Abstract—This paper mainly investigates consensus problem with pull-based event-triggered feedback control. For each agent, the diffusion coupling feedbacks are based on the states of its in-neighbors at its latest triggering time and the next triggering time of this agent is determined by its in-neighbors' information as well. The general directed topologies, including irreducible and reducible cases, are investigated. The scenario of distributed continuous communication is considered firstly. It is proved that if the network topology has a spanning tree, then the event-triggered coupling strategy can realize consensus for the multi-agent system. Then the results are extended to discontinuous communication, i.e., self-triggered control, where each agent computes its next triggering time in advance without having to observe the system's states continuously. The effectiveness of the theoretical results are illustrated by a numerical example finally.

Systems with Directed Topologies

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Keywords: Directed, irreducible, reducible, consensus, multiagent systems, event-triggered, self-triggered.

I. INTRODUCTION

Consensus problem in multi-agent systems has been widely and deeply investigated. The basic idea of consensus lies in that each agent updates its state based on its own state and the states of its neighbors in such a way that the final states of all agents converge to a common value [1]. The model normally is of the following form:

$$\dot{x}(t) = -Lx(t) \tag{1}$$

where the column vector x(t) consists of all nodes' states and L is the corresponding weighted Laplacian matrix. There are many results reported in this field [1]-[4] and the references therein. In these researches, the network topologies vary from fixed topologies to stochastically switching topologies, and the most basic condition to realize a consensus is that the underlying graph of the network system has a spanning tree.

In recent years, with the development of sensing, communications, and computing equipment, event-triggered control [5]-[9] and self-triggered control [10]-[14] have been proposed and studied. Instead of using the continuous state to realize a consensus, the control in event-triggered control strategy is

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piecewise constant between the triggering times which need to be determined. Self-triggered control is a natural extension of the event-triggered control since the derivative of the concern multi-agent system's state is piecewise constant, which is very easy to work out solutions (agents' states) of the system. In particular, each agent predicts its next triggering time at the previous one. Inspired by above idea of event-triggered control and self-triggered control, [17]-[25] considered the consensus problem for multi-agent systems with event-triggered control. In particular, in [17], under the condition that the graph is undirected and strongly connected, the authors provide eventtriggered and self-triggered approaches in both centralized and distributed formulations. It should be emphasized that the approaches cannot be applied to directed graph. In [18], the authors investigate the average-consensus problem of multiagent systems with directed and weighted topologies, but they need an additional assumption that the directed topology must be balanced. In [20], the authors propose a new combinational measurement approach to event design, which will be used in this paper.

In this paper, continuing with previous works, we study event-triggered and self-triggered consensus in multi-agent system with directed, reducible (irreducible) and weighted topology.

Consider the following continuous-time linear multi-agent system with discontinuous diffusions as follows

$$\begin{cases} \dot{x}_i(t) = u_i(t) \\ u_i(t) = -\sum_{j=1}^m L_{ij} x_j(t^i_{k_i(t)}), \ i = 1, \cdots, m \end{cases}$$
(2)

where $k_i(t) = arg \max_k \{t_k^i \leq t\}$, the increasing time sequence $\{t_k^j\}_{k=1}^{\infty}$, $j = 1, \dots, m$, which is named as trigger *times*, is agent-wise and normally assuming $t_1^j = 0$, for all $j \in \mathcal{I}$, where $\mathcal{I} = \{1, 2, \cdots, m\}$. We say agent v_i triggers at time t_k^i means agent v_i renews its control value at time t_k^i and sends t_k^i , $x_i(t_k^i)$ and $u_i(t_k^i)$ to all its out-neighbours immediately. At each t_k^i , each agent v_i "pulls" $x_j(t_k^i)$ from agent v_i if $L_{ij} \neq 0$. (This does not mean that agent v_i has to send a request to its in-neighbours at t_k^i in order to get its in-neighbours' states at t_k^i . Instead, in event-triggered control, agent v_i 's in-neighbours has to send its state to agent v_i continuously. And we will also give an algorithm to avoid such continuous communication later.) In order to distinguish it from others, we name this sort of feedback as *pull-based*.

Let us recall the model

$$x^{i}(t+1) = f(x^{i}(t)) + c_{i} \sum_{j=1}^{m} a_{ij}(f(x^{j}(t)))$$

where $\dot{s}(t) = f(s(t))$ is a chaotic oscillator. It was proposed and investigated in [15] for synchronization of chaotic systems. It can also be considered as nonlinear consensus model.

As a special case, let f(x(t)) = x(t) and $c_i = (t_{k+1}^i - t_k^i)$, then

$$x^{i}(t_{k+1}^{i}) = x^{i}(t_{k}^{i}) + (t_{k+1}^{i} - t_{k}^{i}) \sum_{j=1}^{m} a_{ij} x^{j}(t_{k}^{j})$$

which is just the event triggering (distributed) model for consensus problem, though the term "event triggering" was not used. In centralized control, the bound for $(t_{k+1}^i - t_k^i) = (t_{k+1} - t_k)$ to reach synchronization was given in that paper when the coupling graph is indirected (or in [16] for direct graph), too.

In this paper, the distributed continuous monitoring with pull-based feedback as the event-triggered controller is considered firstly, namely agent can observe its in-neighbours' continuous states by its in-neighbours sending their continuous states to it. It is proved that if the directed network topology is irreducible, then the pull-based event-triggered coupling strategy can realize consensus for the multi-agent system. Then we generalize it to the reducible case. By mathematical induction, it is proved that if the network topology has a spanning tree, then the pull-based event-triggered coupling strategy can realise consensus for the multi-agent system, too. Finally the results are extended to discontinuous monitoring, where each agent computes its next triggering time in advance without having to receive the system's state continuously (selftriggered).

