This paper was originally published in

S. Wang and J. Huang, "Adaptive leader-following consensus for multiple Euler-Lagrange systems with an uncertain leader," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 30, No. 7, July 2019, pp. 2162-2388.

We have spotted some typos in equations (19), (20), (22), (23), (24), (25b), (33c), (42), (43) and (45a) where, in some places, the constant μ_2 should be μ_1.

These typos are now corrected and the modified version is posted here.

Adaptive Leader-Following Consensus for Multiple Euler-Lagrange Systems with an Uncertain Leader System

Shimin Wang and Jie Huang, *Fellow, IEEE*

Abstract—In this paper, we study the leader-following consensus problem of multiple Euler-Lagrange systems subject to an uncertain leader system. We first establish an adaptive distributed observer for a neutrally stable linear leader system whose system matrix is not known exactly. Under standard assumptions, this adaptive distributed observer can estimate and pass the leader's state to each follower through the communication network of the system without knowing the leader's system matrix exactly. Under the additional assumption that the leader's state is persistently exciting, this adaptive distributed observer can also asymptotically learn the parameters of the leader's system matrix. On the basis of this adaptive distributed observer, we further synthesize an adaptive distributed control law to solve our problem via the certainty equivalence principle. Our result allows the leader-following consensus problem of multiple Euler-Lagrange systems to be solved even if none of the followers knows the system matrix of the leader system exactly.

Index Terms—Learning control, Networked control systems, Consensus, Euler-Lagrange multi-agent systems, Distributed observer for uncertain leader system,

I. INTRODUCTION

MMULTI-AGENT control systems arise from many engineering applications such as consensus [10, 14, 17, 32], attitude synchronization of rigid spacecraft systems [3, 19], containment control [27], cooperative synchronization of multiple robots [4, 7], and event-triggered distributed control [13, 26, 29]. A comprehensive coverage of the research on the multi-agent control systems can be found in some monographs [12, 18, 20] and the recent survey paper [34].

An important class of multi-agent control systems is the so-called Euler-Lagrange systems, which describes robot manipulators [22, 23], rigid body systems [22], and so on. In recent years, the interest in studying the consensus of multiple Euler-Lagrange systems is growing. There are two types of consensus problems, i.e., the leaderless consensus problem and the leader-following consensus problem. The leaderless consensus problem is to design a distributed control law to make the states and/or outputs of the closed-loop system asymptotically synchronize to a same trajectory determined by the initial condition of the overall system while the leader-following

consensus problem aims to design a distributed control law to drive the states and/or outputs of every follower subsystem to a prescribed trajectory generated by a leader system. The two consensus problems of multiple Euler-lagrange systems have been extensively studied under various scenarios. Reference [15] studied a leaderless consensus problem of multiple EL systems under the assumption that the network topology was strongly connected, balanced and fixed. Reference [21] further considered the leaderless consensus problem for multiple EL systems over a directed, connected graph by a distributed control law. The leader-following consensus problem for multiple EL systems was studied in [7] assuming that all followers have access to the leader's state. Later in [16], under the assumption that the communication topology is static, directed and connected, the leader-following consensus problem was studied for multiple uncertain EL systems via adaptive control. More recently, reference [30] studied the problem of global output-feedback tracking for a class of multiple Euler-lagrange systems subject to static connected communication network without velocity measurement, and reference [33] considered the control of uncertain Euler-lagrange systems subject to full-state constraints.

The leader-following consensus problem for multiple uncertain EL systems subject to jointly connected switching networks was studied in [2], and [4] by distributed observer approach and adaptive distributed observer approach, respectively. A distributed observer is a dynamic compensator that is able to estimate the state of a leader system and pass it to all followers through the communication network of a multi-agent system. It was first proposed in [24, 25] to handle the cooperative output regulation problem for linear multi-agent systems and then adopted in [2] for dealing with the leader-following consensus problem for multiple EL systems. The distributed observer works under the assumption that every follower knows the system matrix of the leader system. To relax this assumption, a so-called adaptive distributed observer was further proposed in [4, 5], which not only estimates the state but also the system matrix of the leader system. As a result, it only requires that the children of the leader know the leader's system matrix and is thus more practical than the distributed observer. For example, in [31], the adaptive distributed observer was used to develop some optimal control protocols for the distributed output synchronization problem of leader-follower multi-agent systems.

Either the distributed observer or the adaptive distributed observer assumes the dynamics of the leader system is known

by at least some followers. In many practical applications, the dynamics of the leader system may not be known exactly by any follower. For example, a sinusoidal signal can be generated by a two dimensional linear leader system. The parameters of the leader system are determined by the frequency of the sinusoidal signal. If the frequency of the sinusoidal signal is unknown, then the system matrix of the leader system is also unknown.

To cater for this more practical scenario, in this paper, we will first establish an adaptive distributed observer for a neutrally stable linear leader system whose system matrix is not known exactly. Under standard assumptions, this adaptive distributed observer can estimate and pass the leader's state to each follower through the communication network of the system without knowing the leader's system matrix exactly. Under the additional assumption that the leader's state is persistently exciting, this adaptive distributed observer can also asymptotically learn the parameters of the leader's system matrix. On the basis of this adaptive distributed observer, we further synthesize an adaptive distributed control law to solve the leader-following consensus problem of multiple Euler-Lagrange systems subject to an uncertain leader system via the certainty equivalence principle. Since the leader's signal in this paper is a multi-tone sinusoidal signal, our result in this paper allows a multi-tone sinusoidal signal to have arbitrary unknown amplitudes, initial phases and frequencies, thus significantly enhancing the results in [2] and [4].

The rest of this paper is organized as follows: In Section II, we formulate our problem and introduce some standard assumptions. In Section III, we establish an adaptive distributed observer for a neutrally stable linear leader system whose system matrix is uncertain. In Section IV, we apply the adaptive distributed observer for the unknown leader system to synthesize an adaptive distributed control law for solving the leader-following consensus problem of multiple Euler-Lagrange systems subject to an uncertain leader system. An example will be given in Section V. Finally, we close this paper in Section VI.

Notation: $\otimes$ denotes the Kronecker product of matrices. For $X_1, \cdots, X_k \in \mathbb{R}^{n \times m}$, let $\mathrm{col}\,(X_1, \cdots, X_k) = [X_1^T, \cdots, X_k^T]^T$,

$$\mathrm{block\ diag}(X_1, \cdots, X_k) = \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_k \end{bmatrix}.$$

For any $x \in \mathbb{R}^m$, unless described otherwise, x_i denotes the ith component of x.

$$\mathrm{diag}(x) = \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_m \end{bmatrix}.$$

For any $x \in \mathbb{R}^m$ with m an even positive integer, let $\phi : \mathbb{R}^m \mapsto \mathbb{R}^{\frac{m}{2} \times m}$ be such that

$$\phi(x) = \begin{bmatrix} -x_2 & x_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -x_m & x_{m-1} \end{bmatrix}. \quad (1)$$

II. Problem Formulation and Assumptions

As in [4], we consider a group of N systems described by the following Euler-Lagrange equations:

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + G_i(q_i) = \tau_i, \ i = 1, \cdots, N \quad (2)$$

where, for $i = 1, \cdots, N$, $q_i \in \mathbb{R}^n$ is the vector of generalized position vector, $M_i(q_i) \in \mathbb{R}^{n \times n}$ is the symmetric positive definite inertia matrix, $C_i(q_i, \dot{q}_i) \in \mathbb{R}^{n \times n}$ is the coriolis and centripetal forces, and $\tau_i \in \mathbb{R}^n$ is the control torque. For $i = 1, \cdots, N$, (2) has the following three properties [22]:

Property 1. *The inertia matrix $M_i(q_i)$ is symmetric and uniformly positive definite,*

Property 2. *For all $a, b \in \mathbb{R}^n$,*

$$M_i(q_i)\,a + C_i(q_i, \dot{q}_i)\,b + G_i(q_i) = Y_i(q_i, \dot{q}_i, a, b)\,\Theta_i,$$

where $Y_i(q_i, \dot{q}_i, a, b) \in \mathbb{R}^{n \times q}$ is a known regression matrix and $\Theta_i \in \mathbb{R}^p$ is a constant vector consisting of the uncertain parameters of (2).

