arrives when the sampler server is busy and waits for the residual service time R_{i-2}

of update i-2. After this waiting time, update i-2is delivered to the sink, request i is generated at the controller and starts service time X_i , and update i-1simultaneously starts service time Y_{i-1} at the sampler

- If $\overline{B}_i = \{X_i \geq Y_{i-1}\}$ occurs, update packet i arrives at an empty sampler server and immediately starts service time Y_i . In this case, $I_i = R_{i-2} + X_i$ and $T_i = Y_i$.
- If $B_i = \{X_i < Y_{i-1}\}$ occurs, update i arrives at the sampler server before update i-1 departs. In this case, update i waits at the sampler server for the residual service time R_{i-1} of update i-1, implying $I_i = R_{i-2} + X_i$ and $T_i = R_{i-1} + Y_i$.

To summarize this two-step partitioning, we see that

if
$$\overline{B}_{i-1}\overline{B}_i$$
: $I_i = X_i$, $T_i = Y_i$; (25a)
if $\overline{B}_{i-1}B_i$: $I_i = X_i$, $T_i = X_i$, (25b)

if
$$\overline{B}_{i-1}B_i$$
: $I_i = X_i$, $T_i = R_{i-1} + Y_i$; (25b)

if
$$B_{i-1}\overline{B}_i$$
: $I_i = R_{i-2} + X_i$, $T_i = Y_i$; (25c)

if
$$B_{i-1}B_i$$
: $I_i = R_{i-2} + X_i$, $T_i = R_{i-1} + Y_i$. (25d)

Note that the memoryless property of geometric random variables implies that conditioned on B_i , each R_{i-1} is a geometric (μ) random variable, independent of all X_i and ${Y_i: i \neq j-1}.$

Furthermore, careful reading of this two-step partitioning shows that B_i and B_{i-1} are independent events since $B_i = \{X_i < Y_{i-1}\}$, whether $B_{i-1} = \{X_{i-1} < Y_{i-2}\}$ occurred or not in the prior cycle. Since the X_i and Y_i sequences are independent (and iid within each sequence),

$$\Pr(B_i) = \Pr(B_i | B_{i-1}) = \Pr(X_i < Y_{i-1}).$$
 (26)

We further observe that the occurrence of B_i suggests that X_i is shorter than usual. In Appendix VIII-A, we verify the following properties.

Lemma 2. In the Zero-Wait-2 system:

(a) Update i arrives at a busy sampler server with probability

$$\Pr(B_i) = P_B \triangleq \Pr(X_i < Y_{i-1}) = \frac{\gamma \bar{\mu}}{\mu + \gamma \bar{\mu}}.$$
 (27a)

(b) Given event B_i ,

$$\mathbb{E}[X_i|B_i] = \frac{1}{\mu + \gamma\bar{\mu}} = \frac{P_B}{\gamma\bar{\mu}}.$$
 (27b)

Since (25) shows that I_i depends only on the partition $\{B_{i-1}, \overline{B}_{i-1}\}$, it follows from the law of total expectation that

$$\mathbb{E}[I_{i}] = \mathbb{E}[I_{i}|\overline{B}_{i-1}] \operatorname{Pr}(\overline{B}_{i-1}) + \mathbb{E}[I_{i}|B_{i-1}] \operatorname{Pr}(B_{i-1})$$

$$= \mathbb{E}[X_{i}|\overline{B}_{i-1}] \operatorname{Pr}(\overline{B}_{i-1}) + \mathbb{E}[R_{i-2} + X_{i}|B_{i-1}] \operatorname{Pr}(B_{i-1})$$

$$= \mathbb{E}[X_{i}] + \mathbb{E}[R_{i-2}|B_{i-1}] \operatorname{Pr}(B_{i-1})$$

$$= \frac{1}{\gamma} + \frac{P_{B}}{\mu}.$$
(28)

Similarly for the second moment,

$$\mathbb{E}[I_{i}^{2}] = \mathbb{E}[I_{i}^{2}|\overline{B}_{i-1}] \operatorname{Pr}(\overline{B}_{i-1}) + \mathbb{E}[I_{i}^{2}|B_{i-1}] \operatorname{Pr}(B_{i-1})$$

$$= \mathbb{E}[X_{i}^{2}|\overline{B}_{i-1}] \operatorname{Pr}(\overline{B}_{i-1})$$

$$+ \mathbb{E}[R_{i-2}^{2} + 2X_{i}R_{i-2} + X_{i}^{2}|B_{i-1}] \operatorname{Pr}(B_{i-1})$$

$$= \mathbb{E}[X_{i}^{2}] + \mathbb{E}[R_{i-2}^{2} + 2X_{i}R_{i-2}|B_{i-1}] \operatorname{Pr}(B_{i-1}). (29)$$

