Global Exponential Stability for Complex-Valued Recurrent Neural Networks With Asynchronous Time Delays

Xiwei Liu, Member, IEEE, and Tianping Chen, Senior Member, IEEE

Abstract—In this paper, we investigate the global exponential stability for complex-valued recurrent neural networks with asynchronous time delays by decomposing complex-valued networks to real and imaginary parts and construct an equivalent realvalued system. The network model is described by a continuoustime equation. There are two main differences of this paper with previous works: (1), time delays can be asynchronous, i.e., delays between different nodes are different, which makes our model more general; (2), we prove the exponential convergence directly, while the existence and uniqueness of the equilibrium point is just a direct consequence of the exponential convergence. By using three generalized norms, we present some sufficient conditions for the uniqueness and global exponential stability of the equilibrium point for delayed complex-valued neural networks. These conditions in our results are less restrictive because of our consideration of the excitatory and inhibitory effects between neurons, so previous works of other researchers can be extended. Finally, some numerical simulations are given to demonstrate the correctness of our obtained results.

Index Terms—Asynchronous, complex-valued, global exponential stability, recurrent neural networks, time delays.

I. INTRODUCTION

Recurrently connected neural networks, including Hopfield neural networks[1], Cohen-Grossberg neural networks[2], and cellular neural networks [3]-[4], have been extensively studied in past decades and found many applications in different areas, such as signal and image processing, pattern recognition, optimization problems, associative memories, and so on. Until now, many criteria about the stability of equilibrium are obtained in the literature, see [5]-[20] and references therein.

It is natural to generalize the real-valued systems to complex-valued systems [21], which can be used in the nonlinear quantum systems, reaction-advection-diffusion systems, heat equation, petri nets, chaotic systems, etc. Many approaches are also obtained, for example, decomposing complex-valued system to two real-valued systems is applied

Xiwei Liu is with Department of Computer Science and Technology, Tongji University, and with the Key Laboratory of Embedded System and Service Computing, Ministry of Education, Shanghai 200092, P. R. China. E-mail: xwliu@tongji.edu.cn

Corresponding Author Tianping Chen is with the School of Computer Sciences/Mathematical Sciences, Fudan University, 200433, Shanghai, P. R. China. E-mail: tchen@fudan.edu.cn

This work was supported by the National Science Foundation of China under Grant No. 61203149, 61273211, 61233016, the National Basic Research Program of China (973 Program) under Grant No. 2010CB328101, "Chen Guang" project supported by Shanghai Municipal Education Commission and Shanghai Education Development Foundation under Grant No. 11CG22, the Fundamental Research Funds for the Central Universities, and the Program for Young Excellent Talents in Tongji University. in some nonlinear systems and regular networks, see [22]-[23] and references therein. Recently, as an important part of nonlinear complex-valued systems, complex-valued neural network (CVNN) models are proposed as an important part of complex-valued systems, and have attracted more and more attention from various areas in science and technology, see [24]-[47] and references therein. CVNN can be regarded as an extension of real-valued recurrent neural networks, which has complex-valued state, output, connection weight, and activation functions. For example, they are suited to deal with complex state composed of amplitude and phase. This is one of the core concepts in physical systems dealing with electromagnetic, light, ultrasonic, quantum waves, and so on. Moreover, many applications heavily depend on the dynamical behaviors of networks. Therefore, analysis of these dynamical behaviors is a necessary step toward practical design of these neural networks. In [35], a CVNN model on time scales is studied based on delta differential operator. In [36]-[39], discrete-time CVNNs are also discussed. Stability of complexvalued impulsive system is investigated by [40]. Until now, there have been various methods to study the stability of CVNNs, such as the Lyapunov functional method [41], the synthesis method [42], and so on.

1

In particular, in hardware implementation, time delays inevitably occur due to the finite switching speed of the amplifiers and communication time. What's more, to process moving images, one must introduce time delays in the signals transmitted among the cells. Furthermore, time delay is frequently a source of oscillation and instability in neural networks. Therefore, neural networks with time delays have much more complicated dynamics due to the incorporation of delays, and the stability of delayed neural networks has become a hot topic of great theoretical and practical importance, and a great deal of significant results have been reported in the literature. For example, [43] investigates the stability and synchronization for discrete-time CVNNs with time-varying delays; [44] studies the stability of complex-valued impulsive system with delay. The global exponential and asymptotical stability of CVNNs with time-delays is studied by [45] with two assumptions of activation functions, while [46] and [47] point out the mistakes in the proof of [45] and give some new conditions and criteria to ensure the existence, uniqueness, and globally asymptotical stability of the equilibrium point of CVNNs.

In practice, the interconnections are generally asynchronous, that is to say, the inevitable time delays between different nodes are generally different. For example, in order to model vehicular traffic flow [48]-[49], the reaction delays of drivers should be considered, and for different drivers, the reaction delays are different depending on physical conditions, drivers' cognitive and physiological states, etc. Moreover, in the load balancing problem [50], for a computing network consisting of n computers (also called nodes), except for the different communication delays, the task-transfer delays τ_{ik} also should be considered, which depends on the number of tasks to be transferred from node k to node j. More related examples can be found in [51] and references therein. Hence, based on above discussions, it is necessary to study the dynamical behavior of neural networks with asynchronous time (varying) delays. To our best knowledge, there have been few works to report the stability of CVNNs with asynchronous time delays, see [52], [53]. For example, [52] focuses on the existence, uniqueness and global robust stability of equilibrium point for CVNNs with multiple time-delays and under parameter uncertainties with respect to two activation functions: while [53] investigates the dynamical behaviors of CVNNs with mixed time delays. However, all these works ([45]-[47], [52], [53]) apply the homeomorphism mapping approach proposed by [7] to prove the existence, uniqueness and global stability of equilibrium point by two steps: step 1, prove the existence of equilibrium; step 2, prove its stability. In [9] and [10], a direct approach to analyze global and local stability of networks was first proposed. It was revealed that the finiteness of trajectory x(t) under some norms, i.e., $\int_0^\infty \|\dot{x}(t)\| dt < \infty$, is a sufficient condition for the existence, and global stability of the equilibrium point. This idea was also used in [13]. In this paper, we will adopt this approach. Moreover, we give several criteria based on three generalized L_∞ norm, L_1 norm, L_2 norm, respectively. In particular, based on L_{∞} -norm, we can discuss the networks with time-varying delays.

This paper is organized as follows. In Section II, we give the model description, decompose the complex-valued differential equations to real part and imaginary part, and then recast it into an equivalent real-valued differential system, whose dimension is double that of the original complex-valued system. Some definitions, lemmas and notations used in the paper are also given. In Section III, we present some criteria for the uniqueness and global exponential stability of the equilibrium point for recurrent neural networks models with asynchronous time delays by using the generalized ∞ -norm, 1-norm, and 2norm, respectively. Some comparisons with previous M-matrix results are also presented. In Section IV, some numerical simulations under constant and time varying-delays are given to demonstrate the effectiveness of our obtained results. Finally, conclusion is given and some discussions about our future investigation of CVNNs are presented in Section V.

II. PRELIMINARIES

In this section, we give some definitions, lemmas and notations, which will be used throughout the paper.

At first, let us give a definition of asynchronous time delays.

Definition 1: (Synchronous and asynchronous time delays) For any node j in a coupled neural network, the synchronous

time delay means that at time t, node j receives the information from other nodes at the same time $t - \tau_j(t)$; while the asynchronous time delays mean that at time t, node j receives the information from other nodes at different times $t - \tau_{jk}(t)$, i.e., for nodes $k_1 \neq k_2$, $\tau_{jk_1}(t)$ and $\tau_{jk_2}(t)$ can be different.

Obviously, the network models of asynchronous time delays have a larger scope than that of synchronous time delays.

In this paper, we will investigate the CVNN with asynchronous time delays as follows:

$$\dot{z}_{j}(t) = -d_{j}z_{j}(t) + \sum_{k=1}^{n} a_{jk}f_{k}(z_{k}(t)) + \sum_{k=1}^{n} b_{jk}g_{k}(z_{k}(t-\tau_{jk})) + u_{j},$$

$$j = 1, \cdots, n \quad (1)$$

where $z_j \in \mathbb{C}$ is the state of *j*-th neuron, \mathbb{C} is the set of complex numbers; $d_j > 0$ represents the positive rate with which the *j*-th unit will reset its potential to the resting state in isolation when disconnected from the network; $f_j(\cdot) : \mathbb{C} \to \mathbb{C}$ and $g_j(\cdot) : \mathbb{C} \to \mathbb{C}$ are complex-valued activation functions; matrices $A = (a_{jk})$ and $B = (b_{jk})$ are complex-valued connection weight matrices without and with time delays; τ_{jk} are asynchronous constant time delays; $u_j \in \mathbb{C}$ is the *j*-th external input.

Remark 1: When $\tau_{jk} = \tau$, system (1) becomes the model investigated in [45]; when activation functions f_j and g_j are real functions, system (1) becomes the model investigated by [10]. Therefore, this model has a larger scope than previous works, and all the obtained results in the next section can be applied to these special cases.

For any complex number z, we use z^R and z^I to denote its real and imaginary part respectively, so $z = z^R + i \cdot z^I$, where *i* denotes the imaginary unit, that is $i = \sqrt{-1}$.

Now, we introduce some classes of activation functions.

Definition 2: Assume $f_j(z)$ can be decomposed to its real and imaginary part as $f_j(z) = f_j^R(z^R, z^I) + if_j^I(z^R, z^I)$ where $z = z^R + iz^I$, $f_j^R(\cdot, \cdot) : R^2 \to R$ and $f_j^I(\cdot, \cdot) : R^2 \to R$. Suppose the partial derivatives of $f_j(\cdot, \cdot)$ with respect to $z^R, z^I :$ $\partial f_j^R / \partial z^R, \partial f_j^R / \partial z^I, \partial f_j^I / \partial z^R$, and $\partial f_j^I / \partial z^I$ exist. If these partial derivatives are continuous, positive and bounded, i.e., there exist positive constant numbers $\lambda_j^{RR}, \lambda_j^{RI}, \lambda_j^{IR}, \lambda_j^{II}$, such that

$$\begin{array}{l} 0 < \partial f_j^R / \partial z^R \leq \lambda_j^{RR}, \quad 0 < \partial f_j^R / \partial z^I \leq \lambda_j^{RI}, \\ 0 < \partial f_j^I / \partial z^R \leq \lambda_j^{IR}, \quad 0 < \partial f_j^I / \partial z^I \leq \lambda_j^{II}, \end{array}$$

then $f_j(z)$ is said to belong to class $H_1(\lambda_j^{RR}, \lambda_j^{RI}, \lambda_j^{IR}, \lambda_j^{II})$. *Remark 2:* If f_j^R and f_j^I are absolutely continuous, then their partial derivatives exist almost everywhere.

their partial derivatives exist almost everywhere. Definition 3: Assume $g_j(z)$ can be decomposed to its real and imaginary part as $g_j(z) = g_j^R(z^R, z^I) + ig_j^I(z^R, z^I)$, where $z = z^R + iz^I$, $g_j^R(\cdot, \cdot) : R^2 \to R$ and $g_j^I(\cdot, \cdot) : R^2 \to R$. Suppose the partial derivatives of $g_j(\cdot, \cdot)$ with respect to $z^R, z^I : \partial g_j^R / \partial z^R, \partial g_j^R / \partial z^I, \partial g_j^I / \partial z^R$, and $\partial g_j^I / \partial z^I$ exist. If these partial derivatives are continuous and bounded, i.e., there exist positive constant numbers $\mu_j^{RR}, \mu_j^{RI}, \mu_j^{IR}, \mu_j^{II}$, such that

$$|\partial g_j^R/\partial z^R| \leq \mu_j^{RR}, \ |\partial g_j^R/\partial z^I| \leq \mu_j^{RI},$$

$$|\partial g_j^I / \partial z^R| \le \mu_j^{IR}, \ |\partial g_j^I / \partial z^I| \le \mu_j^{II}, \tag{3}$$

then $g_j(z)$ is said to belong to class $H_2(\mu_j^{RR}, \mu_j^{RI}, \mu_j^{IR}, \mu_j^{II})$.

Remark 3: Definition 3 is the usual assumption for activation functions in the literature of CVNNs, which can be found in [45], [52], [53] and references therein. However, the activation functions defined in Definition 2 is more restrictive, which will be useful when considering the signs of entries in connection weights, i.e., there is a trade-off between the assumption on activation functions and obtained final criteria.