Property 3. $\left(\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)\right)$ *is skew symmetric,* $\forall q_i, \dot{q}_i \in \mathbb{R}^n$.

The leader's signal $q_0 \in \mathbb{R}^n$, which represents the desired generalized position vector, is assumed to be generated by the following exosystem:

$$\dot{v} = S(\omega)v \quad (3a)$$
$$q_0 = Cv \quad (3b)$$

where $v \in \mathbb{R}^m$, $S(\omega) \in \mathbb{R}^{m \times m}$ is the system matrix of $(3a)$, and is allowed to rely on some unknown parameter vector $\omega \in \mathbb{R}^l$, and $C \in \mathbb{R}^{n \times m}$ is a known constant matrix.

As in [4], we view the system composed of (2) and (3) as a multi-agent system of $N+1$ agents with (3) as the leader and the N subsystems of (2) as N followers. The network topology of the multi-agent system composed of (2) and (3) is described by a digraph $\bar{\mathcal{G}} = \left(\bar{\mathcal{V}}, \bar{\mathcal{E}}\right)$ with $\bar{\mathcal{V}} = \{0, \cdots, N\}$ and $\bar{\mathcal{E}} \subseteq \bar{\mathcal{V}} \times \bar{\mathcal{V}}$. Here the node 0 is associated with the leader system (3) and the node i, $i = 1, \ldots, N$, is associated with the ith subsystem of system (2), and, for $i = 0, 1, \ldots, N$, $j = 1, \ldots, N$, $(i, j) \in \bar{\mathcal{E}}$ if and only if τ_j can use the information of agent i for control. Let $\bar{\mathcal{N}}_i = \{j, (j, i) \in \bar{\mathcal{E}}\}$ denote the neighbor set of agent i. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote the subgraph of $\bar{\mathcal{G}}$ where $\mathcal{V} = \{1, \ldots, N\}$, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is obtained from $\bar{\mathcal{E}}$ by removing all edges between the node 0 and the nodes in $\mathcal{V}$.

In terms of $\bar{\mathcal{G}}$, we can describe our problem as follows.

***Problem* 1** (Leader-following Consensus Problem). *Given a system consisting of (2) and (3), and a digraph $\bar{\mathcal{G}}$, find a control law of the following form:*

$$\tau_i = f_i\left(q_i, \dot{q}_i, \varphi_i, \varphi_j - \varphi_i, j \in \bar{\mathcal{N}}_i\right)$$
$$\dot{\varphi}_i = g_i\left(\varphi_i, \varphi_j - \varphi_i, j \in \bar{\mathcal{N}}_i\right), \ i = 1, \cdots, N. \quad (4)$$

where $\varphi_0 = v$, such that, for $i = 1, \cdots, N$, for any initial condition $q_i(0)$, $\dot{q}_i(0)$, $v(0)$ and $\varphi_i(0)$, $q_i(t)$, $\dot{q}_i(t)$, $\varphi_i(t)$ exist and are bounded for all $t \geq 0$ and satisfy

$$\lim_{t \to \infty} (q_i(t) - q_0(t)) = 0, \quad \lim_{t \to \infty} (\dot{q}_i(t) - \dot{q}_0(t)) = 0.$$

Since, for each $i = 1, \cdots, N$, $j = 0, \cdots, N$, the right hand side of (4) depends on $\varphi_j(t)$ only if the jth agent is a neighbor of the ith agent at time t, the control law (4) satisfies the communications constraints. Such a control law is called a distributed control law. The specific form of the functions f_i and g_i, $i = 1, \cdots, N$, will be designed later.

Remark 1. *Problem 1 was studied in [2, 4] using the distributed observer approach and the adaptive distributed observer approach, respectively. In either [2] or [4], the matrix S is assumed to be known exactly. However, this assumption may not be desirable. For example, if the matrix S satisfies Assumption 2 to be introduced below, then the leader's signal is a multi-tone sinusoidal signal. In this case, assuming the matrix S is known exactly amounts to assuming all the frequencies of the leader's signal are known exactly. As will be seen later, our result in this paper will allow us to deal with multi-tone sinusoidal signals with arbitrary unknown amplitudes, initial phases and frequencies.*

We end this section by listing two assumptions for solving Problem 1.

Assumption 1. *$\bar{\mathcal{G}}$ contains a spanning tree with the node 0 as the root, and $\mathcal{G}$ is undirected.*

Assumption 2. *For all ω, all the eigenvalues of $S(\omega)$ are semi-simple with zero real part.*

Remark 2. *Assumption 1 is a quite standard assumption for the leader-following consensus problem for multi-agent systems subject to a static network. Assumption 2 is also assumed in [2] and [4]. Under Assumption 2, without loss of generality, we can always assume that $S(\omega)$ is skew-symmetric and takes the following form:*

$$S(\omega) = \begin{bmatrix} 0_{m_0 \times m_0} & \\ & diag(\omega) \otimes a \end{bmatrix}, \qquad (5)$$

where $a = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\omega = col\ (\omega_{01}, \cdots, \omega_{0l}) \in \mathbb{R}^l$, $m_0 + 2l = m$ and $\omega_{0k} > 0$, for $k = 1, \cdots, l$. For convenience, in what follows, we assume $m_0 = 0$. Thus, $l = \frac{m}{2}$.

III. ADAPTIVE DISTRIBUTED OBSERVER WITH AN UNCERTAIN LEADER

Let us start with this section by recalling the distributed observer for the leader system (3) with ω a known vector proposed in [24] which takes the following form:

$$\dot{\eta}_i = S(\omega)\eta_i + \mu_1 \sum_{j \in \bar{\mathcal{N}}_i} (\eta_j - \eta_i) \qquad (6)$$

where $\eta_i \in \mathbb{R}^m$, $\eta_0 = v$ and μ_1 is a positive number. By Lemma 2 of [24], under Assumptions 1 and 2, for $i = 1, \cdots, N$, $\lim_{t \to \infty} (\eta_i - v) = 0$. That is why (6) is called the distributed observer for the leader system. However, (6) is not fully distributed in the sense that, for every $i = 1, \cdots, N$, (6) has to rely on the system matrix $S(\omega)$ of (3). To overcome this difficulty, reference [4] further proposed the following adaptive

distributed observer for the leader system (3) with ω a known vector:

$$\dot{\eta}_i = S(\omega_i)\eta_i + \mu_1 \sum_{j \in \bar{\mathcal{N}}_i} (\eta_j - \eta_i), \qquad (7a)$$

$$\dot{\omega}_i = \mu_2 \left(\sum_{j \in \bar{\mathcal{N}}_i} (\omega_j - \omega_i) \right), \qquad (7b)$$

where, for $i = 1, \cdots, N$, $\eta_i \in \mathbb{R}^m$, $\omega_i \in \mathbb{R}^l$ is the estimation of ω, $\omega_0 = \omega$, $\eta_0 = v$, μ_1 and μ_2 are some positive numbers. By Lemma 2 of [4], under Assumptions 1 and 2, we have both $\lim_{t \to \infty} (\omega - \omega_i) = 0$ and $\lim_{t \to \infty} (\eta_i - v) = 0$, $i = 1, \cdots, N$. That is why we call (7) an adaptive distributed observer for the leader system. The adaptive distributed observer (7) has the property that, for each $i = 1, \cdots, N$, (7) only relies on the information of itself and its neighbors, and is thus a significant enhancement over the distributed observer (6).

