Reservoir Memory Machines as Neural Computers

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Abstract

Differentiable neural computers extend artificial neural networks with an explicit memory without interference, thus enabling the model to perform classic computation tasks such as graph traversal. However, such models are difficult to train, requiring long training times and large datasets. In this work, we achieve some of the computational capabilities of differentiable neural computers with a model that can be trained very efficiently, namely an echo state network with an explicit memory without interference. This extension enables echo state networks to recognize all regular languages, including those that contractive echo state networks provably can not recognize. Further, we demonstrate experimentally that our model performs comparably to its fully-trained deep version on several typical benchmark tasks for differentiable neural computers.

Keywords: Reservoir Computing, Echo State Networks, Finite State Machines, Neural Turing Machines, Differentiable Neural Computers, Memory-Augmented Neural Networks

1 Introduction

Differentiable neural computers (DNCs) are artificial neural networks that combine a recurrent neural network controller with an external memory to store information without interference over long stretches of time [Graves et al., 2016]. Such networks have achieved impressive successes in recent years, solving tasks such as storing inputs losslessly, performing associative memory recalls, up to question-answering and graph traversal [Csordás and Schmidhuber, 2019, Graves et al., 2016, Rae et al., 2016, Giles et al., 1989]. However, DNCs are difficult to train, typically requiring several ten thousand input sequences until convergence [Collier and Beel, 2018]. In this work, we propose a model that can be trained with simple convex optimization techniques and little data, but still retains a lot of the capabilities of DNCs. The only sacrifice we need to make for these advantages is that examples of viable memory access behavior need to be provided as part of the training data.

Architecturally, our proposed model is an echo state network [Jaeger and Haas, 2004] with an explicit external memory to which access is controlled by a convex classifier - in our case a support vector machine. We show that this external memory enhances the computational capabilities of echo state networks to strictly more than finite state machines, whereas contractive echo state networks cannot recognize some regular languages. More generally, we obtain a model with lossless memory over arbitrarily long time spans, which extends beyond the abilities of high but finite capacity memory reservoirs in recent works [Farkaš et al., 2016, Gallicchio et al., 2018, Voelker et al., 2019].

Finally, our work is an extension of our first version of reservoir memory machines [Paaßen and Schulz, 2020]. In particular, we simplify the architecture by merging read and write behavior into a single classifier, thus making it easier to implement and to train, and we provide a novel model variant that can solve associative recall tasks, which was beyond the old version. As we will see in the experiments, a key reason why these changes work is that we now use a reservoir that can losslessly recall past input states over long stretches of time, namely the Legendre delay network [Voelker et al., 2019].

In summary, our contributions in this paper are:

- A novel reservoir neural network architecture namely the reservoir memory machine (RMM) which is equipped with an external memory but can still be trained using convex optimization,
- a proof that RMMs are strictly more powerful than finite state machines, and
- a series of experiments, demonstrating that RMMs can solve many benchmark tasks for differentiable neural computers that are beyond the abilities of standard recurrent models (including deep ones).

2 Background and Related Work

2.1 Differentiable Neural Computers

The original authors define a differentiable neural computer (DNC) as 'a neural network that can read from and write to an external memory matrix, analogous to the random-access memory in a conventional computer' [Graves et al., 2016]. The mechanism to read from and write to memory is typically content-based, i.e. the controller writes to and reads from locations that are similar to a query vector produced by the controller [Graves et al., 2016]. However, not all memory accesses in computing tasks can be implemented in a content-based fashion. Therefore, the model is extended with a linking matrix that connects subsequent write locations in memory and can thus be used during reading to move spatially in the memory [Graves et al., 2016]. While reviewing the full breadth and depth of neural computing history is beyond the scope of this paper, we wish to note at least that many variations of this basic setup exist, such as additional sharpening operations and more effective initialization schemes [Csordás and Schmidhuber, 2019, Collier and Beel, 2018]. Further, we note the long history of neural computing approaches, going back at least to the neural pushdown automaton models of [Giles et al., 1989].

Instead, our aim in this work is to develop a network that is as easy to train as possible while still retaining some of the computational power of DNCs. In particular, we suggest the following changes. First, we observe that most benchmark tasks for DNCs can be solved without content-based addressing, such that our proposed model rather predicts the memory access location directly. We only use content-based addressing for the associative recall task, where it is necessary. Second, after writing something to memory, we do not erase it anymore. This enables us to merge the write and read head: The first access to a memory location is interpreted as writing, all subsequent accesses as reading. Third, all our memory accesses

are strict and discrete instead of smooth and differentiable. This reduces memory access to a straightforward classification problem, provided that training data for memory access behavior is available. We thus also avoid the need for sharpening operations as suggested in some DNC works [Graves et al., 2016, Csordás and Schmidhuber, 2019]. Finally, and most prominently, we do not train the system end-to-end but only train the memory address classifier and the mapping from state to output. This enables training via simple convex optimization, which thus becomes orders of magnitude faster.

While our simplified model can not be expected to achieve the same computational capabilities as a full DNC (which aims at emulating Turing machines [Collier and Beel, 2018]), we can show that our system is strictly more powerful than finite state machines.

2.2 Finite State Machines

We analyze the computational power of our system in comparison to finite state machines (FSMs), in particular Moore Machines [Moore, 1956]. We define a Moore machine as a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, \rho)$, where Q is a finite set called states, Σ is a finite set called input alphabet, Γ is a finite set called output alphabet, Γ is called the state transition function, Γ is called the state state, and Γ is called the output function. A Moore machine transforms an input sequence Γ is called the output sequence Γ is called the dynamical system Γ is a strict generalization of finite state automata, which can be seen as Moore Machines with the output alphabet Γ = $\{0,1\}$ where Γ is an accepting state and Γ of otherwise. By virtue of this mechanism, a Moore machine can recognize any regular/Chomsky-3 language [Sperschneider and Hammer, 1996].

2.3 Relationship of Recurrent Neural Networks and Finite State Machines

Finite state machines are a particularly interesting model for comparison because their dynamics are very similar to recurrent neural networks. In more detail, we define a recurrent neural network with m inputs, n neurons, and K outputs as a 6-tuple $(U, W, \vec{b}, \sigma, \vec{h}_0, g)$ of matrices $U \in \mathbb{R}^{n \times m}$, $W \in \mathbb{R}^{n \times n}$, as well as a bias vector $\vec{b} \in \mathbb{R}^n$, some function $\sigma : \mathbb{R} \to \mathbb{R}$, an initial state $\vec{h}_0 \in \mathbb{R}^n$ (typically the zero vector), and a continuous output function $g : \mathbb{R}^n \to \mathbb{R}^K$. The system dynamics are defined as follows.

$$\vec{h}_t = f(\vec{x}_t, h_t) := \sigma \left(\mathbf{U} \cdot \vec{x}_t + \mathbf{W} \cdot \vec{h}_{t-1} + \vec{b} \right), \tag{1}$$

$$\vec{y_t} = g(\vec{h_t}), \tag{2}$$

where σ is applied element-wise.

Note that a recurrent neural net has the same Markovian structure as a Moore machine with f being related to δ and g to ρ . Indeed, it is well known that recurrent neural network can implement any finite state machine via a correspondence of neurons and states [Šíma and Wiedermann, 1998]. Even more, by carefully selecting the weights, a recurrent neural network, even with rational-valued weights, can simulate a full Turing machine [Siegelmann and Sontag, 1995]. Interestingly, this impressive computational power does not hold for contractive reservoir neural networks, as we will show in the next section.

2.4 Reservoir Neural Networks and their Computational Limits

We define a reservoir with m inputs and n neurons as a 5-tuple $(\boldsymbol{U}, \boldsymbol{W}, \vec{b}, \sigma, \vec{h}_0)$ with $\boldsymbol{U} \in \mathbb{R}^{n \times m}$, $\boldsymbol{W} \in \mathbb{R}^{n \times n}$, $\vec{b} \in \mathbb{R}^n$, $\sigma : \mathbb{R} \to \mathbb{R}$ and $\vec{h}_0 \in \mathbb{R}^n$ which has the same dynamics as in

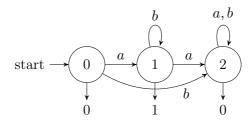


Figure 1: A Moore machine recognizing the language ab^* , which can not be recognized by a contractive echo state network.

Equation 1 but where f ensures the *echo state property*, i.e. that the initial state \vec{h}_0 washes out over time [Jaeger and Haas, 2004, Yildiz et al., 2012]. This property is key to ensure that the state always reacts to the input instead of degenerating to a stable fix point [Jaeger and Haas, 2004] and is typically achieved by designing f as a contractive map [Gallicchio and Micheli, 2011]. In particular, we define a contractive reservoir as follows.

Definition 1 (Contractive reservoir). Let $(U, W, \vec{b}, \sigma, \vec{h}_0)$ be a reservoir. We say that this reservoir is *contractive* with constant $C \in (0,1)$ if for any two $\vec{h}, \vec{h}' \in \mathbb{R}^n$ and any $\vec{x} \in \mathbb{R}^m$ it holds:

$$||f(\vec{x}, \vec{h}) - f(\vec{x}, \vec{h}')|| \le C \cdot ||\vec{h} - \vec{h}'||,$$
 (3)

where f is defined as in Equation 1.

