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The Coevolution of Appraisal and Influence Networks Leads to Structural Balance

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Abstract—In sociology, an appraisal structure, represented by a signed matrix or a signed network, describes an evaluative cognitive configuration among individuals. In this article we argue that interpersonal influences originate from positive interpersonal appraisals and, in turn, adjust individuals' appraisals of others. This mechanism amounts to a coevolution process of interpersonal appraisals and influences. We provide a mathematical formulation of the coevolutionary dynamics, characterize the invariant appraisal structures, and establish the convergence properties for all possible initial appraisals. Moreover, we characterize the implications of our model to the study of signed social networks. Specifically, our model predicts the convergence of the interpersonal appraisal network to a structure composed of multiple factions with multiple followers. A faction is a group of individuals with positive-complete interpersonal appraisals among them. We discuss how this factions-with-followers is a balanced structure with respect to an appropriate generalized model of balance theory.

Index Terms—Appraisal evolution, macro-structural models, structural balance theory, mathematical sociology

1 Introduction

Structural balance, a social psychological theory about the network structure of interpersonal appraisals, has attracted attention recently [1], [2], [3] in the studies of political party networks, large-scale online social networks, and cooperation evolution in social networks. Interpersonal appraisal is a ubiquitous natural relation of evaluative (positive or negative) cognitive orientation of one individual toward another. Cartwright and Harary's seminal work [4] on the signed digraph formalization of Heider's analysis [5], [6] of balanced cognitive configurations is now referred to as the *classical model of structural balance*. This model posits the existence of tensions corresponding to configurations of appraisals among three individuals.

Based on empirical observations, a *generalized model of structural balance* is introduced by Davis and Leinhardt [7] and studied by Johnsen [8]: the tension assumptions are relaxed and more complex realizations of structural balance are allowed. While structural balance theory typically focuses on the static appraisal networks, recent research [1], [3], [9] has concentrated on dynamical models of structural evolution. In what follows, we first review some relevant literature and later state our problem of interest.

Static Structural Balance Theory. In structural balance theory, a complete signed matrix $X \in \{-1,+1\}^{N\times N}$, which we call the *appraisal matrix*, represents the interpersonal ties in a group of N individuals, where x_{ij}, $i,j \in \{1,\dots,N\}$, equals to $+1$ if individual i has a positive appraisal of individual j, and x_{ij} equals to -1 if i has a negative appraisal of individual j. The matrix X is used to describe the interpersonal appraisal structure of the group. Any dyad $\{i,j\}$ in the group associated with X has three possible types: mutual (M), asymmetric (A), or null (N). A dyad $\{i,j\}$ is M-related if $x_{ij}=x_{ji}=1$; it is N-related if $x_{ij}=x_{ji}=-1$; and it is A-related otherwise (i.e., $x_{ij}\times x_{ji}=-1$). Consequently, there are 16 different types of *triads* for an appraisal structure.

The classical model of structural balance posits that an appraisal structure is balanced if the following four statements by Heider [6] are satisfied: "my friend's friend is my friend," "my friend's enemy is my enemy," "my enemy's enemy is my friend," and "my enemy's friend is my enemy." Mathematically, a signed digraph is balanced under these conditions if the value of all cycles (i.e., closed paths beginning and ending with the same node) are positive with respect to the product of all edges of the cycle. A remarkable implication of the classical theory of structural balance is that an appraisal network is tension-free (balanced) if and only if it is positive-complete or partitioned into two positive-complete subgraphs. A positive-complete subgraph, also referred to as a *faction*, is a social subgroup in which each individual has a positive appraisal of each individual in the faction and a non-positive appraisal of any other individual.

The generalized model of structural balance is proposed by Davis and Leinhardt [7] and Johnsen [8], [10]. In this theory, a *micro-model* specifies which subset of the 16 triad types is permitted and, therefore, which resulting structural networks (signed digraphs containing only permitted triad types) are admissible. The resulting structural networks are referred to as *macro-structural models*. In contrast with the classical model, the generalized model of structural balance (in many of its micro-model realizations) allows for arbitrary numbers of factions and directed acyclic graphical structures among them. We review this theory in Section 2 and refer to [8] for a detailed treatment.

Dynamical Models for Social Balance. Classical structural balance theory and its generalizations do not specify the

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mechanisms that transform unbalanced appraisal networks into balanced networks. However, as Marvel et al. [1] noted "... its underlying motivation is dynamic, based on how unbalanced triangles ought to resolve to balanced ones. This situation has led naturally to a search for a full dynamic theory of structural balance."

In Kułakowski's work [9] a continuous-time model of structural balance was presented: given an appraisal matrix $X \in \mathbb{R}^{N \times N}$, the dynamical system

$$dX/dt = X^2, \tag{1}$$

governs the evolution of the appraisal structure over time. Here, the entry x_{ij} denotes the appraisal of individual j held by individual i. Numerical simulations showed that, for almost all initial $X(0)$, the system reaches the structural balance postulated by the classical model in finite time. Kułakowski proved the convergence to this structural balance for an $N = 3$ network. Marvel et al. [1] proved that for a random initial symmetric matrix $X(0)$, the classical structural balance is obtained by the dynamical system (1) in finite time with a probability converging to 1 as the population size goes to infinity. Traag et al. [3] extended the convergence results to normal initial matrices (X is normal if $X^T X = XX^T$). However, for generic initial appraisal matrices, the convergence to structural balance is not necessarily observed [3] in the dynamical system (1). Another interesting continuous-time dynamical model presented by Traag et al. is

$$dX/dt = XX^T. \tag{2}$$

For this model, there exists [3] a dense set of initial conditions $X(0) \in \mathbb{R}^{N \times N}$, such that a balanced structure is achieved generically in finite time. In the first model (1), the appraisal of the individual j held by the individual i is adjusted based on the appraisals of j held by all other individuals. In the second model (2), the appraisal of the individual j held by the individual i is adjusted based on the appraisal of all individuals held by j. In other words, the second model features a cooperative behavior in the appraisal evolution: each individual tends to befriend other individuals who think alike.

The literature on social structural balance also includes a research stream on social energy landscape [2], [4], [11], [12]. These works are motivated by studies which suggest that certain triad types are more stable than others. In these works, an energy landscape is defined to describe structural balance. Numerical experiments show that energy landscapes often feature local minima called jammed states [11], [12]. Antal et al. [11] consider a discrete-time dynamical model, where the signs of the edges of an appraisal network are optimally adjusted under the constraint of a monotonic increase of balanced triads. It is shown that the assumption of an optimal monotonic increase of balanced triads does not suffice to generate the structural balance of the classical model.

In summary, we argue that it is of considerable interest to postulate social psychological mechanisms of appraisal dynamics and to characterize the conditions under which they present outcomes consistent with structural balance theory.

The Coevolution of Interpersonal Appraisals and Influences. We propose a novel sociological model for the evolution of interpersonal appraisals toward generalized structural balance. Our approach treats the evolution of interpersonal appraisals as a special case of opinion dynamics: the evolving opinions are the individuals' bundles of signed cognitive appraisals toward the other individuals. In other words, we ground dynamic structural balance theory in the theory of opinion dynamics and influence networks. As opinion and appraisal evolution mechanism, we adopt the widely-established DeGroot averaging model [13]. Notably, our dynamical model predicts general numbers of factions and rich structure among faction and is consistent with a particular micro-model from generalized balance theory. Our work is also motivated by the coevolution process [14] in which an influence network is associated with an appraisal network, interpersonal influences adjust individuals' appraisals, and these adjusted appraisals lead to an adjusted influence network.