Therefore, by decomposing CVNN (1) to real and imaginary parts, we can get two equivalent real-valued systems:

$$\dot{z}_{j}^{R}(t) = -d_{j}z_{j}^{R}(t) + \sum_{k=1}^{n} a_{jk}^{R}f_{k}^{R}\left(z_{k}^{R}(t), z_{k}^{I}(t)\right) - \sum_{k=1}^{n} a_{jk}^{I}f_{k}^{I}\left(z_{k}^{R}(t), z_{k}^{I}(t)\right) + \sum_{k=1}^{n} b_{jk}^{R}g_{k}^{R}\left(z_{k}^{R}(t-\tau_{jk}), z_{k}^{I}(t-\tau_{jk})\right) - \sum_{k=1}^{n} b_{jk}^{I}g_{k}^{I}\left(z_{k}^{R}(t-\tau_{jk}), z_{k}^{I}(t-\tau_{jk})\right) + u_{j}^{R}, \quad (4)$$

and

$$\dot{z}_{j}^{I}(t) = -d_{j}z_{j}^{I}(t) \\
+ \sum_{k=1}^{n} a_{jk}^{R}f_{k}^{I}\left(z_{k}^{R}(t), z_{k}^{I}(t)\right) + \sum_{k=1}^{n} a_{jk}^{I}f_{k}^{R}\left(z_{k}^{R}(t), z_{k}^{I}(t)\right) \\
+ \sum_{k=1}^{n} b_{jk}^{R}g_{k}^{I}\left(z_{k}^{R}(t-\tau_{jk}), z_{k}^{I}(t-\tau_{jk})\right) \\
+ \sum_{k=1}^{n} b_{jk}^{I}g_{k}^{R}\left(z_{k}^{R}(t-\tau_{jk}), z_{k}^{I}(t-\tau_{jk})\right) + u_{j}^{I}.$$
(5)

Remark 4: The method of decomposing the CVNNs into two real-valued networks makes the network dimension grow two times, which may cause more calculations. However, this expansion of dimension can also bring some benefits. For example, the number (or dimension) of equilibria can be doubled, which enlarges the capacity of neural networks. It is a trade-off.

The following three generalized norms are used throughout the paper.

Definition 4: (See [10]) For any vector $v(t) \in \mathbb{R}^{m \times 1}$,

- 1) $\{\xi, \infty\}$ -norm. $\|v(t)\|_{\{\xi,\infty\}} = \max_j |\xi_j^{-1}v_j(t)|$, where $\xi_j > 0, j = 1, \cdots, m$.
- 2) $\{\xi, 1\}$ -norm. $||v(t)||_{\{\xi, 1\}} = \sum_j |\xi_j v_j(t)|$, where $\xi_j > 0, j = 1, \cdots, m$.
- 3) $\{\xi, 2\}$ -norm. $||v(t)||_{\{\xi, 2\}} = \{\sum_j \xi_j |v_j(t)|^2\}^{1/2}$, where $\xi_j > 0, j = 1, \cdots, m$.

Lemma 1: (See [10]) Let $C = (c_{jk}) \in \mathbb{R}^{m \times m}$ be a nonsingular matrix with $c_{jk} \leq 0, j, k = 1, \dots, m, j \neq k$. Then all the following statements are equivalent.

- 1) C is an M-matrix, i.e., all the successive principal minors of C are equivalent.
- 2) C^T is an M-matrix, where C^T is the transpose of C.
- 3) The real part of all eigenvalues are positive.
- 4) There exists a vector $\xi = (\xi_1, \dots, \xi_m)^T$ with all $\xi_j > 0, j = 1, \dots, m$ such that $\xi^T C > 0$, or $C\xi > 0$.

Notation 1: For any real scalar a, denote $a^+ = \max\{0, a\}$. For any matrix $C = (c_{jk}) \in \mathbb{R}^{n \times n}$, denote $|C| = (|c_{jk}|)$. In the following, we denote $n \times n$ matrices $A^R = (a_{jk}^R)$, $A^I = (a_{jk}^I)$, $B^R = (b_{jk}^R)$, $B^I = (b_{jk}^I)$, and $F^{RR} = \operatorname{diag}\{\lambda_1^{RR}, \dots, \lambda_n^{RR}\}$, $F^{RI} = \operatorname{diag}\{\lambda_1^{RI}, \dots, \lambda_n^{RI}\}$,

 $A^{I} = (a_{jk}^{I}), B^{R} = (b_{jk}^{R}), B^{I} = (b_{jk}^{I}), \text{ and } F^{RR} = \text{diag}\{\lambda_{1}^{RR}, \cdots, \lambda_{n}^{RR}\}, F^{RI} = \text{diag}\{\lambda_{1}^{RI}, \cdots, \lambda_{n}^{RI}\}, F^{IR} = \text{diag}\{\lambda_{1}^{IR}, \cdots, \lambda_{n}^{IR}\}, F^{II} = \text{diag}\{\lambda_{1}^{II}, \cdots, \lambda_{n}^{II}\}, G^{RR} = \text{diag}\{\mu_{1}^{RR}, \cdots, \mu_{n}^{RR}\}, G^{RI} = \text{diag}\{\mu_{1}^{RI}, \cdots, \mu_{n}^{RI}\}, G^{II} = \text{diag}\{\mu_{1}^{II}, \cdots, \mu_{n}^{II}\}, G^{IR} = \text{diag}\{\mu_{1}^{IR}, \cdots, \mu_{n}^{IR}\}, G^{II} = \text{diag}\{\mu_{1}^{II}, \cdots, \mu_{n}^{II}\}, Mototica 2: \text{ For our two non production for } f(t) \ c(t) \ c(t)$

Notation 2: For any two non-negative functions $f(t), g(t) : (-\infty, +\infty) \rightarrow [0, +\infty), f(t) = O(g(t))$ means that for all $t \in R$, there is a positive constant scalar c such that $f(t) \leq c \cdot g(t)$. For any symmetric matric $A, \lambda_{max}(A)$ means its largest eigenvalue. A *n*-dimensional vector $p = (p_1, \dots, p_n)^T$ is called a positive vector, if its all elements are positive, i.e., $p_i > 0, i = 1, \dots, n$.

III. MAIN RESULTS

In this section, we prove some criteria for the uniqueness and global exponential stability of the equilibrium.

A. Criteria with $\{\xi, \infty\}$ -norm

Theorem 1: For dynamical systems (4) and (5), suppose the activation function $f_j(z)$ belongs to class $H_1(\lambda_j^{RR}, \lambda_j^{RI}, \lambda_j^{IR}, \lambda_j^{II})$ and $g_j(z)$ belongs to class $H_2(\mu_j^{RR}, \mu_j^{RI}, \mu_j^{IR}, \mu_j^{II}), j = 1, \dots, n$. If there exists a positive vector $\xi = (\xi_1, \dots, \xi_n, \phi_1, \dots, \phi_n)^T > 0$ and $\epsilon > 0$, such that, for $j = 1, \dots, n$,

$$T1(j) = \xi_{j} \left(-d_{j} + \epsilon + \{a_{jj}^{R}\}^{+} \cdot \lambda_{j}^{RR} + \{-a_{jj}^{I}\}^{+} \cdot \lambda_{j}^{IR} \right)$$
$$+ \sum_{k=1, k \neq j}^{n} \xi_{k} |a_{jk}^{R}| \lambda_{k}^{RR} + \sum_{k=1}^{n} \phi_{k} |a_{jk}^{R}| \lambda_{k}^{RI} + \sum_{k=1, k \neq j}^{n} \xi_{k} |a_{jk}^{I}| \lambda_{k}^{IR}$$
$$+ \sum_{k=1}^{n} \phi_{k} |a_{jk}^{I}| \lambda_{k}^{II} + \left(\sum_{k=1}^{n} \xi_{k} |b_{jk}^{R}| \mu_{k}^{RR} + \sum_{k=1}^{n} \phi_{k} |b_{jk}^{R}| \mu_{k}^{RI} + \sum_{k=1}^{n} \xi_{k} |b_{jk}^{I}| \mu_{k}^{IR} + \sum_{k=1}^{n} \phi_{k} |b_{jk}^{I}| \mu_{k}^{RI} \right) e^{\epsilon \tau_{jk}} \leq 0, \tag{6}$$

and

$$T2(j) = \phi_j \left(-d_j + \epsilon + \{a_{jj}^R\}^+ \cdot \lambda_j^{II} + \{a_{jj}^I\}^+ \cdot \lambda_j^{RI} \right)$$

+ $\sum_{k=1}^n \xi_k |a_{jk}^R| \lambda_k^{IR} + \sum_{k=1, k \neq j}^n \phi_k |a_{jk}^R| \lambda_k^{II} + \sum_{k=1}^n \xi_k |a_{jk}^I| \lambda_k^{RR}$
+ $\sum_{k=1, k \neq j}^n \phi_k |a_{jk}^I| \lambda_k^{RI} + \left(\sum_{k=1}^n \xi_k |b_{jk}^R| \mu_k^{IR} + \sum_{k=1}^n \phi_k |b_{jk}^R| \mu_k^{II} \right)$
+ $\sum_{k=1}^n \xi_k |b_{jk}^I| \mu_k^{RR} + \sum_{k=1}^n \phi_k |b_{jk}^I| \mu_k^{RI} \right) e^{\epsilon \tau_{jk}} \le 0,$ (7)

then dynamical systems (4) and (5) have a unique equilibrium $\overline{Z}^R = (\overline{z}_1^R, \cdots, \overline{z}_n^R)^T$ and $\overline{Z}^I = (\overline{z}_1^I, \cdots, \overline{z}_n^I)^T$, respectively. Moreover, for any solution

$$Z(t) = (z_1^R(t), \cdots, z_n^R(t), z_1^I(t), \cdots, z_n^I(t))^T, \qquad (8)$$

there hold

$$\|\dot{Z}(t)\|_{\{\xi,\infty\}} = O(e^{-\epsilon t}),$$
 (9)

$$||Z(t) - (\overline{Z}^{R^{T}}, \overline{Z}^{I^{T}})^{T}||_{\{\xi, \infty\}} = O(e^{-\epsilon t}).$$
(10)

Its proof can be found in Appendix A.

Corollary 1: For dynamical systems (4) and (5), suppose the activation function $f_j(z)$ belongs to class $H_1(\lambda_j^{RR}, \lambda_j^{RI}, \lambda_j^{IR}, \lambda_j^{II})$ and $g_j(z)$ belongs to class $H_2(\mu_j^{RR}, \mu_j^{RI}, \mu_j^{IR}, \mu_j^{II})$, $j = 1, \dots, n$. If there exists a positive vector $\xi = (\xi_1, \dots, \xi_n, \phi_1, \dots, \phi_n)^T > 0$, such that, for $j = 1, \dots, n$,

$$T3(j) = \xi_{j} \left(-d_{j} + \{a_{jj}^{R}\}^{+} \cdot \lambda_{j}^{RR} + \{-a_{jj}^{I}\}^{+} \cdot \lambda_{j}^{IR} \right)$$
$$+ \sum_{k=1, k \neq j}^{n} \xi_{k} |a_{jk}^{R}| \lambda_{k}^{RR} + \sum_{k=1}^{n} \phi_{k} |a_{jk}^{R}| \lambda_{k}^{RI} + \sum_{k=1, k \neq j}^{n} \xi_{k} |a_{jk}^{I}| \lambda_{k}^{IR}$$
$$+ \sum_{k=1}^{n} \phi_{k} |a_{jk}^{I}| \lambda_{k}^{II} + \sum_{k=1}^{n} \xi_{k} |b_{jk}^{R}| \mu_{k}^{RR} + \sum_{k=1}^{n} \phi_{k} |b_{jk}^{R}| \mu_{k}^{RI}$$
$$+ \sum_{k=1}^{n} \xi_{k} |b_{jk}^{I}| \mu_{k}^{IR} + \sum_{k=1}^{n} \phi_{k} |b_{jk}^{I}| \mu_{k}^{II} < 0,$$
(11)

$$T4(j) = \phi_{j} \left(-d_{j} + \{a_{jj}^{R}\}^{+} \cdot \lambda_{j}^{II} + \{a_{jj}^{I}\}^{+} \cdot \lambda_{j}^{RI} \right)$$

+ $\sum_{k=1}^{n} \xi_{k} |a_{jk}^{R}| \lambda_{k}^{IR} + \sum_{k=1, k \neq j}^{n} \phi_{k} |a_{jk}^{R}| \lambda_{k}^{II} + \sum_{k=1}^{n} \xi_{k} |a_{jk}^{I}| \lambda_{k}^{RR}$
+ $\sum_{k=1, k \neq j}^{n} \phi_{k} |a_{jk}^{I}| \lambda_{k}^{RI} + \sum_{k=1}^{n} \xi_{k} |b_{jk}^{R}| \mu_{k}^{IR} + \sum_{k=1}^{n} \phi_{k} |b_{jk}^{R}| \mu_{k}^{II}$
+ $\sum_{k=1}^{n} \xi_{k} |b_{jk}^{I}| \mu_{k}^{RR} + \sum_{k=1}^{n} \phi_{k} |b_{jk}^{I}| \mu_{k}^{RI} < 0,$ (12)

then any solution of systems (4) and (5) respectively converges to a unique equilibrium exponentially.

If conditions (11) and (12) hold, then we can find a sufficient small constant $\epsilon > 0$, such that inequalities (6) and (7) hold. Therefore, this corollary is a direct consequence of Thm. 1.