For any contractive reservoir it holds that states with the same suffix get arbitrarily close together for longer suffixes. For example, the states representing the input sequences b^T and ab^{T-1} become indistinguishable for any contractive reservoir for high enough T. However, the sequences are easy to distinguish for a Moore machine (refer to Figure 1), showing that there are finite state machines that can not be represented by a recurrent neural net based on a contractive reservoir, no matter the output function g.

Theorem 1. Consider the Moore Machine \mathcal{A}_{ab^*} illustrated in Figure 1. Now, let $(U, W, \vec{b}, \sigma, \vec{h}_0)$ be some contractive reservoir with constant $C \in (0,1)$. Then, it holds: For any continuous function $g: \mathbb{R}^n \to \mathbb{R}$ there exists some $T \in \mathbb{N}$ and some sequence sequence $x_1, \ldots, x_T \in \{a, b\}^*$ such that $g(\vec{h}_T) \neq z_T$, where \vec{h}_T is the state of the reservoir for the one-hot coding sequence of x_1, \ldots, x_T , and where z_T is the final output of the Moore machine \mathcal{A}_{ab^*} for the sequence x_1, \ldots, x_T .

Proof. Let $\vec{x}_a \in \mathbb{R}^m$ and $\vec{x}_b \in \mathbb{R}^m$ be the one-hot encodings of the symbols a and b, respectively. We define the auxiliary function $\tilde{f}(\vec{h}) := f(\vec{x}_b, \vec{h})$ with f as in Equation 1. Because the reservoir is contractive, \tilde{f} is also contractive. Hence, the Banach fix point theorem implies that there exists a unique fix point $\vec{h}^* \in \mathbb{R}^n$ such that for all $\vec{h} \in \mathbb{R}^n$:

$$\|\tilde{f}^T(\vec{h}) - \vec{h}^*\| \le C^T \cdot \|\vec{h} - \vec{h}^*\|,$$

where \tilde{f}^T denotes T applications of \tilde{f} .

Next, any continuous map $g: \mathbb{R}^n \to \mathbb{R}$ is continuous in \vec{h}^* , specifically. Accordingly, for any $\epsilon > 0$ there exists some $\delta_{\epsilon} > 0$, such that for all $\vec{h} \in \mathbb{R}^n$ with $||\vec{h} - \vec{h}^*|| < \delta_{\epsilon}$ it holds $|g(\vec{h}) - g(\vec{h}^*)| < \epsilon$. Let now $\epsilon = \frac{1}{2}$, let $\vec{h}_b = f(\vec{x}_b, \vec{h}_0)$, and let $\vec{h}_a = f(\vec{x}_a, \vec{h}_0)$. Then, there exists some smallest integer $T \in \mathbb{N}$ such that

$$C^{T-1} \cdot \|\vec{h}_b - \vec{h}^*\| < \delta_{\epsilon} \text{ and } C^{T-1} \cdot \|\vec{h}_a - \vec{h}^*\| < \delta_{\epsilon}.$$

Now, consider the two sequences b^T and ab^{T-1} . Note that \mathcal{A}_{ab^*} produces $z_T=0$ for the first and $z_T'=1$ for the second sequence. Further, let \vec{h}_T and \vec{h}_T' be the states obtained by the reservoir via Equation 1 for the first and second sequence, respectively. Note that we can re-write these states as $\vec{h}_T = \tilde{f}^{T-1}(\vec{h}_b)$ and $\vec{h}_T' = \tilde{f}^{T-1}(\vec{h}_a)$, respectively. Therefore, it holds:

$$\|\vec{h}_T - \vec{h}^*\| = \|\tilde{f}^{T-1}(\vec{h}_b) - \vec{h}^*\| \le C^{T-1} \cdot \|\vec{h}_b - \vec{h}^*\| < \delta_{\epsilon}$$

$$\|\vec{h}_T' - \vec{h}^*\| = \|\tilde{f}^{T-1}(\vec{h}_a) - \vec{h}^*\| \le C^{T-1} \cdot \|\vec{h}_a - \vec{h}^*\| < \delta_{\epsilon}.$$

Accordingly, we obtain:

$$|g(\vec{h}_T) - g(\vec{h}_T')| \le |g(\vec{h}_T) - g(\vec{h}^*)| + |g(\vec{h}_T') - g(\vec{h}^*)| < 2\epsilon = 1.$$

However, to ensure $g(\vec{h}_T) = z_T = 0$ and $g(\vec{h}_T') = z_T' = 1$, we would need $|g(\vec{h}_T) - g(\vec{h}_T')| = 1$. Since this is not possible, $g(\vec{h}_T) \neq z_T$ or $g(\vec{h}_T') \neq z_T'$ (or both).

Importantly, this Theorem only applies if words can be made arbitrarily long. If we consider any finite language, it is, in principle, possible to construct a contractive reservoir which can recognize the language. More generally, [Hammer and Tiňo, 2003] have shown that the behavior of any contractive recurrent system can be arbitrarily well approximated in the maximum norm by a definite memory machine, which is strictly less powerful than a finite state machine. Note that this argument relies on C < 1. It is yet unclear whether reservoirs at the edge of chaos, i.e. $C \ge 1$ [Farkaš et al., 2016, Gallicchio et al., 2018, Gallicchio and Micheli, 2011, Boedecker et al., 2012], extend this capability.

Despite the limitation in computational power, reservoirs have the advantage that they can distinguish sequences up to a fixed horizon T very well, without any need to adjust the matrices U or W to the data [Jaeger and Haas, 2004, Gallicchio and Micheli, 2011, Grigoryeva and Ortega, 2018]. Accordingly, one can leave all reservoir parameters as-is and still obtain a well-performing recurrent neural net by simply optimizing an output function g, e.g. via linear regression [Jaeger and Haas, 2004, Gallicchio and Micheli, 2011, Yamane et al., 2019]. We call such a recurrent neural network an echo state network [Jaeger and Haas, 2004, ESN]. In this work, we extend an echo state network with an explicit memory from which we can recall past reservoir states. Strictly speaking, this violates the echo state property, because we can also recall the initial state. However, this violation is controlled, because a recall only occurs if a trained classifier says so. This controlled violation is what raises the computational power beyond finite state machines.

We note that this paper only covers the echo state network perspective on reservoirs. Liquid state machines [Maass et al., 2002] and the Neural Engineering framework [Stewart, 2012] provide a complementary view, using spikes spikes and/or low-rank weight matrices. However, the basic behavior remains: A reservoir computer is focused on short-term memory and can not distinguish sequences that have a sufficiently long shared suffix [Maass and Markram, 2004].

Finally, our proposed model is an extension of a prior version of reservoir memory machines [Paaßen and Schulz, 2020], where the model stores past inputs that align well with the output sequence. In our current work, we store reservoir states instead of past inputs, such that a single memory entry can summarize a long stretch of past inputs. To ensure that we do not lose the input information, we also propose to use Legendre delay networks as reservoirs which support the lossless reconstruction of inputs from reservoir states [Voelker et al., 2019]. We further replace the alignment mechanism with an additional teaching input channel which the user can use to specify when data should be stored or read for the training data. Finally, we simplify the architecture by merging read and write heads into a single state classifier (except for the associative recall task), thus simplifying training.

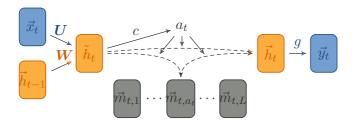


Figure 2: The reservoir memory machine architecture. The preliminary state h_t is computed via Equation 1 as in standard recurrent nets. From \tilde{h}_t we compute the memory address $a_t = c(\tilde{h}_t)$. If $a_t = 0$, the memory is ignored and $\vec{h}_t = \tilde{h}_t$. Otherwise, \tilde{h}_t is written into memory (if $\vec{m}_{t,a_t} = \vec{0}$) and we set $\vec{h}_t = \vec{m}_{t,a_t}$.

3 Method

In this section, we describe our main contribution. In particular, we introduce two mechanisms to extend an echo state network with external memory, prove that this extension suffices to raise the computational power beyond finite state machines, and provide training algorithms for both mechanisms. The first mechanism implements address-based memory with a shared read and write head, whereas the second mechanism implements associative memory with separate read and write heads. We use the first mechanism for most of our experimental tasks and the second only for the associative recall task.

3.1 Standard Reservoir Memory Machines

We define a reservoir memory machine (RMM) with m inputs, n neurons, L rows of memory, and K outputs as an 8-tuple $(\boldsymbol{U}, \boldsymbol{W}, \vec{b}, \sigma, \vec{h}_0, Q, c, g)$, where $(\boldsymbol{U}, \boldsymbol{W}, \vec{b}, \sigma, \vec{h}_0)$ is a reservoir with m inputs and n neurons, $Q = \{0, \dots, L\}$ is called the address set, $c : \mathbb{R}^n \to Q$ is a classifier, mapping the continuous reservoir state to a memory address, and $g : \mathbb{R}^n \to \mathbb{R}^K$ is an output function. The idea behind our RMM architecture is to maintain the usual recurrent neural network dynamic as long as c outputs the zero address, write to the memory whenever a nonzero address is selected the first time, and read from memory whenever a nonzero address is selected a subsequent time.