With this background motivation, we consider a discrete-time dynamical model of structural balance, where the dynamics combine both appraisal and influence structure evolution. For a group of $N \geq 2$ individuals with initial appraisals $X(0) \in \mathbb{R}^{N \times N}$, the evolution of the appraisal matrix $X(t)$ is determined by a discrete-time DeGroot averaging model:

$$X(t+1) = W(X(t))X(t), \quad t = 0, 1, 2 \dots. \tag{3}$$

Here, the row-stochastic *influence matrix* $W(X(t))$ depends on the state $X(t)$ as follows: interpersonal influences for each individual are *proportional* to the *positive* appraisals accorded to the individual by all other individuals. In other words, the entry $w_{ij}(X)$, $i \neq j$, is proportional to x_{ij} if x_{ij} is positive and zero otherwise. We provide a detailed mathematical definition in the modeling section below. As the dynamical processes of interpersonal appraisals and influences are interdependent, we refer to (3) as the *coevolution model of interpersonal appraisal and influence*.

Our setup features several differences with the existing dynamical models on structural balance (e.g., the models (1) and (2)). First, our model (3) considers both appraisal and influence evolution and a novel process to generate and adjust the interpersonal influence network. Note that the appraisal structure plays the role of the influence structure in the previous work [1], [3], [9]. Second, in our model appraisals are modified over time as convex combination of existing appraisals; this guarantees that the appraisals never diverge (by comparison, divergence occurs in the models of (1) and (2)). Third, while positive and negative appraisals with heterogeneous strengths are allowed in our model, our basic assumption is that only positive interpersonal appraisals translate into interpersonal influences. Accordingly, we expect the evolutions of the proposed coevolution model to asymptotically satisfy two statements in the classic balance model: "my friend's enemy is my enemy" and "my friend's friend is my friend." (On the contrary, there are a priori no reasons why the other two structural balance theory statements, "my enemy's enemy is my friend" and "my enemy's friend is my enemy," should be satisfied by the evolutions of our model.) By comparison, the models (1) and (2) satisfy all four statements in the classic balance model and predict only one or two factions of structural balance given certain initial conditions. In other words, the classical model and the models in Equations (1) and (2) are therefore not predictive of the situations, empirically observed in [7], in which more than two factions are often observed. Moreover, for generic initial conditions, the solution of the model (1) converges to a rank-1 matrix which is, in general, not structurally balanced, even in the language of the generalized structural balance. Finally, it is noted that our model (3) is a discrete-time dynamical model, while the Equations (1) and (2) are in continuous time.

Contributions. We propose and analyze the novel coevolution model of interpersonal appraisal and influence given in Equation (3). As a first step, we provide an explicit and concise mathematical formulation for this discrete-time nonlinear system. As the main set of contributions, we study the asymptotic convergence and equilibrium properties of this nonlinear coevolution system. We provide a mathematical analysis on the structural position properties of the individuals; we predict the equilibrium appraisals for individuals in a sink, intermediate or source strongly connected component (SCC). We claim that (1) all individuals in a sink SCC of the equilibrium positive digraph will reach an appraisal consensus on each individual, (2) all intermediate SCCs vanish in the dynamical equilibrium, (3) each source SCC is a singleton and the appraisal of individuals in source SCCs are determined by the appraisals held by the individuals from the sinks, and (4) all equilibrium appraisal networks have a factions-followers-outsiders structure. Here, a faction is a sink SCC with positive appraisals within the component; a follower is a singleton source SCC; and an outsider is an isolated singleton sink SCC with a non-positive self-appraisal.

As the second set of contributions, we demonstrate that all invariant macro-structures are the special cases of the factions-followers-outsiders structure. This finding is related to the concept of core-periphery structure, a prevalent notion in world systems [15], economics [16], and social networks [17], [18]. In other words, the factions-followers-outsiders structure exhibits the properties of a multi-core-periphery structure: dense, cohesive cores (factions) and sparse, unconnected peripheries (followers or outsiders). Furthermore, all equilibrium appraisal structures via the coevolution are discussed and the exclusive set of macro-structural models are predicted to be structurally balanced under our model.

Lastly, we illustrate our results by numerical simulations. In particular, we inspect the different convergence and equilibria performances for different self-influence parameters.

These findings are of sociological interest in their advancement of the dynamical foundations of structural balance in social groups. Our rigorous results for the coevolution model of interpersonal appraisal and influence suggest that interpersonal appraisal networks evolve toward a set of structural equivalent bundles and predict the stable macro-structures under the coevolution model. These findings contribute to the rapidly-growing research on coevolutionary networks [19], [20], to the literature on social network formation and coordination games [21], [22] and, more broadly, to the study of complex networks and evolutionary rules [23], [24].

Finally, we note that the interesting attractor topologies admitted by the generalized balance theory in this paper have important practical implications on the capacity of interpersonal influence systems to resolve social conflicts and disseminate innovations. The hierarchical topological attractors we characterize provide a basis of population consensus generation and diffusion of innovations. The theory suggests that a small set of influential factions, and the chains of positive appraisals that link other followers to them, determine the beliefs, opinions, and behaviors of large numbers of individuals on a variety of issues.

Organization. The rest of the paper is organized as follows. Section 2 introduces some notation and preliminary concepts. Section 3 describes the coevolution model. Sections 4, 5 and 6 discuss, respectively, the topological, asymptotic and structural balance properties of our model. Section 7 contains our conclusions. All technical proofs are in the Appendices, which can be found on the Computer Society Digital Library at http://doi.ieeecomputersociety.org/10.1109/TNSE.2016.2600058.

2 Preliminary Concepts

For a vector $x \in \mathbb{R}^n$, we let $x \geq 0$ and $x > 0$ denote component-wise inequalities. We adopt the abbreviations $\mathbb{1}_n = [1, \dots, 1]^T$ and $\mathbb{0}_n = [0, \dots, 0]^T$. Given a column vector $[x_1, \dots, x_n]^T \in \mathbb{R}^n$, we let $\mathrm{diag}(x)$ denote the diagonal $n \times n$ matrix whose diagonal entries are $x_1, \dots, x_n$. For signed matrix pattern analysis, we let "$-$" represent a block matrix with *non-positive* entries and let "$+$" represent a block matrix with positive entries for the simplicity of presentation. If two matrices $A, B \in \mathbb{R}^{N \times M}$ have the same positive/non-positive entry pattern, we denote $A \sim B$. For example, one matrix A with all positive entries or all non-positive entries is denoted by $A \sim [+]$ or $A \sim [-]$, respectively. For $x \in \mathbb{R}$ and $A = [a_{ij}] \in \mathbb{R}^{N \times M}$, we write $x^+ = \max\{x, 0\} \in \mathbb{R}_{\geq 0}$ and $A^+ := [\max\{a_{ij}, 0\}] \in \mathbb{R}^{N \times M}_{\geq 0}$.

Elements of Graph Theory. An *undirected graph* (in short, *graph*) is an ordered pair $G = (V, E)$, where V is a set of nodes and E is a set of unordered pairs of nodes. A *directed graph* (in short, *digraph*) $G = (V, E)$ consists of a node set V and a set E of ordered pairs of nodes, i.e., $E \subset V \times V$. For $i, j \in V$, the ordered pair (i, j) denotes a directed *edge* from i to j, where i is called an *in-neighbor* of j, and j is called an *out-neighbor* of i. The *in-degree* and *out-degree* of j are the numbers of in-neighbors and out-neighbors of j, respectively. Every node of in-degree (resp. out-degree) 0 is called a *source* (resp. *sink*). A node with both non-zero in-degree and out-degree is an *intermediate* node and a node with both 0 in-degree and out-degree is an *isolated* node.

A *directed path* in a digraph G is an ordered sequence of nodes such that any pair of consecutive nodes in the sequence is a directed edge. A directed path is *simple* if no node appears more than once in it, except possibly for the initial and final node. G is *strongly connected* if there exists a directed path from any node to any other nodes. A node of G is *globally reachable* if it can be reached from any other node by traversing a directed path. A *cycle* in G is a simple directed path that starts and ends at the same node. A *directed acyclic graph (DAG)* is a digraph that has no cycles. G is *aperiodic* if there exists no integer $k > 1$ dividing the length of each cycle in G. Given a digraph $G = (V, E)$, a linear ordering of nodes is an *inverse topological sorting* if, for any edge $(i, j) \in E$, j

precedes i in the ordering. Any DAG has one inverse topological sorting, which may not be unique [25].