Nevertheless, for those subsystems which are the children of the leader, the adaptive distributed observer (7) still needs to know the leader's frequency ω. Thus, it is still incapable of dealing with the case where the system matrix $S(w)$ of the leader system (3) depends on an unknown vector ω. As pointed out in Remark 1, in many applications, the system matrix $S(w)$ is not known exactly. To handle this case, we further define the following distributed dynamic compensator:

$$\dot{\eta}_i = S(\omega_i)\eta_i + \mu_1 \sum_{j \in \bar{\mathcal{N}}_i} (\eta_j - \eta_i) \qquad (8a)$$

$$\dot{\omega}_i = \mu_2 \phi \left(\sum_{j \in \bar{\mathcal{N}}_i} (\eta_j - \eta_i) \right) \eta_i, \quad i = 1, \cdots, N \qquad (8b)$$

where, $\eta_0 = v$, for $i = 1, \cdots, N$, $\eta_i \in \mathbb{R}^m$, $\omega_i \in \mathbb{R}^l$, $S(\omega_i) = (\text{diag}(\omega_i) \otimes a)$, and μ_1 and μ_2 are some positive numbers.

Remark 3. *What makes (8) interesting is that it is independent of the unknown vector ω. If, for any initial condition, the solution of (8) is bounded over $[0, \infty)$, and $\lim_{t \to \infty} (\eta_i - v) = 0$, $i = 1, \cdots, N$, then we call (8) an adaptive distributed observer for the uncertain leader (3).*

Let $e_{vi} = \sum_{j \in \bar{\mathcal{N}}_i} (\eta_j - \eta_i)$, $\tilde{\eta}_i = (\eta_i - v)$ and $\tilde{\omega}_i = (\omega_i - \omega)$. Then, equation (8) can be rewritten as follows:

$$\dot{\tilde{\eta}}_i = S(\omega)\tilde{\eta}_i + \mu_1 e_{vi} + S(\tilde{\omega}_i)\eta_i \qquad (9a)$$

$$\dot{\tilde{\omega}}_i = \mu_2 \phi(e_{vi})\eta_i, \quad i = 1, \cdots, N. \qquad (9b)$$

To put (9) in a compact form, let

$$\begin{aligned}
\eta &= \text{col}\ (\eta_1, \cdots, \eta_N), \quad \tilde{\eta} = \text{col}\ (\tilde{\eta}_1, \cdots, \tilde{\eta}_N), \\
\tilde{\omega} &= \text{col}\ (\tilde{\omega}_1, \cdots, \tilde{\omega}_N), \quad e_v = \text{col}\ (e_{v1}, \cdots, e_{vN}), \\
S_d(\tilde{\omega}) &= \text{block diag}\ (S(\tilde{\omega}_1), \cdots, S(\tilde{\omega}_N)).
\end{aligned}$$

Then, we have the following relation:

$$e_v = -(H \otimes I_m)\tilde{\eta}, \qquad (10)$$

where H consists of the last N rows and the last N columns of the Laplacian matrix $\bar{\mathcal{L}}$ of the graph $\bar{\mathcal{G}}$ [9]. It can be verified that equations (9) can be put in the following form:

$$\dot{\tilde{\eta}} = (I_N \otimes S(\omega) - \mu_1 H \otimes I_m)\tilde{\eta} + S_d(\tilde{\omega})\eta, \qquad (11a)$$

$$\dot{\tilde{\omega}} = \mu_2 \phi_d(e_v)\eta, \tag{11b}$$

where $\phi_d(e_v) = \text{block diag }(\phi(e_{v1}), \cdots, \phi(e_{vN}))$.

Before stating our main lemmas, we need to establish the following proposition.

Proposition 1. *For any $z \in \mathbb{R}^l$ and $x, y \in \mathbb{R}^m$ with $m = 2l$,*

$$x^T S(z) y = z^T \phi(x) y,$$

where $S(z) = (diag(z) \otimes a)$ and the matrix function $\phi(\cdot)$ is defined in (1).

Proof. Denote $z = \text{col }(z_1, \cdots, z_l)$, $x = \text{col }(x_1, \cdots, x_m)$ and $y = \text{col }(y_1, \cdots, y_m)$. From (1), we have

$$\phi(x)y = \begin{bmatrix} -x_2 & x_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -x_m & x_{m-1} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

$$= \begin{bmatrix} x_1 y_2 - x_2 y_1 \\ \vdots \\ x_{m-1} y_m - x_m y_{m-1} \end{bmatrix},$$

$$z^T \phi(x) y = \sum_{k=1}^{l} z_k (x_{2k-1} y_{2k} - x_{2k} y_{2k-1}).$$

Since

$$S(z) = \text{diag}(z) \otimes a = (\text{diag}(z) \otimes I_2)(I_{m_1} \otimes a),$$

we have

$$(I_l \otimes a) y = \text{col }(y_2, -y_1, \cdots, y_m, -y_{m-1}),$$
$$(diag(z) \otimes I_2) x = \text{col }(z_1 x_1, z_1 x_2, \cdots, z_l x_{m-1}, z_l x_m).$$

Hence, we have

$$\begin{aligned} x^T S(z) y &= x^T (\text{diag}(z) \otimes I_2)(I_l \otimes a) y \\ &= ((\text{diag}(z) \otimes I_2) x)^T ((I_l \otimes a) y) \\ &= \sum_{k=1}^{l} z_k ((x_{2k-1} y_{2k} - x_{2k} y_{2k-1})) \\ &= z^T \phi(x) y. \end{aligned}$$

$\square$

We now ready to establish our main technical lemmas.