In comparison to literature, we have three main contributions: (i) different from [17]-[22] and [25], we investigate directed topologies, including irreducible and reducible cases, and we do not make assumption that they are balanced; (ii) different from [19], [22] and [23], the event-triggered principles in our paper are fully distributed in the sense that each agent only needs its in-neighbours' state information, especially does not need any a priori knowledge of any global parameter and the Zeno behaviour can be excluded; (iii) different from [18]-[23], we propose self-triggered principle, by which continuous communication between agents can be avoided.

The paper is organized as follows: in Section II, some necessary definitions and lemmas are given; in Section III, the pull-based event-triggered consensus in multi-agent systems with directed topologies is discussed; in Section IV, the selftriggered formulation of the frameworks provided in Section III is presented; in Section V, one numerical example is provided to show the effectiveness of the theoretical results; the paper is concluded in Section VI.

II. PRELIMINARIES

In this section we first review some relating notations, definitions and results on algebraic graph theory [26], [27] which will be used later in this paper.

Notions: $\|\cdot\|$ represents the Euclidean norm for vectors or the induced 2-norm for matrices. 1 denotes the column vector with each component 1 with proper dimension. $\rho(\cdot)$ stands for the

spectral radius for matrices and $\rho_2(\cdot)$ indicates the minimum positive eigenvalue for matrices having positive eigenvalues. Given two symmetric matrices M, N, M > N (or $M \ge N$) means M - N is a positive definite (or positive semi-definite) matrix.

For a weighted directed graph (or digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with m agents (vertices or nodes), the set of agents $\mathcal{V} = \{v_1, \cdots, v_m\}$, set of links (edges) $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and the weighted adjacency matrix $\mathcal{A} = (a_{ij})$ with nonnegative adjacency elements $a_{ij} > 0$. A link of \mathcal{G} is denoted by $e(i, j) = (v_i, v_j) \in \mathcal{E}$ if there is a directed link from agent v_j to agent v_i with weight $a_{ij} > 0$, i.e. agent v_j can send information to agent v_i while the opposite direction transmission might not exist or with different weight a_{ji} . The adjacency elements associated with the links of the graph are positive, i.e., $e(i, j) \in \mathcal{E} \iff a_{ij} > 0$, for all $i, j \in \mathcal{I}$. It is assumed that $a_{ii} = 0$ for all $i \in \mathcal{I}$. Moreover, the in- and outneighbours set of agent v_i are defined as

$$N_i^{in} = \{ v_j \in \mathcal{V} \mid a_{ij} > 0 \}, \quad N_i^{out} = \{ v_j \in \mathcal{V} \mid a_{ji} > 0 \}$$

The in- and out- degree of agent v_i are defined as follows:

$$deg^{in}(v_i) = \sum_{j=1}^{m} a_{ij}, \quad deg^{out}(v_i) = \sum_{j=1}^{m} a_{ji}$$

The degree matrix of digraph \mathcal{G} is defined as $D = diag[deg^{in}(v_1), \cdots, deg^{in}(v_m)]$. The weighted Laplacian matrix associated with the digraph \mathcal{G} is defined as $L = D - \mathcal{A}$. A directed path from agent v_0 to agent v_k is a directed graph with distinct agents $v_0, ..., v_k$ and links $e_0, ..., e_{k-1}$ such that e_i is a link directed from v_i to v_{i+1} , for all i < k.

Definition 1: We say a directed graph \mathcal{G} is strongly connected if for any two distinct agents v_i , v_j , there exits a directed path from v_i to v_j .

By [27], we know that strongly connectivity of \mathcal{G} is equivalent to the corresponding Laplacian matrix L is irreducible.

Definition 2: We say a directed graph \mathcal{G} has a spanning tree if there exists at least one agent v_{i_0} such that for any other agent v_j , there exists a directed path from v_{i_0} to v_j .

By Perron-Frobenius theorem [28] (for more detail and proof, see [29]), we have

Lemma 1: If L is irreducible, then rank(L) = m - 1, zero is an algebraically simple eigenvalue of L and there is a positive vector $\xi^{\top} = [\xi_1, \dots, \xi_m]$ such that $\xi^{\top}L = 0$ and $\sum_{i=1}^m \xi_i = 1$.

Let $\Xi = diag[\xi_1, \dots, \xi_m]$, by the results first given in [28], we have

Lemma 2: If L is irreducible, then $\Xi L + L^{\top}\Xi$ is a symmetric matrix with all row sums equal to zeros and has zero eigenvalue with algebraic dimension one.

Here we define some matrices, which will be used later. Let $R = [R_{ij}]_{i,j=1}^{m}$, where

$$R = \frac{1}{2} (\Xi L + L^{\top} \Xi)$$

Obviously, R is positive semi-definite. Denote the eigenvalue of R by $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_m$, counting the multiplicities.

We also denote

$$U = \Xi - \xi \xi^{\top}$$

It can also be seen that U has a simple zero eigenvalue and its eigenvalues (counting the multiplicities) can be arranged as $0 = \mu_1 < \mu_2 \le \cdots \le \mu_m$. We also denote the eigenvalues of $L^T L$ by $0 = \gamma_1 < \gamma_2 \le \cdots \le \gamma_m = \rho(L^\top L)$. Then, for all $x \in R^m$ satisfying $x \perp 1$, we have

$$\lambda_2 x^\top x \le x^\top R x$$

and

$$x^{\top}UUx \le \mu_m^2 x^{\top}x$$

Therefore, we have

$$R \ge \frac{\lambda_2}{\mu_m^2} U U \tag{3}$$

$$L^T L \ge \frac{\gamma_2}{\mu_m^2} U U \tag{4}$$

and

$$\frac{\lambda_m}{\gamma_2} L^\top L \ge R \ge \frac{\lambda_2}{\rho(L^\top L)} L^\top L \tag{5}$$

Pick weight function $\mu(t) > 0$ satisfying $\frac{\dot{\mu}(t)}{\mu(t)} \leq \beta$.

III. PULL-BASED EVENT-TRIGGERED PRINCIPLES

In this section, we consider event-triggered control for multi-agent systems with directed and weighted topology.