A digraph (V', E') is a *subgraph* of (V, E) if $V' \subset V$ and $E' \subset E$. A subgraph H is a *strongly connected component* of G if H is strongly connected and any other subgraph of G strictly containing H is not strongly connected. The *condensation digraph* of G, denoted $C(G)$, is a digraph whose nodes are the SCCs of G and in which there exists a directed edge from the SCC H_1 to the SCC H_2 if and only if there exists a directed edge in G from a node of H_1 to a node of H_2. Each $C(G)$ is a DAG and has at least one sink and one source. We say that H_1 *is directly connected to* H_2 in G if there exists a directed edge in $C(G)$ from H_1 to H_2.

Elements of Matrix Theory. A non-negative matrix is *row-stochastic* if all its row sums are equal to 1. Given a square non-negative matrix $M = \{m_{ij}\}_{i,j \in \{1,\dots,n\}}$, its *associated digraph* $G(M)$ is the digraph with node set $\{1, \dots, n\}$ and with edge set defined as follows: (i, j) is a directed edge if and only if $m_{ij} > 0$. M is *irreducible* if $G(M)$ is strongly connected; M is *reducible* if it is not irreducible. M is *aperiodic* if $G(M)$ is aperiodic. M is *primitive* if there exists $k \in \mathbb{N}$ such that M^k is a positive matrix. It is known (e.g., see Example 1.2 in [26]) that M is primitive if and only if M is irreducible and aperiodic.

Generalized Models of Structural Balance. Micro-models and macro-structural models were introduced in Johnsen [8] (see also [27, Section 8.3]) to generalize the structural balance models studied in [4], [6]. A *micro-model* is defined to be a subset of the 16 possible triad types. Associated with a particular micro-model, a *macro-structural model* (or a *macro-structure* in short) is defined to be the set of networks containing only the triad types in the micro-model. We call such a pair of the macro-structure and the micro-model *consistent* and, equivalently, we say that a set of macro-structure networks is *structurally balanced* with respect to a specified micro-model.

We rephrase and generalize the dyad types of interpersonal appraisals (the classical version of which was introduced in [4] and briefly presented in the introduction) as follows: dyad $\{i, j\}$ are *M-related* if $x_{ij} > 0$ and $x_{ji} > 0$; they are *N-related* if $x_{ij} \leq 0$ and $x_{ji} \leq 0$; and they are *A-related* otherwise (i.e., one of $\{x_{ij}, x_{ji}\}$ is strictly positive and the other is non-positive). The triad types are then generalized by the rephrased dyad types such that the values of the appraisal relations are not constrained to -1 or 1. An *M-clique* is a set of individuals who are completely connected by M-links (i.e., links with M-relations).

3 Model of Appraisal and Influence Coevolution

In this section, we present a dynamical model which investigates how an influence network may emerge from interpersonal appraisals in a social network and how the appraisal relations may be modified by interpersonal influences. We adopt the term "structure" for the non-positive/positive (or zero/non-zero, respectively) pattern of the appraisal (or influence, respectively) relations among the network irrespective of their values. We adopt the term "matrix" to describe the exact quantification of the interpersonal appraisals (or influence weights, respectively). In other words, an appraisal structure is a set of appraisal matrices with a certain sign pattern.

We consider a group of $N \geq 2$ individuals with interpersonal appraisals represented by a signed matrix $X \in \mathbb{R}^{N\times N}$. Each entry $x_{ij}, i, j \in \{1, \dots, N\}$, of the *appraisal matrix* X represents individual i's *interpersonal appraisal* of individual j. Additionally, we allow the entries of X equal to 0: $x_{ij} = 0$ implies that individual i has neither positive nor negative appraisal of individual j, or that i does not know j. Thus, we relax the complete digraph or weakly connected digraph assumption on appraisal structures, which was widely adopted in the previous work (e.g., see [1], [3], [4]).

Individuals' appraisals, i.e., signed evaluative orientations of particular strengths, are often automatically generated without conscious effort [28], [29], and these appraisals are important antecedents of displayed cognitive and behavioral orientations toward objects [30]. The available empirical evidence is also consistent with the assumption that individuals update their appraisals as convex combinations of their own and others' displayed appraisals. This convex combination is based on weights that are automatically generated by individuals in their responses to the displayed appraisals held by other individuals. This specification appeared in the literature on opinion dynamics in the early works by French [31], Harary [32], and DeGroot [13]. Especially, in Anderson's seminal information integration theory [33], the convex combination mechanism was regarded as a fundamental "cognitive algebra" of the mind's synthesis of heterogeneous information. Therefore, in this article, we formulate individuals' appraisals about others by a trajectory $t \mapsto X(t)$ that is determined by the seminal DeGroot averaging model:

$$X(t+1) = W(t)X(t), \quad t \geq 0, \tag{4}$$

with initial appraisals $X(0) \in \mathbb{R}^{N\times N}$, and with a sequence of *influence matrices*$\{W(t)\}_{t\geq 0}$. Here, each influence matrix $W(t)$ is assumed to be non-negative and its entry $w_{ij}(t)$, $i, j \in \{1, \dots, N\}$, represents the *interpersonal influence weights* that the individual i accords to individual j at time t.

Our analysis of appraisal evolution (4) depends only on the influence matrices $\{W(t)\}_{t\geq 0}$. The key feature of the proposed model is that $W(t)$ is determined by the appraisal matrix $X(t)$ at each time t. Motivated by Friedkin and Johnsen's work [14], we associate an influence matrix to an appraisal matrix as follows: (i) the interpersonal influence $w_{ij}(t)$ is strictly positive precisely when the corresponding appraisal $x_{ij}(t)$ is strictly positive, (ii) non-positive appraisals lead to zero interpersonal influence weights, and (iii) each individual has a positive self-weight in the influence network. Specifically, given a small *self-appraisal constant* $\epsilon > 0$, individuals' interpersonal influences are determined as functions of the interpersonal appraisals X by

$$w_{\epsilon,ij}(X) = \begin{cases} \dfrac{1}{\sum_{j=1}^n x_{ij}^+ + \epsilon}(x_{ij}^+ + \epsilon), & \text{if } j = i, \\ \dfrac{1}{\sum_{j=1}^n x_{ij}^+ + \epsilon} x_{ij}^+, & \text{if } j \neq i. \end{cases} \tag{5}$$

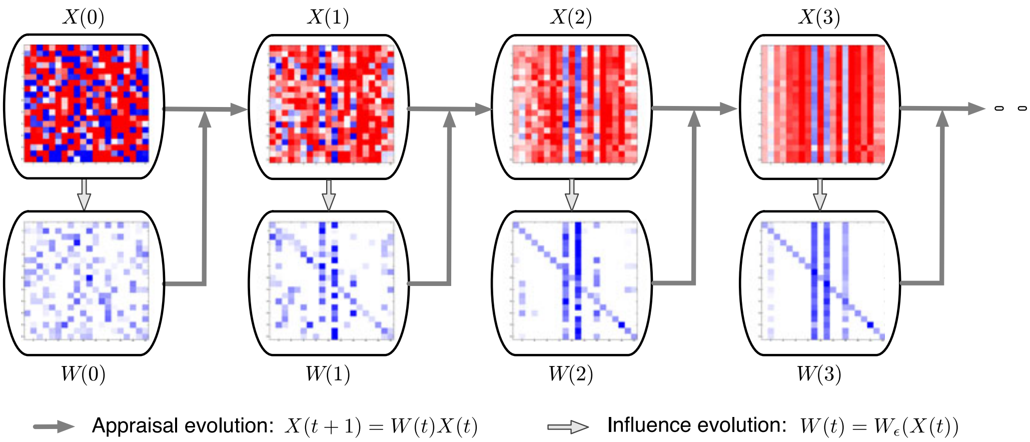

Fig. 1. Coevolution of appraisal and influence: the colormaps correspond to the signed appraisal matrices and influence matrices. In these colormaps and what follows red colors are negative appraisals and blue colors are positive appraisals or influences. The various color depths represent different appraisal or influence values.

