Quantized Minimum Error Entropy Criterion

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Abstract—Comparing with traditional learning criteria, such as mean square error (MSE), the minimum error entropy (MEE) criterion is superior in nonlinear and non-Gaussian signal processing and machine learning. The argument of the logarithm in Renyis entropy estimator, called information potential (IP), is a popular MEE cost in information theoretic learning (ITL). The computational complexity of IP is however quadratic in terms of sample number due to double summation. This creates computational bottlenecks especially for large-scale datasets. To address this problem, in this work we propose an efficient quantization approach to reduce the computational burden of IP, which decreases the complexity from $O(N^2)$ to O(MN)with $M \ll N$. The new learning criterion is called the quantized MEE (QMEE). Some basic properties of QMEE are presented. Illustrative examples are provided to verify the excellent performance of OMEE.

Key Words: Information Theoretic Learning (ITL), Minimum Error Entropy (MEE), Computational Complexity, Quantization.

I. INTRODUCTION

As a well-known learning criterion in *information theoretic learning* (ITL) [1]–[3], the *minimum error entropy* (MEE) finds successful applications in various learning tasks, including regression, classification, clustering, feature selection and many others [4]–[17]. The basic principle of MEE is to learn a model to discover structure in data by minimizing the entropy of error between model and data generating system [1]. Entropy takes all higher order moments into account and hence, is a global descriptor of the underlying distribution. The MEE can perform much better than the traditional mean square error (MSE) criterion that considers only the second order moment of the error, especially in nonlinear and non-Gaussian (multi-peak, heavy-tailed, etc.) signal processing and machine learning.

In practical applications, an MEE cost can be estimated based on a PDF estimator. The most widely used MEE cost in ITL is the *information potential* (IP), which is the argument of the logarithm in Renyis entropy [1]. The IP can be estimated directly from data and computed by a double summation over all samples. This is much different from traditional learning costs that only involve a single summation. Although IP is simpler than many other entropic costs, it is still computationally very expensive due to the pairwise computation (i.e. double summation). This may pose computational bottlenecks for large-scale datasets. To address this issue, we propose in

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this paper an efficient approach to decrease the computational complexity of IP from $O\left(N^2\right)$ to $O\left(MN\right)$ with $M\ll N$. The basic idea is to simplify the inner summation by quantizing the error samples with a simple quantization method. The simplified learning criterion is called the *quantized MEE* (QMEE). Some properties of the QMEE are presented, and the desirable performance of QMEE is confirmed by several illustrative results.

The remainder of the paper is organized as follows. The MEE criterion is briefly reviewed in section II. The QMEE is proposed in section III. The illustrative examples are provided in section IV and finally, the conclusion is given in section V.

II. BRIEF REVIEW OF MEE CRITERION

Consider learning from N examples $Z^N = \{(x_i,y_i) \in \mathcal{X} \times \mathcal{Y}\}, \ i = 1,2,\cdots,N$, which are drawn independently from an unknown probability distribution \mathcal{D} on $\mathcal{Z} := \mathcal{X} \times \mathcal{Y}$. Here we assume $\mathcal{X} \subset \mathbb{R}^d$ and $\mathcal{Y} \subset \mathbb{R}$. Usually, a loss function $\ell(f,(x,y))$ is used to measure the performance of the hypothesis $f: \mathcal{X} \to \mathcal{Y}$. For regression, one can choose the squared error loss $\ell(f,(x,y)) = (y-f(x))^2 = e^2$, where $e = y - f(x) \in \mathbb{R}$ is the prediction error. Then the goal of learning is to find a solution in hypothesis space that minimizes the expected cost function $\mathbf{E}\left[\ell(f,(x,y))\right]$, where the expectation is taken over \mathcal{D} . As the distribution \mathcal{D} is unknown, in general we use the empirical cost function:

$$J = \frac{1}{N} \sum_{i=1}^{N} \ell(f, (x_i, y_i))$$
 (1)

which involves a summation over all samples. Sometimes, a regularization term is added to the above sum to prevent overfitting. Under MSE criterion, the empirical cost function becomes

$$J_{MSE} = \frac{1}{N} \sum_{i=1}^{N} (y_i - f(x_i))^2 = \frac{1}{N} \sum_{i=1}^{N} e_i^2$$
 (2)

where $e_i = y_i - f(x_i)$ is the prediction error for sample (x_i, y_i) . The computational complexity for evaluating the above cost and its gradient with respect to e_i $(i = 1, 2, \dots, N)$ is O(N).

In the context of information theoretic learning (ITL), one can adopt Renyis entropy of order α ($\alpha > 0$, $\alpha \neq 1$) as the cost function [1]:

$$H_{\alpha}(e) = \frac{1}{1 - \alpha} \log \int p^{\alpha}(e) de$$
 (3)

where p(.) denotes the errors PDF. Under MEE criterion, the optimal hypothesis can thus be solved by minimizing the error

entropy $H_{\alpha}(e)$. The argument of the logarithm in $H_{\alpha}(e)$, called information potential (IP), is

$$I_{\alpha}(e) = \int p^{\alpha}(e)de = \mathbf{E}\left[p^{\alpha-1}(e)\right] \tag{4}$$

Since the logarithm function is a monotonically increasing function, minimizing Renyis entropy $H_{\alpha}(e)$ is equivalent to minimizing (for $\alpha < 1$) or maximizing (for $\alpha > 1$) the IP $I_{\alpha}(e)$. In ITL, for simplicity the parameter α is usually set at $\alpha = 2$. In the rest of the paper, without loss of generality we only consider the case of $\alpha = 2$. In this case, we have

$$\min H_2(e) \Leftrightarrow \max I_2(e) = \mathbf{E}[p(e)] \tag{5}$$

According to ITL [1], an empirical version of the quadratic IP can be expressed as

$$\hat{I}_{2}(e) = \frac{1}{N} \sum_{i=1}^{N} \hat{p}(e_{i}) = \frac{1}{N^{2}} \sum_{i=1}^{N} \sum_{j=1}^{N} G_{\sigma}(e_{i} - e_{j})$$
 (6)

where $\hat{p}(.)$ is Parzen's PDF estimator [18]:

$$\hat{p}(x) = \frac{1}{N} \sum_{j=1}^{N} G_{\sigma}(x - e_j)$$
 (7)

with $G_{\sigma}(.)$ being the Gaussian kernel with bandwidth σ :

$$G_{\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) \tag{8}$$

The PDF estimator $\hat{p}(.)$ can be viewed as an adaptive loss function that varies with the error samples $\{e_1, e_2, \cdots, e_N\}$. This is much different from the conventional loss functions that are typically left unchanged after being set. For example, the loss function of MSE is always $\ell(x) = x^2$. The adaptation of loss function is potentially beneficial because the risk is matched to the error distribution. The superior performance of MEE has been shown theoretically as well as confirmed numerically [1]. However, the price we have to pay is that there is a double summation over all samples, which is obviously time consuming especially for large-scale datasets. The computational complexity for evaluating the cost function (6) is $O(N^2)$. The goal of this work is to find an efficient way to simplify the computation of the empirical IP.