Corollary 2: For dynamical systems (4) and (5), suppose the activation function $f_j(z)$ belongs to class $H_2(\lambda_j^{RR}, \lambda_j^{RI}, \lambda_j^{IR}, \lambda_j^{II})$ and $g_j(z)$ belongs to class $H_2(\mu_j^{RR}, \mu_j^{RI}, \mu_j^{IR}, \mu_j^{II}), j = 1, \cdots, n$. If there exists a positive vector $\xi = (\xi_1, \cdots, \xi_n, \phi_1, \cdots, \phi_n)^T > 0$, such that, for $j = 1, \cdots, n$,

$$T5(j) = -\xi_j d_j + \sum_{k=1}^n \xi_k |a_{jk}^R| \lambda_k^{RR} + \sum_{k=1}^n \phi_k |a_{jk}^R| \lambda_k^{RI} + \sum_{k=1}^n \xi_k |a_{jk}^I| \lambda_k^{IR} + \sum_{k=1}^n \phi_k |a_{jk}^I| \lambda_k^{II} + \sum_{k=1}^n \xi_k |b_{jk}^R| \mu_k^{RR} + \sum_{k=1}^n \phi_k |b_{jk}^R| \mu_k^{RI} + \sum_{k=1}^n \xi_k |b_{jk}^I| \mu_k^{IR} + \sum_{k=1}^n \phi_k |b_{jk}^I| \mu_k^{II} < 0,$$
(13)

$$T6(j) = -\phi_j d_j + \sum_{k=1}^n \xi_k |a_{j,k}^R| \lambda_k^{IR} + \sum_{k=1}^n \phi_k |a_{jk}^R| \lambda_k^{II} + \sum_{k=1}^n \xi_k |a_{jk}^I| \lambda_k^{RR} + \sum_{k=1}^n \phi_k |a_{jk}^I| \lambda_k^{RI} + \sum_{k=1}^n \xi_k |b_{jk}^R| \mu_k^{IR}$$

$$+\sum_{k=1}^{n}\phi_{k}|b_{jk}^{R}|\mu_{k}^{II}+\sum_{k=1}^{n}\xi_{k}|b_{jk}^{I}|\mu_{k}^{RR}+\sum_{k=1}^{n}\phi_{k}|b_{jk}^{I}|\mu_{k}^{RI}<0,$$
(14)

then any solution of systems (4) and (5) respectively converges to a unique equilibrium exponentially.

This result is a direct consequence of Corollary 1.

Remark 5: Theorem 1 can be generalized to the system with time-varying delays

$$\dot{z}_{j}(t) = -d_{j}z_{j}(t) + \sum_{k=1}^{n} a_{jk}f_{k}(z_{k}(t)) + \sum_{k=1}^{n} b_{jk}g_{k}(z_{k}(t-\tau_{jk}(t))) + u_{j},$$

$$j = 1, \cdots, n \quad (15)$$

where $\tau_{jk}(t)$ can be bounded or unbounded. In fact, by Theorem 1, system (15) has an equilibrium $\overline{Z} = (\overline{z}_1, \dots, \overline{z}_n)^T$, and

$$\frac{d(z_j(t) - \overline{z}_j)}{dt} = -d_j(z_j(t) - \overline{z}_j) + \sum_{k=1}^n a_{jk}(f_k(z_k(t)) - f_k(\overline{z}_k)) + \sum_{k=1}^n b_{jk}(g_k(z_k(t - \tau_{jk}(t))) - g_k(\overline{z}_k)).$$

Replacing $e^{\epsilon t} \dot{z}(t)$ by $e^{\epsilon t}(z(t) - \overline{Z})$ in the proof of Theorem 1 and with the similar approach, we can prove that under the conditions (6) and (7), system (15) has a unique equilibrium, which is globally μ stable (for the concept of μ stability first proposed in [11] and details, readers can refer to [11], [12]).

B. Criteria with $\{\xi, 1\}$ -norm

Theorem 2: For dynamical systems (4) and (5), suppose the activation function $f_j(z)$ belongs to class $H_1(\lambda_j^{RR}, \lambda_j^{RI}, \lambda_j^{IR}, \lambda_j^{II})$ and $g_j(z)$ belongs to class $H_2(\mu_j^{RR}, \mu_j^{RI}, \mu_j^{IR}, \mu_j^{II})$, $j = 1, \cdots, n$. If there exists a positive vector $\xi = (\xi_1, \cdots, \xi_n, \phi_1, \cdots, \phi_n)^T > 0$ and $\epsilon > 0$, such that, for $k = 1, \cdots, n$,

$$T7(k) = \xi_k (-d_k + \epsilon) + \left[\xi_k a_{kk}^R + \sum_{j=1, j \neq k}^n \xi_j |a_{jk}^R| + \sum_{j=1}^n \phi_j |a_{jk}^I| \right]^+ \lambda_k^{RR} + \left[-\xi_k a_{kk}^I + \sum_{j=1, j \neq k}^n \xi_j |a_{jk}^I| + \sum_{j=1}^n \phi_j |a_{jk}^R| \right]^+ \lambda_k^{IR} + \sum_{j=1}^n \left(\xi_j (|b_{jk}^R| \mu_k^{RR} + |b_{jk}^I| \mu_k^{IR}) \right) \\+ \phi_j (|b_{jk}^R| \mu_k^{IR} + |b_{jk}^I| \mu_k^{RR}) \right) e^{\epsilon \tau_{jk}} \le 0, T8(k) = \phi_k (-d_k + \epsilon) + \left[\phi_k a_{kk}^R + \sum_{j=1, j \neq k}^n \phi_j |a_{jk}^R| + \sum_{j=1}^n \xi_j |a_{jk}^I| \right]^+ \lambda_k^{II} + \left[\phi_k a_{kk}^I + \sum_{j=1, j \neq k}^n \phi_j |a_{jk}^I| + \sum_{j=1}^n \xi_j |a_{jk}^R| \right]^+ \lambda_k^{RI}$$

$$+ \sum_{j=1}^{n} \left(\xi_{j} (|b_{jk}^{R}| \mu_{k}^{RI} + |b_{jk}^{I}| \mu_{k}^{II}) + \phi_{j} (|b_{jk}^{R}| \mu_{k}^{II} + |b_{jk}^{I}| \mu_{k}^{RI}) \right) e^{\epsilon \tau_{jk}} \leq 0$$

then dynamical systems (4) and (5) have a unique equilibrium $\overline{Z}^R = (\overline{z}_1^R, \dots, \overline{z}_n^R)^T$ and $\overline{Z}^I = (\overline{z}_1^I, \dots, \overline{z}_n^I)^T$ respectively. Moreover, for any solution Z(t) defined by (8), equations (9) and (10) hold, while the norm is $\{\xi, 1\}$ -norm.

Its proof can be found in Appendix B.

Corollary 3: For dynamical systems (4) and (5), suppose the activation function $f_j(z)$ belongs to class $H_1(\lambda_j^{RR}, \lambda_j^{RI}, \lambda_j^{IR}, \lambda_j^{II})$ and $g_j(z)$ belongs to class $H_2(\mu_j^{RR}, \mu_j^{RI}, \mu_j^{IR}, \mu_j^{II}), j = 1, \cdots, n$. If there exists a positive vector $\xi = (\xi_1, \cdots, \xi_n, \phi_1, \cdots, \phi_n)^T > 0$, such that, for $k = 1, \cdots, n$,

$$T9(k) = -\xi_k d_k$$

$$+ \left[\xi_k a_{kk}^R + \sum_{j=1, j \neq k}^n \xi_j |a_{jk}^R| + \sum_{j=1}^n \phi_j |a_{jk}^I|\right]^+ \lambda_k^{RR}$$

$$+ \left[-\xi_k a_{kk}^I + \sum_{j=1, j \neq k}^n \xi_j |a_{jk}^I| + \sum_{j=1}^n \phi_j |a_{jk}^R|\right]^+ \lambda_k^{IR}$$

$$+ \sum_{j=1}^n \xi_j (|b_{jk}^R| \mu_k^{RR} + |b_{jk}^I| \mu_k^{IR}) + \phi_j (|b_{jk}^R| \mu_k^{IR} + |b_{jk}^I| \mu_k^{RR})$$

$$< 0,$$

$$T10(k) = -\phi_k d_k$$

$$+ \left[\phi_k a_{kk}^R + \sum_{j=1, j \neq k}^n \phi_j |a_{jk}^R| + \sum_{j=1}^n \xi_j |a_{jk}^R|\right]^+ \lambda_k^{II}$$

$$+ \left[\phi_k a_{kk}^R + \sum_{j=1, j \neq k}^n \phi_j |a_{jk}^I| + \sum_{j=1}^n \xi_j |a_{jk}^R|\right]^+ \lambda_k^{RI}$$

$$+ \sum_{j=1}^n \xi_j (|b_{jk}^R| \mu_k^{RI} + |b_{jk}^I| \mu_k^{II}) + \phi_j (|b_{jk}^R| \mu_k^{II} + |b_{jk}^I| \mu_k^{RI})$$

$$< 0$$

then any solution of systems (4) and (5) respectively converges to a unique equilibrium exponentially.

Corollary 4: For dynamical systems (4) and (5), suppose the activation function $f_j(z)$ belongs to class $H_2(\lambda_j^{RR}, \lambda_j^{RI}, \lambda_j^{IR}, \lambda_j^{II})$ and $g_j(z)$ belongs to class $H_2(\mu_j^{RR}, \mu_j^{RI}, \mu_j^{IR}, \mu_j^{II})$, $j = 1, \cdots, n$. If there exists a positive vector $\xi = (\xi_1, \cdots, \xi_n, \phi_1, \cdots, \phi_n)^T > 0$, such that, for $k = 1, \cdots, n$,

$$T11(k) = -\xi_k d_k + \left[\sum_{j=1}^n \xi_j |a_{jk}^R| + \sum_{j=1}^n \phi_j |a_{jk}^I|\right] \lambda_k^{RR} + \left[\sum_{j=1}^n \xi_j |a_{jk}^I| + \sum_{j=1}^n \phi_j |a_{jk}^R|\right] \lambda_k^{IR} + \sum_{j=1}^n \xi_j (|b_{jk}^R| \mu_k^{RR} + |b_{jk}^I| \mu_k^{IR}) + \phi_j (|b_{jk}^R| \mu_k^{IR} + |b_{jk}^I| \mu_k^{RR}) < 0,$$
(16)

$$T12(k) = -\phi_k d_k + \left[\sum_{j=1}^n \phi_j |a_{jk}^R| + \sum_{j=1}^n \xi_j |a_{jk}^I|\right] \lambda_k^{II} + \left[\sum_{j=1}^n \phi_j |a_{j,k}^I| + \sum_{j=1}^n \xi_j |a_{jk}^R|\right] \lambda_k^{RI} + \sum_{j=1}^n \xi_j (|b_{jk}^R| \mu_k^{RI} + |b_{jk}^I| \mu_k^{II}) + \phi_j (|b_{jk}^R| \mu_k^{II} + |b_{jk}^I| \mu_k^{RI}) < 0,$$
(17)

then any solution of systems (4) and (5) respectively converges to a unique equilibrium exponentially.

C. Some comparisons

The following theorem is a direct consequence of Corollary 2, Corollary 4 and the properties of the M-matrix.

Theorem 3: For dynamical systems (4) and (5), suppose the activation function $f_j(z)$ belongs to class $H_2(\lambda_j^{RR}, \lambda_j^{RI}, \lambda_j^{IR}, \lambda_j^{II})$ and $g_j(z)$ belongs to class $H_2(\mu_j^{RR}, \mu_j^{RI}, \mu_j^{IR}, \mu_j^{II}), j = 1, \cdots, n$. Denote

$$\overline{D} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix},
\overline{A} = \begin{pmatrix} |A^{R}| & |A^{I}| \\ |A^{I}| & |A^{R}| \end{pmatrix}, \overline{F} = \begin{pmatrix} F^{RR} & F^{RI} \\ F^{IR} & F^{II} \end{pmatrix},
\overline{B} = \begin{pmatrix} |B^{R}| & |B^{I}| \\ |B^{I}| & |B^{R}| \end{pmatrix}, \overline{G} = \begin{pmatrix} G^{RR} & G^{RI} \\ G^{IR} & G^{II} \end{pmatrix}.$$
(18)

If $\overline{D} - \overline{AF} - \overline{BG}$ is a nonsingular M-matrix, then any solution of systems (4) and (5) respectively converges to a unique equilibrium exponentially.

Proof: If $\overline{D} - \overline{AF} - \overline{BG}$ is a nonsingular Mmatrix, according to Lemma 1, there exists vector $\xi = (\xi_1, \dots, \xi_n, \phi_1, \dots, \phi_n)^T > 0$, such that $(\overline{D} - \overline{AF} - \overline{BG})\xi > 0$, that is, inequalities (13) and (14) hold. Therefore, the conclusion is a direct consequence of Corollary 2.