An equivalent definition of $W_\epsilon(X) = [w_{\epsilon,ij}]$ is:

$$W_\epsilon(X) = \mathrm{diag}\big((X^+ + \epsilon I_N)\mathbb{1}_N\big)^{-1}\big(X^+ + \epsilon I_N\big). \tag{6}$$

By continuity, we also define $W_0(X) = \lim_{\epsilon\to 0^+} W_\epsilon(X)$. Note that, if each entry of the ith row of X is negative or zero, then $w_{\epsilon,ii}(X) = 1$. This equality implies that the ith individual assigns zero weight to all other individuals and that his appraisals are therefore unchanged after one step of the evolution.

Definition 3.1 (Appraisal and influence coevolution model). *Given a group of $N \geq 2$ individuals, let $X(t) \in \mathbb{R}^{N\times N}$ be the appraisal matrix and $W(t) \in [0,1]^{N\times N}$ be the influence matrix at time $t \geq 0$. For $t \in \mathbb{Z}_{\geq 0}$, the* interpersonal appraisal and influence coevolution system *is defined by:*

(1) *Appraisal evolution:*

$$X(t+1) = W(t)X(t),$$

(2) *Influence evolution:*

$$W(t) = W_\epsilon(X(t)).$$

For simplicity of presentation, the model assumes the positive parameter ϵ to be homogeneous among the individuals. Nevertheless, this assumption is not necessary and is relaxed in Section 5. Fig. 1 illustrates our proposed coevolution model as in Definition 3.1.

Lemma 3.1 (Properties of influence matrices). *For any $\epsilon \geq 0$ and any appraisal matrix $X \in \mathbb{R}^{N\times N}$, the influence matrix $W_\epsilon(X)$ is row-stochastic and, if $\epsilon > 0$, aperiodic.*

4 Topological Properties of the Coevolutionary Dynamics

In this section we study the topological properties of the coevolution model of interpersonal appraisal and influence (4) and (5), and focus on the long-term connectivity properties of the graphs describing interpersonal influences.

We call the digraph $G(W)$ associated to the influence matrix W the *influence digraph*, and call the digraph $G^+(X)$ associated to X the *positive (appraisal) digraph* for which a directed edge (i,j) exists if and only if $x_{ij} > 0$. It is clear that the adjacency matrix X^+ of $G^+(X)$ has the same positive/non-positive entry pattern as $W(X)$ except possibly for the diagonal entries. Consequently, the two digraphs $G^+(X)$ and $G(W)$ have the identical set of nodes and the identical set of edges except possibly for self-loops. In what follows we analyze the evolution of $G^+(X(t))$ along the trajectory $X(t)$ of the dynamical system (4) and (5).

Since $C(G)$ is a DAG, by relabelling its nodes from an inverse topological sorting, the adjacency matrix of $C(G)$ is a block lower triangular non-negative matrix as follows

$$A = \begin{bmatrix} D_1 & 0 & \dots & 0 \\ S_{21} & D_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S_{n1} & S_{n2} & \dots & D_n \end{bmatrix}, \tag{7}$$

where n nodes $\{H_i\}_{i\in\{1,\dots,n\}}$ exist for the condensation digraph $C(G)$. The matrix A can also be regarded as an adjacency matrix of G if we look at D_i and S_{ij} as block matrices and $\{H_i\}_{i\in\{1,\dots,n\}}$ represent the SCCs of G. The ordering of nodes within an SCC H_i is inessential.

As the topology of positive appraisal digraph or influence digraph may vary along the coevolution, we denote the nodes of $C(G^+(X(t)))$ or the SCCs of $G^+(X(t))$ as $\{H_i(t)\}_{i\in\{1,\dots,n(t)\}}$, where $n(t)$ denotes the numbers of SCCs associated with the positive digraphs. With a slight abuse of notation, we refer to $H_i(t)$ as both a SCC of $G^+(X(t))$ at time t and the subset of nodes of $G^+(X(t))$ belonging to that SCC. That is, $H_i(t)$ may represent the same subset of nodes in the digraph $G^+(X(t+1))$ as those in the digraph $G^+(X(t))$ forming the SCC $H_i(t)$ even though $H_i(t)$ may not be an SCC of $G^+(X(t+1))$ any more.

Theorem 4.1 (Finite-Time stability of the SCCs of positive digraphs). *Let $X(t)$ be a trajectory of the coevolution system (4) and (5) with $\epsilon \geq 0$. Pick a time $t \in \mathbb{Z}_{\geq 0}$ and perform an inverse topological sorting of the condensation digraph $C(G^+(X(t)))$. For any two nodes $H_i(t)$ and $H_j(t)$ with labels $i < j$ in $C(G^+(X(t)))$,*

(i) *no directed edge can appear from a node of $H_i(t)$ to a node of $H_j(t)$ in $G^+(X(t+1))$; and*

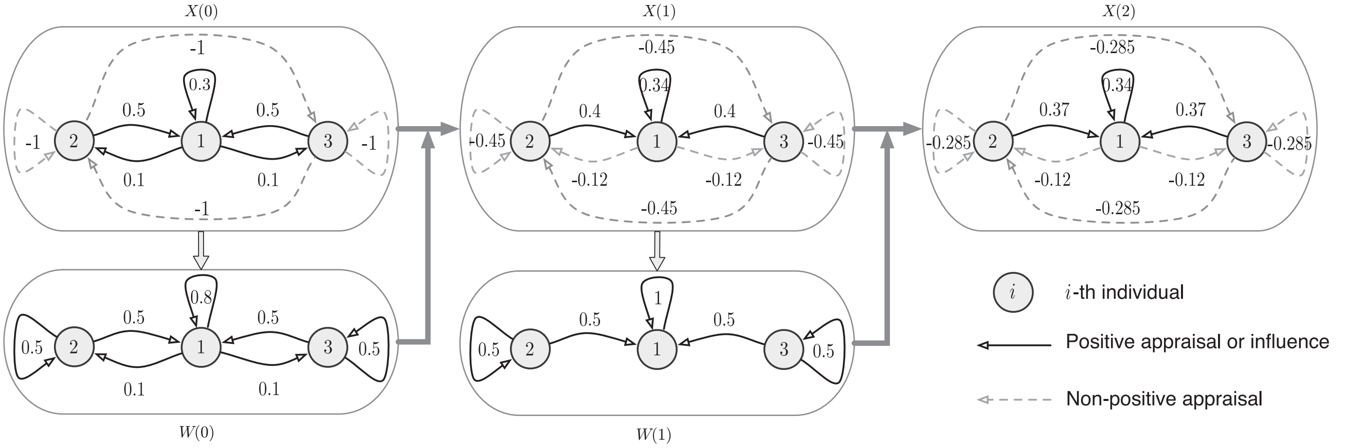

Fig. 2. Appraisal and influence coevolution in a triad: The digraphs formulated by solid directed edges correspond to positive appraisal digraphs (on the top) and influence digraphs (on the bottom). The non-positive appraisals are also shown by dotted lines. The numbers on the edges are the values of the appraisals and influence weights, respectively. As stated in Theorem 4.1, regarding the positive appraisal digraphs, (1) the (positive) edges from individual 1 to the individuals 2 and 3 disappear at time $t = 1$ and these edges will never appear again; (2) the edges between the individuals 2 and 3 never appear. Hence, three SCCs remain unchanged after $t = 1$.