Since X_i is geometric (γ) , independent of the event B_{i-1} , and R_{i-1} is geometric (μ) given B_{i-1} , it follows from (28) and (29) that

$$\mathbb{E}\left[I_i^2\right] = \frac{2-\gamma}{\gamma^2} + \left(\frac{2-\mu}{\mu^2} + \frac{2}{\gamma\mu}\right) P_B$$

$$= \left(\frac{2}{\gamma} - 1\right) \left(\frac{1}{\gamma} + \frac{P_B}{\mu}\right) + 2\frac{P_B}{\mu^2}$$

$$= \left(\frac{2}{\gamma} - 1\right) \mathbb{E}[I] + 2\frac{P_B}{\mu^2}.$$
(30)

We employ the partition $\{\overline{B}_{i-1}\overline{B}_i, \overline{B}_{i-1}B_i, B_{i-1}\overline{B}_i, B_{i-1}B_i\}$ and the law of total expectation to evaluate

$$\mathbb{E}[I_{i}T_{i}] = \mathbb{E}\left[I_{i}T_{i}|\overline{B}_{i-1}\overline{B}_{i}\right] \Pr(\overline{B}_{i-1}\overline{B}_{i}) + \mathbb{E}\left[I_{i}T_{i}|\overline{B}_{i-1}B_{i}\right] \Pr(\overline{B}_{i-1}B_{i}) + \mathbb{E}\left[I_{i}T_{i}|B_{i-1}\overline{B}_{i}\right] \Pr(B_{i-1}\overline{B}_{i}) + \mathbb{E}\left[I_{i}T_{i}|B_{i-1}B_{i}\right] \Pr(B_{i-1}B_{i}).$$
(31)

It follows from (25) and the law of total expectation that

$$\mathbb{E}[I_iT_i]$$

$$= \mathbb{E}\left[X_{i}Y_{i}|\overline{B}_{i-1}\overline{B}_{i}\right] \operatorname{Pr}\left(\overline{B}_{i-1}\overline{B}_{i}\right) + \mathbb{E}\left[X_{i}(R_{i-1} + Y_{i})|\overline{B}_{i-1}B_{i}\right] \operatorname{Pr}\left(\overline{B}_{i-1}B_{i}\right) + \mathbb{E}\left[(R_{i-2} + X_{i})Y_{i}|B_{i-1}\overline{B}_{i}\right] \operatorname{Pr}\left(B_{i-1}\overline{B}_{i}\right) + \mathbb{E}\left[(R_{i-2} + X_{i})(R_{i-1} + Y_{i})|B_{i-1}B_{i}\right] \operatorname{Pr}\left(B_{i-1}B_{i}\right)$$
(32)
$$= \mathbb{E}\left[X_{i}Y_{i}\right] + \mathbb{E}\left[R_{i-1}X_{i}|\overline{B}_{i-1}B_{i}\right] \operatorname{Pr}\left(\overline{B}_{i-1}B_{i}\right) + \mathbb{E}\left[R_{i-2}Y_{i}|B_{i-1}\overline{B}_{i}\right] \operatorname{Pr}\left(B_{i-1}\overline{B}_{i}\right) + \mathbb{E}\left[R_{i-2}(R_{i-1} + Y_{i}) + R_{i-1}X_{i}|B_{i-1}B_{i}\right] \operatorname{Pr}\left(B_{i-1}B_{i}\right).$$
(33)

Note that Y_i is independent of B_{i-1} and B_i while X_i is independent of B_{i-1} but influenced by the occurrence of B_i . Since $\mathbb{E}[X_iY_i] = 1/(\gamma\mu)$, it follows from (33) that

$$\mathbb{E}[I_i T_i] = \frac{1}{\gamma \mu} + \frac{\mathbb{E}[X_i | B_i] \Pr(\overline{B}_{i-1} B_i)}{\mu} + \frac{\Pr(B_{i-1} \overline{B}_i)}{\mu^2}$$

$$+ \left(\frac{1}{\mu^{2}} + \frac{1}{\mu^{2}} + \frac{\mathbb{E}[X_{i}|B_{i}]}{\mu}\right) \Pr(B_{i-1}B_{i})$$
(34)
$$= \frac{1}{\gamma\mu} + \frac{2\Pr(B_{i-1}B_{i}) + \Pr(B_{i-1}\overline{B}_{i})}{\mu^{2}}$$

$$+ \frac{\mathbb{E}[X_{i}|B_{i}](\Pr(\overline{B}_{i-1}B_{i}) + \Pr(B_{i-1}B_{i}))}{\mu}$$
(35)
$$= \frac{1}{\gamma\mu} + \frac{\Pr(B_{i-1}B_{i}) + \Pr(B_{i-1})}{\mu^{2}} + \frac{\mathbb{E}[X_{i}|B_{i}]P_{B}}{\mu}$$

$$= \frac{1}{\gamma\mu} + \frac{P_{B}(1 + P_{B})}{\mu^{2}} + \frac{\mathbb{E}[X_{i}|B_{i}]P_{B}}{\mu}$$
(36)