III. QUANTIZED MEE

Comparing with conventional cost functions for machine learning, the MEE cost (or equivalently, the IP) involves an additional summation operation, namely the computation of the PDF estimator. The basic idea of our approach is thus to reduce the computational burden of the PDF estimation (i.e. the inner summation). We aim to estimate the errors PDF from fewer samples. A natural way is to represent the N error samples $\{e_1, e_2, \cdots, e_N\}$ with a smaller data set by using a simple quantization method. Of course, the quantization will decrease the accuracy of PDF estimation. However, the PDF estimator for an entropic cost function is very different from the ones for traditional density estimation. Indeed, for a cost function for machine learning, ultimately whats going to matter is the extrema (maxima or minima) of the cost function,

not the exact value of the cost. Our experimental results have shown that with quantization the MEE can achieve almost the same (or even better) performance as the original MEE learning.

Let Q[.] denote a quantization operator (or quantizer) with a codebook C containing M (in general $M \ll N$) real valued code words, i.e. $C = \{c_1, c_2, \cdots, c_M \in \mathbb{R}\}$. Then Q[.] is a function that can map the error sample e_j into one of the Mcode words in C, i.e. $Q[e_i] \in C$. In this work, we assume that each error sample is quantized to the nearest code word. With the quantizer Q[.], the empirical IP in (6) can be simplified to

$$\hat{I}_{2}(e) = \frac{1}{N^{2}} \sum_{i=1}^{N} \sum_{j=1}^{N} G_{\sigma} (e_{i} - e_{j})$$

$$\approx \hat{I}_{2}^{Q}(e) = \frac{1}{N^{2}} \sum_{i=1}^{N} \sum_{j=1}^{N} G_{\sigma} (e_{i} - Q[e_{j}])$$

$$= \frac{1}{N^{2}} \sum_{i=1}^{N} \left(\sum_{m=1}^{M} M_{m} G_{\sigma} (e_{i} - c_{m}) \right)$$

$$= \frac{1}{N} \sum_{i=1}^{N} \hat{p}_{Q} (e_{i})$$
(9)

where M_m is the number of error samples that are quantized to the code word c_m , and $\hat{p}_Q(x) = \frac{1}{N} \sum_{m=1}^M M_m G_\sigma(x - c_m)$ is the PDF estimator based on the quantized error samples. Clearly, we have $\sum_{m=1}^M M_m = N$ and $\int \hat{p}_Q(x) \, dx = 1$.

Remark: The computational complexity of the quantized MEE (QMEE) cost $\hat{I}_{2}^{Q}(e)$ is $O\left(MN\right)$, which is much simpler than the original cost of (6) especially for large-scale datasets $(M \ll N)$.

Before designing the quantizer Q[.], we present below some basic properties of the QMEE cost.

Property 1: When the codebook $C = \{e_1, e_2, \dots, e_N\}$, we have $\hat{I}_{2}^{Q}(e) = \hat{I}_{2}(e)$.

Proof: Straightforward since in this case we have $Q[e_i] = e_i$, $j=1,2,\cdots,N$.

Property 2: The QMEE cost $\hat{I}_2^Q(e)$ is bounded, i.e. $\hat{I}_2^Q(e) \leq \frac{1}{\sqrt{2\pi}\sigma}$, with equality if and only if $e_1 = e_2 = \cdots = e_M = c$, where c is an element of C.

Proof: Since $G_{\sigma}(x) \leq \frac{1}{\sqrt{2\pi}\sigma}$ with equality if and only if x = c

0, we have

$$\hat{I}_{2}^{Q}(e) = \frac{1}{N^{2}} \sum_{i=1}^{N} \sum_{j=1}^{N} G_{\sigma} (e_{i} - Q[e_{j}])$$

$$\leq \frac{1}{N^{2}} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} = \frac{1}{\sqrt{2\pi}\sigma}$$
(10)

with equality if and only if $e_i = Q[e_j]$, $\forall i, j$, which means $e_1 = e_2 = \dots = e_M = c .$

Property 3: It holds that $\hat{I}_{2}^{Q}(e) = \frac{1}{M} \sum_{m=1}^{M} \alpha_{m} \hat{p}(c_{m})$, where $\alpha_m = \frac{M_m}{N}$, satisfying $\sum_{m=1}^{M} \alpha_m = 1$.

Proof: One can easily derive

$$\hat{I}_{2}^{Q}(e) = \frac{1}{N^{2}} \sum_{i=1}^{N} \left(\sum_{m=1}^{M} M_{m} G_{\sigma} \left(e_{i} - c_{m} \right) \right)$$

$$= \sum_{m=1}^{M} \frac{M_{m}}{N} \left(\frac{1}{N} \sum_{i=1}^{N} G_{\sigma} \left(e_{i} - c_{m} \right) \right)$$

$$= \sum_{m=1}^{M} \alpha_{m} \hat{p} \left(c_{m} \right)$$
(11)

Remark: By Property 3, the QMEE cost $\hat{I}_2^Q(e)$ is equal to a weighted average of the Parzen's PDF estimator evaluated at the code words. Moreover, when there is only one code word in C, i.e. $C = \{c\}$, we have $\hat{I}_2^Q(e) = \hat{p}(c)$. In particular, when $C = \{0\}$, we have $\hat{I}_2^Q(e) = \hat{V}(e) = \frac{1}{N} \sum_{i=1}^N G_\sigma(e_i) = \hat{p}(0)$, where $\hat{V}(e)$ denotes the empirical correntropy [19]–[23], which is a well-known local similarity measure in ITL. In this sense, the correntropy can be viewed as a special case of the QMEE cost. Actually, the correntropy measures the local similarity about the zero, while QMEE cost $\hat{I}_2^Q(e)$ measures the average similarity about every code word in C.