On the other hand, if $\overline{D} - \overline{AF} - \overline{BG}$ is a nonsingular M-matrix, according to Lemma 1, there exists vector $\xi = (\xi_1, \dots, \xi_n, \phi_1, \dots, \phi_n)^T > 0$, such that $\xi^T (\overline{D} - \overline{AF} - \overline{BG}) > 0$, that is, inequalities (16) and (17) hold. Therefore, the conclusion is also a direct consequence of Corollary 4.

Remark 6: Criterion based on M-matrix was also reported in [45]. However, it neglects the signs of entries in the connection matrices A and B, and thus, the difference between excitatory and inhibitory effects might be ignored. Comparatively, the criteria given in Theorem 1, Theorem 2, Corollary 1, Corollary 3 are more powerful.

In the following, we give a comparison between Corollary 1 and Theorem 3 by using the matrix theory. Denote matrices

$$P_{1} = \operatorname{diag}(|a_{11}^{R}| - \{a_{11}^{R}\}^{+}, \cdots, |a_{nn}^{R}| - \{a_{nn}^{R}\}^{+});$$

$$P_{2} = \operatorname{diag}(|a_{11}^{I}| - \{-a_{11}^{I}\}^{+}, \cdots, |a_{nn}^{I}| - \{-a_{nn}^{I}\}^{+});$$

$$P_{3} = \operatorname{diag}(|a_{11}^{I}| - \{a_{11}^{I}\}^{+}, \cdots, |a_{nn}^{I}| - \{a_{nn}^{I}\}^{+}).$$

Obviously, these matrices are all non-negative definite. Define

$$\overline{\Delta} = \begin{pmatrix} P_1 F^{RR} + P_2 F^{IR} & 0\\ 0 & P_1 F^{II} + P_3 F^{RI} \end{pmatrix}, \quad (19)$$

so it is also non-negative definite. Using this notation, and from Corollary 1, the sufficient condition for global stability is that

$$\overline{D} - \overline{AF} - \overline{BG} + \overline{\Delta} \tag{20}$$

should be a nonsingular M-matrix. Obviously, if $\overline{D} - \overline{AF} - \overline{BG}$ is a nonsingular M-matrix, the above matrix (20) is also a nonsingular M-matrix; instead, if matrix (20) is a nonsingular M-matrix, $\overline{D} - \overline{AF} - \overline{BG}$ may be not.

Therefore, Corollary 1 presents a better criterion than that by previous works, like [45], because it considers the signs of entries in the connection matrix A, whose positive effect is described by the above nonnegative matrix $\overline{\Delta}$ defined in (19). Moreover, from this result, we can also find that in order to make the CVNNs have the stable equilibrium, P_1, P_2, P_3 should be as large as possible, so one way is to make all $a_{jj}^R, j = 1, \cdots, n$ be negative numbers.

Remark 7: The function $M(t) = \max_t \max_{i=1,\dots,m} |u_i(t)|$ proposed in [10] is a powerful tool in dealing with delayed systems. In particular, for the time-varying delays.

Remark 8: It can be seen that in computing the integral $\int_0^\infty ||\dot{Z}(t)|| dt$, the estimation of $\frac{d}{dt} ||\dot{Z}(t)||$ plays an important role.

Let
$$A(t) = (a_{ij})_{i,j=1}^{N}, \xi_i > 0, i = 1, \cdots, N$$
 and
 $\frac{dw}{dt} = Aw(t)$ (21)

It has been shown that (see [9], [10])

$$\max \frac{\frac{d}{dt} \|w(t)\|_{\{\xi,1\}}}{\|w(t)\|_{\{\xi,1\}}} = \max_{j} [a_{jj} + \sum_{i \neq j} \frac{\xi_{i}}{\xi_{j}} |a_{ij}|],$$
$$\max \frac{\frac{d}{dt} \|w(t)\|_{\{\xi,\infty\}}}{\|w(t)\|_{\{\xi,\infty\}}} = \max_{i} [a_{ii} + \sum_{j \neq i} \frac{\xi_{j}}{\xi_{i}} |a_{ij}|],$$
$$\max \frac{\frac{d}{dt} \|w(t)\|_{\{\xi,2\}}^{2}}{\|w(t)\|_{\{\xi,2\}}^{2}} = \lambda_{max} (\Xi A + A^{T} \Xi), \ \Xi = \text{diag}(\xi).$$

$$\frac{d}{dt} \|w(t)\|_{\{\xi,1\}} = \sum_{i=1}^{n} sign(w_i(t))\xi_i \sum_{j=1}^{n} a_{ij}w_j(t)$$
$$= \sum_{j=1}^{n} [\sum_i sign(w_i(t))\frac{\xi_i}{\xi_j}a_{ij}]\xi_jw_j(t)$$
$$\leq \sum_j [a_{jj} + \sum_{i \neq j} \frac{\xi_i}{\xi_j} |a_{ij}|]|\xi_jw_j(t)|$$
$$\leq \max_j [a_{jj} + \sum_{i \neq j} \frac{\xi_i}{\xi_j} |a_{ij}|]\|w(t)\|_{\{\xi,1\}}$$

Therefore,

$$\max \frac{\frac{d}{dt} \|w(t)\|_{\{\xi,1\}}}{\|w(t)\|_{\{\xi,1\}}} = \max_{j} [a_{jj} + \sum_{i \neq j} \frac{\xi_i}{\xi_j} |a_{ij}|]$$

Similarly, we can prove the other two equalities.

These three equalities play very important role in discussing stability of the neural networks or other dynamical systems. For example, if $\max_{j} \left[a_{jj} + \sum_{i \neq j} \frac{\xi_i}{\xi_i} |a_{ij}| \right] \leq 1$ $-\alpha < 0$, then $\frac{d}{dt} \| w(t) \|_{\{\xi,1\}} \le -\alpha \| w(t) \|_{\{\xi,1\}}$, which implies $||w(t)||_{\{\xi,1\}} = O(e^{-\alpha t}).$

It happens that these three equalities are closely relating to the matrix measure of A with respect to three norms.

D. Criteria with $\{\xi, 2\}$ -norm

m t a (t)

Theorem 4: For dynamical systems (4) and (5), suppose the activation function $f_j(z)$ belongs to class $H_1(\lambda_j^{RR}, \lambda_j^{RI}, \lambda_j^{IR}, \lambda_j^{II})$ and $g_j(z)$ belongs to class $H_2(\mu_j^{RR}, \mu_j^{RI}, \mu_j^{IR}, \mu_j^{II})$, $j = 1, \cdots, n$. If there exists a positive vector $\xi = (\xi_1, \cdots, \xi_n, \phi_1, \cdots, \phi_n)^T > 0$ and $\epsilon > 0$, such that, for $j = 1, \dots, n$,

$$\begin{split} T13(j) &= 2\xi_j(-d_j + \epsilon + \{a_{jj}^R\}^+ \lambda_j^{RR} + \{-a_{jj}^I\}^+ \lambda_j^{IR}) \\ &+ \sum_{k=1,k\neq j}^n \xi_j(|a_{jk}^R|\lambda_k^{RR} + |a_{jk}^I|\lambda_j^{IR}) \pi 1_{jk}^{-1} \\ &+ \sum_{k=1,k\neq j}^n \xi_k(|a_{kj}^R|\lambda_k^{RI} + |a_{jk}^I|\lambda_j^{IR}) \pi 1_{kj}^{-1} \\ &+ \sum_{k=1}^n \xi_j(|b_{jk}^R|\mu_k^{RI} + |a_{jk}^I|\lambda_k^{II}) \pi 2_{jk} \\ &+ \sum_{k=1}^n \xi_j(|b_{jk}^R|\mu_k^{RR} + |b_{jk}^I|\mu_k^{IR}) \pi 3_{jk} \\ &+ \sum_{k=1}^n \xi_j(|b_{jk}^R|\mu_k^{RI} + |b_{jk}^I|\mu_j^{RR}) \omega 1_{kj}^{-1} \\ &+ \left(\sum_{k=1}^n \xi_k(|a_{kj}^R|\mu_j^{IR} + |a_{kj}^I|\mu_j^{RR}) \omega 3_{kj}^{-1}\right) e^{2\epsilon\tau_{kj}} \le 0, \\ T14(j) &= 2\phi_j(-d_j + \epsilon + \{a_{jj}^R\}^+ \mu_j^{RR} + \{a_{jj}^I\}^+ \mu_j^{RI}) \\ &+ \sum_{k=1}^n \phi_j(|a_{jk}^R|\mu_k^{IR} + |a_{jk}^I|\mu_k^{RR}) \omega 1_{jk} \\ &+ \sum_{k=1}^n \phi_j(|a_{jk}^R|\mu_k^{IR} + |a_{jk}^I|\mu_k^{RR}) \omega 2_{jk} \\ &+ \sum_{k=1,k\neq j}^n \phi_j(|a_{jk}^R|\mu_k^{IR} + |a_{jk}^I|\mu_k^{RR}) \omega 3_{jk} \\ &+ \sum_{k=1}^n \phi_j(|b_{jk}^R|\mu_k^{IR} + |b_{jk}^I|\mu_k^{RR}) \omega 3_{jk} \\ &+ \sum_{k=1}^n \phi_j(|b_{jk}^R|\mu_k^{IR} + |b_{jk}^I|\mu_k^{RR}) \omega 3_{jk} \\ &+ \sum_{k=1}^n \phi_j(|b_{jk}^R|\mu_k^{IR} + |b_{jk}^I|\mu_k^{RR}) \omega 3_{jk} \\ &+ \sum_{k=1}^n \xi_k(|a_{kj}^R|\lambda_j^{RI} + |a_{kj}^I|\lambda_j^{II}) \pi 2_{kj}^{-1} \\ &+ \sum_{k=1}^n \xi_k(|b_{kj}^R|\mu_k^{RI} + |b_{jk}^I|\mu_j^{II}) \pi 4_{kj}^{-1} \end{split}$$

$$+\sum_{k=1}^{n}\phi_{k}(|b_{kj}^{R}|\mu_{j}^{II}+|b_{kj}^{I}|\mu_{j}^{RI})\omega 4_{kj}^{-1}\bigg)e^{2\epsilon\tau_{kj}}\leq 0$$

where $\pi 1_{jk}, \pi 2_{jk}, \pi 3_{jk}, \pi 4_{jk}, \omega 1_{jk}, \omega 2_{jk}, \omega 3_{jk}, \omega 4_{jk}$ are positive numbers. Then dynamical systems (4) and (5) have a unique equilibrium \overline{Z}^R and \overline{Z}^I respectively. Moreover, for any solution Z(t) defined by (8), equations (9) and (10) hold, where the norm is $\{\xi, 2\}$ -norm.

Its proof can be found in Appendix C.

Corollary 5: For dynamical systems (4) and (5), suppose the activation function $f_j(z)$ belongs to class $H_1(\lambda_j^{RR}, \lambda_j^{RI}, \lambda_j^{IR}, \lambda_j^{II})$ and $g_j(z)$ belongs to class $H_2(\mu_j^{RR}, \mu_j^{RI}, \mu_j^{IR}, \mu_j^{II}), j = 1, \cdots, n$. If there exists a positive vector $\xi = (\xi_1, \cdots, \xi_n, \phi_1, \cdots, \phi_n)^T > 0$, such that, for $j = 1, \cdots, n$,

$$\begin{split} T15(j) &= 2\xi_j(-d_j + \{a_{jj}^R\}^+ \lambda_j^{RR} + \{-a_{jj}^I\}^+ \lambda_j^{IR}) \\ &+ \sum_{k=1,k\neq j}^n \xi_j(|a_{jk}^R|\lambda_k^{RR} + |a_{jk}^I|\lambda_k^{IR})\pi 1_{jk} \\ &+ \sum_{k=1,k\neq j}^n \xi_k(|a_{kj}^R|\lambda_j^{RR} + |a_{jk}^I|\lambda_j^{IR})\pi 1_{kj}^{-1} \\ &+ \sum_{k=1}^n \xi_j(|a_{jk}^R|\lambda_k^{RI} + |a_{jk}^I|\lambda_k^{II})\pi 2_{jk} \\ &+ \sum_{k=1}^n \xi_j(|b_{jk}^R|\mu_k^{RR} + |b_{jk}^I|\mu_k^{IR})\pi 3_{jk} \\ &+ \sum_{k=1}^n \xi_j(|b_{jk}^R|\mu_k^{RI} + |b_{jk}^I|\mu_k^{IR})\pi 4_{jk} \\ &+ \sum_{k=1}^n \xi_k(|a_{kj}^R|\mu_j^{RR} + |b_{kj}^I|\mu_j^{RR})\omega 1_{kj}^{-1} \\ &+ \sum_{k=1}^n \xi_k(|b_{kj}^R|\mu_j^{RR} + |b_{kj}^I|\mu_j^{RR})\omega 3_{kj}^{-1} \\ &+ \sum_{k=1}^n \phi_k(|b_{kj}^R|\mu_j^{RR} + |b_{kj}^I|\mu_j^{RR})\omega 3_{kj}^{-1} < 0, \\ T16(j) &= 2\phi_j(-d_j + \{a_{jj}^R\}^+ \mu_j^{RR} + \{a_{jj}^I\}^+ \mu_j^{RI}) \\ &+ \sum_{k=1}^n \phi_j(|a_{jk}^R|\mu_k^{IR} + |a_{jk}^I|\mu_k^{RR})\omega 1_{jk} \\ &+ \sum_{k=1,k\neq j}^n \phi_k(|a_{kj}^R|\mu_k^{II} + |a_{kj}^I|\mu_k^{RI})\omega 2_{jk} \\ &+ \sum_{k=1}^n \phi_j(|b_{jk}^R|\mu_k^{IR} + |b_{jk}^I|\mu_k^{RR})\omega 3_{jk} \\ &+ \sum_{k=1}^n \phi_j(|b_{jk}^R|\mu_k^{IR} + |b_{jk}^I|\mu_k^{RI})\omega 4_{jk} \\ &+ \sum_{k=1}^n \xi_k(|a_{kj}^R|\lambda_j^{RI} + |a_{kj}^I|\lambda_j^{II})\pi 2_{kj}^{-1} \end{split}$$