Firstly, we consider the case of irreducible *L*. Denote $q(t) = [q_1(t), \dots, q_m(t)]^\top$, where $q_i(t) = -\sum_{j=1}^m L_{ij}x_j(t)$ and $f(t) = [f_1(t), \dots, f_m(t)]^\top$, where $f_i(t) = q_i(t_k^i) - q_i(t), t \in [t_k^i, t_{k+1}^i), k = 1, 2, \dots$

To depict the trigger event, consider the following candidate Lyapunov function (see [28]):

$$V(t) = \frac{1}{2} \sum_{i=1}^{m} \xi_i (x_i(t) - \bar{x}(t))^2 = \frac{1}{2} x^{\top}(t) U x(t) \qquad (6)$$

where $\bar{x}(t) = \sum_{i=1}^{m} \xi_i x_i(t)$.

By the definition, we have

$$\sum_{i=1}^{m} \xi_i(x_i(t) - \bar{x}(t)) = 0$$
(7)

and due to $\xi^{\top}L = 0$, we have

$$\sum_{i=1}^{m} \xi_i L_{ij} x_j(t) = 0$$
(8)

The derivative of V(t) along (2) is

$$\frac{d}{dt}V(t) = \sum_{i=1}^{m} \xi_i(x_i(t) - \bar{x}(t))(\dot{x}_i(t) - \dot{\bar{x}}(t))$$
$$= \sum_{i=1}^{m} \xi_i(x_i(t) - \bar{x}(t))\dot{x}_i(t)$$
$$= -\sum_{i=1}^{m} \xi_i(x_i(t) - \bar{x}(t))\sum_{j=1}^{m} L_{ij}x_j(t_k^i)$$

$$=\sum_{i=1}^{m} \xi_{i}(x_{i}(t) - \bar{x}(t))q_{i}(t_{k}^{i})$$

$$=\sum_{i=1}^{m} \xi_{i}(x_{i}(t) - \bar{x}(t)) \{f_{i}(t) + q_{i}(t)\}$$

$$=\sum_{i=1}^{m} \xi_{i}(x_{i}(t) - \bar{x}(t))[f_{i}(t) - \sum_{j=1}^{m} L_{ij}x_{j}(t)]$$

$$= -\sum_{i=1}^{m} \sum_{j=1}^{m} x_{i}(t)\xi_{i}L_{ij}x_{j}(t) + \sum_{i=1}^{m} \xi_{i}(x_{i}(t) - \bar{x}(t))f_{i}(t)$$

$$= -x^{\top}(t)Rx(t) + x^{\top}(t)Uf(t)$$

$$\leq -x^{\top}(t)Rx(t) + \frac{a}{2}x^{\top}(t)UUx(t) + \frac{1}{2a}f^{\top}(t)f(t)$$

$$\leq -(1 - \frac{a\mu_{m}^{2}}{2\lambda_{2}})x^{\top}(t)Rx(t) + \frac{1}{2a}f^{\top}(t)f(t)$$
(9)

By (5), we have

$$\frac{a}{dt}V(t) \leq -(1-\frac{a\mu_m^2}{2\lambda_2})\frac{\lambda_2}{\rho(L^{\top}L)}x^{\top}(t)L^{\top}Lx(t) + \frac{1}{2a}f^{\top}(t)f(t) \\
= -(1-\frac{a\mu_m^2}{2\lambda_2})\frac{\lambda_2}{\rho(L^{\top}L)}q^{\top}(t)q(t) + \frac{1}{2a}f^{\top}(t)f(t) \\
= \sum_{i=1}^m [-(1-\frac{a\mu_m^2}{2\lambda_2})\frac{\lambda_2}{\rho(L^{\top}L)}q_i^2(t) + \frac{1}{2a}(q_i(t_k^i) - q_i(t))^2] \tag{10}$$

and

$$\frac{d[\mu(t)V(t)]}{dt} = \mu(t)\dot{V}(t) + \dot{\mu}(t)V(t)
\leq \sum_{i=1}^{m} \mu(t) \left\{ \left[-(1 - \frac{a\mu_m^2}{2\lambda_2})\frac{\lambda_2}{\rho(L^{\top}L)} + \frac{\gamma_2\dot{\mu}(t)}{\mu_m\mu(t)} \right] q_i^2(t)
+ \frac{1}{2a}(q_i(t_k^i) - q_i(t))^2 \right\}$$
(11)

Therefore, we have

Theorem 1: Suppose that \mathcal{G} is strongly connected. $\frac{\dot{\mu}(t)}{\mu(t)} \leq \beta(t)$. For $i = 1, \dots, m$, set $0 < a < \frac{2\lambda_2}{\mu_{\infty}^2}$, and

$$b(t) = \left(1 - \frac{a\mu_m^2}{2\lambda_2}\right)\frac{\lambda_2}{\rho(L^\top L)} - \frac{\gamma_2\beta(t)}{\mu_m} > 0$$

$$t_{k+1}^{i} = \max_{\tau \ge t_{k}^{i}} \left\{ \tau : \left| q_{i}(t_{k}^{i}) - q_{i}(t) \right| \\ \le \sqrt{2ab(t)} \left| q_{i}(t) \right|, \ \forall t \in [t_{k}^{i}, \tau] \right\}$$
(12)

Then, system (2) reaches a consensus

$$x_i(t) - \sum_{j=1}^m \xi_j x_j(t) = O\left(\mu^{-1/2}(t)\right)$$
(13)

In addition, for all $i \in \mathcal{I}$, we have and

$$\dot{x}_i(t) = O\left(\mu^{-1/2}(t)\right) \tag{14}$$

Proof: Combining inequalities (10), (15) and (5), we have

$$\frac{d[\mu(t)V(t)]}{dt} \le 0$$

for all $t \ge 0$. It means

$$V(t) \le \mu(0)\mu^{-1}(t)$$

This implies that system (2) reaches consensus and for all $i = 1, \dots, m$,

$$x_i(t) - \sum_{j=1}^m \xi_j x_j(t) = O\left(\mu^{-1/2}(t)\right)$$

and

$$\dot{x}_i(t) = -\sum_{j=1}^m L_{ij} \left(x_j(t_{k_i(t)}^i) - \bar{x}(t_{k_i(t)}^i) \right) = O\left(\mu^{-1/2}(t)\right)$$

This completes the proof.