(ii) *if $C(G^+(X(t)))$ contains no directed path from $H_j(t)$ to $H_i(t)$ with length 1 or 2, then no directed edge can appear from a node of $H_j(t)$ to a node of $H_i(t)$ in $G^+(X(t+1))$.*

Therefore,

(iii) *no two SCCs of $G^+(X(t))$ can merge at time $t+1$ (whereas an SCC of $G^+(X(t))$ may split into multiple SCCs at time $t+1$);*

(iv) *the number $n(t)$ of SCCs of $G^+(X(t))$ is non-decreasing; and*

(v) *there exists a finite time τ such that the SCCs of $G^+(X(t))$ remain unchanged for all subsequent times $t \geq \tau$.*

As $\{H_i(t)\}$ and $n(t)$ remain unchanged for all $t \geq \tau$, we denote $H_i = H_i(\tau)$ and $n = n(\tau)$ for simplicity in the following discussions. It is noted that the blocks below the diagonal in (7) are varying via the coevolution system (4) and (5) even after time τ, and therefore, the directed edges from the node H_i to the node H_j for all $i > j$ do not necessarily remain unchanged in the condensation digraphs $C(G^+(X(t)))$ for $t \geq \tau$. That is, the topology evolution of the positive digraphs may not stabilize at τ. Moreover, due to the discontinuity of $G^+(X(t))$, although the SCCs of $G^+(X(t))$ are unchanged after some finite time τ, they are not necessarily equal to the SCCs of $G^+(\lim_{t\to\infty} X(t))$.

Fig. 2 illustrates one example showing the appraisal digraph evolution and the edge evolution as described in Theorem 4.1. We may verify the claims of the theorem by this example: Given $N = 3$ and $\epsilon = 0.5$, we observe that (1) the positive appraisal digraph $G^+(X(0))$ has only one SCC but this SCC splits into three SCCs at time $t = 1$, i.e., singleton SCC nodes $H_1(1), H_2(1), H_3(1)$; (2) no directed edge can appear between the two SCCs $H_2(t)$ and $H_3(t)$, and no edge can appear from $H_2(t)$ or $H_3(t)$ to $H_1(t)$ for $t \geq 1$; (3) the three SCCs $H_1(t), H_2(t), H_3(t)$ never merge for all $t \geq 1$ and the number of the SCCs are non-decreasing. By simple calculation, we know that the stability time for both the SCC evolution and topology evolution of the positive digraphs considered in Fig. 2 is $\tau = 1$ and the number of the stable SCCs is $n(\tau) = 3$.

5 Asymptotic Properties of the Coevolutionary Dynamics

In this section we study the asymptotic convergence properties of the coevolution model of interpersonal appraisal and influence (4) and (5). Both analytic and numerical results are presented.

5.1 Theoretical Results

We start with arbitrary initial conditions and subject to a non-zero assumption on ϵ and we show that each trajectory converges asymptotically to an equilibrium matrix and we characterize the structure of the equilibrium matrices. Second, we discuss sufficient and necessary conditions such that certain appraisal structures observed in finite time remain unchanged in infinite-time limit.

Definition 5.1 (Factions-followers-outsiders structure). *An appraisal matrix X has a* factions-followers-outsiders structure *if each strongly-connected component of $G^+(X)$ is either*

(i) *a sink in $C(G^+(X))$, called a* faction, *composed of an arbitrary number of nodes, all of which are completely connected and have positive self-loops in $G^+(X)$; or*

(ii) *a source in $C(G^+(X))$, called a* follower, *composed of a single node with directed edges pointing to each node in one or more factions and without self-loop in $G^+(X)$; or*

(iii) *an isolated node in $C(G^+(X))$, called an* outsider, *composed of a single node in $G^+(X)$ with zero in-degree, zero out-degree and no self-loop.*

The node in a follower component with directed edges toward one or more factions is called a *follower of that or those factions*. If there are only factions and followers associated with $G^+(X)$, it is called a *factions-followers structure*. This definition is illustrated in Fig. 3: note that a faction can have one or multiple nodes and can have zero, one or more followers. A faction-follower-outsiders structure may includes one or multiple copies of the set or a subset of the structures shown in figure.

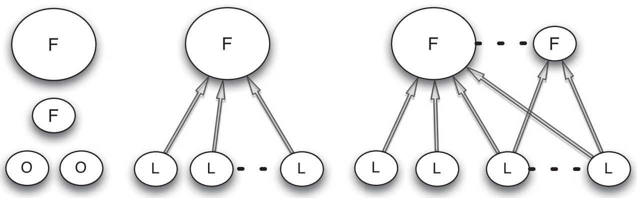

Fig. 3. Faction-follower-outsiders structure: "F" components are factions, "L" components are followers, "O" components are outsiders. The small size components are singletons and the large size components are SCCs with two or more nodes. On the left: Two factions and two outsiders. In the middle: A factions-followers structure with a single faction. On the right: A factions-followers structure with multiple factions.

Theorem 5.1 (Asymptotic appraisal matrices). *For the coevolution system (4) and (5) with $\epsilon > 0$, each trajectory $X(t)$ converges asymptotically to an equilibrium X^* (function of $X(0)$ and ϵ). Moreover, each equilibrium matrix X^* has the following properties:*

(i) X^* has a factions-followers-outsiders structure;

(ii) for a faction of $G^+(X^*)$, the appraisals of each individual in the network held by all individuals in the faction are the same: this faction's appraisal of one individual is positive if the individual belongs to this faction and is non-positive otherwise;

(iii) for a follower of $G^+(X^*)$, the appraisal of each individual in the network held by the follower is a convex combination of the appraisals of that individual held by the factions the follower follows. In particular, if the follower follows only one faction, then its appraisal of each individual in the network is identical to that held by the faction;

(iv) for an outsider of $G^+(X^*)$, the appraisal of each individual in the network held by the outsider is non-positive.

Theorem 5.1 says that, subject to the coevolution (4) and (5), the factions-followers-outsiders structure is the only possible equilibrium structure of the appraisal matrix X^*. In particular, we know that: (i) the rows of X^*, corresponding to all individuals' appraisals in a faction, are identical and are equal to $\mathbb{1}_{N_s} v^T$ for some $v = [v_j] \in \mathbb{R}^N$, where N_s is the cardinality of the faction, $v_j > 0$ if node j is in the faction and $v_j \leq 0$ otherwise; (ii) all followers, i.e., source strongly-connected components, are singletons and their appraisals are determined by the appraisals held by the factions, of which the followers hold positive-complete appraisals in the equilibria; and (iii) all outsiders have non-positive appraisals of each individual in the group. The examples of the convergence in Theorem 5.1 are presented in Figs. 4, 5, and 6. The nodes of the right graphs are M-cliques. $\mathrm{M}_{(1)}$ in these figures is a special M-clique with only one node. More discussions of M-clique are referred to Section 6.

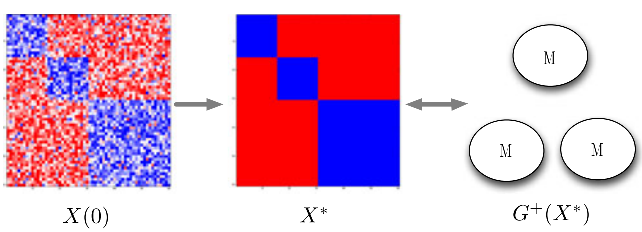

Fig. 4. Convergence to three factions.

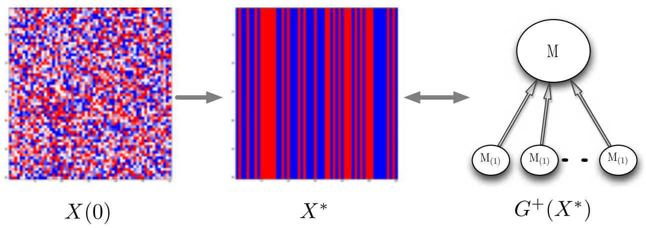

Fig. 5. Convergence to a factions-followers structure with a single faction.

Next, we analyze when and what finite-time structures (i.e., strongly-connected components of the positive digraph) remain unchanged in the asymptotic limit. For simple presentation, we denote the matrix corresponding to interpersonal appraisals in a sink SCC H_s by $X_s \in \mathbb{R}^{N_s \times N_s}$, where N_s is the node cardinality of the SCC.