Property 4: When σ is large enough, we have $\hat{I}_2^Q(e) \approx \frac{1}{\sqrt{2\pi}\sigma} - \frac{1}{2\sqrt{2\pi}\sigma^3} \sum_{m=1}^M \alpha_m \mu_m$, where $\mu_m = \frac{1}{N} \sum_{i=1}^N \left(e_i - c_m\right)^2$ is the second order moment of error about the code word c_m . Proof: As $\sigma \to \infty$, we have $G_\sigma\left(e_i - c_m\right) \approx \frac{1}{\sqrt{2\pi}\sigma} \left(1 - \frac{\left(e_i - c_m\right)^2}{2\sigma^2}\right)$. It follows easily that

$$\hat{I}_{2}^{Q}(e) = \frac{1}{N^{2}} \sum_{i=1}^{N} \left(\sum_{m=1}^{M} M_{m} G_{\sigma} \left(e_{i} - c_{m} \right) \right)$$

$$\approx \frac{1}{N^{2} \sqrt{2\pi} \sigma} \sum_{i=1}^{N} \left(\sum_{m=1}^{M} M_{m} \left(1 - \frac{\left(e_{i} - c_{m} \right)^{2}}{2\sigma^{2}} \right) \right)$$

$$= \frac{1}{\sqrt{2\pi} \sigma} - \frac{1}{2N^{2} \sqrt{2\pi} \sigma^{3}} \sum_{i=1}^{N} \left(\sum_{m=1}^{M} M_{m} (e_{i} - c_{m})^{2} \right)$$

$$= \frac{1}{\sqrt{2\pi} \sigma} - \frac{1}{2\sqrt{2\pi} \sigma^{3}} \sum_{m=1}^{M} \frac{M_{m}}{N} \left(\frac{1}{N} \sum_{i=1}^{N} \left(e_{i} - c_{m} \right)^{2} \right)$$

$$= \frac{1}{\sqrt{2\pi} \sigma} - \frac{1}{2\sqrt{2\pi} \sigma^{3}} \sum_{m=1}^{M} \alpha_{m} \mu_{m}$$

Remark: By Property 4, as $\sigma \to \infty$, the second order moments tend to dominate the QMEE cost $\hat{I}_2^Q(e)$. In this case, maximizing the QMEE cost is equivalent to minimizing a weighted average of the second order moments about the code words.

Property 5: If $\forall j, |e_j - Q[e_j]| \leq \varepsilon$ with ε being a positive number, then $|\hat{I}_2^Q(e) - \hat{I}_2(e)| \leq \frac{\varepsilon \exp(-1/2)}{\sigma}$.

Proof: Because the Gaussian function $G_{\sigma}(.)$ is continuously differentiable over \mathbb{R} , according to the *Mean Value Theorem*, $\forall i,j$, there exists a point $\xi_{ij} \in (\min\{e_i-Q[e_j],e_i-e_j\},\max\{e_i-Q[e_j],e_i-e_j\})$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

$$G_{\sigma}(e_{i} - Q[e_{j}]) - G_{\sigma}(e_{i} - e_{j})$$

$$= G_{\sigma}(e_{i} - e_{j} + (e_{j} - Q[e_{j}])) - G_{\sigma}(e_{i} - e_{j})$$

$$= (e_{j} - Q[e_{j}]) G'_{\sigma}(\xi_{ij})$$
(13)

(11) where $G'_{\sigma}(.)$ denotes the derivative of $G_{\sigma}(.)$ with respect to the argument. Then we have

$$|G_{\sigma}(e_{i} - Q[e_{j}]) - G_{\sigma}(e_{i} - e_{j})|$$

$$= |e_{j} - Q[e_{j}]| \times |G'_{\sigma}(\xi_{ij})|$$

$$\stackrel{(a)}{\leq} \frac{\varepsilon \exp(-1/2)}{\sigma}$$
(14)

where (a) comes from $|e_{j}-Q[e_{j}]|\leq \varepsilon$ and $|G'_{\sigma}\left(x\right)|\leq \frac{\exp\left(-1/2\right)}{\sigma}$ for any $x\in\mathbb{R}.$ It follows that

$$\left| \hat{I}_{2}^{Q}(e) - \hat{I}_{2}(e) \right| = \left| \frac{1}{N^{2}} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(G_{\sigma} \left(e_{i} - Q[e_{j}] \right) - G_{\sigma} \left(e_{i} - e_{j} \right) \right) \right|$$

$$\leq \frac{1}{N^{2}} \sum_{i=1}^{N} \sum_{j=1}^{N} \left| G_{\sigma} \left(e_{i} - Q[e_{j}] \right) - G_{\sigma} \left(e_{i} - e_{j} \right) \right|$$

$$\leq \frac{1}{N^{2}} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\varepsilon \exp(-1/2)}{\sigma}$$

$$= \frac{\varepsilon \exp(-1/2)}{\sigma}$$
(15)

Remark: From Property 5, when ε is very small or σ is very large, the difference between the values of $\hat{I}_2^Q(e)$ and $\hat{I}_2(e)$ will be very small.

Property 6: For a linear regression model $f(x) = \omega^T x$, with $\omega \in \mathbb{R}^d$ being the weight vector to be estimated, the optimal solution under QMEE criterion satisfies

$$\omega = R_{QMEE}^{-1} P_{QMEE} \tag{16}$$

where $R_{QMEE} = \sum_{i=1}^{N} \sum_{m=1}^{M} M_m G_{\sigma} \left(e_i - c_m \right) x_i x_i^T$ and

 $P_{QMEE} = \sum_{i=1}^{N} \sum_{m=1}^{M} M_m G_{\sigma} (e_i - c_m) (y_i - c_m) x_i.$

Proof: The derivative of the QMEE cost $\hat{I}_2^Q(e)$ with respect to ω is

$$\frac{\partial}{\partial \omega} \hat{I}_{2}^{Q}(e) = \frac{1}{N^{2}} \sum_{i=1}^{N} \left(\sum_{m=1}^{M} M_{m} \frac{\partial}{\partial \omega} G_{\sigma} \left(e_{i} - c_{m} \right) \right)
= \frac{1}{N^{2} \sigma^{2}} \sum_{i=1}^{N} \left(\sum_{m=1}^{M} M_{m} G_{\sigma} \left(e_{i} - c_{m} \right) \left(y_{i} - \omega^{T} x_{i} - c_{m} \right) x_{i} \right)
= \frac{1}{N^{2} \sigma^{2}} \sum_{i=1}^{N} \sum_{m=1}^{M} M_{m} G_{\sigma} \left(e_{i} - c_{m} \right) \left(y_{i} - c_{m} \right) x_{i}
- \frac{1}{N^{2} \sigma^{2}} \left(\sum_{i=1}^{N} \sum_{m=1}^{M} M_{m} G_{\sigma} \left(e_{i} - c_{m} \right) x_{i} x_{i}^{T} \right) \omega
= \frac{1}{N^{2} \sigma^{2}} \left\{ P_{QMEE} - R_{QMEE} \omega \right\}$$
(17)

Setting $\frac{\partial}{\partial\omega}\hat{I}_2^Q(e)=0,$ we get $\omega=R_{QMEE}^{-1}P_{QMEE}.$ It completes the proof.