As special cases, we have

Corollary 1: Suppose that \mathcal{G} is strongly connected. $\mu(t) = e^{\beta t}$. For $i = 1, \dots, m$, set $0 < a < \frac{2\lambda_2}{\mu_m^2}$, and

$$b = \left(1 - \frac{a\mu_m^2}{2\lambda_2}\right) \frac{\lambda_2}{\rho(L^\top L)} - \frac{\gamma_2 \beta}{\mu_m} > 0$$
$$t_{k+1}^i = \max_{\tau \ge t_k^i} \left\{ \tau : \left| q_i(t_k^i) - q_i(t) \right| \\ \le \sqrt{2ab} \left| q_i(t) \right|, \ \forall t \in [t_k^i, \tau] \right\}$$
(15)

Then, system (2) reaches a consensus

$$x_{i}(t) - \sum_{j=1}^{m} \xi_{j} x_{j}(t) = O\left(e^{-\beta t/2}\right)$$
(16)

In addition, for all $i \in \mathcal{I}$, we have

$$\dot{x}_i(t) = O(e^{-\beta t/2}) \tag{17}$$

Corollary 2: Suppose that \mathcal{G} is strongly connected. Set

$$t_{k+1}^{i} = \max_{\tau \ge t_{k}^{i}} \left\{ \tau : \left| q_{i}(t_{k}^{j}) - q_{i}(t) \right| \\ \le c \left| q_{i}(t) \right|, \ \forall t \in [t_{k}^{i}, \tau] \right\}$$
(18)

or

$$t_{k+1}^{i} = \max_{\tau \ge t_{k}^{i}} \left\{ \tau : \frac{|q_{i}(t_{k}^{j})|}{1+c} \le \left| q_{i}(t) \right| \\ \le \frac{|q_{i}(t_{k}^{j})|}{1-c}, \ \forall t \in [t_{k}^{i}, \tau] \right\}$$
(19)

for some sufficient small constant c. Then, system (2) reaches a consensus.

Theorem 1 shows such a constant c does exist.

Remark 1: To utilize event-triggering algorithm, two issues should be addressed. Firstly, for any initial condition, at any time $t \ge 0$, under the condition and the event-triggered principle in Theorem 1, there exists at least one agent v_{j_1} , of which the next inter-event time is strictly positive before consensus is reached.

In fact, suppose that there is no trigger event when t > T. Then, we have

$$\dot{x}_i(t) = \sum_{j=1}^m L_{ij} x_j(T^i_{k_i(T)}), \ t > T, \ i = 1, \cdots, m$$
(20)

which implies

$$x_i(t) - x_i(T) = (t - T) \sum_{j=1}^m L_{ij} x_j(T^i_{k_i(T)}).$$

By Theorem 1, we have $x_{i_1}(t) - x_{i_2}(t) \rightarrow 0$. Therefore, for all $i_1, i_2 = 1, \dots, m$, we have

$$\sum_{j=1}^{m} L_{i_1 j} x_j(T_{k_{i_1}(T)}^{i_1}) = \sum_{j=1}^{m} L_{i_2 j} x_j(T_{k_{i_2}(T)}^{i_2})$$

and

$$x_{i_1}(T) = x_{i_2}(T)$$

which implies $x_{i_1}(t) = x_{i_2}(t)$ for all $t \ge T$ and $i_1, i_2 = 1, \dots, m$. It means that in case there is no triggering time for t > T, the consensus has reached at time T.

Secondly, it should be addressed that in any finite interval $[t_1, t_2, t_2]$, there are only finite triggers. It would be discussed in the following algorithms.

In the following, we propose another event-triggering setting. Denote $\delta x_{i}(t) = x_{i}(t) - \bar{x}(t)$ and rewrite

Denote
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$$\frac{d}{dt}V(t) = \sum_{i=1}^m \xi_i(x_i(t) - \bar{x}(t))[f_i(t) - \sum_{j=1}^m L_{ij}x_j(t)]$$

$$= -\sum_{i=1}^m \sum_{j=1}^m \delta x_i(t)\xi_i L_{ij}\delta x_j(t) + \sum_{i=1}^m \xi_i\delta x_i(t)f_i(t)$$

$$\leq (-\frac{\lambda_2}{\max\{\xi_i\}} + \frac{a}{2}) \sum_{i=1}^m \xi_i (\delta x_i(t))^2 + \frac{1}{2a} \sum_{i=1}^m \xi_i (f_i(t))^2$$
(21)

and

$$\frac{d[\mu(t)V(t)]}{dt} \leq \left(-\frac{\lambda_2}{\max\{\xi_i\}} + \frac{a}{2} + \frac{\dot{\mu}(t)}{\mu(t)}\right)\mu(t)V(t) + \frac{\mu(t)}{2a}\sum_{i=1}^m \xi_i(f_i(t))^2$$
(22)

Firstly, we give a simple lemma.

Lemma 3: Suppose a function $V_1(t)$ with $V_1(0) > 0$ and satisfies

$$\dot{V}_1(t) \le -c_1 V_1(t) + c_2$$

for some constants $c_1 > 0$ and $c_2 > 0$. Then, $V_1(t)$ is bounded. In fact, if $V_1(t) > c_2/c_1$, then $\dot{V}_1(t) < 0$.