$$+\sum_{k=1}^{n} \xi_{k}(|b_{kj}^{R}|\mu_{j}^{RI} + |b_{kj}^{I}|\mu_{j}^{II})\pi 4_{kj}^{-1} + \sum_{k=1}^{n} \phi_{k}(|b_{kj}^{R}|\mu_{j}^{II} + |b_{kj}^{I}|\mu_{j}^{RI})\omega 4_{kj}^{-1} < 0,$$

where $\pi 1_{jk}$, $\pi 2_{jk}$, $\pi 3_{jk}$, $\pi 4_{jk}$, $\omega 1_{jk}$, $\omega 2_{jk}$, $\omega 3_{jk}$, $\omega 4_{jk}$ are positive numbers. Then any solution of systems (4) and (5) respectively converges to a unique equilibrium exponentially.

Corollary 6: For dynamical systems (4) and (5), suppose the activation function $f_j(z)$ belongs to class $H_2(\lambda_j^{RR}, \lambda_j^{RI}, \lambda_j^{IR}, \lambda_j^{II})$ and $g_j(z)$ belongs to class $H_2(\mu_j^{RR}, \mu_j^{RI}, \mu_j^{IR}, \mu_j^{II})$, $j = 1, \cdots, n$. If there exists a positive vector $\xi = (\xi_1, \cdots, \xi_n, \phi_1, \cdots, \phi_n)^T > 0$, such that, for $j = 1, \cdots, n$,

$$\begin{split} T17(j) &= -2\xi_j d_j \\ &+ \sum_{k=1}^n \xi_j (|a_{jk}^R|\lambda_k^{RR} + |a_{jk}^I|\lambda_k^{IR}) + \sum_{k=1}^n \xi_k (|a_{kj}^R|\lambda_j^{RR} + |a_{kj}^I|\lambda_j^{IR}) \\ &+ \sum_{k=1}^n \xi_j (|a_{jk}^R|\lambda_k^{RI} + |a_{jk}^I|\lambda_k^{II}) + \sum_{k=1}^n \xi_j (|b_{jk}^R|\mu_k^{RR} + |b_{jk}^I|\mu_k^{IR}) \\ &+ \sum_{k=1}^n \xi_j (|b_{jk}^R|\mu_k^{RI} + |b_{jk}^I|\mu_k^{II}) + \sum_{k=1}^n \phi_k (|a_{kj}^R|\mu_j^{IR} + |a_{kj}^I|\mu_j^{RR}) \\ &+ \sum_{k=1}^n \xi_k (|b_{kj}^R|\mu_k^{RR} + |b_{kj}^I|\mu_j^{IR}) + \sum_{k=1}^n \phi_k (|b_{kj}^R|\mu_k^{II} + |b_{kj}^I|\mu_k^{RI}) \\ &< 0, \\ T18(j) &= -2\phi_j d_j \\ &+ \sum_{k=1}^n \phi_j (|a_{jk}^R|\mu_k^{IR} + |a_{kj}^I|\mu_k^{RI}) + \sum_{k=1}^n \phi_j (|b_{jk}^R|\mu_k^{II} + |a_{jk}^I|\mu_k^{RI}) \\ &+ \sum_{k=1}^n \phi_k (|a_{kj}^R|\mu_j^{II} + |a_{kj}^I|\mu_j^{RI}) + \sum_{k=1}^n \phi_j (|b_{jk}^R|\mu_k^{II} + |b_{jk}^I|\mu_k^{RI}) \\ &+ \sum_{k=1}^n \phi_j (|b_{jk}^R|\mu_k^{II} + |b_{jk}^I|\mu_k^{RI}) + \sum_{k=1}^n \phi_k (|a_{kj}^R|\lambda_j^{II} + |a_{kj}^I|\lambda_j^{II}) \\ &+ \sum_{k=1}^n \xi_k (|b_{kj}^R|\mu_k^{RI} + |b_{kj}^I|\mu_j^{II}) + \sum_{k=1}^n \phi_k (|b_{kj}^R|\mu_j^{II} + |a_{kj}^I|\lambda_j^{II}) \\ &+ \sum_{k=1}^n \xi_k (|b_{kj}^R|\mu_j^{RI} + |b_{kj}^I|\mu_j^{II}) + \sum_{k=1}^n \phi_k (|b_{kj}^R|\mu_j^{II} + |b_{kj}^I|\mu_j^{RI}) \\ &< 0, \end{split}$$

then any solution of systems (4) and (5) respectively converges to a unique equilibrium exponentially.

Remark 9: As for how to use the norms $|| \cdot ||_{\{\xi,1\}}$ and $|| \cdot ||_{\{\xi,2\}}$ to discuss the time-varying delayed networks, readers can refer to the papers [15], [16].

IV. NUMERICAL EXAMPLE

In this section, some numerical simulations are presented to show the effectiveness of our obtained results.

Consider a two-neuron complex-valued recurrent neural network described as follows:

$$\begin{cases} \dot{z}_1(t) = -dz_1(t) + a_{11}f_1(z_1(t)) + a_{12}f_2(z_2(t)) \\ +b_{11}g_1(z_1(t-1)) + b_{12}g_2(z_2(t-2)) + u_1 \\ \dot{z}_2(t) = -dz_2(t) + a_{21}f_1(z_1(t)) + a_{22}f_2(z_2(t)) \\ +b_{21}g_1(z_1(t-3)) + b_{22}g_2(z_2(t-4)) + u_2 \end{cases}$$
(22)

where $z_k = z_k^R + i z_k^I$, k = 1, 2, $D = \text{diag}(d, d) = 19I_2$, and

$$A = (a_{jk})_{2 \times 2} = \begin{pmatrix} -2 - 3i & 3 - i \\ 4 - 2i & -1 + 2i \end{pmatrix},$$

$$B = (b_{jk})_{2 \times 2} = \begin{pmatrix} -1 + 2i & 2 + i \\ 3 - 4i & -3 + 2i \end{pmatrix},$$

$$u = (u_1, u_2)^T = (-3 + i, 2 + 4i)^T,$$

$$f_k(z_k) = \frac{1 - \exp(-2z_k^R - z_k^I)}{1 + \exp(-2z_k^R - z_k^I)} + i\frac{1}{1 + \exp(-z_k^R - 2z_k^I)},$$

$$g_k(z_k) = \frac{1}{1 + \exp(-z_k^R - 2z_k^I)} + i\frac{1 - \exp(-2z_k^R - z_k^I)}{1 + \exp(-2z_k^R - z_k^I)}.$$

From simple calculations, we have, for j = 1, 2,

$$\begin{split} 0 &< \frac{\partial f_j^R}{\partial z_j^R} \leq 1 = \lambda_j^{RR}; \qquad 0 < \frac{\partial f_j^R}{\partial z_j^I} \leq 0.5 = \lambda_j^{RI}; \\ 0 &< \frac{\partial f_j^I}{\partial z_j^R} \leq 0.25 = \lambda_j^{IR}; \qquad 0 < \frac{\partial f_j^I}{\partial z_j^I} \leq 0.5 = \lambda_j^{II}; \end{split}$$

therefore, $f_j(z)$ belongs to class $H_1(1, 0.5, 0.25, 0.5), j = 1, 2$. Similarly, we can prove that $g_j(z)$ belongs to class $H_2(0.25, 0.5, 1, 0.5), j = 1, 2$.

From the notations defined in (18), we have $\overline{D} = 19I_4$,

$$\overline{A} = \begin{pmatrix} 2 & 3 & 3 & 1 \\ 4 & 1 & 2 & 2 \\ 3 & 1 & 2 & 3 \\ 2 & 2 & 4 & 1 \end{pmatrix}, \overline{F} = \begin{pmatrix} 1 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0.5 \\ 0.25 & 0 & 0.5 & 0 \\ 0 & 0.25 & 0 & 0.5 \end{pmatrix},$$
$$\overline{B} = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 3 & 3 & 4 & 2 \\ 2 & 1 & 1 & 2 \\ 4 & 2 & 3 & 3 \end{pmatrix}, \overline{G} = \begin{pmatrix} 0.25 & 0 & 0.5 & 0 \\ 0 & 0.25 & 0 & 0.5 \\ 1 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0.5 \end{pmatrix}.$$

Calculations show that eigenvalues of $\overline{D} - \overline{AF} - \overline{BG}$ are: -0.7655, 18.6670, 20.9701, 19.8784, so it is not an M-matrix, which means that Theorem 3 is not satisfied. However, according to Corollary 1 and Remark 6, we have $P_1 = \text{diag}\{2,1\}, P_2 = \text{diag}\{0,2\}, P_3 = \text{diag}\{3,0\}$, and

$$\overline{\Delta} = \operatorname{diag}\{P_1 + 0.25P_2, 0.5(P_1 + P_3)\}$$

then eigenvalues of $\overline{D} - \overline{AF} - \overline{BG} + \overline{\Delta}$ are 0.8488, 20.0717, 22.7947, 21.5348, therefore Corollary 1 holds, so the above system can achieve its equilibrium exponentially.

The following simulations present the correctness of our claim. We choose five cases for initial values. Case 1: $z_1(t) = -4 + 3i, z_2(t) = -5 - i, t \in [-4, 0]$; Case 2: $z_1(t) = 2 + i, z_2(t) = -3 + 2.5i, t \in [-4, 0]$; Case 3: $z_1(t) = 3 - 5i, z_2(t) = 6 + 3i, t \in [-4, 0]$; Case 4: $z_1(t) = -2 - 4i, z_2(t) = -7 + 4i, t \in [-4, 0]$; Case 5: $z_1(t) = 1 + 4i, z_2(t) = -5 - 1.5i, t \in [-4, 0]$. Figures 1-4 depict the trajectories of $z_1^R(t), z_1^I(t), z_2^R(t), z_2^I(t)$ respectively. For different initial values, they converge to the same equilibrium $(-0.0351, 0.1423, 0.0912, 0.2239)^T$, i.e., the unique equilibrium has the global exponential stability property.

Moreover, if we choose the initial values as Case 1, and only the external control u are different, i.e., different external controls $(u_1, u_2)^T = (-3 + i, 2 + 4i)^T$ and $(u'_1, u'_2)^T = (3 + i)^T$

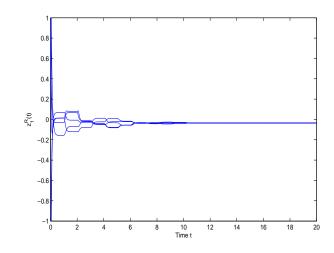


Fig. 1. Trajectories of $z_1^R(t)$ for different initial values, which show the global exponential stability of equilibrium

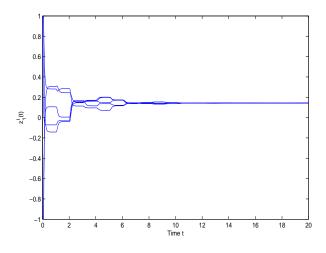


Fig. 2. Trajectories of $z_1^I(t)$ for different initial values, which show the global exponential stability of equilibrium

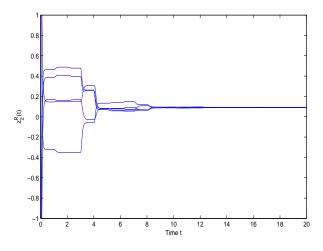


Fig. 3. Trajectories of $z_2^R(t)$ for different initial values, which show the global exponential stability of equilibrium

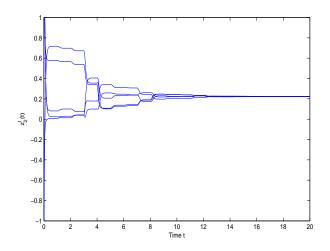


Fig. 4. Trajectories of $z_2^I(t)$ for different initial values, which show the global exponential stability of equilibrium

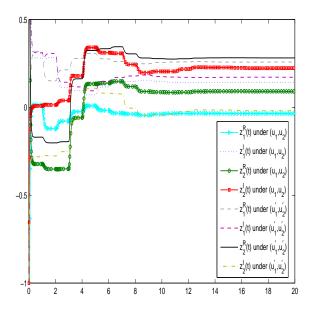


Fig. 5. Trajectories of all $z_i^R(t)$ and $z_i^I(t)$, i = 1, 2 under different external controllers (u_1, u_2) and (u'_1, u'_2) , which means that the equilibrium is impacted by external control

 $2i, 4-i)^T$ are added on the CVNNs, figure 5 shows that the equilibriums are different, therefore, the equilibrium is heavily impacted by the external control.