In more detail, we adjust the system dynamics from Equation 1 as follows, where $\vec{m}_{t,l}$ denotes the *l*th memory entry at time t.

$$\tilde{h}_{t} = f(\vec{x}_{t}, \vec{h}_{t-1}) = \sigma \left(\boldsymbol{U} \cdot \vec{x}_{t} + \boldsymbol{W} \cdot \vec{h}_{t-1} + \vec{b} \right),$$

$$a_{t} = c(\tilde{h}_{t})$$

$$\vec{m}_{t,l} = \begin{cases}
\tilde{h}_{t} & \text{if } l = a_{t} \text{ and } \vec{m}_{t-1,l} = \vec{0} \\
\vec{m}_{t-1,l} & \text{otherwise}
\end{cases}$$

$$\vec{h}_{t} = \begin{cases}
\tilde{h}_{t} & \text{if } a_{t} = 0 \\
\vec{m}_{t,a_{t}} & \text{otherwise}
\end{cases}$$
(4)

where all memory entries are initialized as $\vec{m}_{0,l} = \vec{0}$, except if $c(\vec{h}_0) > 0$. In that case, $\vec{m}_{0,c(\vec{h}_0)} = \vec{h}_0$.

The output is generated as $\vec{y}_t = g(\vec{h}_t)$ as in Equation 2.

The architecture is illustrated in Figure 2. The dynamic is further illustrated in Figure 3. In particular, the reservoir memory machine behaves like a regular echo state neural network

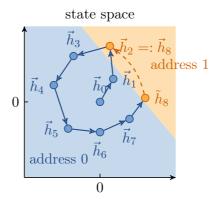


Figure 3: An illustration of the reservoir memory machine dynamics. The first time the continuous state \vec{h}_t crosses into the receptive field of a memory address $a_t > 0$ (orange region), it is stored in memory. Whenever the continuous state re-enters this receptive field, the stored state is recovered (here, for example, $\vec{h}_8 = \vec{h}_2$).

until the classifier c outputs a nonzero memory address. In that case, we record the current state \vec{h}_t in memory, which we recover whenever c outputs the same memory address another time (here in time step 8, where we recover \vec{h}_2). Note that this is a strict generalization over a standard echo state network because we recover Equation 1 if c is constantly zero.

3.2 Computational Power

In this section, we analyze the computational power of our model in more detail. We first introduce the notion of cycles, both in a Moore machine and in an RMM. Then, we show that the state of an RMM resulting from any sequence is equivalent to the state resulting from its cycle-free version. Further, we introduce the notion of a (τ, ϵ) -distinguishing reservoir and finally prove our main result, namely that any Moore machine can be implemented by a reservoir memory machine if the reservoir is (τ, ϵ) -distinguishing for sufficiently large τ . We will further show that there exist tasks for which reservoir memory machines need exponentially less memory compared to finite state machines.

Definition 2 (Cycles). Given a Moore machine \mathcal{A} and an input sequence $x_1, \ldots, x_T \in \Sigma^*$, we define an \mathcal{A} -cycle as a subsequence $x_{t'}, \ldots, x_t$ where $0 \le t' < t \le T$ and $q_{t'} = q_t$.

Similarly, given an RMM \mathcal{M} and an input sequence $\vec{x}_1, \ldots, \vec{x}_T \in (\mathbb{R}^m)^*$, we define an \mathcal{M} -cycle as a subsequence $\vec{x}_{t'}, \ldots, \vec{x}_t$ where $0 \leq t' < t \leq T$ and $a_{t'} = a_t > 0$ (with $a_0 := c(\vec{h}_0)$). Further, we define the \mathcal{M} -cycle-reduced version of $\vec{x}_1, \ldots, \vec{x}_T$ as the result of the following recursive procedure: Identify the largest t such that there exists a t' with $\vec{x}_{t'}, \ldots, \vec{x}_t$ being an \mathcal{M} -cycle. If no such t exists, return the sequence itself. Otherwise, take the lowest such t' and return the \mathcal{M} -cycle-reduced version of $\vec{x}_1, \ldots, \vec{x}_{t'}, \vec{x}_{t+1}, \ldots, \vec{x}_T$.

As an example, consider Figure 3. There, $\vec{x}_2, \ldots, \vec{x}_8$ is an \mathcal{M} -cycle because $a_2 = a_8 = 1 > 0$. The cycle-reduced version of the sequence would be \vec{x}_1, \vec{x}_2 .

Lemma 1. Let \mathcal{M} be an RMM, let $\vec{x}_1, \ldots, \vec{x}_T \in \Sigma^*$, and let $\vec{x}_1', \ldots, \vec{x}_\tau' \in \Sigma^*$ be its \mathcal{M} -cycle reduced version. Then, $\vec{h}_T = \vec{h}_T'$, i.e. the final states for both input sequences are the same.

Proof. We prove this statement via an induction over the number of cycles in $\vec{x}_1, \ldots, \vec{x}_T$. First, assume that $\vec{x}_1, \ldots, \vec{x}_T$ contains no cycles. Then, the claim holds trivially because the sequence is equal to its cycle-reduced version.

Second, assume that $\vec{x}_1,\ldots,\vec{x}_T$ contains at least one cycle and let $\vec{x}_{t'},\ldots,\vec{x}_t$ be the cycle with largest t and smallest t', i.e. the first cycle that would be removed in Definition 2. Then, consider the sequence $\vec{x}_1,\ldots,\vec{x}_{t'}$ and let $\vec{x}'_1,\ldots,\vec{x}'_{\tau}$ be its cycle-reduced version. Because $\vec{x}_1,\ldots,\vec{x}_{t'}$ contains at least one cycle less than $\vec{x}_1,\ldots,\vec{x}_T$, it follows by induction that $\vec{h}_{t'}=\vec{h}'_{\tau}$. Further, because $a_{t'}=a_t$, we know that the RMM recalls the same state in t as in t', which yields $\vec{h}_t=\vec{h}_{t'}=\vec{h}'_{\tau}$. Finally, we know that $\vec{x}_{t+1},\ldots,\vec{x}_T$ does not add any cycles, otherwise t would not have been maximal. Therefore, $\vec{x}'_1,\ldots,\vec{x}'_{\tau},\vec{x}'_{\tau+1},\ldots,\vec{x}'_{\tau+T-t}$ with $\vec{x}'_{\tau+1},\ldots,\vec{x}'_{\tau+T-t}=\vec{x}_{t+1},\ldots,\vec{x}_T$ is exactly the cycle-reduced version of $\vec{x}_1,\ldots,\vec{x}_T$ and we obtain $\vec{h}_T=\vec{h}'_{\tau+T-t}$ as desired. This concludes the proof.

This lemma forms one pillar of our main result. The other pillar is a sufficiently rich reservoir, which we define as follows.

Definition 3 $((\tau, \epsilon)$ -distinguishing reservoirs). Let Σ be a subset of \mathbb{R}^m . Further, let $\mathcal{R} = (U, W, \vec{b}, \sigma, \vec{h}_0)$ be a reservoir with m inputs and n neurons for some n. Then, we call \mathcal{R} a (τ, ϵ) -distinguishing reservoir on Σ if for any two sequences $\vec{x}_1, \ldots, \vec{x}_T \in \Sigma^*$ with $T \leq \tau$ and $\vec{x}'_1, \ldots, \vec{x}'_{T'} \in \Sigma^*$ with $T' \leq \tau$ and $\vec{x}_1, \ldots, \vec{x}_T \neq \vec{x}'_1, \ldots, \vec{x}'_{T'}$ it holds: $||\vec{h}_T - \vec{h}'_{T'}|| \geq \epsilon$, where \vec{h}_T is the state representing the first and $\vec{h}'_{T'}$ is the state representing the second sequence.

We note that our notion of (τ, ϵ) -distinguishing reservoirs is related to the concept of memory capacity [Farkaš et al., 2016] in the sense that a memory capacity of τ requires that input stimuli τ steps in the past can still be distinguished. Past research has demonstrated that any finite memory capacity can be achieved with sufficiently many neurons and a properly constructed reservoir [Rodan and Tino, 2011], such that we assume in the following that a (τ, ϵ) -distinguishing reservoir is available for sufficiently distinct input stimuli Σ (like one-hot codings). Importantly, for any finite τ , this property can be achieved by contractive reservoirs [Rodan and Tino, 2011].

Now follows our main result regarding the computational power of RMMs.

Theorem 2. Let $\mathcal{A} = (Q, \Sigma, \Gamma, \delta, \rho, q_0)$ be a Moore machine with $\Sigma \subset \mathbb{R}^m$, $Q = \{1, \ldots, L\}$, and $\Gamma = \{1, \ldots, K\}$ for some $m, L, K \in \mathbb{N}$. Further, let $(\mathbf{U}, \mathbf{W}, \vec{b}, \sigma, \vec{h}_0)$ be a reservoir with m inputs and n neurons (for some $n \in \mathbb{N}$) that is (L, ϵ) -distinguishing on Σ for some $\epsilon > 0$.

Then, there exist functions $c: \mathbb{R}^n \to \{0, \dots, L\}$ and $g: \mathbb{R}^n \to \{1, \dots, K\}$ such that $\mathcal{M}_{\mathcal{A}} = (\boldsymbol{U}, \boldsymbol{W}, \vec{b}, \sigma, \vec{h}_0, \{0, \dots, L\}, c, g)$ is a reservoir memory machine with the following property: For all sequences $\vec{x}_1, \dots, \vec{x}_T \in \Sigma^*$ it holds: $a_T = q_T$ and $y_T = z_T$, where a_T is the memory address of $\mathcal{M}_{\mathcal{A}}$ at time T, q_T is the state of \mathcal{A} at time T, y_T is the output of $\mathcal{M}_{\mathcal{A}}$ at time T, and z_T is the output of \mathcal{A} at time T.