Theorem 2: Suppose that \mathcal{G} is strongly connected, function $\mu(t) > 0$ satisfies $\dot{\mu}(t) \leq \beta \mu(t)$ for some $\beta > 0$ and

$$-\frac{\lambda_2}{\max\{\xi_i\}} + \frac{a}{2} + \beta < 0$$

for some small numbers a and β . For agent v_i , if trigger times $t_1^i = 0, \dots, t_k^i$ are known, then use the following trigger strategy to find t_{k+1}^i :

$$t_{k+1}^{i} = \max\left\{\tau \ge t_{k}^{i}: |q_{i}(t_{k}^{i}) - q_{i}(t)|^{2} \le \mu^{-1}(t), \forall t \in [t_{k}^{i}, \tau] \right.$$
(23)

Then, system (2) reaches consensus

$$|x_i(t) - \sum_{j=1}^m \xi_j x_j(t)| \le \frac{\mu^{-1/2}(t)}{\sqrt{2a\xi_i(\frac{\lambda_2}{\max\{\xi_i\}} - \frac{a}{2} - \beta)}}$$
(24)

In addition, for all $i \in \mathcal{I}$, we have

$$\dot{x}_i(t) = O(\mu^{-1/2}(t))$$
 (25)

and the Zeno behaviour could be excluded.

Proof: By previous derivations, it is clear that there are two constants $c_1 > 0$ and $c_2 > 0$ such that

$$\frac{d\{\mu(t)V(t)\}}{dt} \le -\left\{\frac{\lambda_2}{\max\{\xi_i\}} - \frac{a}{2} - \beta\right\}\mu(t)V(t) + \frac{1}{2a}$$

By Lemma 3, $\mu(t)V(t)$ are bounded. Therefore, for sufficient large t, we have

$$\mu(t)V(t) \le \frac{1}{2a(\frac{\lambda_2}{\max\{\xi_i\}} - \frac{a}{2} - \beta)}$$

and

$$V(t) \le \frac{\mu^{-1}(t)}{2a(\frac{\lambda_2}{\max\{\xi_i\}} - \frac{a}{2} - \beta)}$$

$$|x_i(t) - \bar{x}(t)| \le \frac{\mu^{-1/2}(t)}{\sqrt{2a\xi_i(\frac{\lambda_2}{\max\{\xi_i\}} - \frac{a}{2} - \beta)}}$$
(26)

In addition, for all $i \in \mathcal{I}$ and $t \in [t_{k_i}, t_{k_i+1}]$, we have

$$\begin{aligned} |\dot{x}_{i}(t)| &= |\sum_{j=1}^{m} L_{ij} \left(x_{j}(t_{k_{i}}^{i}) - \bar{x}(t_{k_{i}}^{i}) \right)| \\ &\leq 2L_{ii} \frac{\mu^{-1/2}(t_{k_{i}})}{\sqrt{2a\xi_{i}(\frac{\lambda_{2}}{\max\{\xi_{i}\}} - \frac{a}{2} - \beta)}} \end{aligned}$$

Furthermore, in any finite length interval [0,T], $||\dot{q}_i(t)||^2 \leq M$, and $\mu(t)$ is bounded. Thus, $M(t_{k+1}^i - t_k^i)^2 \geq |q_i(t_{k+1}^i) - q_i(t_k^i)|^2 = \mu^{-1}(t_{k+1}^i)$ is lower bounded. Then,

$$(t_{k+1}^i - t_k^i)^2 \ge \frac{\mu^{-1}(t_{k+1}^i)}{M} \ge \frac{\mu^{-1}(T)}{M}$$

Therefore, in any finite length interval [0,T], there are only finite triggers. It means that Zeno behavior is avoided.

Theorem 3: Suppose that \mathcal{G} is strongly connected, and

$$-\frac{\lambda_2}{\max\{\xi_i\}} + \frac{a}{2} + \beta < 0$$

for some small numbers a and β . For agent v_i , if trigger times $t_1^i = 0, \dots, t_k^i$ are known, then use the following trigger strategy to find t_{k+1}^i :

$$t_{k+1}^{i} = \max\left\{\tau \ge t_{k}^{i}: |q_{i}(t_{k}^{i}) - q_{i}(t)|^{2} \le e^{-\beta t}, \forall t \in [t_{k}^{i}, \tau]\right\}$$
(27)

Then, system (2) reaches consensus

$$|x_i(t) - \bar{x}(t)| \le \frac{e^{-\beta t/2}}{\sqrt{2a\xi_i(\frac{\lambda_2}{\max\{\xi_i\}} - \frac{a}{2} - \beta)}}$$
(28)

^J and the Zeno behaviour could be excluded; In addition,

$$\dot{x}_i(t) = O(e^{-\beta t/2})$$

Remark 2: (i) In Theorem 2, in order to determine the trigger times, each agent only needs its in-neighbours' state information, especially do not need any a priori knowledge of any global parameter. (ii) By a little more detail analysis, we can show that there exists a constant c such that for each agent $v_i, t_{k+1}^i - t_k^i \ge c > 0$. We omit detail proof here.

Remark 3: By picking different function $\mu(t)$, we can obtain different convergence rate. It can be seen that if $\mu(t)$ increases fast, then the interval $t_{k+1}^i - t_k^i$ can be larger, which means less triggers are needed. Instead, if $\mu(t)$ increases slowly, then the interval $t_{k+1}^i - t_k^i$ should be smaller, which means more triggers are needed.

Remark 4: The event-triggered principle used in Theorem 2 may be costly since each agent has to continuously send its state information to its out-neighbours. In the next section, we will give an algorithm to avoid this.

IV. DISTRIBUTED SELF-TRIGGERED PRINCIPLES

In this section, we extend the pull-based event-triggered principle discussed in **Section III** to self-triggered case in order to avoid continuous communication between agents.

Self-triggered approach means that one can predict next triggering time t_k^i based on the information at previous triggering time t_k^i .

Recall again the model

$$x^{i}(t_{k+1}^{i}) = x^{i}(t_{k}^{i}) + (t_{k+1}^{i} - t_{k}^{i}) \sum_{j=1}^{m} a_{ij} x^{j}(t_{k}^{j})$$

In centralized control, the bound for $(t_{k+1}^i - t_k^i) = (t_{k+1} - t_k)$ to reach consensus was given in that paper [15] when the coupling graph is indirected (or in [16] for direct graph), too. It means that the idea of self-triggering has been considered in these two papers.