Lemma 1. *Consider systems (3) and (11). Under Assumptions 1 and 2, for any $v(0)$, any $\eta(0)$ and $\omega(0)$, any $\mu_1 > 0$ and $\mu_2 > 0$, $\eta(t)$ and $\omega(t)$ exist and are bounded for all $t \geq 0$ and satisfy*

$$\lim_{t \to \infty} \tilde{\eta}(t) = 0, \tag{15}$$
$$\lim_{t \to \infty} \dot{\tilde{\omega}}(t) = 0, \tag{16}$$
$$\lim_{t \to \infty} S_d(\tilde{\omega}) \eta = 0. \tag{17}$$

Proof. Consider the following Lyapunov function candidate for (11):

$$V = \frac{1}{2} \left(\tilde{\eta}^T (H \otimes I_m) \tilde{\eta} + \mu_2^{-1} \tilde{\omega}^T \tilde{\omega} \right). \tag{18}$$

By Lemma 4 of [9], under Assumption 1, the matrix H is symmetric and positive definite. Differentiating (18) along the trajectory of (11) gives

$$\begin{aligned} \dot{V} &= \tilde{\eta}^T (H \otimes I_m) \dot{\tilde{\eta}} + \mu_2^{-1} \tilde{\omega}^T \dot{\tilde{\omega}} \\ &= \tilde{\eta}^T (H \otimes S(\omega)) \tilde{\eta} - \mu_1 \tilde{\eta}^T (H^2 \otimes I_m) \tilde{\eta} \\ &\quad + \tilde{\eta}^T (H \otimes I_m) S_d \eta + \mu_2^{-1} \tilde{\omega}^T \dot{\tilde{\omega}}. \end{aligned} \tag{19}$$

Since $S(\omega)$ is skew symmetric and H is symmetric, $H \otimes S(\omega)$ is skew symmetric. Thus, we have

$$\begin{aligned} \dot{V} &= -\mu_1 \tilde{\eta}^T (H^2 \otimes I_m) \tilde{\eta} - e_v^T S_d \eta + \mu_2^{-1} \tilde{\omega}^T \dot{\tilde{\omega}} \\ &= -\mu_1 \tilde{\eta}^T (H^2 \otimes I_m) \tilde{\eta} \\ &\quad - \sum_{i=1}^{N} e_{vi}^T S(\tilde{\omega}_i) \eta_i + \mu_2^{-1} \tilde{\omega}^T \dot{\tilde{\omega}}. \end{aligned} \tag{20}$$

By Proposition 1, we have, for $i = 1, \cdots, N$,

$$e_{vi}^T S(\tilde{\omega}_i) \eta_i = \tilde{\omega}_i^T \phi(e_{vi}) \eta_i,$$

which can be put in the following compact form:

$$e_v^T S_d(\tilde{\omega}) \eta = \tilde{\omega}^T \phi_d(e_v) \eta. \tag{21}$$

Thus, from (20), we have

$$\begin{aligned} \dot{V} &= -\mu_1 \tilde{\eta}^T (H^2 \otimes I_m) \tilde{\eta} - \tilde{\omega}^T \phi_d(e_v) \eta + \mu_2^{-1} \tilde{\omega}^T \dot{\tilde{\omega}} \\ &= -\mu_1 \tilde{\eta}^T (H^2 \otimes I_m) \tilde{\eta} - \tilde{\omega}^T (\phi_d(e_v) \eta - \mu_2^{-1} \dot{\tilde{\omega}}) \end{aligned} \tag{22}$$

Using (11b) gives

$$\dot{V} = -\mu_1 \tilde{\eta}^T (H^2 \otimes I_m) \tilde{\eta} \leq 0. \tag{23}$$

Since V is positive definite and $\dot{V}$ is negative semi-definite, V is bounded, which means $\tilde{\eta}$ and $\tilde{\omega}$ are bounded. From (11a), $\dot{\tilde{\eta}}$ is bounded, which implies $\ddot{V}$ is bounded. By Barbalat's Lemma, we have

$$\lim_{t \to \infty} \dot{V}(t) = 0,$$

which implies (15). Thus, by (10), we have

$$\lim_{t \to \infty} e_v(t) = 0,$$

which together with (11b) yields (16). To show (17), differentiating $\dot{\tilde{\eta}}$ gives,

$$\begin{aligned} \ddot{\tilde{\eta}} &= (I_N \otimes S(\dot{\omega})) \tilde{\eta} + (I_N \otimes S(\omega) - \mu_1 H \otimes I_m) \dot{\tilde{\eta}} \\ &\quad + S_d(\dot{\tilde{\omega}}) \eta + S_d(\tilde{\omega}) \dot{\eta}. \end{aligned} \tag{24}$$

We have shown $\tilde{\eta}$, $\tilde{\omega}$, and $\dot{\eta}_i$ are all bounded. From (11), $\dot{\tilde{\eta}}$ and $\dot{\tilde{\omega}}$ are also bounded. Thus, $\ddot{\tilde{\eta}}$ is bounded. By Barbalat's Lemma again, we have $\lim_{t \to \infty} \dot{\tilde{\eta}} = 0$, which together with (15) implies (17). $\square$

Remark 4. *As a result of Lemma 1, we have, for $i = 1, \cdots, N$,*

$$\lim_{t \to \infty} (C\eta_i - q_0) = \lim_{t \to \infty} C(\eta_i - v) = 0, \tag{25a}$$

$$\begin{aligned} \lim_{t \to \infty} (C\dot{\eta}_i - \dot{q}_0) &= \lim_{t \to \infty} C(S(\omega_i)\eta_i + \mu_1 e_{vi} - \dot{v}) \\ &= \lim_{t \to \infty} C(S(\tilde{\omega}_i)\eta_i + \mu_1 e_{vi} + S(\omega)\tilde{\eta}_i) \\ &= 0. \end{aligned} \tag{25b}$$

Lemma 1 does not guarantee $\lim_{t\to\infty} \tilde{\omega} = 0$. It is possible to make $\lim_{t\to\infty} \tilde{\omega} = 0$ if the signal $v(t)$ is persistently exciting in the following sense.

Definition 1. *[22] A bounded piecewise continuous function $f : [0, +\infty) \mapsto \mathbb{R}^n$ is said to be persistently exciting (PE) if there exist positive constants ϵ, t_0, T_0 such that,*

$$\frac{1}{T_0} \int_t^{t+T_0} f(s)f^T(s)ds \geq \epsilon I_n, \quad \forall t \geq t_0$$

Remark 5. *A piecewise continuous function $f : [0, \infty) \mapsto \mathbb{R}^n$ is said to have spectral lines at frequencies $\omega_1, \cdots, \omega_n$, if, for $k = 1, \cdots, n$,*

$$\lim_{\delta\to\infty} \frac{1}{\delta} \int_t^{t+\delta} f(s)e^{-jw_k s}ds = \hat{f}(w_k) \neq 0,$$

uniformly in t [1]. It was established in Lemma 3.4 of [1] that $f(t)$ is PE if $\hat{f}(w_k)$, $k = 1, \cdots, n$, are linearly independent over $\mathbb{C}^n$.

We also need the following result which is taken from Lemma 2.4 of [6].

Lemma 2. *Consider a continuously differentiable function $g : [0, +\infty) \mapsto \mathbb{R}^n$ and a bounded piecewise continuous function $f : [0, +\infty) \mapsto \mathbb{R}^n$, which satisfy*

$$\lim_{t\to\infty} g^T(t)f(t) = 0.$$

Then, $\lim_{t\to\infty} g(t) = 0$ holds under the following two conditions:

(i) $\lim_{t\to\infty} \dot{g}(t) = 0$
(ii) $f(t)$ is persistently exciting.

To ensure that the leader's state is PE, we need to strengthen Assumption 2 to the following:

Assumption 3. *For all ω, all the eigenvalues of $S(\omega)$ are simple with zero real part.*

Remark 6. *Under Assumption 3, if m is even, then we can assume that $S(\omega)$ takes the following form:*

$$S(\omega) = diag(\omega) \otimes a; \quad (26)$$

and, if m is odd, then we can assume that $S(\omega)$ takes the following form:

$$S(\omega) = \begin{bmatrix} 0 & 0 \\ 0 & diag(\omega) \otimes a \end{bmatrix}, \quad (27)$$

For convenience, we assume m is even. The case where m is odd can be studied in a way similar to the case where m is even.