Applying Lemma 2 to (36) yields

$$\mathbb{E}[I_{i}T_{i}] = \frac{1}{\gamma\mu} + \frac{P_{B}(1+P_{B})}{\mu^{2}} + \frac{P_{B}^{2}}{\gamma\mu\bar{\mu}}$$

$$= \frac{1}{\mu} \left(\frac{1}{\gamma} + \frac{P_{B}}{\mu}\right) + \frac{P_{B}^{2}}{\mu} \left(\frac{1}{\mu} + \frac{1}{\gamma\bar{\mu}}\right)$$

$$= \frac{\mathbb{E}[I_{i}]}{\mu} + \frac{P_{B}^{2}}{\mu} \left(\frac{1}{\mu} + \frac{1}{\gamma\bar{\mu}}\right)$$

$$= \frac{\mathbb{E}[I_{i}]}{\mu} + \frac{P_{B}^{2}}{\mu} \left(\frac{\mu + \gamma\bar{\mu}}{\gamma\mu\bar{\mu}}\right)$$

$$= \frac{\mathbb{E}[I_{i}]}{\mu} + \frac{P_{B}}{\mu^{2}}.$$
(37)

Combining (7), (30) and (37), we obtain

$$\Delta^{\text{Zero-Wait-2}} = \frac{1}{\gamma} + \frac{1}{\mu} - 1 + 2\frac{P_B}{\mu^2 \mathbb{E}[I]}.$$
 (38)

Applying Lemma 2(a) and (28), we have the following claim.

Theorem 3. The average AoI under the Zero-Wait-2 policy is

$$\Delta^{\text{Zero-Wait-2}} = \frac{1}{\gamma} + \frac{1}{\mu} - 1 + \frac{2\gamma^2 \bar{\mu}}{\mu(\gamma \bar{\mu}(\gamma + \mu) + \mu^2)}.$$
 (39)

By applying algebraic manipulations to the difference $\Delta^{\text{Zero-Wait-2}} - \Delta^{\text{Zero-Wait-1}}$, we obtain the following corollary that specifies the ranges of service time parameters γ and μ for which the Zero-Wait-2 policy outperforms the Zero-Wait-1 policy.

Corollary 1. The Zero-Wait-2 policy outperforms the Zero-Wait-1 policy, i.e., $\Delta^{Zero-Wait-2} \leq \Delta^{Zero-Wait-1}$, if the service rates satisfy

$$\mu \ge \frac{\gamma(\bar{\gamma} + \sqrt{(\gamma + 1)^2 + 4})}{2(\gamma + 1)} \quad or \tag{40a}$$

$$\gamma \le \frac{\mu\left(-\bar{\mu} + \sqrt{\bar{\mu}(5-\mu)}\right)}{2 + \bar{\mu}}.\tag{40b}$$

The regions specified by are illustrated in Fig. 5. As one observation from (40a), when $\mu \geq \frac{\sqrt{2}}{2} \approx 0.7071$, we have $\Delta^{\text{Zero-Wait-1}} \leq \Delta^{\text{Zero-Wait-1}}$ for any value of γ .

C. Wait-1: AoI Analysis

In this section, we derive the average AoI under the Wait-1 policy. Differently to the zero-wait policies, when the controller receives status update packet i-1, the Wait-1 policy permits the controller to assign a *waiting time* prior to sending request message i. Such appropriately placed waiting time has

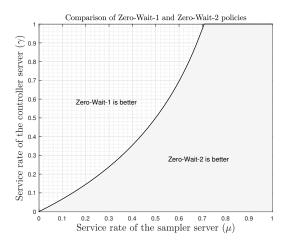


Fig. 5: Average AoI performance comparison between the Zero-Wait-1 policy and the Zero-Wait-2 policy with respect to different service rate pairs (μ, γ) .

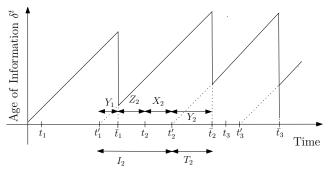


Fig. 6: AoI as a function of time under the Wait-1 policy.

potential to lead to lower average AoI than that of the zero-wait policy³, as will be shown by our subsequent analysis.