Remark: It is worth noting that the solution $\omega=R_{QMEE}^{-1}P_{QMEE}$ is not a closed-form solution as the matrix R_{QMEE} and the vector P_{QMEE} on the right side of the equation depend on the weight vector ω through the error samples (i.e. $e_i=y_i-\omega^Tx_i$). Actually, the equation $\omega=R_{QMEE}^{-1}P_{QMEE}$ is a fixed-point equation.

A key problem in QMEE is how to design a simple and efficient quantizer Q[.], including how to build the codebook and how to assign the code words to the data. In this work, we will use a method proposed in our recent papers, to quantize the error samples. In [24], [25], we proposed a simple online vector quantization (VQ) to curb the network growth in kernel adaptive filters, such as kernel least mean square (KLMS) and kernel recursive least squares (KRLS). The main advantage of this quantization method lies in its simplicity and online feature. The pseudocode of this online VQ algorithm is presented in *Algorithm 1*.

Algorithm 1

Input: error samples $\{e_i\}_{i=1}^N$ **Output:** quantized errors $\{Q[e_i]\}_{i=1}^N$

- 1: Parameters setting: quantization threshold ε
- 2: Initialization: Set $C_1 = \{e_1\}$, where C_i denotes the codebook at the iteration i
- 3: for i = 2, ..., N do
- 4: Compute the distance between e_i and C_{i-1} : $dis\ (e_i,C_{i-1}) = |e_i-C_{i-1}(j^*)|$ where $j^* = \underset{1 \leq j \leq |C_{i-1}|}{\operatorname{argmin}} |e_i-C_{i-1}(j)|, \, C_{i-1}(j)$ denotes the jth element of C_{i-1} , and $|C_{i-1}|$ stands for the cardinality of C_{i-1} .
- 5: **if** $dis(e_i, C_{i-1}) \le \varepsilon$ **then**
- Keep the codebook unchanged: $C_i = C_{i-1}$ and quantize e_i to the closest code word $Q[e_i] = C_{i-1}(j^*)$;
- 7: else
- 8: Update the codebook: $C_i = \{C_{i-1}, e_i\}$ and quantize e_i to itself: $Q[e_i] = e_i$;
- 9: end if
- 10: end for

Remark: The online VQ method in Algorithm 1 creates the codebook sequentially from the samples, which is computationally very simple, with computational complexity that is linear in the number of samples.

IV. ILLUSTRATIVE EXAMPLES

In the following, we present some illustrative examples to demonstrate the desirable performance of the proposed QMEE criterion.

A. Linear Regression

In the first example, we use the QMEE criterion to perform the linear regression. According to Property 6, the optimal solution of the linear regression model $f(x) = \omega^T x$ can easily be solved by the following fixed-point iteration:

$$\omega_k = \left[R_{QMEE}(\omega_{k-1}) \right]^{-1} P_{QMEE}(\omega_{k-1}) \tag{18}$$

in which the matrix $R_{QMEE}(\omega_{k-1})$ and vector $P_{QMEE}(\omega_{k-1})$ are

$$\begin{cases}
R_{QMEE}(\omega_{k-1}) = \sum_{m=1}^{M} \mathbf{X} \mathbf{\Lambda}_{m} \mathbf{X}^{T} \\
P_{QMEE}(\omega_{k-1}) = \sum_{m=1}^{M} \mathbf{X} \mathbf{\Lambda}_{m} \mathbf{Y}_{m}
\end{cases} (19)$$

where $\mathbf{X} = [x_1, x_2, \cdots, x_N] \in \mathbb{R}^{d \times N}, \ \mathbf{Y}_m = [y_1 - c_m, y_2 - c_m, \cdots, y_N - c_m]^T \in \mathbb{R}^N, \ \text{and} \ \mathbf{\Lambda}_m \ \text{is a} N \times N \ \text{diagonal matrix with diagonal elements} \ \mathbf{\Lambda}_m(ii) = M_m G_\sigma \ (e_i - c_m), \ \text{with} \ .$ The detailed procedure of the linear regression under QMEE is summarized in *Algorithm 2*.

Algorithm 2

Input: samples $\{x_i, y_i\}_{i=1}^N$ Output: weight vector ω

- 1: Parameters setting: iteration number K, kernel width σ , quantization threshold ε
- 2: Initialization: Set $\omega_0 = \mathbf{0}$
- 3: **for** k = 2, ..., K **do**
- 4: Compute the error samples based on ω_{k-1} : $e_i = y_i \omega_{k-1}^T x_i$, $i = 1, 2, \dots, N$;
- 5: Create the quantization codebook C and quantize the N error samples by Algorithm 1;
- 6: Compute the matrix $R_{QMEE}(\omega_{k-1})$ and the vector $P_{QMEE}(\omega_{k-1})$ by (19);
- 7: Update the weight vector by (18);
- 8: end for

We now consider a simple scenario where the data samples are generated by a two-dimensional linear system y_i = $\omega^{*T} x_i + v_i$, where $\omega^* = [2,1]^T$, and v_i is an additive noise. The input vectors $\{x_i\}$ are assumed to be uniformly distributed over $[-2,2] \times [-2,2]$. In addition, the noise v_i is assumed to be generated by $v_i = (1 - a_i) A_i + a_i B_i$, where a_i is a binary process with probability mass $Pr\{a_i = 1\} = c$, $\Pr\{a_i = 0\} = 1 - c$, with $0 \le c \le 1$ being an occurrence probability. The processes A_i and B_i represent the background noises and the outliers respectively, which are mutually independent and both independent of a_i . In the simulations below, c is set at 0.1 and B_i is assumed to be a white Gaussian process with zero-mean and variance 10000. For the distribution of A_i , we consider four cases: 1) symmetric Gaussian mixture density: $0.5\mathcal{N}(3,1) + 0.5\mathcal{N}(-3,1)$, where $\mathcal{N}(\mu,\sigma^2)$ denotes the Gaussian density with mean μ and variance σ^2 ; 2) asymmetric Gaussian mixture density: $\frac{2}{3}\mathcal{N}(-5,1) + \frac{1}{3}\mathcal{N}(2,1)$; 3) binary distribution with probability mass $Pr\{x=-2\}=$ $\Pr\{x=2\} = 0.5$; 4) Gaussian distribution with zero-mean and unit variance. The root mean squared error (RMSE) is employed to measure the performance, computed by

$$RMSE = \sqrt{\frac{1}{2} \|\omega_k - \omega^*\|^2}$$
 (20)

where ω_k and ω^* denote the estimated and the target weight vectors respectively.