Theorem 5.2 (Finite-time properties determining asymptotic structures). *For the coevolution system (4) and (5) with $\epsilon > 0$, let the trajectory $X(t)$ satisfy $X^* = \lim_{t\to\infty} X(t)$. Then*

(i) a sink SCC H_s of $G^+(X(t))$ exponentially converges to a faction in X^* if and only if there exists a time $t_1 \geq t$ such that $X_s(t_1)$ has one column with all positive entries;

(ii) $G^+(X^*)$ has a globally reachable node if and only if there exists a time t such that $X(t)$ has one column with all positive entries;

(iii) if all entries of $X(t)$ are non-negative and $G^+(X(t))$ is irreducible, then $G^+(X^*)$ is one faction and all individuals have the same appraisal of each individual in the group; and

(iv) an outsider of $G^+(X(t))$ remains an outsider for all following times and in the limit $G^+(X^*)$.

The statement (ii) of Theorem 5.2 extends its statement (i) to an appraisal structure of which the positive digraph is at least weakly connected and has only one sink SCC. The positive digraph could be either reducible or irreducible. Moreover, in the equilibrium, such a structure has only one faction, an arbitrary number of followers and no outsider. Regarding the irreducibility assumption of the third statement of Theorem 5.2, we can show that if $G^+(X(0))$ is irreducible, then $G^+(X(t))$ is irreducible for all $t \geq 0$ and $G^+(\lim_{t\to\infty} X(t))$ is irreducible. This statement can be extended such that a sink SCC (with at least two nodes) becomes a faction in the equilibrium.

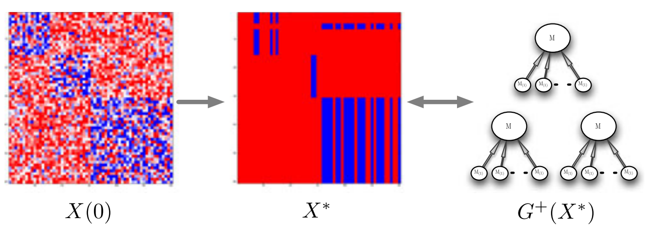

Fig. 6. Convergence to a factions-followers structure: Three disconnected factions-followers structures each of which has a single faction.

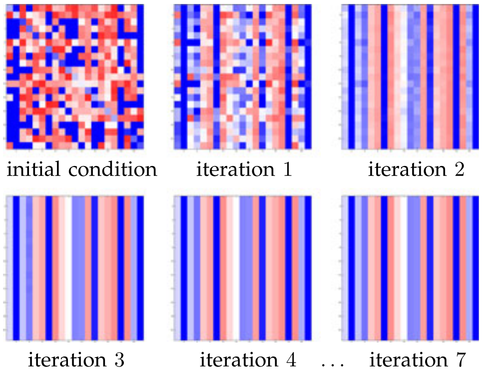

Fig. 7. Evolution of appraisal matrices on Krackhardt's advice network: The interpersonal appraisals converge to a rank-1 matrix of the form $\mathbb{1}_N v^T$.

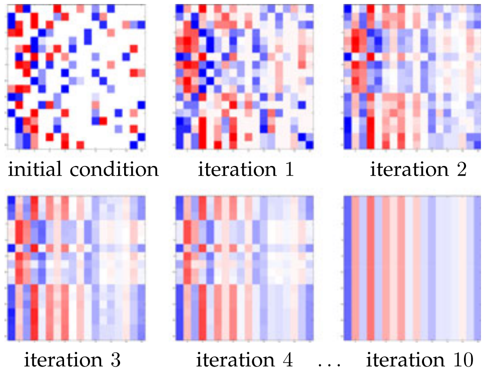

Fig. 8. Evolution of appraisal matrices on Sampson's monastery network: The appraisal matrix converges to a rank-1 matrix of the form $\mathbb{1}_N v^T$.

5.2 Numerical Study on Empirical Networks

We now apply our results to empirical social network examples. To minimize the impact of the self-appraisal constant ϵ on the trajectory of the coevolution system, we select small values of ϵ in the simulations. We report trajectories computed for strictly-positive small ϵ, but we comment that essentially identical trajectories are generated by setting ϵ to zero.

In our first example, we consider the appraisal evolution on a Krackhardt's advice network [34]; this network describes a manufacturing organization with 21 managers and 128 relationships in which a manager sought advice from another manager. Because the available data about this advice network does not include a complete set of interpersonal appraisals, we set up an initial appraisal matrix based on two ancillary assumptions: (1) each individual holds initial interpersonal appraisals equal to 1 of the individuals she seeks advice from and has non-positive appraisals (uniformly randomly-selected from $[-0.5, 0]$) of all other individuals; (2) the initial self-appraisals are equal to the normalized in-degree of the individuals in the advice network. Note that the second ancillary assumption is grounded in the theory of reflected appraisal as presented in [35]. Given $\epsilon = 0.1$, the trajectory of the appraisal evolution is shown in Fig. 7. The equilibrium structure is a factions-followers structure with a single faction. Appraisal consensus of each individual in the network is observed. Note that this convergence to an appraisal structure with a single faction is robust with respect to the initial non-positive appraisal assignment among $[-0.5, 0]$. Indeed, our initial appraisal assignment guarantees that at least two (the 2nd and the 21st) columns of $X(1)$ are positive. Based on statement (ii) in Theorem 5.2, there always exists a globally reachable node in the equilibrium structure. That is to say, the equilibrium must have a factions-followers structure with a single faction, with or without followers. However, this claim is not true for an arbitrary initial appraisal assignment.

The second example considers the social interactions among a group of monks in an isolated contemporary American monastery observed by Sampson [36]. Based on observations and experiments, Sampson collected a variety of experimental information on four types of relations: Affect, Esteem, Influence, and Sanctioning. Each of 18 respondent monks ranked their three first choices on these relations, where 3 indicates the highest or first choice and 1 the last choice in the presented interaction matrices. Some subjects offered tied ranks for their top five choices. Here we focus on the monastery appraisal structures based on the ranking of the affection ("like" and "dislike") relations in Sampson's empirical data. Note that we apply data collected directly from original PhD dissertation [36], where "like" and "dislike" relations were both collected for three times, while most Sampson's "dislike" dataset available online are incomplete.

Because the available data about this network does not include a complete set of interpersonal appraisals, we set up an initial appraisal matrix for our simulation based on one ancillary assumption: the initial self-appraisal of each individual is equal to the mean value of the appraisals of this individual held by all other group members. Given $\epsilon = 0.1$, the trajectories of the appraisal evolution on Sampson's affection network measured for the third time are shown in Fig. 8. The equilibrium structure illustrated in Fig. 8 is still a factions-followers structure with a single faction. Appraisal consensus of each individual in the network is also observed along the trajectories. Moreover, we observe a quick consolidation to approximately two clusters (two factions-followers structures) on a short time-scale, but then on a long time-scale, the bridging ties with positive appraisals and influences bring the whole group together slowly, and eventually, one faction with followers emerges in the equilibrium.

The third example considers Zachary's karate club network. The interactions among the members of a university karate club were recorded for 2 years by Zachary [37]. During observation, a conflict between the administrator and the instructor of the club developed and eventually the club broke into two clubs.

Because the available data about this network does not include a complete set of interpersonal appraisals, we set up an initial appraisal matrix for our simulation based on four ancillary assumptions: (1) each individual in the group has positive initial appraisals of the individuals that she interacted with and the appraisal values are proportional to the number of contexts in which interaction took place between the two individuals; (2) each individual has non-positive appraisals of the remaining individuals and the appraisal values are uniformly randomly-selected from $[-0.5, 0]$, while the administrator and the instructor have -1 appraisal of each other; (3) the initial self-appraisal of each individual is equal to the mean value of the positive appraisals of this individual held by the others; moreover, (4) the self-influence

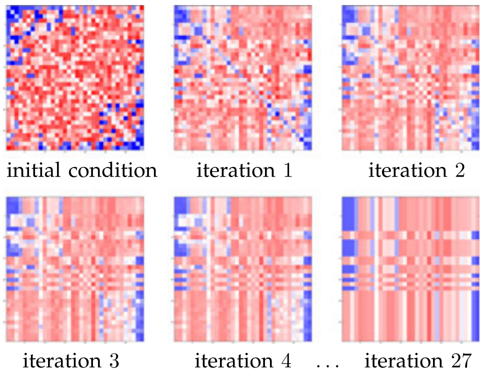

Fig. 9. Evolution of appraisal matrices on Karate club network: The equilibrium positive digraph includes two factions-followers structures. Node 1 corresponds to the instructor and Node 34 corresponds to the administrator. The appraisal submatrices associated with two structures converge to two rank-1 matrices.

parameter for the two intransigent individuals (i.e., the administrator and the instructor) is $\epsilon = 1$, and $\epsilon = 0$ for the remaining individuals. The trajectory of the appraisal evolution on Zachary's karate club network is shown in Fig. 9. Consistent with Zachary's analysis, we observe two factions-followers structures emerged in the equilibrium and each structure has a single faction consisting of either the instructor (node 1) or the administrator (node 34).