Formally, let Z_i be a random variable for the waiting time assigned to request message i after receiving update i-1. In this work, in line with [5], we consider that $Z_i=f(Y_{i-1})$, i.e., the waiting time is a deterministic function of the service time of status update packet i-1 (at the sampler server). AoI evolution under the Wait-1 policy is illustrated in Fig. 6.

We restrict to threshold-type waiting functions of form

$$Z_i = f(Y_{i-1}) = (\beta - Y_{i-1})^+, \tag{41}$$

where β is a positive integer; $\beta=1$ corresponds to the Zero-Wait-1 policy. While we have not been able to prove the optimality of this waiting function for our considered system, the structure of an optimal policy (obtained by the MDP approach), as visualized in Section VI, has such threshold-type structure. In prior works, the optimality has been proven in some different status updating systems, as elaborated next.

Remark 2. The threshold-type waiting function (41) with appropriately adjusted β , was proven to be the optimal waiting function in a stop-and-wait continuous-time system with a non-delayed control channel in [5, Theorem 1] and [6, Theorem 4] and in its discrete-time equivalent in [49, Theorem 3]. Under a delayed control channel, the optimality of the threshold-type

waiting in continuous-time systems has been shown in [55, Proposition 2] and [56, Theorem 1].

To characterize the average AoI as given by (7), we need to calculate $\mathbb{E}[I_i]$, $\mathbb{E}[I_i^2]$, and $\mathbb{E}[I_iT_i]$. As it can be seen in Fig. 6, the interarrival time of status update packets i-1 and i is the summation of the service time of status update packet i-1, Y_{i-1} , the waiting time of request message i, Z_i , and the service time of request message i, X_i , i.e., $I_i = Y_{i-1} + Z_i + X_i$. The mean interarrival time, $\mathbb{E}[I_i]$, is given as

$$\mathbb{E}[I_i] = \mathbb{E}[Y_{i-1}] + \mathbb{E}[Z_i] + \mathbb{E}[X_i] = \frac{1}{\mu} + \mathbb{E}[Z_i] + \frac{1}{\gamma}, \quad (42)$$

where $\mathbb{E}[Z_i]$ is given by the following lemma.

Lemma 3. The mean waiting time $\mathbb{E}[Z_i]$ is given as

$$\mathbb{E}\left[Z_{i}\right] = \frac{1}{\mu} \left(\beta \mu + \bar{\mu}^{\beta} - 1\right). \tag{43}$$

Proof. See Appendix VIII-B.

The second moment of the interarrival time, $\mathbb{E}[I_i^2]$, is derived as

$$\begin{split} \mathbb{E}[I_i^2] &= \mathbb{E}[(Y_{i-1} + Z_i + X_i)^2] \\ &= \mathbb{E}[(Y_{i-1} + X_i)^2] + 2\mathbb{E}[Z_i(Y_{i-1} + X_i)] + \mathbb{E}[Z_i^2] \\ &\stackrel{(a)}{=} \mathbb{E}[Y^2] + \mathbb{E}[X^2] + 2\mathbb{E}[X]\mathbb{E}[Y] + 2\mathbb{E}[Z_iY_{i-1}] \\ &+ 2\mathbb{E}[Z_i]\mathbb{E}[X_i] + \mathbb{E}[Z_i^2], \end{split}$$

where equality (a) comes from the fact that X_i and Y_{i-1} are independent and Z_i and X_i are independent; the expectations $\mathbb{E}[Z_iY_{i-1}]$ and $\mathbb{E}[Z_i^2]$ are given by the following lemmas.

Lemma 4. The expectation $\mathbb{E}[Z_iY_{i-1}]$ is given as

$$\mathbb{E}[Z_i Y_{i-1}] = \frac{1}{\mu^2} (\mu + \beta \mu + \bar{\mu}^{\beta} (2 - \mu + \beta \mu) - 2)$$
 (44)

Proof. See Appendix VIII-C.

Lemma 5. The second moment of the waiting time, $\mathbb{E}\left[Z_i^2\right]$, is given as

$$\mathbb{E}\left[Z_i^2\right] = -\frac{1}{\mu^2} \left(\mu + 2\beta\mu + \bar{\mu}^\beta (2-\mu) - \beta^2 \mu^2 - 2\right). \tag{45}$$

As it can be seen in Fig. 6, the system time of packet i, T_i , is equal to the service time of status update packet i, Y_i . Thus, the expectation $\mathbb{E}[I_iT_i]$ is derived as

$$\mathbb{E}[I_i T_i] = \mathbb{E}[(Y_{i-1} + Z_i + X_i) Y_i]$$

$$\stackrel{(a)}{=} \mathbb{E}[Y_i] \mathbb{E}[Y_{i-1} + Z_i + X_i] = \frac{1}{\mu} \left(\frac{1}{\mu} + \mathbb{E}[Z_i] + \frac{1}{\gamma}\right),$$

where equality (a) follows from Y_i being independent of Y_{i-1} , Z_i , and X_i . Substituting (42), (44), and (46) into (7), we have Theorem 4.