We compare the performance of four learning criteria, namely MSE, MCC [19]-[23], MEE and QMEE. For the MSE criterion, there is a closed-form solution, so no iteration is needed. For other three criteria, a fixed-point iteration is used to solve the model (see [22], [26] for the details of the fixed-point algorithms under MCC and MEE). The parameter settings of MCC, MEE and QMEE are given in Table I. The simulations are carried out with MATLAB 2014a running in i5-4590, 3.30 GHZ CPU. The mean deviation results of the RMSE and the training time over 100 Monte Carlo runs are presented in Table II. In the simulations, the sample number is N=200 and the iteration number is K=100. From Table II, we observe: i) the MCC, MEE and OMEE can significantly outperform the traditional MSE criterion although they have no closed-form solution; ii) the MEE and QMEE can achieve much better performance than the MCC criterion, except the case of Gaussian background noise, in which they achieve almost the same performance; iii) the QMEE can achieve almost the same (or even better) performance as the original MEE criterion, but with much less computational cost. Fig. 1 shows the average training time of QMEE and MEE with increasing number of samples.

Further, we show in Fig. 2 the contour plots of the performance surfaces (i.e. the cost surfaces over the parameter space), where the background noise distribution is assumed to be symmetric Gaussian mixture. In Fig. 2, the target weight vector and the optimal solutions of the performance surfaces are denoted by the red crosses and blue circles, respectively. As one can see, the optimal solutions under MEE and QMEE are almost identical to the target value, while the solutions under MSE and MCC (especially the MSE solution) are apart from the target.

TABLE I PARAMETER SETTINGS OF THREE CRITERIA

	MCC	MEE	QM	IEE
	σ	σ	σ	ϵ
Case 1)	10	1.1	1.5	0.3
Case 2)	15	1.1	1.5	0.3
Case 3)	8	0.7	1.0	0.3
Case 4)	2.8	0.6	4.0	0.1

B. Extreme Learning Machines

The second example is about the training of *Extreme Learning Machine* (ELM) [27]–[31], a single-hidden-layer feedforward neural network (SLFN) with random hidden nodes.

Given N distinct training samples $\{\mathbf{x}_i, t_i\}_{i=1}^N$, with $\mathbf{x}_i = [x_{i1}, x_{i2}, \dots, x_{id}]^T \in \mathbb{R}^d$ being the input vector and $t_i \in \mathbb{R}$ the target response, the output of a standard SLFN with L hidden nodes is

$$y_i = \sum_{j=1}^{L} \beta_j f\left(\mathbf{w}_j \mathbf{x}_i + b_j\right)$$
 (21)

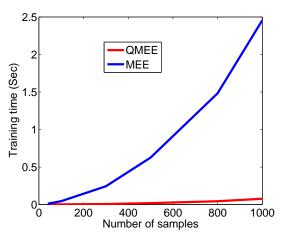


Fig. 1. Training time versus the number of samples

where f(.) is an activation function, $\mathbf{w}_j = [w_{j1}, w_{j2}, ..., w_{jd}] \in \mathbb{R}^d$ and $b_j \in \mathbb{R}$ (j = 1, 2, ..., L) are the randomly generated parameters of the L hidden nodes, and $\boldsymbol{\beta} = (\beta_1, ..., \beta_L)^T \in \mathbb{R}^L$ represents the output weight vector. Since the hidden parameters are determined randomly, we only need to solve the output weight vector $\boldsymbol{\beta}$. To this end, we express (22) in a vector form as

$$\mathbf{Y} = \mathbf{H}\boldsymbol{\beta} \tag{22}$$

where $\mathbf{Y} = (y_1, ..., y_N)^T$, and

$$\mathbf{H} = \begin{pmatrix} \mathbf{h}_1 \\ \vdots \\ \mathbf{h}_N \end{pmatrix} = \begin{pmatrix} f(\mathbf{w}_1 \mathbf{x}_1 + b_1), & \dots & f(\mathbf{w}_L \mathbf{x}_1 + b_L) \\ \vdots & \ddots & \vdots \\ f(\mathbf{w}_1 \mathbf{x}_N + b_1), & \dots & f(\mathbf{w}_L \mathbf{x}_N + b_L) \end{pmatrix}$$
(23)

Usually, the output weight vector β can be solved by minimizing the following squared (MSE based) and regularized loss function:

$$J_{MSE}(\boldsymbol{\beta}) = \sum_{i=1}^{N} e_i^2 + \lambda \|\boldsymbol{\beta}\|_2^2 = \|\mathbf{H}\boldsymbol{\beta} - \mathbf{T}\|_2^2 + \lambda \|\boldsymbol{\beta}\|_2^2$$
 (24)

where $e_i = t_i - y_i = t_i - \mathbf{h}_i \boldsymbol{\beta}$ is the *i*th error between the target response and actual output, $\lambda \geq 0$ represents the regularization factor, and $\mathbf{T} = (t_1, ..., t_N)^T$. Applying the pseudo inversion operation, one can obtain a unique solution under the loss function (24), that is

$$\boldsymbol{\beta} = \left[\mathbf{H}^T \mathbf{H} + \lambda \mathbf{I} \right]^{-1} \mathbf{H}^T \mathbf{T}$$
 (25)

Here, we propose the following QMEE based loss function:

 $\label{table II} \textbf{RMSE} \ \textbf{and} \ \textbf{training time of different criteria}$