Proof. First, we introduce two auxiliary functions which compute the state of \mathcal{A} and $\mathcal{M}_{\mathcal{A}}$ respectively, in particular: $\Delta(\vec{x}_1,\ldots,\vec{x}_T) := \delta(\vec{x}_T,\Delta(\vec{x}_1,\ldots,\vec{x}_{T-1}))$ with $\Delta(\varepsilon) = q_0$ and $F(\vec{x}_1,\ldots,\vec{x}_T) = \sigma(\boldsymbol{U}\cdot\vec{x}_T + \boldsymbol{W}\cdot F(\vec{x}_1,\ldots,\vec{x}_{T-1}) + \vec{b})$ with $F(\varepsilon) = \vec{h}_0$.

Next, we define two auxiliary sets, namely a) the set \mathcal{X}_0 of all sequences $\vec{x}_1, \ldots, \vec{x}_T$ which contain no \mathcal{A} -cycle, and b) the set \mathcal{X}_1 of all sequences $\vec{x}_1, \ldots, \vec{x}_T$ where $\vec{x}_1, \ldots, \vec{x}_{T-1}$ contains no \mathcal{A} -cycle but there exists some t < T such that $\Delta(\vec{x}_1, \ldots, \vec{x}_t) = \Delta(\vec{x}_1, \ldots, \vec{x}_T)$, i.e. the sequence contains exactly one cycle that ends in T. Note that $T \leq |Q| = L$, otherwise $\vec{x}_1, \ldots, \vec{x}_{T-1}$ would contain a cycle.

Now, consider the set of tuples $\mathcal{H} = \{ (F(\bar{x}), \Delta(\bar{x})) | \bar{x} \in \mathcal{X}_0 \cup \mathcal{X}_1 \}$. Because we required that the reservoir $(U, W, \vec{b}, \sigma, \vec{h}_0)$ is (L, ϵ) -distinguishing, we obtain for any two $(\vec{h}, q), (\vec{h}', q') \in \mathcal{H}$: If $||\vec{h} - \vec{h}'|| < \epsilon$, then q = q'. Accordingly, we can construct c as a one-nearest neighbor classifier and obtain $c(F(\bar{x})) = \Delta(\bar{x})$ for all $\bar{x} \in \mathcal{X}_0 \cup \mathcal{X}_1$.

We next show via induction over the number of A-cycles that this classifier suffices to ensure $a_T = q_T$ for any input sequence. First let $\vec{x}_1, \ldots, \vec{x}_T \in \Sigma^*$ be \mathcal{A} -cycle free. Then, $a_T =$ $c(F(\vec{x}_1,\ldots,\vec{x}_T)) = \Delta(\vec{x}_1,\ldots,\vec{x}_T) = q_T$ follows immediately from our classifier construction above. Now, assume that $\vec{x}_1, \ldots, \vec{x}_T$ contains at least one \mathcal{A} -cycle and let $\vec{x}_r, \ldots, \vec{x}_s$ with r < sbe the \mathcal{A} -cycle with largest s and smallest r. Because $\vec{x}_1, \ldots, \vec{x}_{s-1}$ contains at least one \mathcal{A} cycle less than before, we know by induction that $q_t = a_t$ for t < s. Accordingly, the equality also holds for the \mathcal{M} -cycle reduced version $\vec{x}'_1, \ldots, \vec{x}'_{\tau}$ of $\vec{x}_1, \ldots, \vec{x}_{s-1}$, which hence implies that $\vec{x}_1', \dots, \vec{x}_\tau'$ is also \mathcal{A} -cycle free. Now, consider the sequence $\vec{x}_1', \dots, \vec{x}_\tau', \vec{x}_s$. Since $\vec{x}_1', \dots, \vec{x}_\tau'$ is \mathcal{A} -cycle free, $\vec{x}'_1, \dots, \vec{x}'_{\tau}, \vec{x}_s$ must lie either in \mathcal{X}_0 or \mathcal{X}_1 . Accordingly, our classifier construction ensures that $q_s = q'_{\tau+1} = a'_{\tau+1}$. Further, thanks to Lemma 1, we know that the state \vec{h}'_{τ} is equal to \vec{h}_{s-1} , which in turn implies that $a'_{\tau+1} = a_s$. Next, because $\vec{x}_{s+1}, \ldots, \vec{x}_T$ adds no \mathcal{A} -cycles (otherwise s would not have been maximal), $\vec{x}'_1, \ldots, \vec{x}'_{\tau}, \vec{x}_{s+1}, \ldots, \vec{x}_T$ is \mathcal{A} -cycle free, which means that our classifier construction ensures that the states $q_{s+t} = q'_{\tau+t} = a'_{\tau+t}$ for all $t \in \{1, \ldots, T-s\}$. Lemma 1 then yields $a'_{\tau+t} = a_{s+t}$, which concludes our proof by induction. It remains to show that $y_T = z_T$. Consider the set of tuples $\mathcal{Y} = \{(F(\bar{x}), \rho(\Delta(\bar{x}))) | \bar{x} \in$ \mathcal{X}_0 . With the same reasoning as before, this yields a well-defined training data set for a 1-nearest neighbor classifier g, which ensures $g(F(\bar{x})) = \rho(\Delta(\bar{x}))$ for all $\bar{x} \in \mathcal{X}_0$.

Furthermore, this construction suffices to ensure $y_T = z_T$ for any input sequence $\vec{x}_1, \ldots, \vec{x}_T$. In particular, Lemma 1 guarantees that $\vec{h}_T = F(x'_1, \ldots, x'_{\tau})$ where $\vec{x}'_1, \ldots, \vec{x}'_{\tau}$ is the \mathcal{M} -cycle free version of $\vec{x}_1, \ldots, \vec{x}_T$. Further, because $q'_t = a'_t$ for all $t \in \{1, \ldots, \tau\}$, $\vec{x}'_1, \ldots, \vec{x}'_{\tau}$ is also \mathcal{A} -cycle free, which ensures that $x'_1, \ldots, x'_{\tau} \in \mathcal{X}_0$ and, hence, $y_T = g(\vec{h}_T) = g(F(\vec{x}'_1, \ldots, \vec{x}'_{\tau})) = \rho(\Delta(\vec{x}'_1, \ldots, \vec{x}'_T)) = z_T$. This concludes the proof.

We note in passing that a one-nearest neighbor classifier is only one possible implementation of c and g. In practice, we use support vector machines (SVMs). SVMs fit well because they are maximum margin classifiers and, as such, can exploit the (τ, ϵ) -distinguishing property of reservoirs [Schölkopf et al., 1997].

We have shown that RMMs are at least as powerful as finite state machines. We now proceed to show that there is at least one task which an RMM can solve for which a finite state machine would require exponentially or infinitely many states.

Theorem 3. Let $\Sigma \subset \mathbb{R}^m$ be a finite set with $\vec{0}, \vec{1} \in \Sigma$. We define the (Σ, τ) -copy task for some $\tau \in \mathbb{N}$ as follows: For any input sequence $\vec{x}_1, \ldots, \vec{x}_T, \vec{1}, \vec{0}, \ldots, \vec{0} \in \Sigma^*$ with $\vec{x}_1, \ldots, \vec{x}_T \notin \{\vec{0}, \vec{1}\}^*$, a suffix of T-1 zeros, and $T \leq \tau$ we define the desired output sequence $\vec{x}_1, \ldots, \vec{x}_T, \vec{x}_1, \ldots, \vec{x}_T$, i.e. the input is copied once.

- a) Let $(\mathbf{U}, \mathbf{W}, \vec{b}, \sigma, \vec{h}_0)$ be a reservoir with m inputs and n neurons that is $(\tau + 1, \epsilon)$ -distinguishing on Σ for some $\epsilon > 0$. Then, there exist two functions $c : \mathbb{R}^n \to \{1, \ldots, \tau\}$ and $g : \mathbb{R}^n \to \Sigma$ such that the reservoir memory machine $(\mathbf{U}, \mathbf{W}, \vec{b}, \sigma, \vec{h}_0, \{0, \ldots, \tau\}, c, g)$ solves the copy task.
 - b) Any Moore machine that solves the copy task has at least $|\Sigma|^T$ states

Proof. a) Because the reservoir is $(\tau+1,\epsilon)$ -distinguishing, the sets $\mathcal{H}_T=\{\vec{h}_T\}$ containing states resulting from sequences of length $T\leq \tau$ are distinguishable with a margin of at least ϵ . Accordingly, we can construct a classifier $c:\mathbb{R}^n\to\{1,\ldots,\tau\}$ (e.g. a one-nearest neighbor-classifier) that maps $c(\vec{h}_T)=T$ for all $T\in\{1,\ldots,\tau\}$. Further, because the reservoir is $(\tau+1,\epsilon)$ -distinguishing, we can also ensure that the classifier maps $c(\vec{h}_{T+1})=1$ if \vec{h}_{T+1} is a state representing a sequence $\vec{x}_1,\ldots,\vec{x}_T,\vec{1}$ with $T\leq \tau$. Finally, also by the $(\tau+1,\epsilon)$ -distinguishing property, we can construct a classifier $g:\mathbb{R}^n\to\Sigma$ which maps the state representing each sequence up to length T to its last symbol, i.e. for all $\vec{x}_1,\ldots,\vec{x}_T\in\Sigma^*$ with $T\leq \tau$, we obtain $g(\vec{h}_T)=\vec{x}_T$.