Theorem 4. For waiting functions $Z_i = (\beta - Y_{i-1})^+$, the average AoI under the Wait-1 policy is given as

$$\Delta^{wau-1} = \frac{\beta\mu(-\beta\gamma + \gamma - 2) - 2(\beta\gamma + 1)}{2(\gamma(\bar{\mu}^{\beta} + \beta\mu) + \mu)} + \beta + \frac{1}{\gamma} + \frac{2}{\mu} - 1.$$
 (46)

³This is not unforeseen: a waiting-based policy, instead of a zero-wait policy, has been shown to realize optimal status updating in various other systems, as detailed in Remark 2.

For a given service rate pair (γ, μ) , the *optimal* Wait-1 policy can easily be found by searching for the value of $\beta \in \{1, 2, 3, \ldots\}$ that minimizes (46); this involves only a simple search over the positive integers.

By applying algebraic manipulations to the difference $\Delta^{\text{Wait-1}} - \Delta^{\text{Zero-Wait-1}}$ leads to following corollary.

Corollary 2. In the 1-Packet system, it is optimal to wait, i.e., to set $\beta \geq 2$ for the waiting function $Z_i(Y_{i-1}) = (\beta - Y_{i-1})^+$, if the service rates μ and γ satisfy

$$\mu \le \frac{-\gamma + \sqrt{\gamma^2 + 2\gamma}}{2} \quad or \quad \gamma \ge \frac{2\mu^2}{1 - 2\mu}.$$
 (47)

Corollary 2 reveals a threshold: a waiting scheme that reduces the average AoI cannot be realized if $\mu>\frac{\sqrt{3}-1}{2}\approx 0.366$, regardless of the value of γ .

VI. NUMERICAL RESULTS

In this section, we evaluate the performance of the optimal and fixed control policies proposed for the 1-Packet and 2-Packet systems as well as visualize the structure of the optimal policies. The optimal policies are obtained via the MDP-based approaches presented in Section IV-A and IV-B. The fixed policies are the Zero-Wait-1 policy (Section V-A), Zero-Wait-2 policy (Section V-B), and Wait-1 policy (Section V-C). Algorithm 1 is run with the maximum value of the AoI $\bar{\Delta}=50$ and the stopping criterion threshold $\epsilon=0.0005$, unless otherwise stated.

A. Structure of the Optimal Policies

In this section, the structure of the optimal policy for the 1-Packet and 2-packet systems for service rate values $\mu \in \{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$ and $\gamma \in \{0.4, 0.7, 1\}$ are illustrated.

1) 1-Packet System: Fig. 7 illustrates the structure of the optimal policy in the 1-Packet system by showing the optimal action at state $s = (\delta, 0, 0, \star)$ (i.e., empty system) for the different values of μ and γ . The figure reveals a threshold structure with respect to the AoI, δ : when the current AoI has a low value (e.g., when $\delta \leq 2$ for $\gamma = 0.4$ and $\mu = 0.2$), the optimal action is to stay idle (a = 0). Similarly, when the AoI exceeds this threshold, the optimal action is to send a request packet. When the controller server's service rate γ increases, the AoI threshold increases. These behaviors are due to the fact that sending a new request packet would put the same burden on the network, regardless of its contribution to reducing the AoI, as any other request packet. Thus, if sending a new request packet cannot significantly reduce the AoI, it is optimal to wait. Finally, we observe that Corollary 2 holds in Fig. 7: for the sampler server's service rate μ exceeding $\frac{\sqrt{3-1}}{2} \approx 0.366$, the optimal action is always a=1.

To further analyze the threshold structure, Fig. 8 illustrates the average AoI obtained by the Wait-1 policy (Theorem 4) as a function of β for different values of γ with $\mu=0.1$. The average AoI obtained by the optimal policy in the 1-Packet system (obtained with RVI parameters $\bar{\Delta}=100$ and $\epsilon=10^{-6}$) is also depicted. As it can be seen, when the

value of γ increases, the optimal value of β increases, i.e., the optimal policy induces longer waiting times. This is because when the request packets are served faster in the controller server, the corresponding generated status update packets will provide small AoI reduction if the AoI upon sending the new request was already small. Thus, it is beneficial to purposely postpone sending the requests. Note that the same trend is visible in Fig. 7. Importantly, Fig. 8 shows that the average AoI obtained by the optimal Wait-1 policy (i.e., the Wait-1 policy with the optimal β) and the optimal policy in the 1-Packet system coincide. This shows evidence on the optimality of the waiting-based scheme, as conjectured in Section V-C.