		MSE	MCC	MEE	QMEE
Case 1)	RMSE	1.1649 ± 0.6587	0.1493 ± 0.0756	0.0468 ± 0.0205	0.0473 ± 0.0203
Case 1)	Training Time (sec)	N/A	$3.0000\!\!\times\!\!10^{-4} \pm 2.6000\!\!\times\!\!10^{-4}$	$0.2963 \pm 3.5300\!\!\times\!\!10^{-3}$	$9.1410\!\!\times\!\!10^{-3} \pm 6.1800\!\!\times\!\!10^{-4}$
Case 2)	RMSE	1.2951 ± 0.6701	0.1987 ± 0.1111	0.0455 ± 0.0226	0.0460 ± 0.0227
Case 2)	Training Time (sec)	N/A	$3.3900 \times 10^{-4} \pm 2.6000 \times 10^{-4}$	$0.3013 \pm 8.4110 \!\!\times\!\! 10^{-3}$	$9.0140 \times 10^{-3} \pm 8.6000 \times 10^{-4}$
Case 3)	RMSE	1.0939 ± 0.6407	0.0928 ± 0.0480	$7.7500 \times 10^{-4} \pm 0.0010$	$7.8940 \times 10^{-4} \pm 0.0010$
Case 3)	Training Time (sec)	N/A	$3.6700\!\!\times\!\!10^{-4} \pm 2.6500\!\!\times\!\!10^{-4}$	$0.2932 \pm 3.4600\!\!\times\!\!10^{-3}$	$7.3230\!\!\times\!\!10^{-3} \pm 5.2700\!\!\times\!\!10^{-4}$
Case 4)	RMSE	1.2031 ± 0.6531	0.0422 ± 0.0224	0.0452 ± 0.0262	0.0422 ± 0.0231
Case 4)	Training Time (sec)	N/A	$3.5300 \times 10^{-4} \pm 2.6100 \times 10^{-4}$	$0.2999 \pm 2.4750 \!\!\times\!\! 10^{-3}$	$7.9500 \times 10^{-3} \pm 6.4300 \times 10^{-4}$

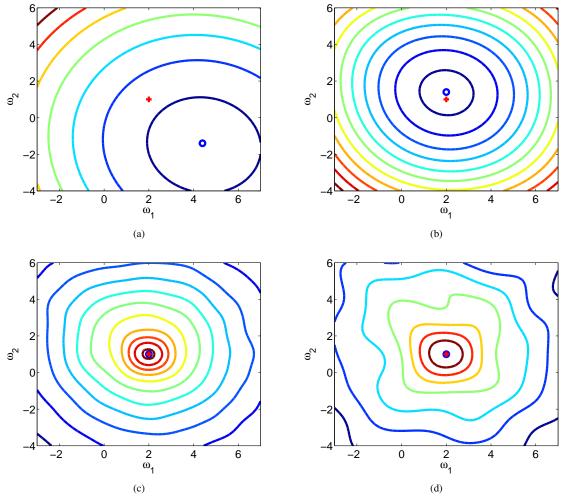


Fig. 2. Contour plots of the performance surfaces (a) MSE; (b) MCC; (c) MEE; (d) QMEE

$$J_{QMEE}(\beta) = -\hat{I}_{2}(e) + \lambda \|\beta\|_{2}^{2}$$

$$= -\frac{1}{N^{2}} \sum_{i=1}^{N} \left(\sum_{m=1}^{M} M_{m} G_{\sigma}(e_{i} - c_{m}) \right) + \lambda \|\beta\|_{2}^{2}$$

$$= -\frac{1}{N^{2}} \sum_{i=1}^{N} \left(\sum_{m=1}^{M} M_{m} \exp\left(-\frac{(e_{i} - c_{m})^{2}}{2\sigma^{2}}\right) \right) + \lambda \|\beta\|_{2}^{2}$$
(26)

Setting $\frac{\partial J_{QMEE}(\beta)}{\partial \beta} = 0$, one can obtain

$$\boldsymbol{\beta} = [A + \lambda' \mathbf{I}]^{-1} B \tag{27}$$

where
$$A = \sum\limits_{m=1}^{M} \mathbf{H}^T \mathbf{\Lambda_m} \mathbf{H}$$
, $B = \sum\limits_{m=1}^{M} \mathbf{H}^T \mathbf{\Lambda_m} \mathbf{T}_m$, $\lambda' = 2\lambda N^2 \sigma^2$, $\mathbf{T}_m = [t_1 - c_m, \cdots, t_N - c_m]^T$, and $\mathbf{\Lambda_m}$ is a $N \times N$ diagonal matrix with diagonal elements $\mathbf{\Lambda_m}(ii) = M_m G_\sigma \left(e_i - c_m \right)$.

Similar to the linear regression case, the equation (27) is a fixed-point equation since the matrix $\Lambda_{\mathbf{m}}$ depends on the weight vector $\boldsymbol{\beta}$ through $e_i = t_i - \mathbf{h}_i \boldsymbol{\beta}$. Thus, one can solve $\boldsymbol{\beta}$ by using the following fixed-point iteration:

$$\boldsymbol{\beta}_k = [A(\boldsymbol{\beta}_{k-1}) + \lambda' \mathbf{I}]^{-1} B(\boldsymbol{\beta}_{k-1})$$
 (28)

where $A(\beta_{k-1})$ and $B(\beta_{k-1})$ denote, respectively, the matrix A and vector B evaluated at β_{k-1} . The learning procedure of the ELM under QMEE is described in *Algorithm 3*. This algorithm is called the ELM-QMEE in this paper.

Algorithm 3 ELM-QMEE

Input: samples $\{x_i, y_i\}_{i=1}^N$ Output: weight vector $\boldsymbol{\beta}$

- 1: Parameters setting: number of hidden nodes L, regularization parameter λ' , iteration number K, kernel width σ , quantization threshold ε
- 2: Initialization: set $\beta_0 = \mathbf{0}$ and randomly initialize the hidden parameters \mathbf{w}_j and b_j (j=1,...,L)
- 3: **for** k = 2, ..., K **do**
- 4: Compute the error samples based on β_{k-1} : $e_i = t_i \mathbf{h}_i \beta_{k-1}$, $i = 1, 2, \dots, N$;
- 5: Create the quantization codebook C and quantize the N error samples by Algorithm 1;
- 6: Compute the matrix $A(\beta_{k-1})$ and the vector $B(\beta_{k-1})$ by (19);
- 7: Update the weight vector β by (28);
- 8: end for

In the following, we consider the regression problem with five benchmark datasets from the UCI machine learning repository [32]. The details of the datasets are shown in Table III. For each dataset, the training and testing samples are randomly selected form the dataset. Particularly, the data are normalized to the range [0, 1]. Five algorithms are compared here, including ELM [27], RELM [28], ELM-RCC [30], ELM-MEE and ELM-QMEE. The ELM-MEE can be viewed as the ELM-QMEE with $\varepsilon=0$. The parameter settings of the five ELM algorithms are presented in Table IV, which are experimentally chosen by fivefold cross-validation.