In the final simulation, we will consider the time-varying delays, thus we choose the equation as

$$\begin{pmatrix}
\dot{z}_{1}(t) = -dz_{1}(t) + a_{11}f_{1}(z_{1}(t)) + a_{12}f_{2}(z_{2}(t)) \\
+ b_{11}g_{1}(z_{1}(t - 1 - \sin(t))) \\
+ b_{12}g_{2}(z_{2}(t - 2 - \cos(t))) + u_{1} \\
\dot{z}_{2}(t) = -dz_{2}(t) + a_{21}f_{1}(z_{1}(t)) + a_{22}f_{2}(z_{2}(t)) \\
+ b_{21}g_{1}(z_{1}(t - 3 + \sin(t))) \\
+ b_{22}g_{2}(z_{2}(t - 4 + \cos(t))) + u_{2}
\end{cases}$$
(23)

All the parameters, including the external control,

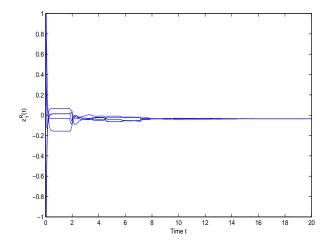


Fig. 6. Trajectories of $z_1^R(t)$ for different initial values, which show the global exponential stability of equilibrium

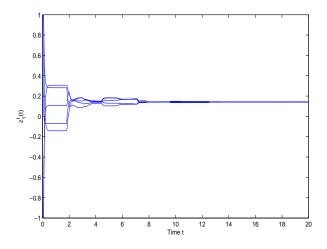


Fig. 7. Trajectories of $z_1^I(t)$ for different initial values, which show the global exponential stability of equilibrium

are the same as defined in the above simulations. Similarly, according to Corollary 1, Remark 5 and Remark 6, this system can achieve its equilibrium exponentially. Figures 6-9 depict the trajectories of $z_1^R(t), z_1^I(t), z_2^R(t), z_2^I(t)$ respectively. Moreover, the equilibrium is also $(-0.0351, 0.1423, 0.0912, 0.2239)^T$, i.e., the equilibriums are the same for system (22) and system (23) even though they have different time delays.

V. CONCLUSION AND DISCUSSIONS

In this paper, we first propose a complex-valued recurrent neural network model with asynchronous time delays. This feature is the first difference of this paper with previous works. Then under the assumptions of activation functions, we prove the exponential convergence directly by using the ∞ -norm, 1-norm and 2-norm respectively, the existence and uniqueness of the equilibrium point is a direct consequence of the exponential convergence; while previous works in the literature

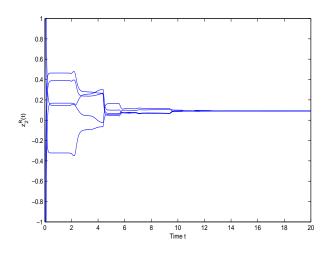


Fig. 8. Trajectories of $z_2^R(t)$ for different initial values, which show the global exponential stability of equilibrium

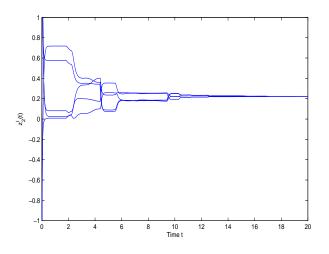


Fig. 9. Trajectories of $z_2^{I}(t)$ for different initial values, which show the global exponential stability of equilibrium

always use two proving steps: step 1, prove the existence of equilibrium; step 2, prove the stability. This is also a novelty of this paper for investigating the equilibrium of CVNNs. Moreover, considering the signs of coupling matrix, some sufficient conditions for the uniqueness and global exponential stability of the equilibrium point are presented, which are more general and less restrictive than previous works, i.e., the *M*-matrix property of $\overline{D} - \overline{AF} - \overline{BG}$ is just a special case of criteria for exponential stability. These are our main theoretical results. Finally, three numerical examples are given to show the correctness of our obtained results.

In the end, we give some discussions about future directions of the complex-valued neural networks:

 This paper deals with complex-valued neural network by decomposing it to real and imaginary parts and constructing an equivalent real-valued system. To ensure this decomposition, we assume the partial derivatives of activation functions exist and bounded, see Definition

- 2) In this paper, we consider the asynchronous time delays, which can be regarded as discrete delays. However, a distribution of propagation delays can exist for neural networks due to the multitude of parallel pathways with a variety of axon sizes and lengths. Therefore, continuously distributed delays can be a good choice, so investigation of stability under distributed delays will also be our future direction.
- 3) As for the dynamical behaviors of complex-valued neural networks, the global existence and exponential stability is just an aspect, there are also many other interesting dynamical behaviors for future research, for example, the multistability, the robustness of uncertain neural networks, the existence and stability of periodic (or almost periodic) solutions, chaotic behaviors for (delayed) complex-valued neural networks, etc.

APPENDIX A: THE PROOF OF THEOREM 1

Proof: Define

$$x_j(t) = e^{\epsilon t} \dot{z}_j^R(t), \quad y_j(t) = e^{\epsilon t} \dot{z}_j^I(t), j = 1, 2, \cdots, n.$$
 (24)

Then we have

$$\begin{aligned} \dot{x}_{j}(t) &= \left(-d_{j} + \epsilon\right)x_{j}(t) \\ &+ \sum_{k=1}^{n} a_{jk}^{R} \left[\frac{\partial f_{k}^{R}}{\partial z_{k}^{R}} x_{k} + \frac{\partial f_{k}^{R}}{\partial z_{k}^{I}} y_{k}\right] - \sum_{k=1}^{n} a_{jk}^{I} \left[\frac{\partial f_{k}^{I}}{\partial z_{k}^{R}} x_{k} + \frac{\partial f_{k}^{I}}{\partial z_{k}^{I}} y_{k}\right] \\ &+ \sum_{k=1}^{n} b_{jk}^{R} e^{\epsilon \tau_{jk}} \left[\frac{\partial g_{k}^{R}}{\partial z_{k}^{R}(\underline{\tau_{jk}})} x_{k}(\underline{\tau_{jk}}) + \frac{\partial g_{k}^{R}}{\partial z_{k}^{I}(\underline{\tau_{jk}})} y_{k}(\underline{\tau_{jk}})\right] \\ &- \sum_{k=1}^{n} b_{jk}^{I} e^{\epsilon \tau_{jk}} \left[\frac{\partial g_{k}^{I}}{\partial z_{k}^{R}(\underline{\tau_{jk}})} x_{k}(\underline{\tau_{jk}}) + \frac{\partial g_{k}^{I}}{\partial z_{k}^{I}(\underline{\tau_{jk}})} y_{k}(\underline{\tau_{jk}})\right], \end{aligned}$$

$$(25)$$

and

$$\begin{split} \dot{y}_{j}(t) &= (-d_{j} + \epsilon)y_{j}(t) \\ &+ \sum_{k=1}^{n} a_{jk}^{R} \left[\frac{\partial f_{k}^{I}}{\partial z_{k}^{R}} x_{k} + \frac{\partial f_{k}^{I}}{\partial z_{k}^{I}} y_{k} \right] + \sum_{k=1}^{n} a_{jk}^{I} \left[\frac{\partial f_{k}^{R}}{\partial z_{k}^{R}} x_{k} + \frac{\partial f_{k}^{R}}{\partial z_{k}^{I}} y_{k} \right] \\ &+ \sum_{k=1}^{n} b_{jk}^{R} e^{\epsilon \tau_{jk}} \left[\frac{\partial g_{k}^{I}}{\partial z_{k}^{R}(\underline{\tau_{jk}})} x_{k}(\underline{\tau_{jk}}) + \frac{\partial g_{k}^{I}}{\partial z_{k}^{I}(\underline{\tau_{jk}})} y_{k}(\underline{\tau_{jk}}) \right] \\ &+ \sum_{k=1}^{n} b_{jk}^{I} e^{\epsilon \tau_{jk}} \left[\frac{\partial g_{k}^{R}}{\partial z_{k}^{R}(\underline{\tau_{jk}})} x_{k}(\underline{\tau_{jk}}) + \frac{\partial g_{k}^{R}}{\partial z_{k}^{I}(\underline{\tau_{jk}})} y_{k}(\underline{\tau_{jk}}) \right], \end{split}$$

$$(26)$$

where $\partial f_k^a / \partial z_k^b$ denotes $\partial f_k^a (z_k^R(t), z_k^I(t)) / \partial z_k^b, a, b = R, I; \quad x_k(\underline{\tau_{jk}}) = x_k(t - \tau_{jk}), \text{ and } y_k(\underline{\tau_{jk}}) = y_k(t - \overline{\tau_{jk}}); \text{ while } \partial f_k^a / \partial z_k^b(\underline{\tau_{jk}}) \text{ denotes } \partial f_k^a (z_k^R(t - \tau_{jk}), z_k^I(t - \tau_{jk})) / \partial z_k^b(t - \tau_{jk}), a, b = R, I.$ Let

$$X(t) = (x_1(t), \cdots, x_n(t), y_1(t), \cdots, y_n(t))^T \in \mathbb{R}^{2n \times 1},$$

so $X(t) = e^{\epsilon t} \dot{Z}(t)$, and $||X(t)||_{\{\xi,\infty\}}$ $\max\{\max_j\{|\xi_j^{-1}x_j(t)|\}, \max_j\{|\phi_j^{-1}y_j(t)|\}\}.$

Case 1: For X(t), if $j_0 = j_0(t)$, which depends on t, is such an index that $|\xi_{j_0}^{-1}x_{j_0}(t)| = ||X(t)||_{\{\xi,\infty\}}$, then