2) 2-Packet System: In the 2-Packet system, the three states of interest where the optimal action needs to be determined are: 1) $s = (\delta, 0, 0, 0, 0, \star, \star)$ (i.e., empty system), 2) $\underline{s} = (\delta, 0, 0, 0, 1, \star, \Delta_s)$ (i.e., there is a status update packet at the sampler server), and 3) $s = (\delta, 0, 1, 0, 0, \star, \Delta_s)$ (i.e., there is a request packet at the controller server). Regarding the first case, the structure of the optimal policy in the 2-Packet system at state $s = (\delta, 0, 0, 0, 0, \star, \star)$ is the same as that for the 1-Packet system; this threshold-type policy with respect to the AoI was illustrated in Fig. 7. Regarding the third case, the optimal action at state $\underline{s} = (\delta, 0, 1, 0, 0, \star, \star)$ is to stay idle (a = 0), regardless of the current value of the AoI. This stems from the perfect knowledge on the system's occupancy (Remark 1): as the controller will immediately know when the controller server becomes empty, it can send a new request message just in time, should this action be optimal. The second case is elaborated next.

Interestingly, Fig. reveals that at state $\underline{s} = (\delta, 0, 0, 0, 1, \star, \Delta_s)$, the optimal policy has a threshold with respect to the age of the packet at the sampler server, $\Delta_{\rm s}$. This behavior can be interpreted as follows. First, due to the memoryless property of the geometric random variable, the average residual service time of the packet in the sampler server is $\mathbb{E}[Y] = 1/\mu$, regardless of Δ_s . Consequently, a newly inserted request message will lead its associated status update packet to enter the sampler queue with the same probability, regardless of Δ_s . Thus, when at state $\underline{s} = (\delta, 0, 0, 0, 1, \star, \Delta_s)$, the controller cannot do measures to avoid queueing delays in the sampler server, which always are detrimental to the age performance. However, the controller can look at the value of Δ_s to optimize its action. If the status update packet under service has a low value of Δ_s , it will provide large age reduction once completing service. If a new request was sent in such state, it would, in turn, provide relatively low age reduction, because the preceding packet set the AoI to a low value. In this case, it is better to wait prior to sending a request, and hence the threshold in Fig. 9. Furthermore, based on the same grounds as for Fig. 7, when the value of γ increases, the optimal policy induces longer waiting times before sending a new request at state $s = (\delta, 0, 0, 0, 1, \star, \Delta_s).$

B. Average AoI Value for Different Policies

Fig. 10 illustrates the value of the average AoI as a function of μ for different values of γ under the Zero-Wait-1 policy, the

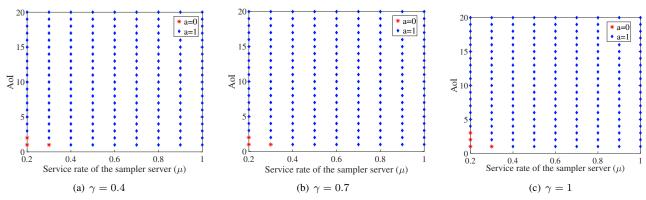


Fig. 7: Structure of the optimal policy for the 1-Packet system at state $s = (\delta, 0, 0, 0, \star)$ and for the 2-Packet system at state $\underline{s} = (\delta, 0, 0, 0, 0, \star, \star)$; a = 1 indicates that the optimal action is to generate a request packet, whereas a = 0 indicates that the optimal action is to stay idle.

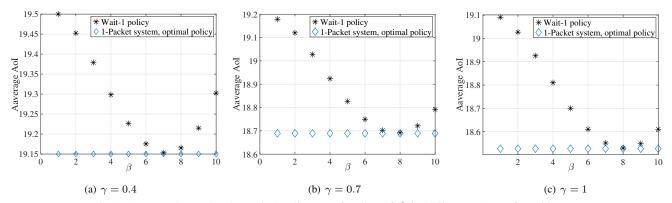


Fig. 8: Average AoI under the Wait-1 policy as a function of β for different values of γ with $\mu=0.1$.

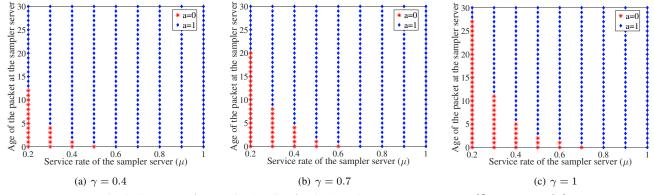


Fig. 9: Structure of the optimal policy for the 2-Packet system at state $\underline{s} = (\delta, 0, 0, 0, 1, \star, \Delta_s)$.