Lemma 3. *Under Assumptions 1 and 3, if, for all $i = 1, \cdots, l$ with $l = \frac{m}{2}$, col $(v_{2i-1}(0), v_{2i}(0)) \neq 0$, then,*
(i) $v(t)$ is PE; and
(ii)

$$\lim_{t\to\infty} \tilde{\omega} = 0. \quad (28)$$

Proof. Part (i)
For each $i = 1, \cdots, \frac{m}{2}$,

$$v(t) = \begin{bmatrix} C_1 \sin(\omega_{01}t + \psi_1) \\ C_1 \cos(\omega_{01}t + \psi_1) \\ \vdots \\ C_l \sin(\omega_{0l}t + \psi_l) \\ C_l \cos(\omega_{0l}t + \psi_l) \end{bmatrix}$$

where, for $i = 1, \cdots, l$, $C_i = \sqrt{v_{2i-1}^2(0) + v_{2i}^2(0)}$ and $\tan \psi_i = \frac{v_{2i-1}(0)}{v_{2i}(0)}$. Let $f_i(t) = $ col $(v_{2i-1}(t), v_{2i}(t))$. Then, we have, for $i = 1, \cdots, \frac{m}{2}$,

$$\begin{aligned} f_i(t) &= \begin{bmatrix} v_{2i-1}(t) \\ v_{2i}(t) \end{bmatrix} \\ &= \begin{bmatrix} C_i \sin(\omega_{0i}t + \psi_i) \\ C_i \cos(\omega_{0i}t + \psi_i) \end{bmatrix} \\ &= \begin{bmatrix} C_i \frac{e^{(\omega_{0i}t+\psi_i)j} - e^{-(\psi_i + \omega_{0i}t)j}}{2j} \\ C_i \frac{e^{(\omega_{0i}t+\psi_i)j} + e^{-(\omega_{0i}t+\psi_i)j}}{2} \end{bmatrix} \\ &= \frac{C_i}{2} \begin{bmatrix} e^{-\psi_i j}e^{-\omega_{0i}jt}j - e^{\psi_i j}e^{\omega_{0i}jt}j \\ e^{\psi_i j}e^{\omega_{0i}jt} + e^{-\psi_i j}e^{-\omega_{0i}jt} \end{bmatrix}, \end{aligned}$$

where $j = \sqrt{-1}$. It can be verified that, for $i = 1, \cdots, \frac{m}{2}$,

$$\begin{aligned} \hat{f}_i(\omega_{0i}) &= \lim_{\delta\to\infty} \frac{1}{\delta} \int_\epsilon^{\epsilon+\delta} f_i(t)e^{-\omega_{0i}jt}dt \\ &= \frac{C_i}{2}\text{col } (-e^{\psi_i j}j, e^{\psi_i j}) = \frac{C_i e^{\psi_i j}}{2}\text{col } (-j, 1), \end{aligned}$$

$$\begin{aligned} \hat{f}_i(-\omega_{0i}) &= \lim_{\delta\to\infty} \frac{1}{\delta} \int_\epsilon^{\epsilon+\delta} f_i(t)e^{\omega_{0i}jt}dt \\ &= \frac{C_i}{2}\text{col } (e^{-\psi_i j}j, e^{-\psi_i j}) = \frac{C_i e^{-\psi_i j}}{2}\text{col } (j, 1). \end{aligned}$$

Under Assumption 3, for all $i = 1, \cdots, l$, ω_i are distinct. Thus, for $i = 1, \cdots, \frac{m}{2}$, $\hat{f}_i(\omega_{0i})$ and $\hat{f}_i(-\omega_{0i})$ are linearly independent over $\mathbb{C}^2$ if and only if $C_i \neq 0$. Since, none of C_i is equal to zero due to our assumption on the initial condition $v(0)$, by Remark 5, for $i = 1, \cdots, \frac{m}{2}$, $f_i(t) = $ col $(v_{2i-1}(t), v_{2i}(t))$ is PE. Finally, noting that, for all $i, j = 1, \cdots, \frac{m}{2}$ and $i \neq j$,

$$\frac{1}{T_0} \int_t^{t+T_0} f_i(s)f_j^T(s)ds = 0_{2\times 2}, \quad \forall t \geq t_0$$

concludes that $v(t) = $ col $(f_1(t), \cdots, f_l(t))$ is PE.
Part (ii)
Note that the satisfaction of Assumption 3 implies that of Assumption 2. Thus, (15) and (17) hold, which implies, for $i = 1, \cdots, N$,

$$\begin{aligned} &\lim_{t\to\infty} S(\tilde{\omega}_i) v(t) \\ &= \lim_{t\to\infty} S(\tilde{\omega}_i)(\eta_i - \tilde{\eta}_i) \\ &= \lim_{t\to\infty} S(\tilde{\omega}_i)\eta_i = 0. \end{aligned}$$

Since $v(t)$ is PE, by Lemma 2, we have (28). $\quad\square$

Remark 7. *Reference [28] introduced an adaptive reference generator for the uncertain exosystem (3) with*

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix},$$

and

$$S(\rho) = \left[\begin{array}{c|c} 0 & I_{m-1} \\ \hline -s_1 & -s_2, \cdots, -s_m \end{array}\right],$$

where $\rho = col\,(s_1 s_2, \cdots, s_m)$ is in a known compact set $\mathcal{Q} \subset \mathbb{R}^m$, $S(\rho) \in \mathbb{R}^{m \times m}$ satisfies Assumption 2 for all $\rho \in \mathcal{Q}$.

Under the assumption (Assumption 9 of [28]) that there exists a positive definite matrix $P \in \mathbb{R}^{m \times m}$, independent of ρ, that satisfies, for all $\rho \in \mathcal{Q}$, the following equation

$$P^T S(\rho) + S(\rho)P - 2xPC^TCP + 2bI_m < 0, \quad (30)$$

where b, x are some positive numbers, reference [28] proposed, in terms of our notation, a dynamic compensator of the following form:

$$\dot{\eta}_i = \left[\begin{array}{c|c} 0 & I_{m-1} \\ \hline -\hat{s}_{1,i} & -\hat{s}_{2,i}, \cdots, -\hat{s}_{m,i} \end{array}\right] \eta_i + PC^TCe_{vi}, \quad (31a)$$

$$\dot{\hat{s}}_{i,k} = -\left(e_{vi}^T P^{-1}\right)^{(m)} \eta_i^{(k)}, \quad (31b)$$

where, for $k = 1, \cdots, m$ and $i = 1, \cdots, N$, $\left(e_{vi}^T P^{-1}\right)^{(m)}$ denotes the last element of the row vector $e_{vi}^T P^{-1}$, the $\eta_i^{(k)}$ denotes the kth element in the column vector η_i. It was shown that, under some additional assumptions, the dynamic compensator (31) can also estimate the state of (3) and the unknown parameters s_i, $i = 1, \cdots, m$. Nevertheless, since such matrix P is generally a function of ρ, it is unlikely that Assumption 9 of [28] can be satisfied for all ρ in a given compact set $\mathcal{Q}$.

Other advantages of our adaptive distributed observer (8) include that we only need to estimate $\frac{m}{2}$ unknown parameters by $\frac{mN}{2}$ equations instead of m parameters by $m \times N$ equations as in (31)), we do not limit our unknown parameters to lie in some known compact set $\mathcal{Q}$, and our C matrix can have multiple rows.