Now, let $\vec{x}_1,\ldots,\vec{x}_T,\vec{1},\vec{0},\ldots,\vec{0}\in\Sigma^*$ with $T\leq\tau$ be an input sequence for the (Σ,τ) -copy task. By construction, we obtain the memory addresses $a_t=c(\tilde{h}_t)=t$ for $t\in\{1,\ldots,T\}$ and $a_{T+1}=1$ because the sequence $\vec{x}_1,\ldots,\vec{x}_T,\vec{1}$ ends in $\vec{1}$. Accordingly, the RMM re-sets the state to $\vec{h}_{T+1}=\vec{h}_1$ and, hence, the preliminary state \tilde{h}_{T+2} corresponds to the input sequence $\vec{x}_1,\vec{0}$, which in turn results in $c(\tilde{h}_{T+2})=2$, such that we recover $\vec{h}_{T+2}=\vec{h}_2$ and so forth, i.e. the state sequence is $\vec{h}_1,\ldots,\vec{h}_T,\vec{h}_1,\ldots,\vec{h}_T$. Further, $g(\vec{h}_t)=\vec{x}_t$, which implies that the RMM solves the copy task.

b) Assume there exists a Moore machine with less than $|\Sigma|^T$ states that solves the copy task. Because there are $|\Sigma|^T$ different sequences of length T over Σ , there must thus exist two sequences $\vec{x}_1, \ldots, \vec{x}_T \neq \vec{x}'_1, \ldots, \vec{x}'_T$ over Σ which lead to the same state. Accordingly, if we now input T zeros, the output of the Moore Machine will be the same as well, which is incorrect for at least one of the two sequences.

We note that, for fixed sequence length T, this task is simple to solve for a regular echo state network by training g to return $g(\vec{h}_t) = \vec{x}_t + \vec{x}_{t-T}$. However, we define the copy task with variable sequence length, which is not obviously solvable for an ESN but remains simple for an RMM. Further, by extending Σ to an infinite set (e.g. \mathbb{R}^m), the same construction yields that no Moore machine can solve the copy task, whereas an RMM still can (as we show empirically in the experiments).

3.3 Associative Memory

To implement associative memory, we separate write and read access to the memory. Roughly speaking, we change c to a binary classifier that only decides whether we want to write to memory or not, and we read from memory whenever a) we don't write to it and b) the current state is similar enough to some state stored in memory. More precisely, we define an associative RMM (aRMM) with m inputs, n neurons, L rows of memory, d latent dimensions and K outputs as an 11-tuple $(\boldsymbol{U}, \boldsymbol{W}, \vec{b}, \sigma, \vec{h}_0, Q, c, \phi, \psi, \theta, g)$ where $(\boldsymbol{U}, \boldsymbol{W}, \vec{b}, \sigma, \vec{h}_0)$ is a reservoir with m inputs and n neurons as before, $Q = \{1, \dots, L\}$ is an address set as before, $g: \mathbb{R}^n \to \mathbb{R}^K$ is an output function as before, but $c: \mathbb{R}^n \to \{0,1\}$ is now a binary classifier called write head, $\theta \in \mathbb{R}^+$ is a threshold, and $\phi: \mathbb{R}^n \to \mathbb{R}^d$ as well as $\psi: \mathbb{R}^n \to \mathbb{R}^d$ are auxiliary mappings into some latent space in which we measure distance for the purpose of association. In particular, the aRMM dynamic is:

$$\tilde{h}_{t} = f(\vec{x}_{t}, \vec{h}_{t-1}) = \sigma \left(\boldsymbol{U} \cdot \vec{x}_{t} + \boldsymbol{W} \cdot \vec{h}_{t-1} + \vec{b} \right),$$

$$a_{t} = a_{t-1} + c(\tilde{h}_{t})$$

$$\vec{m}_{t,l} = \begin{cases}
\tilde{h}_{t} & \text{if } l = a_{t} \text{ and } c(\tilde{h}_{t}) = 1 \\
\vec{m}_{t-1,l} & \text{otherwise}
\end{cases}$$

$$\vec{h}_{t} = \begin{cases}
\tilde{h}_{t} & \text{if } c(\tilde{h}_{t}) = 1 \text{ or } \forall l : d_{t,l}^{2} > \theta \\
\vec{m}_{t,l^{*}} & \text{otherwise, with } l^{*} = \arg\min_{l} d_{t,l}^{2}
\end{cases}$$

$$d_{t,l}^{2} = \|\phi(\tilde{h}_{t}) - \psi(\vec{m}_{t,l})\|^{2}$$
(5)

where all $\vec{m}_{0,l} = \vec{0}$ and $a_0 = 0$. Note that, once the memory is full, additional writes are ignored.

The system dynamic is illustrated in Figure 4. The orange region in the left part of the figure corresponds to the receptive field of the write head, i.e. where $c(\tilde{h}_t) = 1$. All states in that region are written to memory until the memory is full. The association mechanism works as follows. We map the current state \tilde{h}_t to a latent space (right part of the figure) via the

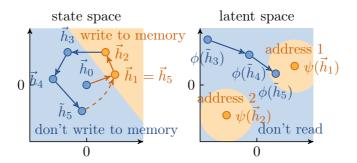


Figure 4: An illustration of the aRMM dynamics. Left: Any state in the orange region is written to memory. The stored states are then mapped via ψ to a latent space (right) and compared with the current latent state $\phi(\vec{h}_t)$. Whenever $\phi(\vec{h}_t)$ enters a θ -ball around $\psi(\vec{m})$ for some memory state \vec{m} , \vec{m} is read from memory and overrides \vec{h}_t .

mapping ϕ and all memory states to the same space via a mapping ψ . Whenever $\phi(\tilde{h}_t)$ enters a θ -ball (orange regions) around some memory state $\psi(\vec{m}_{t,l})$, we set $\vec{h}_t = \vec{m}_{t,l}$. If multiple θ -balls are entered, we take the closest memory state.

3.4 Training Reservoir Memory Machines

The first step in setting up a reservoir memory machine is to initialize a reservoir $(\boldsymbol{U}, \boldsymbol{W}, \vec{b}, \sigma, \vec{h}_0)$ that is expressive enough to enable address classification as well as the output mapping. Although our approach is agnostic regarding the choice of reservoir, we generally recommend Legendre delay networks as reservoirs because they are designed to losslessly extract past states via linear operations, have few hyperparameters, and are deterministically constructed, which avoids issues of unfortunate random initializations [Voelker et al., 2019]. We will also see that this type of reservoir performs best on our benchmarks.

Once a reservoir is set up, we require training data in the form of three sequences $\vec{x}_1, \ldots, \vec{x}_T \in (\mathbb{R}^m)^*$, $a_1, \ldots, a_T \in \{0, \ldots, L\}^*$, and $\vec{y}_1, \ldots, \vec{y}_T \in (\mathbb{R}^K)^*$, where $\vec{x}_1, \ldots, \vec{x}_T$ are the inputs, a_1, \ldots, a_T are the desired memory addresses, and $\vec{y}_1, \ldots, \vec{y}_T$ are the desired outputs. The fact that the memory addresses are part of the training data makes our approach less autonomous compared to differentiable neural computers, which can learn the memory addresses [Graves et al., 2016]. However, we would argue that the memory access patterns for the training data are typically straightforward to construct, at least for all tasks in our experiments (see there). If this is not the case, one would develop data-driven heuristics to recognize special states which need to be stored and recovered, such as a clustering based on the output or an alignment of input and output states as recommended in our past work [Paaßen and Schulz, 2020].

Once such training data is constructed, we can compute the state sequence $\vec{h}_1, \ldots, \vec{h}_T$ by applying the dynamics in Equation 4 or 5, where a_t is given by the training data instead of a classifier, i.e. we perform teacher forcing. We can then train the output function g via linear regression or any other fast optimization, using the training data pairs (\vec{h}_t, \vec{y}_t) .

The final step of our training mechanism is the state classifier c. If we wish to train a standard RMM, we can directly use the pairs (\tilde{h}_t, a_t) as training data. Again, our approach is agnostic to the choice of classifier but we use support vector machines in practice because they are swift to train and, by virtue of being maximum margin classifiers, can exploit the (τ, ϵ) -distinguishing property of reservoirs. This yields the final RMM $(\boldsymbol{U}, \boldsymbol{W}, \vec{b}, \sigma, \vec{h}_0, \{0, \dots, L\}, c, g)$. The exact training scheme is shown in Algorithm 1.