5.3 Theoretical and Numerical Analysis of the Self-Influence Parameter

To complete the analysis, we discuss the self-influence parameter ϵ and show its impact on the appraisal and influence coevolution. In particular, for $\epsilon = 0$, if we additionally assume that the influence (and equivalently positive appraisal) submatrices associated with the sink SCCs are aperiodic for all time, then all results in Theorems 5.1 and 5.2 hold true. The proofs are similar and skipped here. Moreover, even without an aperiodicity assumption, given $\epsilon = 0$, $X_{\mathrm{s}}(t)$ in the statement (i) of Theorem 5.2 still exponentially converges to a rank-1 positive matrix of the form $\mathbb{1}_{N_{\mathrm{s}}} v^T$, and consequentially the statement (ii) holds. That is, aperiodicity is implicitly satisfied for these two cases if there exists a positive column for the considered appraisal submatrix $X_{\mathrm{s}}(t)$ or matrix $X(t)$. In addition, ϵ does not need to be homogeneous for each individual and one may verify that all results in this article hold for positive and heterogeneous $\{\epsilon_i\}_{i \in \{1,\dots,N\}}$. Heterogeneous self-appraisal constants are adopted in the simulation of Fig. 9.

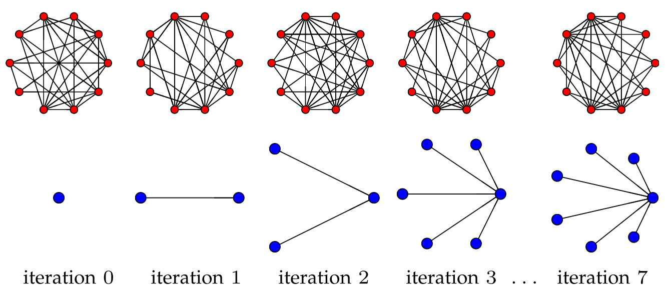

Fig. 10. Topology evolution of an appraisal structure with $\epsilon = 0$: In this and following three figures, the digraphs above (resp. below) correspond to the evolution of $G^+(X(t))$ (resp. $C(G^+(X(t)))$). The equilibrium positive digraph includes one faction (consisting of four nodes) and six followers.

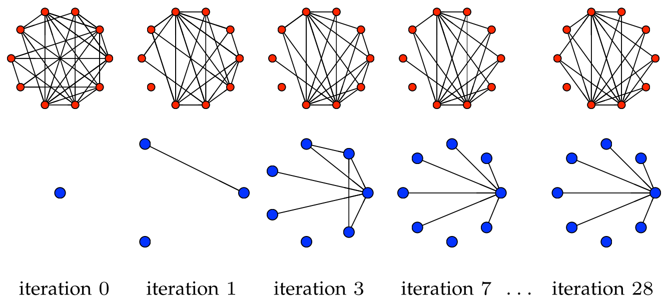

Fig. 11. Topology evolution of an appraisal structure with $\epsilon = 0.5$. The equilibrium positive digraph has two disconnected components. The component with nine nodes has one faction and six followers. Another disconnected component includes an outsider.

By the following numerical simulations, we claim that the equilibrium appraisal structure and the convergence rate may vary for different ϵ. Consider a coevolution system with 10 individuals. Given a constant initial state, we show the dynamical trajectories for three different ϵ. For $\epsilon = 0$, the dynamical system converges to an $O(10^{-5})$-neighborhood of the equilibrium in 7 iterations, and the topology evolutions of the positive appraisal digraphs and their condensation digraphs are shown in Fig. 10. For $\epsilon = 0.5$, the system converges in 28 iterations to an $O(10^{-5})$-neighborhood of the equilibrium as shown in Fig. 11. For $\epsilon = 0.9$, the topology evolutions of the digraphs are referred to Fig. 12. It takes 41 iterations in this case to reach an $O(10^{-5})$-neighborhood of the equilibrium. The simulations illustrate that a larger ϵ essentially corresponds to a slower convergence rate. It is easy to understand as ϵ represents the self-influence parameter of individuals, which measures the stubbornness of each individual on its previous opinion. From Figs. 10, 11, and 12, we also observe different trajectories of the appraisal structure evolutions for different ϵ.

Moreover, the number of factions at equilibrium may also vary for different ϵ. As illustrated in Fig. 13, given a coevolution system with 10 individuals and a constant initial state, the equilibrium appraisal structure has one factions-followers structure with a single faction for $\epsilon = 0.1$ and has two disconnected factions-followers structures each of which has a single faction for $\epsilon = 0.9$.

6 Structural Balance Properties of the Coevolutionary Dynamics

In this section we study the structural balance properties of the coevolution model of interpersonal appraisal and influ-

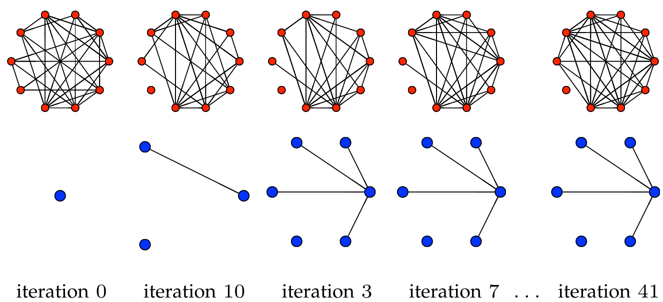

Fig. 12. Topology evolution of an appraisal structure with $\epsilon = 0.9$. The equilibrium positive digraph has one outsider and one factions-followers component. The factions-followers component includes nine nodes: One faction (consisting of five nodes) and four followers.

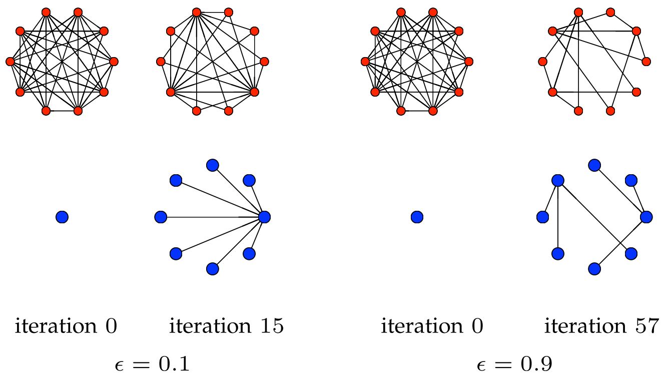

Fig. 13. Topology evolutions of an appraisal structure with $\epsilon = 0.1$ and $\epsilon = 0.9$, respectively. The equilibrium positive digraph has one faction for $\epsilon = 0.1$ but two factions for $\epsilon = 0.9$.

ence (4) and (5). In the previous two sections we have illustrated the topology evolution for the positive appraisal digraphs and the convergence properties for the appraisal and influence coevolution. Now we are able to combine these results with macro-structural models: it is interesting to study what macro-structural models the equilibrium factions-followers-outsiders structure is related to and which class of macro-structures are invariant under the coevolution. In what follows the word macro-structure is a synonym for an appraisal structure, i.e., a set of all appraisal matrices with a certain sign pattern. A macro-structure is *invariant* under the coevolution if, given an initial appraisal matrix belonging to the macro-structure, all trajectory matrices via the coevolution system (4) and (5) remain in the macro-structure.