IV. LEADER-FOLLOWING CONSENSUS OF MULTIPLE EL SYSTEMS

In this section, we will apply our adaptive distributed observer (8) to synthesize a distributed control law to solve the leader-following consensus problem of multiple Euler-Lagrange systems with an uncertain leader system. As in [4], for $i = 1, \cdots, N$, let

$$\dot{q}_{ri} = CS\,(\omega_i)\,\eta_i - \alpha\,(q_i - C\eta_i), \quad (32a)$$

$$s_i = \dot{q}_i - \dot{q}_{ri}, \quad (32b)$$

where α is positive number.

We now propose our control law as follows:

$$\tau_i = -K_i s_i + Y_i \hat{\Theta}_i, \quad (33a)$$

$$\dot{\hat{\Theta}}_i = -\Lambda_i^{-1} Y_i^T s_i, \quad (33b)$$

$$\dot{\eta}_i = S\,(\omega_i)\,\eta_i + \mu_1 \sum_{j \in \bar{\mathcal{N}}_i} (\eta_j - \eta_i), \quad (33c)$$

$$\dot{\omega}_i = \mu_2 \phi(e_{vi})\eta_i, \quad (33d)$$

where, for $i = 1, \cdots, N$, K_i and Λ_i are positive definite matrices, $\hat{\Theta}_i$ are vectors for estimating Θ_i, Λ_i are diagonal matrices with positive diagonal entries which determine the

update rates of $\hat{\Theta}_i$, and the matrix function $\phi(\cdot)$ is as defined in (1).

Remark 8. *Since the system matrix S of our leader system is not known exactly, the third equations of our control law contain N unknown vectors ω_i, which are updated by the fourth equations of (33). In contrast, in [4], the system matrix $S(\omega)$ of the leader system must be known to all the children of the leader. Therefore, the control law in [4] relies on the exact knowledge of the matrix $S(\omega)$.*

Theorem 1. *Consider systems (2), (3) and the graph $\bar{\mathcal{G}}$. Under Assumptions 1 and 2, Problem 1 is solvable by the control law (33).*

Proof. Differentiating (32) gives, for $i = 1, \cdots, N$,

$$\ddot{q}_{ri} = CS\,(\omega_i)\,\dot{\eta}_i + CS\,(\dot{\omega}_i)\,\eta_i - \alpha\,(\dot{q}_i - C\dot{\eta}_i), \quad (34a)$$

$$\dot{s}_i = \ddot{q}_i - \ddot{q}_{ri}. \quad (34b)$$

By Property 3, there exists a known matrix $Y_i = Y_i\,(q_i, \dot{q}_i, \ddot{q}_{ri}, \dot{q}_{ri})$ and an unknown constant vector Θ_i such that,

$$Y_i \Theta_i = M_i\,(q_i)\,\ddot{q}_{ri} + C_i\,(q_i, \dot{q}_i)\,\dot{q}_{ri} + G_i\,(q_i). \quad (35)$$

Substituting $Y_i \Theta_i$ into (2) gives

$$M_i\,(q_i)\,\ddot{q}_i + C_i\,(q_i, \dot{q}_i)\,\dot{q}_i + G_i\,(q_i) - M_i\,(q_i)\,\ddot{q}_{ri}$$
$$-C_i\,(q_i, \dot{q}_i)\,\dot{q}_{ri} - G_i\,(q_i) + Y_i \Theta_i = \tau_i. \quad (36)$$

Using (32b) and (34b) in (36) gives

$$M_i\,(q_i)\,\dot{s}_i = -C_i\,(q_i, \dot{q}_i)\,s_i + \tau_i - Y_i \Theta_i. \quad (37)$$

Substituting (33a) and (33b) into (37) gives, for $i = 1, \cdots, N$,

$$M_i\,(q_i)\,\dot{s}_i = -C_i\,(q_i, \dot{q}_i)\,s_i - K_i s_i + Y_i \tilde{\Theta}_i, \quad (38a)$$

$$\dot{\tilde{\Theta}}_i = \Lambda_i^{-1} Y_i^T s_i, \quad (38b)$$

where $\tilde{\Theta}_i = \left(\hat{\Theta}_i - \Theta_i\right)$.

For $i = 1, \cdots, N$, let

$$V_i = \frac{1}{2}\left(s_i^T M_i\,(q_i)\,s_i + \tilde{\Theta}_i^T \Lambda_i \tilde{\Theta}_i\right), \quad (39)$$

Then, along the trajectory of (38),

$$\begin{aligned} \dot{V}_i &= s_i^T M_i\,(q_i)\,\dot{s}_i + \frac{1}{2}s_i^T \dot{M}_i\,(q_i)\,s_i + \tilde{\Theta}_i^T \Lambda_i \dot{\tilde{\Theta}}_i \\ &= s_i^T \left(-C_i\,(q_i, \dot{q}_i)\,s_i - K_i s_i + Y_i \tilde{\Theta}_i\right) \\ &\quad + \frac{1}{2}s_i^T \dot{M}_i\,(q_i)\,s_i + \tilde{\Theta}_i^T \Lambda_i \dot{\tilde{\Theta}}_i. \end{aligned}$$

Noting $\left(\dot{M}_i\,(q_i) - 2C_i\,(q_i, \dot{q}_i)\right)$ is skew symmetric gives

$$\dot{V}_i = -s_i^T K_i s_i, \quad i = 1, \cdots, N. \quad (40)$$

Since K_i are positive definite, the vectors s_i and $\tilde{\Theta}_i$ are bounded. We now further show $s_i \to 0$ as $t \to \infty$ using Barbalat's lemma. For this purpose, substituting (32a) into (32b) gives

$$\dot{q}_i + \alpha q_i = s_i + CS\,(\omega_i)\,\eta_i + \alpha C\eta_i. \quad (41)$$

Since s_i is already shown to be bounded, $S(\omega_i)$ and η_i are also bounded by Lemma 1, and α is positive, equation (41) can be viewed as a stable first order linear system in q_i with a bounded input. Thus, both q_i and $\dot{q}_i$ are bounded. Thus, from (32a) and (34a), $\dot{q}_{ri}$ and $\ddot{q}_{ri}$ are bounded. By equation (34b), $\dot{s}_i$ are also bounded. Thus $\ddot{V}_i$ are bounded, which implies $\dot{V}_i$ is uniformly continuous. By Barbalat's lemma, we have, for $i = 1, \cdots, N$, $\dot{V}_i \to 0$ as $t \to \infty$, which implies $s_i \to 0$ as $t \to \infty$.