Algorithm 1 The training algorithm for a reservoir memory machine given a reservoir $(\boldsymbol{U}, \boldsymbol{W}, \vec{b}, \sigma, \vec{h}_0)$ with m inputs and n neurons, as well as training inputs $\vec{x}_1, \ldots, \vec{x}_T$, training addresses a_1, \ldots, a_T , and training outputs $\vec{y}_1, \ldots, \vec{y}_T$. For multiple training sequences, lines 3-14 need to be repeated.

```
1: function TRAIN RMM(Reservoir (\boldsymbol{U}, \boldsymbol{W}, \vec{b}, \sigma, \vec{h}_0), training data \vec{x}_1, \dots, \vec{x}_T, a_1, \dots, a_T,
      and \vec{y}_1, \ldots, \vec{y}_T
           L \leftarrow \max\{a_1, \ldots, a_T\}.
 2:
           Initialize M as an L \times n matrix.
 3:
           for t \leftarrow 1, \dots, T do
 4:
                 \tilde{h}_t \leftarrow \sigma \left( \boldsymbol{U} \cdot \vec{x}_t + \boldsymbol{W} \cdot \vec{h}_{t-1} + \vec{b} \right).
 5:
                 if a_t > 0 then
 6:
                       if \vec{m}_{at} = \vec{0} then
 7:
                             \vec{m}_{a_t} \leftarrow \tilde{h}_t.
 8:
                       end if
 9:
                       \vec{h}_t \leftarrow \vec{\vec{m}}_{a_t}.
10:
11:
12:
                 end if
13:
           end for
14:
           Train function g via linear regression with training data \{(\vec{h}_t, \vec{y}_t) | t \in \{1, \dots, T\}\}.
15:
           Train classifier c (e.g. an SVM) with training data \{(\tilde{h}_t, a_t)|t \in \{1, \dots, T\}\}.
16:
           return (U, W, \vec{b}, \sigma, \vec{h}_0, \{0, ..., L\}, c, g).
17:
18: end function
```

By contrast, training an associative RMM is more complicated. We begin with the write head c. In particular, we create a new address sequence $\hat{a}_1, \ldots, \hat{a}_T$ where $\hat{a}_t = 1$ if $a_t > 0$ and there is no t' < t with $a_t = a_{t'}$, and $\hat{a}_t = 0$ otherwise. Then, we train the write head c as a binary classifier (e.g. a Gauss-kernel SVM) on the training data (\tilde{h}_t, \hat{a}_t) . Now, the read head remains, i.e. we need to find ϕ and ψ as well as θ , such that $\|\phi(\tilde{h}_t) - \psi(\vec{m}_{t,l})\|^2$ smaller θ if state \tilde{h}_t and memory line $\vec{m}_{t,l}$ should be associated and larger or equal to θ otherwise. For this purpose, we re-write the data as triples $(\tilde{h}_i, \vec{m}_i, z_i)$ with $z_i = +1$ if \tilde{h}_i and \vec{m}_i should be associated and $z_i = -1$ otherwise. This can be seen as a metric learning problem [Kulis, 2013] or a transfer learning problem [Weiss et al., 2016], either of which is hard to solve in general. Fortunately, our specific problem is simpler because any state \tilde{h}_t can only encode the past inputs. Accordingly, we construct linear operators $\Phi_1, \ldots, \Phi_{\tau} \in (\mathbb{R}^{m \times n})^*$ which reconstruct the input t steps before the current state (for example by using Legendre delay units [Voelker et al., 2019]). Then, we solve the problem:

$$\min_{\mathbf{A} \in \mathbb{R}_{+}^{\tau \times \tau}, \theta \geq 0} \quad \sum_{i} \left[(d_{i}^{2} - \theta) \cdot z_{i} + 1 \right]_{+}$$
s.t.
$$d_{i}^{2} = \sum_{t=1}^{\tau} \sum_{t'=1}^{\tau} \alpha_{t,t'} \cdot \|\mathbf{\Phi}_{t} \cdot \vec{h}_{i} - \mathbf{\Phi}_{t'} \cdot \vec{m}_{i}\|^{2} \qquad \forall i$$

where $[x]_+ = \max\{0, x\}$ denotes the hinge loss. Note that this loss is zero if and only if $d_i^2 - \theta < -1$ if $z_i = +1$ and $d_i^2 - \theta > 1$ if $z_i = -1$, i.e. if and only if all associations are correct and a margin of safety for the associations is maintained. The d_i^2 term in this problem further corresponds exactly to the squared distance $\|\phi(\vec{h}_i) - \psi(\vec{m}_i)\|^2$ by setting $\phi(\vec{h}_i) = (\sqrt{\alpha_{1,1}} \cdot \mathbf{\Phi}_1, \dots, \sqrt{\alpha_{1,\tau}} \cdot \mathbf{\Phi}_1, \dots, \sqrt{\alpha_{\tau,\tau}} \cdot \mathbf{\Phi}_\tau) \cdot \vec{h}_i$ and $\psi(\vec{m}_i) = (\sqrt{\alpha_{1,1}} \cdot \mathbf{\Phi}_1, \dots, \sqrt{\alpha_{1,\tau}} \cdot \mathbf{\Phi}_1, \dots, \sqrt{\alpha_{1,\tau}} \cdot \mathbf{\Phi}_\tau)$

Algorithm 2 The training algorithm for an associative reservoir memory machine given a reservoir $(U, W, \vec{b}, \sigma, \vec{h}_0)$ with m inputs and n neurons, as well as training inputs $\vec{x}_1, \ldots, \vec{x}_T$, training addresses a_1, \ldots, a_T , and training outputs $\vec{y}_1, \ldots, \vec{y}_T$.

```
1: function TRAIN_ARMM(Reservoir (\boldsymbol{U}, \boldsymbol{W}, \vec{b}, \sigma, \vec{h}_0), training data \vec{x}_1, \dots, \vec{x}_T, a_1, \dots, a_T,
      and \vec{y}_1, \ldots, \vec{y}_T)
            L \leftarrow \max\{a_1, \ldots, a_T\}.
 2:
            Initialize M as an L \times n matrix.
 3:
 4:
            Initialize \mathcal{H} as empty set.
           for t \leftarrow 1, ..., T do
\tilde{h}_t \leftarrow \sigma \left( \boldsymbol{U} \cdot \vec{x}_t + \boldsymbol{W} \cdot \vec{h}_{t-1} + \vec{b} \right).
 5:
 6:
 7:
                 if a_t > 0 and \vec{m}_{a_t} = \vec{0} then
 8:
                       \vec{m}_{a_t} \leftarrow h_t.
                                                                                                                            ▶ Write to memory
 9:
                       \hat{a}_t \leftarrow 1.
10:
                 else
11:
                       for l \in \{1, \ldots, L\} \setminus \{a_t\} do
12:
                             Add (\tilde{h}_t, \vec{m}_l, -1) to \mathcal{H}.
13:
                       end for
14:
                       if a_t > 0 then
15:
                             Add (\tilde{h}_t, \vec{m}_{a_t}, +1) to \mathcal{H}.
16:
                             \vec{h}_t \leftarrow \vec{m}_{a_t}.
                                                                                                                        ▶ Read from memory
17:
18:
                       else
                            \vec{h}_t \leftarrow \tilde{h}_t.
19:
                       end if
20:
                 end if
21:
            end for
22:
           Train function g via linear regression with training data \{(\vec{h}_t, \vec{y}_t) | t \in \{1, \dots, T\}\}.
23:
            Train classifier c (e.g. an SVM) with training data \{(\tilde{h}_t, \hat{a}_t) | t \in \{1, \dots, T\}\}.
24:
            Train \phi, \psi, and \theta via problem 6 for training data \mathcal{H}.
25:
            return (\boldsymbol{U}, \boldsymbol{W}, b, \sigma, h_0, Q, c, \phi, \psi, \theta, g).
26:
27: end function
```

 $\Phi_{\tau}, \ldots, \sqrt{\alpha_{\tau,\tau}} \cdot \Phi_{\tau}) \cdot \vec{m}_i$, where the commas indicate row-wise concatenation and $\alpha_{i,j}$ is the entry of the *i*th row and *j*th column of A. This is a highly sparse linear program, which can thus be solved efficiently with standard LP solvers. We emphasize that the same scheme works for pre-defined non-linear operators, similar to multiple kernel learning [Gönen and Alpaydın, 2011]. Further note that we can omit operators in the concatenation where $\alpha_{t,t'}=0$, which we can incentivize with an L1 regularization term. This completes the construction of the associative RMM $(U, W, \vec{b}, \sigma, \vec{h}_0, Q, c, \phi, \psi, \theta, g)$. The full training algorithm for an associative RMM is shown in Algorithm 2.

4 Experiments and Results

In this section, we introduce our seven benchmark tasks, strategies for generating the required memory addresses, explain the experimental setup, and present our results.

4.1 Benchmark Tasks

We evaluate reservoir memory machines (RMMs) on the following seven benchmark tasks:

latch [Paaßen and Schulz, 2020] The input is a one-dimensional time series of length 9-200 that is always zero except for ones at 3 random time points. The desired output is zero until the first one in the input, where it switches to one, then back to zero at the next one, and back to one at the third one.

copy [Collier and Beel, 2018] The input is a nine-dimensional time series of 1-20 random vectors from $\{0,1\}^8$ on the first eight channels, followed by zeros. The last input channel is zero except for a one right before and after the input vectors. The desired output is a copy of the first eight channels as in Theorem 3.

repeat copy [Collier and Beel, 2018] The input is a nine-dimensional time series of 1-10 random vectors from $\{0,1\}^8$ on the first eight channels, followed by zeros. The desired output are 1-10 copies of the first eight input channels. Each copy is preceded by a one on the last input channel, which is zero otherwise. Refer to Figure 5 (left) for an example.

associative recall [Collier and Beel, 2018] The input is a seven-dimensional time series with 2-6 random blocks á 3 random vectors from $\{0,1\}^6$ on the first six channels, followed by a one on the seventh channel and then a random repeated block from the previous input. The desired output is the block *after* the presented element. Refer to Figure 5 (center left) for an example.

smooth associative recall The input is a two-dimensional time series of two smooth random wavelets of length 256 on the first channel, followed by 1-10 blocks of 256 zeros each. On the second channel, each block of 256 time steps ends with one of two marker wavelets of length 32. The first block with marker one, the second with marker two, and a random marker (one or two) otherwise. The desired output responds to each marker with its associated wavelet in the respective next 256 time steps. Refer to Figure 5 (center right) for an example.