TABLE III SPECIFICATION OF THE DATASETS

Datasets	Features	Observations			
Datasets	Teatures	Training	Testing		
Servo	5	83	83		
Yacht	6	154	154		
Computer Hardware	8	105	104		
Price	16	80	79		
Machine-CPU	6	105	104		

The RMSE is used as the performance measure for regression. The "mean \pm standard deviation" results of Testing RMSE and the Training time over 100 runs are shown in Table V and VI. In addition, since the MEE and QMEE criteria are shift-invariant, the RMSE of MEE and QMEE are calculated by adding a bias value to the testing errors. This bias value was adjusted so as to yield zero-mean error over the training set. As one can see, in all the cases the proposed ELM-QMEE can outperform other algorithms except the ELM-MEE, and the results of ELM-QMEE is very close to those of ELM-MEE. Besides, compared with ELM-MEE, the computational complexity of ELM-QMEE is much smaller.

C. Echo State Networks

In the last example, we apply the QMEE to train an echo state network (ESN) [33]–[35], a new paradigm in recurrent neural network (RNN) [36], [37]. The ESN randomly builds a large sparse reservoir to replace the hidden layer of RNN, which overcomes the shortcomings of complicated computation and difficulties in determining the network topology of a traditional RNN.

We consider a discrete-time ESN with P input units, L internal network units and Q output units. The dynamic and output equations of the standard ESN can be written as follows:

$$\begin{cases} x(k+1) = f\left(\mathbf{W}^{x}x(k) + \mathbf{W}^{in}u(k+1) + \mathbf{W}^{fb}y(k)\right) \\ y(k) = g\left(\mathbf{W}^{out}\varphi(k)\right) \end{cases}$$
(29)

where $\varphi(k) = \begin{pmatrix} u(k) \\ x(k) \end{pmatrix}$, $f = (f_1 \dots f_L)$ is the nonlinear activation function of reservoir units, $g = (g_1 \dots g_Q)$ is a linear or nonlinear activation function of the output layer, \mathbf{W}^{in} is an $L \times P$ input weight matrix, \mathbf{W}^x is an $L \times L$ internal connection weight matrix of the reservoir, \mathbf{W}^{fb} is an $L \times Q$ weight matrix that feeds back the output to the reservoir units, and \mathbf{W}^{out} is an $Q \times (P+L)$ output weight matrix. To establish an ESN described above, with the property of echo states, the weight matrix \mathbf{W}^x must satisfy the condition $\sigma_{\max} < 1$, with σ_{\max} being the largest singular value of \mathbf{W}^x . In this article we assume that $\mathbf{W}^{fb} = 0$. The weight matrices \mathbf{W}^{in} and \mathbf{W}^x are randomly determined. Then the nonlinear system can be converted to:

TABLE IV
PARAMETER SETTINGS OF FIVE ELM ALGORITHMS

Datasets	ELM		RELM		ELM-	-RCC		ELM-	-MEE		EI	LM-QMEE	_
Datasets	L	L	λ	L	σ	λ	L	σ	λ	L	σ	λ'	γ
Servo	25	90	1×10^{-5}	65	0.8	1×10^{-4}	55	0.1	8×10^{-7}	75	0.2	4×10^{-4}	0.05
Yacht	90	187	2.5×10^{-5}	195	0.4	1×10^{-7}	225	0.1	1×10^{-7}	210	0.2	5×10^{-7}	0.6
Computer Hardware	20	35	9×10^{-6}	40	0.1	8×10^{-6}	95	0.2	5×10^{-6}	90	0.1	6×10^{-5}	0.009
Price	20	20	4×10^{-6}	15	0.3	4×10^{-5}	15	0.3	7×10^{-6}	15	0.3	8×10^{-6}	0.02
Machine-CPU	10	30	7×10^{-5}	20	0.2	7×10^{-5}	25	0.3	5×10^{-6}	25	0.4	0.06	0.08

Datasets	ELM	RELM	ELM-RCC	ELM-MEE	ELM-QMEE
Servo	0.1199 ± 0.0200	0.1046 ± 0.0178	0.1029 ± 0.0158	0.1014 ± 0.0194	0.1014±0.0196
Yacht	0.0596 ± 0.0171	0.0490 ± 0.0058	0.1029 ± 0.0158	0.0327 ± 0.0080	$0.0223 \!\pm\! 0.0108$
Computer Hardware	$0.0262 {\pm} 0.0198$	0.0170 ± 0.0110	0.0162 ± 0.0125	$0.0140 {\pm} 0.0081$	0.0147 ± 0.0114
Price	0.1036 ± 0.0182	0.1031 ± 0.0168	0.1006 ± 0.0142	$0.0985{\pm}0.0137$	0.0997 ± 0.0161
Machine-CPU	0.0646 ± 0.0260	0.0573 ± 0.0182	0.0544 ± 0.0156	0.0530 ± 0.0163	0.0534 ± 0.0164

TABLE VI
TRAINING TIME(SEC) OF FIVE ALGORITHMS

Datasets	ELM	RELM	ELM-RCC	ELM-MEE	ELM-QMEE
Servo	0.0020 ± 0.0082	0.0011 ± 0.0040	0.0127 ± 0.0184	1.0286 ± 0.0116	0.0314±0.0181
Yacht	0.0056 ± 0.0125	0.0048 ± 0.0103	0.0641 ± 0.0325	59.9422 ± 2.1326	$0.1086 {\pm} 0.0340$
Computer Hardware	0.0022 ± 0.0067	0.0014 ± 0.0067	0.0050 ± 0.0125	4.7689 ± 0.2338	0.0716 ± 0.0271
Price	$1.5625 \times 10^{-4} \pm 0.0016$	0.0651 ± 0.0086	$7.8125 \times 10^{-4} \pm 0.0034$	0.5859 ± 0.0093	0.0233 ± 0.0129
Machine-CPU	0.0011 ± 0.0056	$3.1250 \times 10^{-4} \pm 0.0022$	0.0027 ± 0.0063	1.0913 ± 0.0121	$0.0223 {\pm} 0.0114$

$$\mathbf{Y} = \mathbf{W}^{out}\mathbf{X} \tag{30}$$

where the kth column of the matrix \mathbf{X} is $\varphi(k)$. The optimal solution of \mathbf{W}^{out} under MSE criterion can be obtained by $\mathbf{W}^{out} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{Y}$. Here, we use the following QMEE cost function to train the ESN:

$$J_{QMEE}\left(\mathbf{W}^{out}\right) = \frac{1}{N^2} \sum_{i=1}^{N} \left(\sum_{m=1}^{M} M_m G_{\sigma}\left(e_i - c_m\right)\right)$$
 (31)