Using (33c) in (32) gives

$$\dot{q}_i - C\dot{\eta}_i + \alpha(q_i - C\eta_i) = s_i - \mu_1 C \sum_{j \in \bar{\mathcal{N}}_i}(\eta_j - \eta_i). \quad (42)$$

With $e_i = (q_i - C\eta_i)$, equation (42) can be rewritten as

$$\dot{e}_i + \alpha e_i = s_i - \mu_1 C e_{vi}. \quad (43)$$

Since, by Lemma 1, $\lim_{t \to \infty} e_{vi} = 0$, for $i = 1, \cdots, N$, equation (43) can be viewed as a stable first order differential equation in e_i with $(s_i - \mu_1 C e_{vi})$ as the input, which are bounded over $t \geq 0$ and tend to zero as $t \to \infty$. Thus, we conclude that both $e_i = (q_i - C\eta_i)$ and $\dot{e}_i = (\dot{q}_i - C\dot{\eta}_i)$ are bounded over $t \geq 0$ and will tend to zero as $t \to \infty$. These facts together with (25a) and (25b) complete the proof. □

V. AN EXAMPLE

Consider a group of six Euler-Lagrange systems, each of which describes a two-link robot arm whose motion equations taken from [11] are as follows:

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + G_i(q_i) = \tau_i, i = 1, \cdots, 6, \quad (44)$$

where $q_i = \text{col }(q_{i1}, q_{i2})$,

$$M_i(q_i) = \begin{pmatrix} a_{i1} + a_{i2} + 2a_{i3}\cos q_{i2} & a_{i2} + a_{i3}\cos q_{i2} \\ a_{i2} + a_{i3}\cos q_{i2} & a_{i2} \end{pmatrix},$$

$$C_i(q_i, \dot{q}_i) = \begin{pmatrix} -a_{i3}(\sin q_{i2})\dot{q}_{i2} & -a_{i3}(\sin q_{i2})(\dot{q}_{i1} + \dot{q}_{i2}) \\ a_{i3}(\sin q_{i2})\dot{q}_{i1} & 0 \end{pmatrix},$$

$$G_i(q_i) = \begin{pmatrix} a_{i4}g\cos q_{i1} + a_{i5}g\cos(q_{i1} + q_{i2}) \\ a_{i5}g\cos(q_{i1} + q_{i2}) \end{pmatrix},$$

and the unknown vector $\Theta_i = \text{col }(a_{i1}, a_{i2}, a_{i3}, a_{i4}, a_{i5})$. The communication topology is shown in Figure.1, which satisfies Assumption 1. The leader's signal is

$$q_0 = \text{col }(A_1\sin(\omega_{01}t + \phi_1), A_2\sin(\omega_{02}t + \phi_2)),$$

where A_1, A_2, ω_{01} and ω_{02} are arbitrary unknown positive real numbers; and ϕ_1 and ϕ_2 are arbitrary unknown real numbers. This leader's signal can be generated by (3) with

$$S = \begin{bmatrix} 0 & \omega_{01} & 0 & 0 \\ -\omega_{01} & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_{02} \\ 0 & 0 & -\omega_{02} & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Thus, Assumption 3 is also satisfied.

By Lemma 1, for $i = 1, \cdots, 6$, we can design an adaptive distributed observer for (3) as follows:

$$\dot{\eta}_i = S(\omega_i)\eta_i + \mu_1 \sum_{j \in \bar{\mathcal{N}}_i}(\eta_j - \eta_i), \quad (45a)$$

Figure 1. Communication topology $\bar{G}$

$$\dot{\omega}_i = \mu_2\phi(e_{vi})\eta_i, \ i = 1, \cdots, 6, \quad (45b)$$

where $\mu_1 = 10$, $\mu_2 = 20$ and $\omega_i \in \mathbb{R}^2$.

Based on this observer, we can further synthesize a control law of the form (33) with the following parameters: $K_i = 20I_2$, $\alpha = 10$, and $\Lambda_i = 10$. For $i = 1, \cdots, N$, the actual value of Θ_i are given as follows:

$$\Theta_1 = \text{col }(0.64, 1.10, 0.08, 0.64, 0.32),$$
$$\Theta_2 = \text{col }(0.76, 1.17, 0.14, 0.93, 0.44),$$
$$\Theta_3 = \text{col }(0.91, 1.26, 0.22, 1.27, 0.58),$$
$$\Theta_4 = \text{col }(1.10, 1.36, 0.32, 1.67, 0.73),$$
$$\Theta_5 = \text{col }(1.21, 1.16, 0.12, 1.45, 1.03),$$
$$\Theta_6 = \text{col }(1.31, 1.56, 0.22, 1.65, 1.33).$$

Simulation is conducted with the following initial condition: $q_i(0) = 0$, $\hat{\Theta}_i(0) = 0$, randomly chosen $\omega_i(0)$ in the range $[0, 1]$, $i = 1, \cdots, 6$. The leader's initial condition is $v_0(0) = \text{col }(1, 0, 1, 0)$, and the actual unknown frequencies are $\omega_{01} = 4$ and $\omega_{02} = 2$. Figure 2 shows the trajectories of q_i and $\dot{q}_i$, respectively, for $i = 1, \cdots, 6$. Figure 3 shows the tracking performance of q_i and $\dot{q}_i$, for $i = 1, \cdots, 6$.

Since $v_0(0) = \text{col }(1, 0, 1, 0)$, by Lemma 3, $v(t)$ is PE. Thus, the distributed observer (8) will guarantee $\tilde{\omega}_i \to 0$ as $t \to \infty$, for $i = 1, \cdots, 6$. Figures 4 and 5 show the trajectories of the two components of ω_i and $\tilde{\omega}_i$, respectively, for $i = 1, \cdots, 6$.

VI. CONCLUSION

In this paper, we have studied the leader-following consensus problem of multiple Euler-Lagrange systems subject an uncertain leader system. For this purpose, we have first established an adaptive distributed observer for a neutrally stable linear leader system whose system matrix is not known exactly. Under standard assumptions, this adaptive distributed observer can estimate and pass the leader's state to each follower through the communication network of the system without knowing the leader's system matrix. Further, if the leader's state is persistently exciting, this adaptive distributed observer can also asymptotically learn the unknown parameters of the leader's system matrix. On the basis of this adaptive distributed observer, we can synthesize an adaptive distributed control law to solve our problem via the certainty equivalence

Figure 2. Trajectory of q_i and $\dot{q}_i$, $i = 1, \cdots, 6$.

Figure 3. Tracking Performance of q_i and $\dot{q}_i$, $i = 1, \cdots, 6$.

Figure 4. Trajectory of $\tilde{\omega}_i$, $i = 1, \cdots, 6$.

Figure 5. Trajectory of ω_i, $i = 1, \cdots, 6$.

principle. Our result allows the leader-following consensus problem of multiple Euler-Lagrange systems to be solved even if none of our followers knows the exact information of the system matrix of the leader system.

The adaptive distributed observer for the uncertain leader system can also be applied to other problems such as the cooperative output regulation problem as studied in [5]. In the future, we will further consider establishing an adaptive distributed observer for an uncertain leader system subject to a switching topology.

VII. APPENDIX

A brief introduction of graph theory is shown in the following which can be found in [8]:

A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set of nodes $\mathcal{V} = \{1, 2, \cdots, N\}$ and an edge set $\mathcal{E} = \{(i, j), i, j \in$

$\mathcal{V}, i \neq j\}$. Weighted directed graph $\mathcal{G}$ is used to model communication among the N systems. Graph $\mathcal{G}$ consists of a node set $\mathcal{V} = \{1, \cdots, N\}$, and edge set $\mathcal{E} \subseteq (\mathcal{V} \times \mathcal{V})$ and a weighted adjacency matrix $\mathcal{C} = [c_{ij}] \in \mathbb{R}^{N \times N}$ with $c_{ij} \geq 0$. If $c_{ij} > 0$, then $(j, i) \in \mathcal{E}$. The in degree of node i is defined as $d_i = \sum_{j \in \mathcal{N}_i} c_{ij}$. Let $D = diag\{d_1, \cdots, d_N\}$ be the degree matrix of $\mathcal{G}$. The lapalcian matrix of $\mathcal{G}$ is defined as $H = D - \mathcal{C}$. A node i is called a neighbor of a node j if the edge $(i, j) \in \mathcal{E}$. $\mathcal{N}_i$ denotes the subset of $\mathcal{V}$ that consists of all the neighbors of the node i. If the graph $\mathcal{G}$ contains a sequence of edges of the form $(i_1, i_2), (i_2, i_3), \cdots, (i_k, i_{k+1})$, then the set $\{(i_1, i_2), (i_2, i_3), \cdots, (i_k, i_{k+1})\}$ is called a path of $\mathcal{G}$ from i_1 to i_{k+1}, and node i_{k+1} is said to be reachable from node i_1.