Zero-Wait-2 policy, and the optimal policies in the 1-Packet and 2-packet systems. As it can be seen, the optimal policy in the 2-Packet system, in general, outperforms the other policies, or, for certain service rates, coincides with another policy. This is as expected, because the policy optimization is over a set of policies for the 2-Packet system includes these other policies.

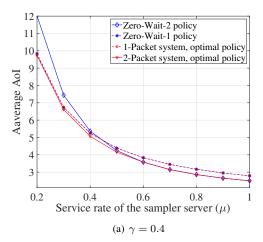
Fig. 10 shows that for $\mu \geq \sqrt{2}/2 \approx 0.7071$, Zero-Wait-2 outperforms Zero-Wait-1 regardless of the value of γ , as shown in Corollary 1. Moreover, we can see that for $\mu \geq \frac{\sqrt{3}-1}{2} \approx 0.366$, the Zero-Wait-1 policy and the optimal policy in the 1-Packet system provide the same average AoI regardless of the value of γ , as shown in Corollary 2.

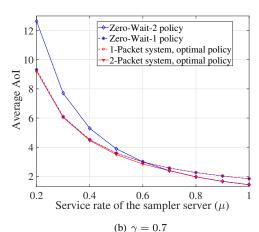
We can also see from Fig. 10 that as μ increases, the

average AoI decreases for all policies, as expected, because the status update packets are served faster. However, maybe a bit surprisingly, such monotonic behavior is not seen with respect to γ ; increasing the value of γ is not always beneficial. As it can be seen, for the low values of μ , the Zero-Wait-2 policy results in a higher average AoI value when the value of γ increases. This occurs because a large value of γ causes the requests to be served too fast in relation to the sampler service rate μ , leading to queueing of update packets in the sampler buffer, deteriorating the AoI performance.

VII. CONCLUSIONS

We studied status updating under two-way delay in a discrete-time system. We developed AoI-optimal control poli-





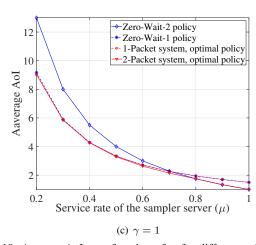


Fig. 10: Average AoI as a function of μ for different values of γ .

cies using the tools of MDPs in two scenarios. We began with the system having at most one active request. Then, we initiated pipelining-type status updating by studying a system having at most two active requests. In addition, we conducted AoI analysis by deriving closed-form expressions of the average AoI under the Zero-Wait-1, Zero-Wait-2 and Wait-1 policies which employ only the controller's local information.

Numerical results showed that for both 1-Packet and 2-Packet systems, the optimal policy is a threshold-type policy. In addition, it was seen that for high values of the update

service rate μ , Zero-Wait-1 is an optimal policy for the 1-Packet system, and Zero-Wait-2 is an optimal policy for the 2-Packet system, regardless of the service rate of the controller server. Moreover, the results showed that increasing the request message service rate γ is not necessarily beneficial for all policies, e.g., when the update service rate μ is low, the average AoI under the Zero-Wait-2 policy is increasing γ .

VIII. APPENDIX

A. Proof of Lemma 2

(a) The probability of event $B_i = \{X_i < Y_{i-1}\}$ is

$$P_{B} = \sum_{j=1}^{\infty} \Pr(X_{i} < Y_{i-1} \mid X_{i} = j) \Pr(X_{i} = j)$$

$$\stackrel{(a)}{=} \sum_{j=1}^{\infty} \bar{\mu}^{j} \gamma \bar{\gamma}^{j-1} = \frac{\gamma}{\bar{\gamma}} \sum_{j=1}^{\infty} (\bar{\mu} \bar{\gamma})^{j} \stackrel{(b)}{=} \frac{\gamma \bar{\mu}}{\mu + \gamma \bar{\mu}}.$$
(48)

where (a) holds because $\Pr(X_i = j) = \gamma \bar{\gamma}^{j-1}$ and $\Pr(j < Y_{i-1}) = \bar{\mu}^j$, and (b) follows since $\sum_{j=1}^{\infty} a^j = a/(1-a)$ for all |a| < 1, and the identity $1 - \bar{\mu}\bar{\gamma} = \mu + \gamma\bar{\mu}$.