For agent v_i , given $t_1^i = 0, \dots, t_k^i$, its state at $t \in [t_k^i, t_{k+1}^i]$ (t_{k+1}^i) to be determined) can be written as:

$$x_i(t) = x_i(t_k^i) + (t - t_k^i)q_i(t_k^i), \ t \in [t_k^i, t_{k+1}^i]$$
(29)

Since each agent $v_p \in N_i^{in}$ sends trigger information to agent v_i whenever agent v_p triggers, then at any given time point r, agent v_i can predict agent v_p 's state at time $t \ge r$ as

$$x_p(t) = x_p(t_{k_p(r)}^p) + (t - t_{k_p(r)}^p)q_p(t_{k_p(r)}^p)$$
(30)

until agent v_p ' next triggering after s.

Then, from Theorem 3, we have the following result

Theorem 4: Suppose that \mathcal{G} has spanning trees and L is written in the form of (32). For agent v_i , pick $\phi_i > 0$, $\alpha_i > 0$. If trigger times $t_1^i = 0, \dots, t_k^i$ are known, then use the following trigger strategy to find t_{k+1}^i :

1) At time $s = t_k^i$, substituting (29) and (30) into (40), and solve the following maximizing problem to find out τ_{k+1}^i :

$$\tau_{k+1}^{i} = \max\left\{\tau \ge s : |q_{i}(t_{k}^{i}) - q_{i}(t)| \le e^{-\beta t}, \forall t \in [s, \tau]\right\}$$
(31)

- 2) In case that some in-neighbours of agent v_i triggers at time $t_0 \in (s, \tau_{k+1}^i)$, i.e., agent v_i received the renewed information form some of its in-neighbours, then updating $s = t_0$ and go to step (1);
- 3) In case that any of v_i 's in-neighbours does not trigger during (s, τ_{k+1}^i) , then v_i triggers at time $t_{k+1}^i = \tau_{k+1}^i$. The agent v_i renews its state at $t = t_{k+1}^i$ and sends the renewed information, including t_{k+1}^i , $x_i(t_{k+1}^i)$ and $q_i(t_{k+1}^i)$, to all its out-neighbours immediately.

then, system (2) reaches consensus exponentially and the Zeno behaviour could be excluded.

Remark 5: Obviously, Theorem 4 can be regarded as an algorithm of Theorem 3, by which the continuous communications between different states can be avoided.

Secondly, we consider the case L is reducible. The following mathematical methods are inspired by [31]. By proper permutation, we rewrite L as the following Perron-Frobenius form:

$$L = \begin{bmatrix} L^{1,1} & L^{1,2} & \cdots & L^{1,K} \\ 0 & L^{2,2} & \cdots & L^{2,K} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L^{K,K} \end{bmatrix}$$
(32)

where $L^{k,k}$ is with dimension n_k and associated with the k-th strongly connected component (SCC) of \mathcal{G} , denoted by SCC_k , $k = 1, \dots, K$. Accordingly, define $x = [x^{1T}, \dots, x^{KT}]^T$, where $x^k = [x_1^k, \dots, x_{n_k}^k]^\top$.

For agent $v_i \in SCC_k$, i.e., $i = M_{k-1} + 1, \cdots, M_k$, where $M_0 = 0$, $M_k = \sum_{i=1}^k n_i$, denote the combinational state measurement $q_i^k(t) = -\sum_{j=M_{k-1}+1}^m L_{i+M_{k-1},j}x_j(t) = -\sum_{j=1}^m L_{i+M_{k-1},j}x_j(t) = q_{i+M_{k-1}}(t)$. And denote the combinational measurement error by $f_i^k(t) = q_i^k(t_i^{i+M_{k-1}}) - q_i^k(t)$ and $u_i^k(t) = q_i^k(t_l^{i+M_{k-1}})$, $t \in [t_l^{i+M_{k-1}}, t_{l+1}^{i+M_{k-1}}]$.

If \mathcal{G} has spanning trees, then each $L^{k,k}$ is irreducible or has one dimension and for each k < K, $L^{k,q} \neq 0$ for at least one q > k. Define an auxiliary matrix $\tilde{L}^{k,k} = [\tilde{L}^{k,k}_{ij}]_{i,j=1}^{n_k}$ as

$$\tilde{L}_{ij}^{k,k} = \begin{cases} L_{ij}^{k,k} & i \neq j \\ -\sum_{p=1, p \neq i}^{n_k} L_{ip}^{k,k} & i = j \end{cases}$$

Then, let $D^k = L^{k,k} - \tilde{L}^{k,k} = diag[D_1^k, \cdots, D_{n_k}^k]$, which is a diagonal semi-positive definite matrix and has at least one diagonal positive (nonzero).

Let $\xi^{k^{\perp}}$ be the positive left eigenvector of the irreducible $\tilde{L}^{k,k}$ corresponding to the eigenvalue zero and has the sum of components equaling to 1. Denote $\Xi^k = diag[\xi^k]$. By the structure, it can be seen that $R^k = \frac{1}{2}[\Xi^k \tilde{L}^{k,k} + (\Xi^k \tilde{L}^{k,k})^{\top}]$ has zero row sums and has zero eigenvalue with algebraic dimension one. Then, we have

Property 1: Under the setup above, $Q^k = \frac{1}{2} [\Xi^k L^{k,k} + (\Xi^k L^{k,k})^\top] = R^k + \Xi^k D^k$ is positive definite and $\Xi^k \leq \frac{\rho(\Xi^k)}{\rho_2(Q^k)} Q^k$ for all k < K.

And let $U^K = \Xi^K - \xi^K (\xi^K)^\top$ and in order to facilitate the presentation, also denote $U^k = \Xi^k$, $k = 1, \dots, K-1$.