where $\mathbf{e}_i = \mathbf{t}_i - \mathbf{y}_i$, with \mathbf{t}_i and \mathbf{y}_i being respectively, the *i*th rows of the target matrix \mathbf{T} and output matrix \mathbf{Y} . Different approaches can be used to solve the above optimization problem. Here, the Root Mean Square Propagation (RMSProp) is used. The RMSProp as a variant of stochastic gradient descent (SGD) has been widely used in deep learning. With RMSProp the output weights can be updated by

$$w_i^{out}(k+1) = w_i^{out}(k) - \frac{\eta}{\sqrt{v+r}} \nabla_{w_i^{out}} J_{QMEE}(k) \quad (32)$$

$$v = \rho v + (1 - \rho) \left(\nabla_{w_i^{out}} J_{QMEE}(k) \right)^2$$
 (33)

where η is the learning rate parameter, r is a small positive constant and ρ is the forgetting factor. The gradient term $\nabla_{w_{*}^{out}}J_{QMEE}\left(n\right)$ can be computed as

$$\nabla_{w_{i}^{out}} J_{QMEE}(k) = \sum_{m=1}^{M} M_{m} \exp\left(-\frac{(t_{i}(k) - w_{i}^{out}(k) x(k) - c_{m})^{2}}{2\sigma^{2}}\right)$$

$$(t_{i}(k) - w_{i}^{out}(k) x(k) - c_{m}) x'(k)$$
(34)

The learning algorithm of the ESN under QMEE is given in *Algorithm 4*, called ESN-QMEE in this paper.

Next, we apply the proposed ESN-QMEE to the short-term prediction of the Mackey-Glass (MG) chaotic time series, compared to some other ESN algorithms. The MG dynamic system is governed by the following time-delay ordinary differential equation [38]

$$\frac{dx}{dt} = ax(t) + \frac{bx(t-\tau)}{1 + x(t-\tau)^{10}}$$
(35)

with a=-0.1, b=0.2, $\tau=17$. This system has a chaotic attractor if $\tau>16.8$. In this article, we choose the delay time and the embedded dimension as six and four, which are determined by the mutual information [39], i.e. the vector

Algorithm 4 ESN-QMEE

Input: samples $\{\mathbf{u}_i, \mathbf{t}_i\}_{i=1}^N$ Output: weight matrix \mathbf{W}^{out}

- 1: Parameters setting: learning rate η , constant r, forgetting factor ρ , iteration number K, kernel width σ , quantization threshold ϵ
- 2: Initialization: set the number and the sparseness of reservoir units, randomly initialize \mathbf{W}^x and \mathbf{W}^{in} , and compute the matrix \mathbf{X}
- 3: **for** k = 2, ..., K **do**
- 4: Compute the gradient term $\nabla_{w_{\cdot}^{out}} J_{QMEE}(k)$ by (34)
- 5: Compute the term v by (33)
- 6: Update the weight matrix \mathbf{W}^{out} by (32)
- 7: end for

TABLE VII
PARAMETER SETTINGS OF FIVE ESN ALGORITHMS IN CHAOTIC TIME
SERIES PREDICTION

α	RESN	LESN [40]	CESN [41]	ESN-MEE	ESN-QMEE	
<u> </u>	λ	L	σ	σ	σ	γ
0.1	0.01	0.94	6.3	0.06	0.8	0.07
0.2	0.01	0.92	4.0	0.07	0.7	0.01
0.3	0.1	0.93	3.0	0.08	0.7	0.02
0.4	0.1	0.93	3.0	0.08	0.7	0.03

 $[x(t-24), x(t-18), x(t-12), x(t-6)]^T$ is used as the input to predict the present value x(t) that is the desired response in this example. In the simulation, the number of reservoir units is set to 400. The spectral radius and the sparseness of \mathbf{W}^x are 0.95 and 0.01. A segment of 900 samples are used as the training data and another 400 samples as the testing data. The noise model $v_i = (1-a_i) A_i + a_i B_i$ mentioned in the subsection A is used to generate the noises added to the training data, where the occurrence probability is c=0.2, B_i is a white Gaussian process with zero-mean and variance 0.01, and A_i is a mixture Gaussian process with density $0.5\mathcal{N}(\alpha, 0.01) + 0.5\mathcal{N}(-\alpha, 0.01)$. Further, the normalized root mean squared error (NRMSE) is used to measure the performance of different algorithms, given by

$$NRMSE = \sqrt{\frac{1}{N\sigma_{target}^{2}} \sum_{n=0}^{N-1} (t(n) - y(n))^{2}}$$
 (36)

which σ_{target}^2 denotes the variance of the target signal. Similar to the previous example, the NRMSE of the MEE and QMEE will be calculated by adding a bias value to the testing errors. The parameter settings of five ESN algorithms are given in Table VII. The NRMSEs of six ESN algorithms over 10 Monte Carlo runs for different values of α are illustrated in Fig. 3, and the corresponding training times are shown in Table VIII. Once again, the QMEE based algorithm can outperform other algorithms, whose performance is very close to that of the MEE based algorithm but with much less computational cost.

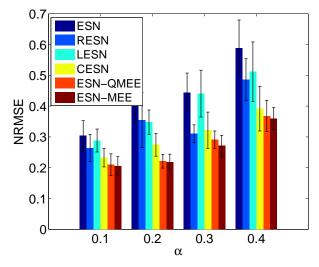


Fig. 3. NRMSE with different values of α

V. CONCLUSION

Minimum error entropy (MEE) criterion can outperform traditional MSE criterion in non-Gaussian signal processing and machine learning. However, it is computationally much more expensive due to the double summation operation in the objective function, resulting in computational expense scaling as $O(N^2)$, where N is the number of samples. In this paper, we proposed a simplified MEE criterion, called quantized MEE (QMEE), whose computational complexity is O(MN), with $M \ll N$. The basic idea is to reduce the number of the inner summations by quantizing the error samples. Some important properties of the QMEE are presented. Experimental results with linear and nonlinear models (such as ELM and ESN) confirm that the proposed QMEE can achieve almost the same performance as the original MEE criterion, but needs much less computational time.

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 $\label{thm:thm:chaotic} \textbf{TABLE VIII}$ Parameter settings of five ESN algorithms in chaotic time series prediction

	ESN	RESN	LESN	CESN	ESN-MEE	ESN-QMEE
Training time (Sec)	0.0336 ± 0.0029	$0.0330 \pm 5.5606 \times 10^{-4}$	0.0328 ± 0.0010	1.4505 ± 0.1640	$2.9903 \times 10^3 \pm 10.3857$	58.3307 ± 1.1894

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