(b) For the conditional expectation, similar facts imply

$$\mathbb{E}[X_{i}|B_{i}] = \sum_{j=1}^{\infty} j \Pr(X_{i} = j|X_{i} < Y_{i-1})$$

$$= \frac{1}{P_{B}} \sum_{j=1}^{\infty} j \Pr(X_{i} = j, Y_{i-1} > j)$$

$$= \frac{1}{P_{B}} \sum_{j=1}^{\infty} j \bar{\gamma}^{j-1} \gamma \bar{\mu}^{j}$$

$$= \frac{\gamma \bar{\mu}}{P_{B} (1 - \bar{\gamma} \bar{\mu})} \sum_{j=1}^{\infty} j (\bar{\gamma} \bar{\mu})^{j-1} (1 - \bar{\gamma} \bar{\mu})$$

$$= \frac{1}{1 - \bar{\gamma} \bar{\mu}} = \frac{1}{\mu + \gamma \bar{\mu}} = \frac{P_{B}}{\gamma \bar{\mu}}.$$
(49)

B. Proof of Lemma 3

The mean waiting time $\mathbb{E}[Z_i]$ is derived as

$$\mathbb{E}[Z_{i}] = \sum_{j=1}^{\infty} j \Pr(Z_{i} = j) \stackrel{(a)}{=} \sum_{j=1}^{\beta-1} (\beta - j) \Pr(Y_{i-1} = j)$$

$$= \beta \sum_{j=1}^{\beta-1} \bar{\mu}^{j-1} \mu - \sum_{j=1}^{\beta-1} j \bar{\mu}^{j-1} \mu$$

$$= \frac{\beta \mu}{\bar{\mu}} \sum_{j=1}^{\beta-1} \bar{\mu}^{j} - \frac{\mu}{\bar{\mu}} \sum_{j=1}^{\beta-1} j \bar{\mu}^{j} \stackrel{(b)}{=} \frac{1}{\mu} (\beta \mu + \bar{\mu}^{\beta} - 1),$$
(50)

where (a) follows from $Z_i = f(Y_{i-1}) = (\beta - Y_{i-1})^+$ and (b) follows from the identities [72, Page 484]

$$\sum_{j=1}^{m} \alpha^j = \frac{\alpha(1 - \alpha^m)}{1 - \alpha},\tag{51}$$

$$\sum_{j=1}^{m} j\alpha^{j} = \frac{\alpha(1 - \alpha^{m}(1 + m(1 - \alpha)))}{(1 - \alpha)^{2}}.$$
 (52)

C. Proof of Lemma 4

Since (41) implies
$$Pr(Z_i = k \mid Y_{i-1} = j) = 1_{\{k = (\beta - j)^+\}}$$
,

$$\mathbb{E}[Z_{i}Y_{i-1}] = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} jk \Pr(Y_{i-1} = j) \Pr(Z_{i} = k \mid Y_{i-1} = j)$$
$$= \sum_{j=1}^{\beta-1} j(\beta - j) \Pr(Y_{i-1} = j). \tag{53}$$

This implies

$$\mathbb{E}[Z_{i}Y_{i-1}] = \beta \sum_{j=1}^{\beta-1} j\bar{\mu}^{j-1}\mu - \sum_{j=1}^{\beta-1} j^{2}\bar{\mu}^{j-1}\mu$$

$$= \frac{\beta\mu}{\bar{\mu}} \sum_{j=1}^{\beta-1} j\bar{\mu}^{j} - \frac{\mu}{\bar{\mu}} \sum_{j=1}^{\beta-1} j^{2}\bar{\mu}^{j}$$

$$\stackrel{(a)}{=} \frac{1}{\mu^{2}} (\mu(1+\beta) + \bar{\mu}^{\beta}(2-\mu+\beta\mu) - 2), \quad (54)$$

where (a) follows from (52) and the identity

$$\sum_{j=1}^{m} j^{2} \alpha^{j} = \frac{\alpha^{2} + \alpha + \alpha^{m+2} (2m^{2} + 2m - 1)}{(1 - \alpha)^{3}} - \frac{\alpha^{m+1} [(m+1)^{2} + m^{2} \alpha^{2}]}{(1 - \alpha)^{3}}.$$
 (55)

Note that (55) can be verified by taking derivatives with respect to α of both sides of (52).

D. Proof of Lemma 5

It follows from (41) that

$$\mathbb{E}[Z_i^2] = \sum_{j=1}^{\infty} j^2 \Pr(Z_i = j) \stackrel{(a)}{=} \sum_{j=1}^{\beta-1} (\beta - j)^2 \Pr(Y_{i-1} = j)$$

$$= \beta^2 \sum_{j=1}^{\beta-1} \bar{\mu}^{j-1} \mu - 2\beta \sum_{j=1}^{\beta-1} j \bar{\mu}^{j-1} \mu \sum_{j=1}^{\beta-1} j^2 \bar{\mu}^{j-1} \mu$$

$$\stackrel{(a)}{=} -\frac{1}{\mu^2} \left(\mu(1+2\beta) + \bar{\mu}^{\beta}(2-\mu) - \beta^2 \mu^2 - 2 \right). \tag{56}$$

Note that (a) follows from (51), (52), and (55).

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