Now we are going to determine the triggering times for the system (2) to reach consensus. Firstly, applying Theorem 1 to the K-th SCC, we can conclude that the K-th SCC can reach a consensus with the agreement value $\nu(t) = \sum_{p=1}^{n_K} \xi_p^K x_p^K(t)$ and $\lim_{t\to\infty} \dot{\nu}(t) = 0$ exponentially.

Then, inductively, consider the K-1-th SCC. We will prove that $\lim_{t\to\infty} |x_p^{K-1}(t) - \nu(t)| = 0$, for all $p = 1, \dots, n_{K-1}$. Construct a candidate Lyapunov function as follows

$$V_{K-1}(t) = \frac{1}{2} (x^{K-1}(t) - \nu(t)\mathbf{1})^{\top} \Xi^{K-1} (x^{K-1}(t) - \nu(t)\mathbf{1})$$
(33)

Differentiate $V_{K-1}(t)$ along (2), we have

$$\frac{d}{dt}V_{K-1}(t) = (x^{K-1}(t) - \nu(t)\mathbf{1})^{\top} \Xi^{K-1} \left\{ f^{K-1}(t) + q^{K-1}(t) - \dot{\nu}(t)\mathbf{1} \right\} = (x^{K-1}(t) - \nu(t)\mathbf{1})^{\top} \Xi^{K-1} \left\{ f^{K-1}(t) - \dot{\nu}(t)\mathbf{1} - L^{K-1,K-1}(x^{K-1}(t) - \nu(t)\mathbf{1}) - L^{K-1,K}(x^{K}(t) - \nu(t)\mathbf{1}) \right\} = (x^{K-1}(t) - \nu(t)\mathbf{1})^{\top} \Xi^{K-1} f^{K-1}(t) - [x^{K-1}(t) - \nu(t)\mathbf{1}]^{\top} \Xi^{K-1} L^{K-1,K-1}[x^{K-1}(t) - \nu(t)\mathbf{1}] - (x^{K-1}(t) - \nu(t)\mathbf{1})^{\top} \Xi^{K-1} \left\{ L^{K-1,K}(x^{K}(t) - \nu(t)\mathbf{1}) \right\} - (x^{K-1}(t) - \nu(t)\mathbf{1})^{\top} \Xi^{K-1} \left\{ \dot{\nu}(t)\mathbf{1} \right\} = W_{0}^{K-1}(t) - W_{1}^{K-1}(t) - W_{2}^{K-1}(t) - W_{3}^{K-1}(t) \quad (34)$$

where

$$W_0^{K-1}(t) = (x^{K-1}(t) - \nu(t)\mathbf{1})^\top \Xi^{K-1} f^{K-1}(t)$$

$$\leq a_{K-1} V_{K-1}(t) + \frac{1}{2a_{K-1}} \sum_{i=1}^{n_{K-1}} \xi_i^{K-1} [f_i^{K-1}]^2 \qquad (35)$$

with any $a_{K-1} > 0$,

$$W_{1}^{K-1}(t) = [x^{K-1}(t) - \nu(t)\mathbf{1}]^{\top} \Xi^{K-1} L^{K-1,K-1} [x^{K-1}(t) - \nu(t)\mathbf{1}]$$

$$= [x^{K-1}(t) - \nu(t)\mathbf{1}]^{\top} Q^{K-1,K-1} [x^{K-1}(t) - \nu(t)\mathbf{1}] \quad (36)$$

$$W_{2}^{K-1}(t) = (x^{K-1}(t) - \nu(t)\mathbf{1})^{\top} \Xi^{K-1} L^{K-1,K} (x^{K}(t) - \nu(t)\mathbf{1})$$

$$W_{3}^{K-1}(t) = (x^{K-1}(t) - \nu(t)\mathbf{1})^{\top} \Xi^{K-1} (\dot{\nu}(t)\mathbf{1})$$

By Cauchy inequality, for any $v_2^{K-1} > 0$, $v_3^{K-1} > 0$, we have

$$-W_2^{K-1}(t) \le v_2^{K-1}V_{K-1}(t) + F_2^{K-1}(t) -W_3^{K-1}(t) \le v_3^{K-1}V_{K-1}(t) + F_3^{K-1}(t)$$
(37)

where

$$F_2^{K-1}(t) = \frac{1}{4v_2^{K-1}} \sum_{i=1}^{n_{K-1}} \xi_i^{K-1} \left\{ \sum_{p=1}^{n_K} L_{i,p}^{K-1,K} [x_p^K(t) - \nu(t)] \right\}^2$$
$$F_3^{K-1}(t) = \frac{1}{4v_2^{K-1}} \sum_{i=1}^{n_{K-1}} \xi_i^{K-1} [\dot{\nu}(t)]^2 = \frac{1}{2v_3^{K-1}} [\dot{\nu}(t)]^2$$

According to the discussion of SCC_K and Theorem ??, for all $p = 1, \dots, n_K$, we have $\lim_{t\to\infty} x_p^K(t) - \nu(t) = 0$, $\lim_{t\to\infty} \dot{\nu}(t) = 0$ exponentially. Thus

$$\lim_{t \to \infty} F_2^{K-1}(t) = 0, \quad \lim_{t \to \infty} F_3^{K-1}(t) = 0$$
(38)

exponentially.

Thus, (34) can be rewritten as follows

$$\frac{d}{dt}V_{K-1}(t) \leq a_{K-1}V_{K-1}(t) + \frac{1}{2a_{K-1}}\sum_{i=1}^{n_{K-1}}\xi_i^{K-1}[f_i^{K-1}]^2
- W_1^{K-1}(t) - W_2^{K-1}(t) - W_3^{K-1}(t)
\leq -\left[1 - \frac{a_{K-1}\rho(\Xi^{K-1})}{2\rho_2(Q^{K-1})}\right]W_1^{K-1}(t) - W_2^{K-1}(t)
+ \frac{1}{2a_{K-1}}\sum_{i=1}^{n_{K-1}}\xi_i^{K-1}[f_i^{K-1}]^2 - W_3^{K-1}(t)$$
(39)

Thus, we have

Theorem 5: Suppose that \mathcal{G} has spanning trees and L is written in the form of (32). For agent v_i , if trigger times $t_1^i = 0, \dots, t_k^i$ are known, then use the following trigger strategy to find t_{k+1}^i :

$$t_{k+1}^{i} = \max\left\{\tau \ge t_{k}^{i}: |q_{i}(t_{k}^{i}) - q_{i}(t)| \le e^{-\beta t}, \forall t \in [t_{k}^{i}, \tau]\right\}$$
(40)

system (2) reaches consensus exponentially and the Zeno behaviour could be excluded.