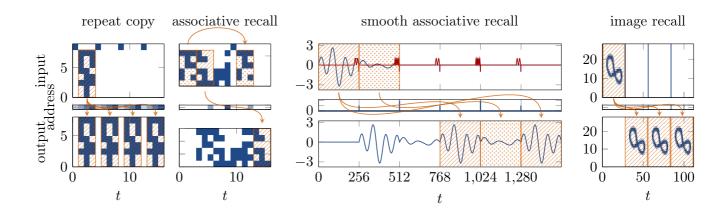


Figure 5: An example from the repeat copy (left), associative recall (center left), smooth associative recall (center right), and image recall (right) data sets, each with input in the first, memory addresses in the second, and output in the third row. Task-relevant blocks are highlighted with stripes/dots and arrows.

Image recall The input is a random 28×28 grayscale image from the MNIST data set, followed by 1-10 vectors of ones. After each such vector, the output should be a copy of the input image. Refer to Figure 5 (right) for an example.

FSMs For finite state machine learning we construct Moore machines with $\Sigma = \Gamma = \{(1,0)^T, (0,1)^T\}$, $Q = \{1,2,3,4\}$ and randomly sampled transition as well as output functions. As training data, we construct all sequences with exactly one repeated state in the Moore machine as suggested in Theorem 2. The test data consists of much longer sequences of length 256 over Σ and the output the Moore machine would predict.

We note that, among these tasks, only latch and FSM can be solved straightforwardly with Moore machines. Copy, repeat copy, image recall, and associative recall require exponentially many states by the same argument as in Theorem 3, and smooth associative recall is not solvable with a Moore machine because there are infinitely many smooth wavelets that could be generated.

4.2 Generating Memory Addresses

To train RMMs, we require example memory address sequences for the training data. The optimal strategy for such example address sequences depends on the task. We generally recommend the following strategies:

For tasks with discrete and abstract outputs such as latch, we suggest to apply a clustering on the target outputs. More specifically, for latch the memory address sequence becomes the training output +1 (i.e. naming the clusters 1 and 2).

For tasks which aim at storing information and recalling it in the same order, such as copy and repeat copy, we suggest to use the state sequence $1, \ldots, T, 1, \ldots, T, \ldots, T$, i.e. to enumerate each input vector and then recall each vector to generate the copy. This is equivalent to the proof of Theorem 3.

For block-wise recall tasks, such as smooth associative recall, image recall, and associative recall we suggest to store the state after each block in memory and load it when recalling the block. Otherwise, the memory address should be zero. More specifically, for smooth associative recall the state sequence is zero except after each block of 256 where it is 1 for marker one and 2 for marker two. For image recall, the address is 1 at positions 29, 57, etc.

dataset	rand	CRJ	LDN	GRU	GRU-MM	rand-RMM	CRJ-RMM	LDN-RMM
latch	0.66 ± 0.29	0.50 ± 0.04	0.53 ± 0.02	0.05 ± 0.04	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
copy	0.39 ± 0.02	0.45 ± 0.01	0.34 ± 0.01	0.39 ± 0.01	0.03 ± 0.00	0.43 ± 0.03	0.48 ± 0.02	0.09 ± 0.06
repeat copy	0.44 ± 0.01	0.46 ± 0.01	0.44 ± 0.02	0.45 ± 0.02	0.02 ± 0.00	0.37 ± 0.02	0.43 ± 0.01	0.01 ± 0.02
smooth recall	12.21 ± 22.73	12.07 ± 22.76	11.06 ± 19.69	11.49 ± 11.90	11.50 ± 11.92	12.21 ± 22.72	12.05 ± 22.76	4.79 ± 20.21
image recall	69.92 ± 2.92	88.60 ± 47.37	97.83 ± 193.09	-	-	70.22 ± 3.29	103.48 ± 60.74	26.91 ± 9.59
FSMs	0.41 ± 0.20	0.72 ± 0.39	0.56 ± 0.17	-	-	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
assoc. recall	0.42 ± 0.01	0.31 ± 0.01	0.31 ± 0.01	_	_	0.43 ± 0.02	0.20 ± 0.05	0.10 ± 0.08

Table 1: The root mean square error for all data sets and all models (\pm std. dev.).

and zero otherwise. For associative recall the state sequence is $0, 0, 0, 0, 0, 1, 0, 0, 2, \ldots$, i.e. we mark each block of 3, except the first one, and recall the index of the state to be recalled after the presented item.

For FSM learning we use the ground-truth FSM to provide the state labels. We also evaluate the alternative of first learning an FSM from data and using that to provide labels.

For repeat copy, associative recall, smooth associative recall, and image recall, example inputs (top), outputs (bottom), and address sequences (center) are shown in Figure 5.

4.3 Experimental Setup

We use a standard RMM with 64 neurons for latch, smooth associative recall, and FSM learning, a standard RMM with 256 neurons for copy and repeat copy, a standard RMM with 512 neurons for image recall, and an associative RMM with 256 neurons for associative recall. We compare against echo state networks [Jaeger and Haas, 2004, ESN] without an external memory and the same number of neurons. We compare three kinds of reservoir, namely Gaussian random numbers normalized to a spectral radius < 1 (rand), cycle reservoirs with jumps [Rodan and Tiňo, 2012, CRJ], and Legendre delay networks [Voelker et al., 2019, LDN]. In each case we optimize the reservoir hyperparameters via random search with 20 trials and 3 repeats per trial. An exception is the time horizon T of a Legendre delay network because it follows logically from the task, namely the time series length for latch, 20 for copy, 10 for repeat copy, $6 \cdot 3$ for associative recall, 256 for smooth associative recall, 28 for image recall, and 4 for FSMs. For RMMs, we add the kernel of the SVM classifiers (linear or RBF with automatic bandwidth choice) to the hyperparameters to be trained.

On the first four tasks, we additionally compare against deep learning models with the same number of neurons, namely a gated recurrent unit [Cho et al., 2014, GRU], and a deep version of our reservoir memory machine with a GRU as a recurrent controller (GRU-MM), a softmax state classifier, and linear output. We train this GRU-MM to minimize the mean squared error on the output plus the crossentropy loss on the state predictions. We train the deep models with an ADAM optimizer with learning rate 10^{-3} , weight decay of 10^{-8} , and minibatch size of 32. We stop the training after 1000 minibatches or if the loss is below 10^{-3} .

All models are trained on 90 training sequences and 10 test sequences from the data sets. We repeat all experiments 20 times for the reservoir models and 3 times for the deep models. All experiments were performed on a desktop PC with Intel core i9-10900 CPU and 32 GB RAM.

All experimental source code and reference implementations are available at https://gitlab.com/bpaass

4.4 Results

The average root mean square errors (\pm std.) for all models on all data sets is shown in Table 1. As can be seen, all RMM variants outperform ESNs on latch and FSM learning, but only the LDN-RMM variant also achieves better results on copy, repeat copy, associative

Table 2: The average runtime (without hyperparameter optimization) in seconds for all data sets and all models (\pm std. dev.).

dataset	rand	CRJ	LDN	GRU	GRU-MM	$\operatorname{rand-RMM}$	CRJ-RMM	LDN-RMM
latch	0.09 ± 0.00	0.09 ± 0.00	0.11 ± 0.00	510.27 ± 8.95	1180.03 ± 17.25	0.19 ± 0.02	0.18 ± 0.02	0.20 ± 0.02
copy	0.09 ± 0.01	0.03 ± 0.00	0.04 ± 0.00	448.03 ± 6.22	475.95 ± 8.50	1.83 ± 0.17	2.78 ± 0.12	0.65 ± 0.03
repeat copy	0.23 ± 0.04	0.06 ± 0.01	0.07 ± 0.01	1063.18 ± 29.97	869.85 ± 20.84	1.88 ± 0.25	6.20 ± 0.63	1.81 ± 0.27
smooth recall	1.91 ± 0.05	1.77 ± 0.04	1.98 ± 0.04	12147.15 ± 87.73	27377.17 ± 102.16	2.96 ± 0.08	2.73 ± 0.06	2.93 ± 0.08
image recall	1.81 ± 0.13	0.53 ± 0.02	0.50 ± 0.03	-	-	18.63 ± 0.66	1.77 ± 0.11	28.84 ± 32.88
FSMs	0.06 ± 0.01	0.04 ± 0.01	0.06 ± 0.01		-	0.26 ± 0.01	0.23 ± 0.01	0.25 ± 0.02
assoc, recall	0.10 ± 0.01	0.03 ± 0.01	0.05 ± 0.02	=	_	1.39 ± 0.14	1.70 ± 0.27	1.37 ± 0.17

recall, smooth associative recall, and image recall. This is likely due to the fact that LDNs guarantee lossless reconstruction of past inputs, which is required for these five tasks. In comparison to the deep learning models we note that the standard GRU performs well on latch, but comparable to standard ESNs on copy, repeat copy, and smooth associative recall, which replicates earlier results on differentiable neural computers [Graves et al., 2016]. By contrast, a GRU-MM can solve all tasks but smooth associative recall, but the error remains comparable to the LDN-RMM.