$$\begin{split} &\xi_{j_0} \frac{d||X(t)||_{\{\xi,\infty\}}}{dt} = \frac{d|x_{j_0}(t)|}{dt} \\ = & \operatorname{sign}\{x_{j_0}(t)\} \left\{ \xi_{j_0}(-d_{j_0} + \epsilon) \xi_{j_0}^{-1} x_{j_0}(t) \right. \\ &+ \sum_{k=1}^n \xi_k a_{j_0k}^R \frac{\partial f_k^R}{\partial z_k^R} \xi_k^{-1} x_k + \sum_{k=1}^n \phi_k a_{j_0k}^R \frac{\partial f_k^R}{\partial z_k^R} \phi_k^{-1} y_k \\ &- \sum_{k=1}^n \xi_k a_{j_0k}^R \frac{\partial f_k^R}{\partial z_k^R} \xi_k^{-1} x_k - \sum_{k=1}^n \phi_k a_{j_0k}^I \frac{\partial f_k^I}{\partial z_k^I} \phi_k^{-1} y_k \\ &+ \sum_{k=1}^n \xi_k b_{j_0k}^R \frac{\partial g_k^R}{\partial z_k^R(\underline{\tau}_{j_0k})} \cdot e^{\epsilon \tau_{j_0k}} \xi_k^{-1} x_k(\underline{\tau}_{j_0k}) \\ &+ \sum_{k=1}^n \phi_k b_{j_0k}^R \frac{\partial g_k^R}{\partial z_k^R(\underline{\tau}_{j_0k})} \cdot e^{\epsilon \tau_{j_0k}} \phi_k^{-1} y_k(\underline{\tau}_{j_0k}) \\ &- \sum_{k=1}^n \xi_k b_{j_0k}^I \frac{\partial g_k^I}{\partial z_k^R(\underline{\tau}_{j_0k})} \cdot e^{\epsilon \tau_{j_0k}} \xi_k^{-1} x_k(\underline{\tau}_{j_0k}) \\ &- \sum_{k=1}^n \phi_k b_{j_0k}^I \frac{\partial g_k^I}{\partial z_k^R(\underline{\tau}_{j_0k})} \cdot e^{\epsilon \tau_{j_0k}} \phi_k^{-1} y_k(\underline{\tau}_{j_0k}) \\ &+ \sum_{k=1, k \neq j_0}^n \xi_k |a_{j_0k}^R| \lambda_k^{RR} \cdot ||X(t)||_{\{\xi,\infty\}} \\ &+ \sum_{k=1, k \neq j_0}^n \xi_k |a_{j_0k}^R| \lambda_k^{RR} \cdot ||X(t)||_{\{\xi,\infty\}} \\ &+ \sum_{k=1}^n \phi_k |a_{j_0k}^R| \lambda_k^{RR} \cdot e^{\epsilon \tau_{j_0k}} ||X(t - \tau_{j_0k})||_{\{\xi,\infty\}} \\ &+ \sum_{k=1}^n \xi_k |b_{j_0k}^R| \mu_k^{RR} \cdot e^{\epsilon \tau_{j_0k}} ||X(t - \tau_{j_0k})||_{\{\xi,\infty\}} \\ &+ \sum_{k=1}^n \xi_k |b_{j_0k}^R| \mu_k^{RR} \cdot e^{\epsilon \tau_{j_0k}} ||X(t - \tau_{j_0k})||_{\{\xi,\infty\}} \\ &+ \sum_{k=1}^n \phi_k |b_{j_0k}^R| \mu_k^{RR} \cdot e^{\epsilon \tau_{j_0k}} ||X(t - \tau_{j_0k})||_{\{\xi,\infty\}} \\ &+ \sum_{k=1}^n \phi_k |b_{j_0k}^R| \mu_k^{RR} \cdot e^{\epsilon \tau_{j_0k}} ||X(t - \tau_{j_0k})||_{\{\xi,\infty\}} \\ &+ \sum_{k=1}^n \xi_k |b_{j_0k}^R| \mu_k^{RR} \cdot e^{\epsilon \tau_{j_0k}} ||X(t - \tau_{j_0k})||_{\{\xi,\infty\}} \\ &+ \sum_{k=1}^n \phi_k |b_{j_0k}^R| \mu_k^{RR} \cdot e^{\epsilon \tau_{j_0k}} ||X(t - \tau_{j_0k})||_{\{\xi,\infty\}} \\ &= \left\{ \xi_{j_0} \left(-d_{j_0} + \epsilon + \{a_{j_{j_0j}}^R \}^+ \lambda_{j_0}^R + \{-a_{j_{0j_0}}^R \}^+ \lambda_{j_0}^{RR} \right\} - \left\{ \sum_{k=1, k \neq j_0}^n \xi_k |a_{j_0k}^R| \lambda_k^{RR} + \sum_{k=1}^n \phi_k |a_{j_0k}^R| \lambda_k^{RI} \\ &+ \sum_{k=1, k \neq j_0}^n \xi_k |a_{j_0k}^R| \lambda_k^{RR} + \sum_{k=1}^n \phi_k |a_{j_0k}^R| \lambda_k^{RI} \\ \\ &+ \sum_{k=1, k \neq j_0}^n \xi_k |a_{j_0k}^R| \lambda_k^{RR} + \sum_{k=1}^n \phi_k |a_{j_0k}^R| \lambda_k^{RI} \right\} + \left\{ \sum_{k=1}^n \xi_k |b_{j_0k}^R| \xi_k^R + \sum_{k=1}^n \xi_k |b_{j_0k}^R| \xi_k^R + \xi_k^R + \xi_k^R \right\} \right\}$$

$$+ \left\{ \sum_{k=1}^{n} \xi_{k} | b_{j_{0}k}^{R} | \mu_{k}^{RR} + \sum_{k=1}^{n} \phi_{k} | b_{j_{0}k}^{R} | \mu_{k}^{RI} + \sum_{k=1}^{n} \xi_{k} | b_{j_{0}k}^{I} | \mu_{k}^{IR} + \sum_{k=1}^{n} \phi_{k} | b_{j_{0}k}^{I} | \mu_{k}^{II} \right\} e^{\epsilon \tau_{j_{0}k}} \| X(t - \tau_{j_{0}k}) \|_{\{\xi, \infty\}}.$$

Furthermore, define

=

$$M(t) = \sup_{t - \tau \le s \le t} \|X(s)\|_{\{\xi, \infty\}},$$
(27)

where $\tau=\max_{jk}\tau_{jk}.$ Then $\|X(t)\|_{\{\xi,\infty\}}\leq M(t),$ and if $\|X(t)\|_{\{\xi,\infty\}}=M(t),$ we have

$$\xi_{j_0} \frac{d \|X(t)\|_{\{\xi,\infty\}}}{dt} \le T1(j_0) \cdot M(t) \le 0.$$
(28)

Case 2: For X(t), if $j_0 = j'_0(t)$, which depends on t, is such an index that $|\phi_{j'_0}^{-1}y_{j'_0}(t)| = ||X(t)||_{\{\xi,\infty\}}$, then

$$\begin{split} \phi_{j'_{0}} \frac{d\|X(t)\|_{\{\xi,\infty\}}}{dt} &= \frac{d|y_{j'_{0}}(t)|}{dt} \\ = & \operatorname{sign}\{y_{j'_{0}}(t)\} \left\{ \phi_{j'_{0}}(-d_{j'_{0}} + \epsilon) \phi_{j'_{0}}^{-1} y_{j'_{0}}(t) \\ &+ \sum_{k=1}^{n} \xi_{k} a_{j'_{0}k}^{R} \frac{\partial f_{k}^{I}}{\partial z_{k}^{R}} \xi_{k}^{-1} x_{k} + \sum_{k=1}^{n} \phi_{k} a_{j'_{0}k}^{R} \frac{\partial f_{k}^{I}}{\partial z_{k}^{I}} \phi_{k}^{-1} y_{k} \\ &+ \sum_{k=1}^{n} \xi_{k} a_{j'_{0}k}^{I} \frac{\partial f_{k}^{R}}{\partial z_{k}^{R}} \xi_{k}^{-1} x_{k} + \sum_{k=1}^{n} \phi_{k} a_{j'_{0}k}^{I} \frac{\partial f_{k}^{R}}{\partial z_{k}^{I}} \phi_{k}^{-1} y_{k} \\ &+ \sum_{k=1}^{n} \xi_{k} b_{j'_{0}k}^{R} \frac{\partial g_{k}^{I}}{\partial z_{k}^{R}} (\underline{\gamma_{j'_{0}k}}) e^{\epsilon \tau_{j'_{0}k}} \xi_{k}^{-1} x_{k} (\underline{\gamma_{j'_{0}k}}) \\ &+ \sum_{k=1}^{n} \phi_{k} b_{j'_{0}k}^{R} \frac{\partial g_{k}^{R}}{\partial z_{k}^{I} (\underline{\gamma_{j'_{0}k}})} e^{\epsilon \tau_{j'_{0}k}} \phi_{k}^{-1} y_{k} (\underline{\gamma_{j'_{0}k}}) \\ &+ \sum_{k=1}^{n} \xi_{k} b_{j'_{0}k}^{I} \frac{\partial g_{k}^{R}}{\partial z_{k}^{R} (\underline{\gamma_{j'_{0}k}})} e^{\epsilon \tau_{j'_{0}k}} \xi_{k}^{-1} x_{k} (\underline{\gamma_{j'_{0}k}}) \\ &+ \sum_{k=1}^{n} \phi_{k} b_{j'_{0}k}^{I} \frac{\partial g_{k}^{R}}{\partial z_{k}^{R} (\underline{\gamma_{j'_{0}k}})} e^{\epsilon \tau_{j'_{0}k}} \xi_{k}^{-1} x_{k} (\underline{\gamma_{j'_{0}k}}) \\ &+ \sum_{k=1}^{n} \xi_{k} b_{j'_{0}k}^{I} \frac{\partial g_{k}^{R}}{\partial z_{k}^{I} (\underline{\gamma_{j'_{0}k}})} e^{\epsilon \tau_{j'_{0}k}} \xi_{k}^{-1} x_{k} (\underline{\gamma_{j'_{0}k}}) \\ &+ \sum_{k=1}^{n} \phi_{k} b_{j'_{0}k}^{I} \frac{\partial g_{k}^{R}}{\partial z_{k}^{I} (\underline{\gamma_{j'_{0}k}})} e^{\epsilon \tau_{j'_{0}k}} \xi_{k}^{-1} x_{k} (\underline{\gamma_{j'_{0}k}}) \\ &+ \sum_{k=1}^{n} \xi_{k} |a_{j'_{0}k}| \lambda_{k}^{IR} + \sum_{k=1, k \neq j'_{0}}^{n} \phi_{k} |a_{j'_{0}k}^{R}| \lambda_{k}^{II} + \sum_{k=1}^{n} \xi_{k} |a_{j'_{0}k}^{I}| \lambda_{k}^{RR} \\ &+ \sum_{k=1}^{n} \xi_{k} |b_{j'_{0}k}^{R}| \mu_{k}^{IR} + \sum_{k=1}^{n} \phi_{k} |b_{j'_{0}k}^{R}| \mu_{k}^{II} + \sum_{k=1}^{n} \xi_{k} |b_{j'_{0}k}^{I}| \mu_{k}^{RR} \\ &+ \sum_{k=1}^{n} \phi_{k} |b_{j'_{0}k}^{I}| \mu_{k}^{RI} \right\} e^{\epsilon \tau_{j'_{0}k}} \|X(t - \tau_{j'_{0}k})\|_{\{\xi,\infty\}}. \end{split}$$

From the definition of (27), we have $||X(t)||_{\{\xi,\infty\}} \le M(t)$, and if $||X(t)||_{\{\xi,\infty\}} = M(t)$,

$$\phi_{j'_0} \frac{d \|X(t)\|_{\{\xi,\infty\}}}{dt} \le T2(j'_0) \cdot M(t) \le 0.$$
(29)

Therefore, for the above two cases, according to (28) and (29), one can get that M(t) decreases monotonely, which

implies $||X(t)||_{\{\xi,\infty\}} = O(1)$ and

$$|\dot{Z}(t)||_{\{\xi,\infty\}} = O(e^{-\epsilon t}),$$

i.e., $\dot{z}_j^R(t) = O(e^{-\epsilon t})$ and $\dot{z}_j^I(t) = O(e^{-\epsilon t}), j = 1, 2, \cdots, n.$

Consequently, for any $t_1, t_2 \in R, t_1 > t_2$, there exists a constant C > 0, such that

$$\begin{aligned} \|Z(t_1) - Z(t_2)\|_{\{\xi,\infty\}} &= \|\int_{t_2}^{t_1} \dot{Z}(t) dt\| \le \int_{t_2}^{t_1} \|\dot{Z}(t)\| dt\\ &\le \int_{t_2}^{t_1} Ce^{-\epsilon t} dt = \frac{C}{\epsilon} (e^{-\epsilon t_2} - e^{-\epsilon t_1}) \le \frac{C}{\epsilon} e^{-\epsilon t_2}. \end{aligned}$$

By Cauchy convergence principle, we conclude that $\lim_{t\to+\infty} Z(t) = \overline{Z}$, for some $\overline{Z} = (\overline{Z}^{R^T}, \overline{Z}^{I^T})^T$. It is easy to get that \overline{Z} is an equilibrium point of the systems (4) and (5).

Next, we prove that the equilibrium point is unique. Let \overline{Z} be any equilibrium point of the systems (4) and (5). By the same arguments, we can prove that

$$||Z(t) - \overline{Z}||_{\{\xi,\infty\}} = ||\int_t^\infty \dot{Z}(t)dt|| \le \frac{C}{\epsilon}e^{-\epsilon t}$$

which means that any solution Z(t) converges to \overline{Z} exponentially and the equilibrium point is unique.

APPENDIX B: PROOF OF THEOREM 2

Proof: Recall the definition of $x_j(t)$ and $y_j(t)$ defined in (24), we can define a Lyapunov function as

$$L_{1}(t) = \sum_{j=1}^{n} \xi_{j} |x_{j}(t)| + \sum_{j=1}^{n} \phi_{j} |y_{j}(t)|$$

+
$$\sum_{j,k=1}^{n} \alpha_{jk} e^{\epsilon \tau_{jk}} \int_{t-\tau_{jk}}^{t} |x_{k}(s)| ds$$

+
$$\sum_{j,k=1}^{n} \beta_{jk} e^{\epsilon \tau_{jk}} \int_{t-\tau_{jk}}^{t} |y_{k}(s)| ds,$$

where

$$\begin{split} \alpha_{jk} &= \xi_j (|b_{jk}^R| \mu_k^{RR} + |b_{jk}^I| \mu_k^{IR}) + \phi_j (|b_{jk}^R| \mu_k^{IR} + |b_{jk}^I| \mu_k^{RR}); \\ \beta_{jk} &= \xi_j (|b_{jk}^R| \mu_k^{RI} + |b_{jk}^I| \mu_k^{II}) + \phi_j (|b_{jk}^R| \mu_k^{II} + |b_{jk}^I| \mu_k^{RI}). \end{split}$$

Differentiating $L_1(t)$ along equations (25) and (26), using some calculations (the details are left to interested readers), we have

$$\dot{L}_1(t) \le \sum_{k=1}^n T7(k) \cdot |x_k(t)| + \sum_{k=1}^n T8(k) \cdot |y_k(t)| \le 0.$$

By similar arguments used in the proof of Theorem 1, it is easy to see that the equilibrium point is unique.