REFERENCES

[1] S. Boyd and S. Sastry, "On parameter convergence in adaptive control," *Systems and Control Letters*, vol. 3, pp. 311-319, 1983.

[2] H. Cai and J. Huang, "Leader-following consensus of multiple uncertain Euler-Lagrange systems under switching network topology," *International Journal of General Systems*, vol. 43, no. 3-4, pp. 294–304, Jan, 2014.

[3] H. Cai and J. Huang, "Leader-following attitude consensus of multiple rigid body systems by attitude feedback control," *Automatica*, vol. 69, pp. 87-92, July, 2016.

[4] H. Cai and J. Huang, "The leader-following consensus for multiple uncertain Euler-Lagrange systems with an adaptive distributed observer," *IEEE Transactions on Automatic Control*, vol. 61, no. 10, pp. 3152–3157, Oct, 2016.

[5] H. Cai, F. L. Lewis, G. Hu, and J. Huang, "The adaptive distributed observer approach to the cooperative output regulation of linear multi-agent systems," *Automatica*, vol. 75, pp. 299–305, Nov, 2017.

[6] Z. Chen and J. Huang, *Stabilization and Regulation of Nonlinear Systems.* Springer, 2015.

[7] S.-J. Chung and J.-J. E. Slotine, "Cooperative robot control and concurrent synchronization of Lagrangian systems," *IEEE Transactions on Robotics*, vol. 25, no. 3, pp. 686–700, Jun, 2009.

[8] C. Godsil and G. F. Royle, *Algebraic Graph Theory.* Springer Science & Business Media, 2013, vol. 207.

[9] J. Hu and Y. Hong, "Leader-following coordination of multi-agent systems with coupling time delays," *Physica A: Statistical Mechanics and its Applications*, vol. 374, no. 2, pp. 853-863, 2007.

[10] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol 48, no. 6, pp. 988-1001, Jun. 2003.

[11] F. L. Lewis, C. T. Abdallah, and D. M. Dawson, *Control of Robot Manipulators.* Macmillan New York, 1993, vol. 236.

[12] F. L. Lewis, H. Zhang, K. Hengster-Movric and A. Das, *Cooperative Control of Multi-Agent Systems: Optimal and Adaptive Design Approaches.* London, UK: Springer-Verlag, 2014.

[13] W. Liu and J. Huang, "Cooperative adaptive output regulation for second-order nonlinear multiagent systems with jointly connected switching networks," *IEEE Transactions on Neural Networks and Learning Systems*, Vol. 29, No. 3, pp. 695-705, March 2018.

[14] X. Liu, J. Lam, W. Yu and G. Chen, "Finite-time consensus of multiagent systems with a switching protocol," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 27, no. 4, pp. 853-862, May, 2016.

[15] H. Min, F. Sun, S. Wang, and H. Li, "Distributed adaptive consensus algorithm for networked Euler-Lagrange systems," *IET Control Theory and Applications*, Vol. 5, Issue. 1, 145-154. Jan, 2011.

[16] E. Nuno, R. Ortega, L. Basanez, and D. Hill, "Synchronization of networks of nonidentical Euler-Lagrange systems with uncertain parameters and communication delays," *IEEE Transactions on Automatic Control*, vol. 56, no. 4, pp. 935–941, Jan, 2011.

[17] R. Olfati-Saber, J. A. Fax, and R. M. Murray, Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1), 215-233, 2007.

[18] Z. Qu, *Cooperative Control of Dynamical Systems: Applications to Autonomous Vehicles.* London, UK: Springer-Verlag, 2009.

[19] W. Ren and R. W. Beard, "Decentralized scheme for spacecraft formation flying via the virtual structure approach," *Journal of Guidance, Control, and Dynamics*, vol. 27, no. 1, pp. 73–82, 2004.

[20] W. Ren and R. Beard, *Distributed Consensus in Multivehicle Cooperative Control.* London, UK: Springer-Verlag, 2008.

[21] W. Ren, "Distributed leaderless consensus algorithms for networked Euler-Lagrange systems," *International Journal of Control*, vol. 82, no. 11, pp. 2137–2149, Sep, 2009.

[22] J.J. Slotine and W. Li, *Applied Nonlinear Control*, New Jersey: Prentice Hall, 1991.

[23] M. W. Spong and M. Vidyasagar, *Robot Dynamics and Control*, New York: John Wiley and Sons, 1989.

[24] Y. Su and J. Huang, "Cooperative output regulation of linear multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 4, pp. 1062–1066, Sep, 2012.

[25] Y. Su and J. Huang, "Cooperative output regulation with application to multi-agent consensus under switching network," *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, vol. 42, no. 3, pp. 864–875, Jan, 2012.

[26] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1680-1685, Sep, 2007.

[27] Y. Wang, Y. Song, and W. Ren, "Distributed adaptive finite-time approach for formation-containment control of networked nonlinear systems under directed topology," *IEEE Transactions on Neural Networks and Learning systems*, DOI: 10.1109/TNNLS.2017.2714187, July, 2017.

[28] Y. Wu, R. Lu, P. Shi, H. Su, and Z.-G. Wu, "Adaptive

output synchronization of heterogeneous network with an uncertain leader," *Automatica*, vol. 76, pp. 183–192, Feb, 2017.

[29] X. Yi, W. Lu, and T. Chen, "Pull-based distributed event-triggered consensus for multiagent systems with directed topologies," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 28, no. 1, pp. 71-79, Dec, 2017.

[30] Q. Yang, H. Fang, J. Chen, Z. P. Jiang, and M. Cao, "Distributed global output-feedback control for a class of Euler-Lagrange systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 9, pp. 4855 - 4861, Sept, 2017.

[31] Y. Yang, H. Modares, D. C. Wunsch, and Y. Yin, "Leader-follower output synchronization of linear heterogeneous systems with active leader using reinforcement learning," *IEEE Transactions on Neural Networks and Learning Systems*, DOI: 10.1109/TNNLS.2018.2803059, March, 2018.

[32] L. Zhu, Z. Chen, R. Middleton, "A general framework for robust output synchronization of heterogeneous nonlinear networked systems," *IEEE Transactions on Automatic Control*, vol.61, no.8, pp. 2092–2107, 2016.

[33] K. Zhao, Y. Song, T. Ma, L. He, "Prescribed performance control of uncertain Euler-Lagrange systems subject to full-state constraints," *IEEE Transactions on Neural Networks and Learning Systems*, vol. PP, no. 99, pp. 1-12, Aug. 2017.

[34] B. Zhu, L. Xie, D. Han, X. Meng, and R. Teo, "A survey on recent progress in control of swarm systems," *Science China Information Sciences*, vol. 60, 070201:1–070201:24, 2017.