Regarding FSM learning, we also consider a setting where we first learn a finite state machine via Gold's algorithm [de la Higuera, 2010] on the training sequences and then use the states of this surrogate FSM as training memory address sequence. This retains a perfect result in 19 out of 20 repeats but has one repeat with 0.32 error, yielding an error of 0.02 ± 0.07 for all RMM variants, which is still close to optimal.

Table 2 shows the time needed for training and prediction (without hyperparameter optimization; averaged across experimental repeats) as measured by Pythons time function. Unsurprisingly, we observe that CRJ-ESNs are the fastest because they operate with a highly sparse matrix that is fast to initialize. We also note that LND-RMMs are roughly 20 times slower than LND-ESNs but remain in the second range. The overhead is driven by the SVM training, which is less costly on data sets with small explicit memory and, thus, fewer classes (such as latch and smooth associative recall). By contrast, GRU-MMs are roughly 1000 times slower compared to LND-RMMs and take minutes to hours to train.

5 Conclusion

We introduced a novel reservoir memory machine (RMM) architecture which extends echo state networks with an external memory and thereby becomes computationally strictly more powerful than finite state machines, whereas contractive echo state networks cannot recognize some regular languages. In addition to these theoretical results, we showed that RMMs can solve several benchmark tasks of differentiable neural computers which are out of reach for standard recurrent neural networks. These successes require a sacrifice, namely that examples of memory access behavior need to be supplied as part of the training data. Future research should investigate how optimal memory access behavior can instead be learned from data, perhaps via heuristics. Still, our model makes neural computation much more efficient to train, requiring both less time and less training data compared to differentiable neural computers.

Acknowledgment

Funding by the German Research Foundation (DFG) under grant numbers PA 3460/1-1 and PA 3460/2-1 is gratefully acknowledged.

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A Conversion of FSMs to RNNs

As an appendix to our main paper, we provide here a variant of a proof that any finite state machine can be simulated with a recurrent neural net, using neurons as representations of the states. Note that this is opposed to our simulation via reservoir memory machines, where we use the entire state vector, and not just a single neuron, as representation of a Moore machine state.

Theorem 4. Let $(Q, \Sigma, \Gamma, \delta, \rho, q_0)$ be a Moore machine with $Q = \{1, \ldots, L\}$, $\Sigma = \{e_1, \ldots, e_m\}$ and $\Gamma = \{e_1, \ldots, e_K\}$, where e_i is the ith unit basis vector. Further, let x_1, \ldots, x_T be any sequence over Σ . Then, there exists a recurrent neural network $(\boldsymbol{U}, \boldsymbol{W}, \boldsymbol{V}, \vec{b}, \sigma, \vec{h}_0)$ with $\boldsymbol{U} \in \mathbb{R}^{n \times m}$, $\boldsymbol{W} \in \mathbb{R}^{n \times n}$, $\boldsymbol{V} \in \mathbb{R}^{K \times n}$, \vec{b} , $\vec{h} \in \mathbb{R}^n$ and $n = L \cdot (m+1)$.

Further, let \vec{h}_t be the state and \vec{y}_t be the output of the recurrent neural net at step t of processing the input sequence $x_1, \vec{0}, x_2, \ldots, \vec{0}, x_T$ according to Equation 1, and let q_t be the state and y_t be the output of the Moore machine at step t of processing the input sequence x_1, \ldots, x_T . Then it holds for all $t \in \{1, \ldots, T\}$: $\vec{h}_{2t} = e_{q_t}$ and $\vec{y}_{2t} = y_t$.

Proof. The idea of our proof is quite simple. Our aim is to set up \vec{h}_{2t-1} such that the coordinates $L+1,\ldots,(m+1)\cdot L$ represent the tuple (q_{t-1},x_t) via one-hot coding, which then makes it trivial to map \vec{h}_{2t-1} to $\vec{h}_{2t}=e_{q_t}$.

In particular, we set the weight matrices U, W, and V as well as the bias vector \vec{b} as follows.

$$w_{k,l} = \begin{cases} 1 & \text{if } \lfloor (l-1)/L \rfloor = i, \mod(l,L) = j \text{ and } \delta(e_i,j) = k \\ 1 & \text{if } k > L \text{ and } \mod(k,L) = l \\ 0 & \text{otherwise} \end{cases}$$

$$u_{k,i} = \begin{cases} 1 & \text{if } 0 < \lfloor (k-1)/L \rfloor = i \le m \\ 0 & \text{otherwise} \end{cases}$$

$$b_k = \begin{cases} -\frac{1}{2} & \text{if } k \le L \\ -\frac{3}{2} & \text{otherwise} \end{cases}$$

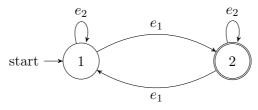
$$v_{j,k} = \begin{cases} 1 & \text{if } \rho(k) = e_j \\ 0 & \text{otherwise} \end{cases}$$

We finally set σ as the Heaviside function, i.e. $\sigma(x)=1$ if x>0 and $\sigma(x)=0$ otherwise, and \vec{h}_0 as e_{q_0} . An example of this translation is visualized in Figure 6 for the Moore machine $(\{1,2\},\{e_1,e_2\},\{e_1,e_2\},\delta,\rho,0)$ with $\delta(e_1,1)=\delta(e_2,2)=2$ and $\delta(e_1,2)=\delta(e_2,1)=1$ as well as $\rho(1)=e_1$ and $\rho(2)=e_2$. An example processing of the input sequence e_1,e_2,e_2,e_1 would be processed by the neural net as shown in Table 3.

Now, consider the claim $\vec{h}_{2t} = e_{qt}$. We prove this claim via induction over t. First, for t = 0, the claim holds because we set $\vec{h}_0 = e_{q_0}$. Next, consider t > 0 and let's inspect the state \vec{h}_{2t-1} . We wish to show that $h_{2t-1,k} = 1$ if $e_{\lfloor (k-1)/L \rfloor} = x_t$ and $\mod(k,L) = q_{t-1}$ and $h_{2t-1,k} = 0$ otherwise. Due to induction we know that $\vec{h}_{2t-2} = e_{q_{t-1}}$. Hence, we know that the sum $\sum_l w_{k,l} \cdot h_{2t-2,l}$ is 1 if k > L as well as $\mod(k,L) = q_{t-1}$. In any other case, this sum is zero because all coordinates $h_{2t-2,l}$ for l > L are zero.

Further, we know that the sum $\sum_i u_{k,i} \cdot x_{t,i}$ is 1 if there exists an $i \in \{1, \ldots, m\}$ such that $\lfloor (k-1)/L \rfloor = i$ and $e_i = x_t$. Otherwise, the sum is zero. Plugging these results together we obtain that the sum $\sum_l w_{k,l} \cdot h_{2t-2,l} + \sum_i u_{k,i} \cdot x_{t,i}$ is zero if $k \leq L$, is 1 if k > L and either mod $(k, L) = q_{t-1}$ or $\lfloor (k-1)/L \rfloor = i$, but not both, and is 2 if both $\mod(k, L) = q_{t-1}$

Moore Machine



Recurrent neural net

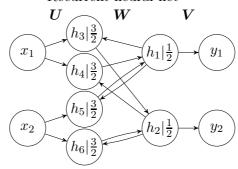


Figure 6: An example for the translation from a Moore machine (top) to a recurrent neural network (bottom). For simplicity, we do not show weight values because all weights are 1. The negative bias value for each neuron is denoted after a vertical bar.

and $\lfloor (k-1)/L \rfloor = i$. Combining this with the definition of b_k and the Heaviside function, we obtain indeed that $h_{2t-1,k} = 1$ if $e_{\lfloor (k-1)/L \rfloor} = x_t$ and $\mod(k,L) = q_{t-1}$ and $h_{2t-1,k} = 0$ otherwise.

Now, consider the coordinate h_{2t,q_t} . This coordinate must be one because the term $w_{q_t,L\cdot i+q_{t-1}}\cdot h_{2t-1,L\cdot i+q_{t-1}}$ for $e_i=x_t$ is 1 and, hence, the sum $\sum_l w_{q_t,l}\cdot h_{2t-1,l}-\frac{1}{2}$ is positive. For all other coordinates k, note that the conditions $\lfloor (l-1)/L \rfloor = i$, mod (l,L)=j and $\delta(e_i,j)=k$ do not hold and that all coordinates $h_{2t-1,l}$ for $l\leq L$ are zero. Hence, the sum $\sum_l w_{k,l}\cdot h_{2t-1,l}-b_k$ for all $k\neq q_t$ is negative. This concludes the proof by induction.

Once this result is established, showing that $\vec{y}_{2t} = y_t$ is straightforward. We simply notice that $y_t = \rho(q_t)$ and $\vec{y}_{2t} = \mathbf{V} \cdot \vec{h}_{2t} = \mathbf{V} \cdot e_{q_t}$, such that the sum $\sum_k v_{j,k} \cdot h_{2t,k}$ is 1 if $k = q_t$ and $\rho(k) = e_j$ and zero otherwise, which concludes the proof.

Table 3: The processing of the example sequence e_1, e_2, e_2, e_1 via the recurrent neural network shown in Figure 6. Note that the input symbol at time t is put into the network at time 2t.

t	0	1	2	3	4	5	6	7	8
\vec{x}_t	-	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$ec{h}_t$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$
$ec{y}_t$	-	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$