APPENDIX C: PROOF OF THEOREM 4

Proof: Recall the definition of $x_j(t)$ and $y_j(t)$ defined in (24), we can define a Lyapunov function as

$$L_{2}(t) = \sum_{j=1}^{n} \xi_{j} x_{j}^{2}(t) + \sum_{j=1}^{n} \phi_{j} y_{j}^{2}(t) + \sum_{j,k=1}^{n} \alpha'_{jk} e^{2\epsilon\tau_{jk}} \int_{t-\tau_{jk}}^{t} x_{k}^{2}(s) ds + \sum_{j,k=1}^{n} \beta'_{jk} e^{2\epsilon\tau_{jk}} \int_{t-\tau_{jk}}^{t} y_{k}^{2}(s) ds,$$

where

$$\begin{split} \alpha'_{jk} &= \sum_{k=1}^{n} \xi_{j} (|b_{jk}^{R}| \mu_{k}^{RR} + |b_{jk}^{I}| \mu_{k}^{IR}) \pi 3_{jk}^{-1} \\ &+ \sum_{k=1}^{n} \phi_{j} (|b_{jk}^{R}| \mu_{k}^{IR} + |b_{jk}^{I}| \mu_{k}^{RR}) \omega 3_{jk}^{-1}; \\ \beta'_{jk} &= \sum_{k=1}^{n} \xi_{j} (|b_{jk}^{R}| \mu_{k}^{RI} + |b_{jk}^{I}| \mu_{k}^{II}) \pi 4_{jk}^{-1} \\ &+ \sum_{k=1}^{n} \phi_{j} (|b_{jk}^{R}| \mu_{k}^{II} + |b_{jk}^{I}| \mu_{k}^{RI}) \omega 4_{jk}^{-1}. \end{split}$$

Differentiating $L_2(t)$ along equations (25) and (26), using some calculations (the details are left to interested readers), one can get that

$$\dot{L}_2(t) \le \sum_{j=1}^n T13(j)x_j^2(t) + \sum_{j=1}^n T14(j)y_j^2(t) \le 0$$

By similar arguments used in the proof of Theorem 1, it is easy to see that the equilibrium point is unique.

REFERENCES

- J. J. Hopfield, "Neural networks and physical systems with emergent collective computational abilities," *Proc. Nat. Academy Sci.*, vol. 79, no. 8, pp. 2554-2558, 1982.
- [2] M. A. Cohen and S. Grossberg, "Absolute stability of global pattern formation and parallel memory storage by competitive neural networks," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 13, no. 5, pp. 815-826, Sep. 1983.
- [3] L. O. Chua and L. Yang, "Cellular Neural Networks: Theory," *IEEE Trans. Circuits Syst.*, vol. 35, no. 10, pp. 1257-1272, Oct. 1988.
- [4] L. O. Chua and L. Yang, "Cellular Neural Networks: Applications" IEEE Trans. Circuits Syst., vol. 35, no. 10, pp. 1273-1290, Oct. 1988.
- [5] L. Gopalsamy and X. Z. He, "Stability in asymmetric Hopfield nets with transmission delays," *Physica D*, vol. 76, no. 4, pp. 344-358, 1994.
- [6] M. Forti, "On global asymptotic stability of a class of nonlinear systems arising in neural network theory," J. Differential Equ., vol. 113, no. 1, pp. 246-264, 1994.
- [7] M. Forti and A. Tesi, "New conditions for global stability of neural networks with application to linear and quadratic programming problems," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 42, no. 7, pp. 354-366, Jul. 1995.
- [8] T. P. Chen and S. Amari, "Exponential convergence of delayed dynamical systems," *Neural Comput.*, vol. 13, no. 3, pp. 621-635, 2001.
- [9] T. P. Chen and S. Amari, "Stability of asymmetric Hopfield networks," *IEEE Trans. Neural Netw.*, vol. 12, no. 1, pp. 159-163, 2001.
- [10] T. P. Chen, "Global exponential stability of delayed Hopfield neural networks," *Neural Netw.*, vol. 14, pp. 977-980, 2001.
- [11] T. P. Chen and L. L. Wang, "Global μ-stability of delayed neural networks with unbounded time-varying delays," *IEEE Trans. Neural Netw.*, vol. 18, no. 6, pp. 1836-1840, Nov. 2007.

- [12] B. Liu, W. L. Lu, and T. P. Chen, "Generalized halanay inequalities and their applications to neural networks with unbounded time-varying delays," *IEEE Trans. Neural Netw.*, vol. 22, no. 9, pp. 1508-1513, Sep. 2011.
- [13] M. Forti and A. Tesi, "Absolute stability of analytic neural networks: An approach based on finite trajectory length," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 51, no. 12, pp. 2460-2469, Dec. 2004.
- [14] T. P. Chen and L. B. Rong, "Delay-independent stability analysis of Cohen-Grossberg neural networks," *Phys. Lett. A*, vol. 317, nos. 5-6, 436-449, 2003.
- [15] T. P. Chen and L. B. Rong, "Robust global exponential stability of Cohen-Grossberg neural networks with time delays," *IEEE Trans. Neural Netw.*, vol. 15, no. 1, pp. 203-206, Jan. 2004.
- [16] T. P. Chen and L. L. Wang, "Power-rate global stability of dynamical systems with unbounded time-varying delays," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 54, no. 8, pp. 705-709, Aug. 2007.
- [17] W. L. Lu and T. P. Chen, "Synchronization of coupled connected neural networks with delays," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 51, no. 12, pp. 2491-2503, Dec. 2004.
- [18] W. L. Lu and T. P. Chen, "Dynamical behaviors of Cohen-Grossberg neural networks with discontinuous activation functions," *Neural Netw.*, vol. 18, no. 3, pp. 231-242, 2005.
- [19] W. L. Lu and T. P. Chen, "Dynamical behaviors of delayed neural network systems discontinuous activation functions," *Neural Comput.*, vol. 18, no. 3, pp. 683-708, 2006.
- [20] H. G. Zhang, Z. S. Wang, and D. R. Liu, "A comprehensive review of stability analysis of continuous-time recurrent neural networks," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 25, no. 7, pp. 1229-1262, Jul. 2014.
- [21] R. Manasevich, J. Mawhin, and F. Zanolin, "Periodic solutions of complex-valued differential equations and systems with periodic coefficients," J. Differ. Equations, vol. 126, no. 2, pp. 355-373, 1996.
- [22] G. M. Mahmoud and E. E. Mahmoud, "Lag synchronization of hyperchaotic complex nonlinear systems," *Nonlinear Dynam.*, vol. 67, no. 2, pp. 1613-1622, 2012.
- [23] H. Zhang, X. Y. Wang, and X. H. Lin, "Combination synchronisation of different kinds of spatiotemporal coupled systems with unknown parameters," *IET Control Theory Appl.*, vol. 8, no. 7, pp. 471-478, May 2014.
- [24] S. Jankowski, A. Lozowski, and J. M. Zurada, "Complex-valued multistate neural associative memory," *IEEE Trans. Neural Netw.*, vol. 7, no. 6, pp. 1491-1496, Nov. 1996.
- [25] A. Hirose, Complex-Valued Neural Networks: Theories and Applications. Singapore: World Scientific, 2003.
- [26] A. Hirose, "Recent progress in applications of complex-valued neural networks," in *Proc. 10th Int. Conf. Artif. Intell. Soft Comput.*, Jun. 2010, pp. 42-46.
- [27] A. Hirose, Complex-Valued Neural Networks, 2nd ed. Heidelberg, Berlin: Springer-Verlag, 2012.
- [28] I. Aizenberg, "MLMVN with soft margins learning," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 25, no. 9, pp. 1632-1644, Sep. 2014.
- [29] T. B. Ding and A. Hirose, "Fading channel prediction based on combination of complex-valued neural networks and chirp Z-transform," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 25, no. 9, pp. 1686-1695, Sep. 2014.
- [30] D. L. Lee, "Improving the capacity of complex-valued neural networks with a modified gradient descent learning rule," *IEEE Trans. Neural Netw.*, vol. 12, no. 2, pp. 439-443, Mar. 2001.
- [31] D. L. Lee, "Relaxation of the stability condition of the complex-valued neural networks," *IEEE Trans. Neural Netw.*, vol. 12, no. 5, pp. 1260-1262, Sep. 2001.
- [32] D. L. Lee, "Improvements of complex-valued Hopfield associative memory by using generalized projection rules," *IEEE Trans. Neural Netw.*, vol. 17, no. 5, pp. 1341-1347, Sep. 2006.
- [33] S. L. Goh and D. P. Mandic, "A complex-valued RTRL algorithm for recurrent neural networks," *Neural Comput.*, vol. 16, no. 12, pp. 2699-2713, 2004.
- [34] S. L. Goh and D. P. Mandic, "An augmented extended Kalman filter algorithm for complex-valued recurrent neural networks," *Neural Comput.*, vol. 19, pp. 1039-1055, 2007.
- [35] M. Bohner, V. S. H. Rao, and S. Sanyal, "Global stability of complexvalued neural networks on time scales," *Differ. Equ. Dyn. Syst.*, vol. 19, nos. 1-2, pp. 3-11, 2011.
- [36] W. Zhou and J. M. Zurada, "Discrete-time recurrent neural networks with complex-valued linear threshold neurons," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 56, no. 8, pp. 669-673, Aug. 2009.

- [37] C. J. Duan and Q. K. Song, "Boundedness and stability for discrete-time delayed neural network with complex-valued linear threshold neurons," *Discrete Dyn. Nature Soc.*, vol. 2010, pp. 1-19, 2010.
- [38] M. Mohamad, T. Werner, and L. Juegen, "Local stability analysis of discrete-time, continuous-state, complex-valued recurrent neural networks with inner state feedback," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 25, no. 4, pp. 830-836, Apr. 2014.
- [39] V. S. H. Rao and G. R. Murthy, "Global dynamics of a class of complex valued neural networks," *Int. J. Neural Syst.*, vol. 18, no. 2, pp. 165-171, 2008.
- [40] T. Fang and J. T. Sun, "Stability analysis of complex-valued impulsive system," *IET Control Theory Appl.*, vol. 7, no. 8, pp. 1152-1159, 2013.
- [41] N. Özdemir, B. B. İskender, and N. Y. Özgür, "Complex valued neural network with Möbius activation function," *Commun. Nonlinear Sci. Numer. Simulat.*, vol. 16, pp. 4698-4703, 2011.
- [42] X. Y. Liu, K. L. Fang, and B. Liu, "A synthesis method based on stability analysis for cocmplex-valued Hopfield neural networks," in *Proc. 7th Asian Control Conf.*, 2009, pp. 1245-1250.
- [43] H. Zhang, X. Y. Wang, X. H. Lin, and C. X. Liu, "Stability and synchronization for discrete-time complex-valued neural networks with time-varying delays," *Plos one*, e93838, 2014.
- [44] T. Fang and J. T. Sun, "Stability of complex-valued impulsive system with delay," *Appl. Math. Comput.*, vol. 240, pp. 102-108, 2014.
- [45] J. Hu and J. Wang, "Global stability of complex-valued recurrent neural networks with time-delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 23, no. 6, pp. 853-865, Jun. 2012.
- [46] Z. Y. Zhang, C. Lin, and B. Chen, "Global stability criterion for delayed complex-valued recurrent neural networks," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 25, no. 9, pp. 1704-1708, Sep. 2014.
- [47] T. Fang and J. T. Sun, "Further investigate the stability of complexvalued recurrent neural networks with time-delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 25, no. 9, pp. 1709-1713, Sep. 2014.
- [48] D. Helbing, "Traffic and related self-driven many-particle systems," *Rev. Modern Phys.*, vol. 73, no. 4, pp. 1067-1141, 2001.
- [49] A. Bose and P. A. Ioannou, "Analysis of traffic flow with mixed manual and semiautomated vehicles," *IEEE Trans. Intell. Transport. Syst.*, vol. 4, no. 4, pp. 173-188, Dec. 2003.
- [50] J. Chiasson, Z. Tang, J. Ghanem, C. T. Abdallah, J. D. Birdwell, M. M. Hayat, and H. Jerez, "The effects of time delay systems on the stability of load balancing algorithms for parallel computations," *IEEE Trans. Contr. Syst. Technol.*, vol. 13, no. 6, pp. 932-942, Nov. 2005.
- [51] R. Sipahi, S. I. Niculescu, C. T. Abdallah, W. Michiels, and K. Q. Gu, "Stability and stabilization of systems with time delay limitations and opportunities," *IEEE Control Syst. Mag.*, vol. 31, no. 1, pp. 38-65, Feb. 2011.
- [52] W. Zhang, C. D. Li, and T. W. Huang, "Global robust stability of complex-valued recurrent neural networks with time-delays and uncertainties," *Int. J. Biomath.*, vol. 7, no. 2, 1450016, 2014.
- [53] X. H. Xu, J. Y. Zhang, and J. Z. Shi, "Exponential stability of complexvalued neural networks with mixed delays," *Neurocomputing*, vol. 128, pp. 483-490, 2014.