Proof: If $v_i \in K$ -th SCC, the event-triggered rule (40) is the same as (27) in Theorem 3, since L is written in the form of (32). By Theorem 3, we can conclude that under the updating rule of $\{t_l^{j+M_{K-1}}\}$ for all $j = 1, \dots, n_K$ and $\lim_{t\to\infty} \dot{\nu}(t) =$ 0, the subsystem restricted in SCC_K reaches a consensus. Additionally, $\lim_{t\to\infty} |x_i^K(t) - \nu(t)| = 0$ for all $i = 1, \dots, n_K$ and $\lim_{t\to\infty} \dot{\nu}(t) = 0$ as well.

In the following, we are to prove that the state of the agent $v_{p+M_{K-2}} \in SCC_{K-1}$ converges to $\nu(t)$. The remaining can be proved similarly by induction.

From (39) and the inequality (40), we have

$$\frac{d}{dt}V_{K-1}(t) \le -\left[1 - \frac{a_{K-1}\rho(\Xi^{K-1})}{2\rho_2(Q^{K-1})}\right] \frac{\rho_2(Q^{K-1})}{\rho(U^{K-1})} V_{K-1}(t) + (v_2^{K-1} + v_3^{K-1}) V_{K-1}(t) + W_4^{K-1}(t)$$

where

$$W_4^{K-1}(t) = F_2^{K-1}(t) + F_3^{K-1}(t) + \frac{1}{2a_{K-1}} \sum_{j=1}^{n_{K-1}} \xi_j^{K-1} \delta_j^2(t)$$

Picking $a_{K-1} = \frac{\rho_2(Q^{K-1})}{\rho(\Xi^{K-1})}$ and sufficiently small v_2^{K-1} and v_3^{K-1} , there exists some $\varepsilon_{K-1} > 0$ such that

$$\frac{d}{dt}V_{K-1}(t) \le -\varepsilon_{K-1}V_{K-1}(t) + W_4^{K-1}(t)$$

Thus

$$V_{K-1}(t) \le e^{-\varepsilon_{K-1}t} \left\{ V_{K-1}(0) + \int_0^t e^{\varepsilon_{K-1}s} W_4^{K-1}(s) ds \right\}$$

From (38), we have $\lim_{t\to\infty} W_4^{K-1}(t) = 0$ exponentially. Thus, we have $\lim_{t\to\infty} V_{K-1}(t) = 0$ exponentially. This implies that system (2) reaches a consensus and $\lim_{t\to\infty} |x_p^{K-1}(t) - \nu(t)| = 0$ exponentially for all $p = 1, \dots, n_{K-1}$.

Similar to the proof in Theorem 3, we can prove that the Zeno behaviour can be excluded for agent $v_i \in K-1$ -th SCC.

Then, we can complete the proof by induction to SCC_k for k < K - 1.

V. EXAMPLES

In this section, one numerical example is given to demonstrate the effectiveness of the presented results.

Consider a network of seven agents with a directed reducible Laplacian matrix

$$L = \begin{vmatrix} -12 & 0 & 5 & 2 & 5 & 0 & 0 \\ 3 & -8 & 3 & 0 & 0 & 0 & 2 \\ 0 & 4 & -12 & 3 & 0 & 5 & 0 \\ 0 & 0 & 6 & -11 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & -7 & 2 & 5 \\ 0 & 0 & 0 & 0 & 5 & -6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 8 & -8 \end{vmatrix}$$

with a spanning tree described by Figure 1. The seven agents can be divided into two strongly connected components, i.e. the first four agents form a strongly connected component and the rest form anther. The initial value of each agent is also randomly selected within the interval [-5,5]in our simulation. Figure 2 (a) shows the evolution of the Lyapunov function $V(t) = V_1(t) + V_2(t)$ (see (33)), and Figure 2 (b) illustrates the trigger times of each agents under the self-triggered principles provided in Theorem 4 with $\phi_i = 20$ and $\alpha_i = 1.5$, $i = 1, \dots, 7$, and initial value $[2.192, -3.699, -2.982, 4.726, 3.575, 4.074, -3.424]^{\top}$. It can be seen that under the self-triggering principle in Theorem 4, V(t) approaches 0 exponentially and the inter-event times of each agent are strictly bigger than some positive constants.

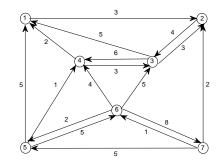


Fig. 1. The communication graph.

VI. CONCLUSIONS

In this paper, we present distributed event-triggered and self-triggered principles in for multi-agent systems with general directed topologies. We derive pull-based event-triggered

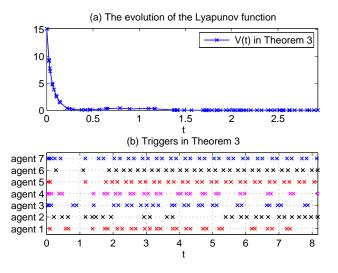


Fig. 2. The evolution of the Lyapunov function and the trigger times of each agents..

principles: in case the graph is reducible with a spanning tree, the triggering time of each agent given by the inequality (40) only depends on the states of each agent's in-neighbors. It is shown that with those principles, consensus can be reached exponentially, and Zeno behavior can be excluded. The results then are extended to discontinuous monitoring, where each agent computes its next triggering time in advance without having to observe the systems state continuously. The effectiveness of the theoretical results are verified by one numerical example.

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