Our coevolution model approach does not pre-specify a particular micro-model. Instead, it pre-specifies the conditions of interpersonal influence relations and addresses the implications of the model. Recall that our coevolution model satisfies the two statements in the classic balance model: "my friend's enemy is my enemy" and "my friend's friend is my friend", whereas the other two statements: "my enemy's enemy is my friend" and "my enemy's friend is my enemy" are not intuitively necessary for the coevolution of appraisal and influence. By examining all 16 types of triads in an appraisal structure, the deduced micro-model of permitted triad types by the first two statements is $\{300, 120D, 102, 021U, 012, 003\}$ (see Fig. 14 and [8] and [27, Section 8.3] for the detailed description of these triad types). Moreover, as we allow the interpersonal appraisal relation to be 0, the triad type $021D$ is also permitted in our model if the two bottom nodes of the positive digraph of $021D$ in Fig. 14 have 0 appraisal of each other. Overall, the micro-model associated with the coevolution model (4) and (5) is $P_{\text{co-evolv}} = \{300, 120D, 102, 021D, 021U, 012, 003\}$.

Furthermore, we examine the equilibrium structures described in Theorem 5.1. It is clear that the factions-followers-outsiders structure is the macro-structure defined by the micro-model $P_{\text{co-evolv}}$, where only triad types in $P_{\text{co-evolv}}$ appear in the structure and all remaining triad types are forbidden. In particular, triad type 300 is a one-faction structure, $120D$ is a one-faction-one-follower structure, 102 is a one-faction-one-outsider structure or a two-faction structure (depending on the top individual's self-appraisal), $021D$ is a two-faction-one-follower structure (where the interpersonal appraisals between the factions are 0), $021U$ is a one-faction-two-follower structure, 012 is a one-faction-one-follower-one-outsider/faction structure, and finally 003 includes three factions or outsiders. Consequently, any appraisal network including only triad types in $P_{\text{co-evolv}}$ has a factions-followers-outsiders structure.

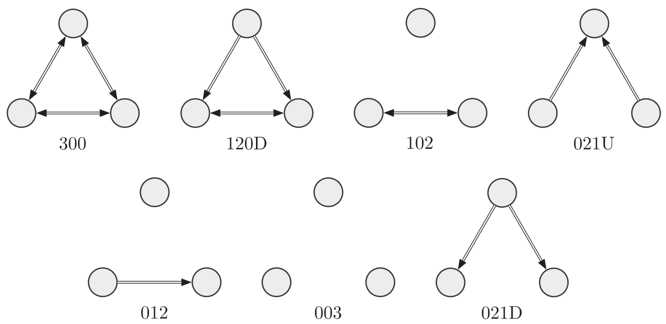

Fig. 14. Permitted triads: The positive appraisal digraph representations of the permitted triad types by the coevolution model.

Proposition 6.1. *The factions-followers-outsiders structure, i.e., the equilibrium macro-structure of the coevolution system (4) and (5), is consistent with the micro-model* $P_{\text{co-evolv}}$.

In other words, the coevolution system bridges the static micro-model and the dynamical convergence of the macro-structure networks, and the factions-followers-outsiders networks are then structurally balanced with respect to the micro-model $P_{\text{co-evolv}}$.

6.1 Invariant Macro-Structures

We have studied the macro-structure associated with the coevolution equilibrium appraisal networks. In the following, we analyze the macro-structures which are invariant in the coevolution system (4) and (5).

The implication of the *classical model of structural balance* is a class of appraisal macro-structures where either all individuals have strictly positive appraisal relations, or there are at most two subgroups such that individuals have strictly positive appraisal relations in the same subgroup but strictly negative appraisal relations between two subgroups. A classical balance structure has two possible block matrix patterns: (1) $[+]$ or (2) $\begin{bmatrix} D_1 & - \\ - & D_2 \end{bmatrix}$, where $\{D_i\}_{i\in\{1,\dots,2\}}$ are M-cliques. Moreover, $D_i \sim [+]$ if there are $N_i \geq 2$ individuals in this M-clique, and D_i can be either "−" or "+" for $N_i = 1$. It is noted that "M-clique" and "faction" are two similar but different concepts in this paper: the differences lie on (1) an outsider is a stand-alone M-clique but not a faction, (2) there may exist A-relations between two M-cliques but never between two factions. The classical balance structure is a special case of a factions-followers-outsiders structure, which includes at most two factions, may include outsiders, but does not include any followers. The classical balance macro-structure has been intensively studied, see e.g., in [1], [2]. We also consider other two macro-structural models: *clustering structure* and *ranked clusters of M-clique structure* in the following analysis.

Lemma 6.2 (Invariance of classical balanced structure). *The classical balanced structure is invariant under the coevolution system (4) and (5).*

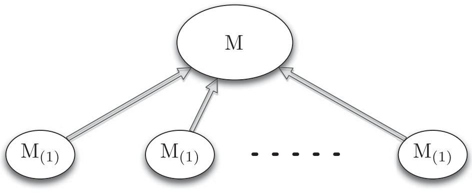

Fig. 15. Positive digraph of the invariant ranked clusters of M-clique structure: The nodes of the graph are M-cliques. The top M-clique is a faction and all $M_{(1)}$ are followers. The structure is a factions-followers structure with a single faction.

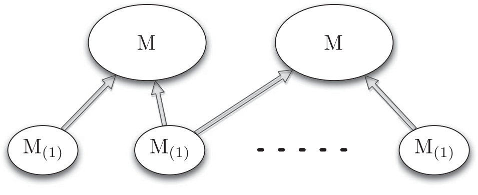

Fig. 16. Positive digraph of another equilibrium ranked clusters of M-clique structure: each sink M-clique is a faction; all source singleton $M_{(1)}$, are followers that hold (complete) positive appraisals of one or multiple factions. The structure is a factions-followers structure with multiple factions.

Define a *clustering structure* as an appraisal structure with a representative matrix

$$X \sim \begin{bmatrix} D_1 & - & \dots & - \\ - & D_2 & \dots & - \\ \vdots & \vdots & \ddots & \vdots \\ - & - & \dots & D_n \end{bmatrix}.$$

Here D_i, $i \in \{1, \dots, n\}$ are M-cliques (clusters), and all "$-$" block submatrices represent complete N-relations among these M-cliques. That is to say, the clustering structure extends the classical balanced structure to a structure with $n > 2$ M-cliques. This structure is also a factions-followers-outsiders structure, with an arbitrary number of factions.

Lemma 6.3 (Invariance of clustering structure). *The clustering structure is invariant under the coevolution system (4) and (5).*

A *ranked clusters of M-clique structure* is defined by a block matrix form: $X \sim \begin{bmatrix} D_1 & - & \dots & - \\ S_{21} & D_2 & \dots & - \\ \vdots & \vdots & \ddots & \vdots \\ S_{n1} & S_{n2} & \dots & D_n \end{bmatrix}$, with n M-cliques for $n \geq 2$. Here, without loss of generality, if $i \geq j$, the rank of the ith M-clique is higher than or equal to the rank of the jth M-clique. S_{ij} is strictly positive if and only if the ith M-clique ranks strictly higher than the jth M-clique; otherwise, if the ith and the jth M-cliques have the same rank, then S_{ij} is non-positive. One may check the ranked clusters of M-clique structure is not invariant under the coevolution system (4) and (5) in general. However, we will show that a subset of this structure is invariant under the coevolution.

Lemma 6.4 (Invariance of ranked clusters of M-Clique structure). *A ranked clusters of M-clique structure is not invariant under the coevolution system (4) and (5) in general. But, if an appraisal matrix X has both a ranked clusters of M-clique structure and a factions-followers structure with only one faction, then the structure of X is invariant under the coevolution.*

One may check that X in Lemma 6.4 satisfies

$$X \sim \begin{bmatrix} + & \dots & + & - & \dots & - \\ + & \dots & + & - & \dots & - \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ + & \dots & + & - & \dots & - \end{bmatrix}, \tag{8}$$

that is, all entries in the same columns of X have the same sign. It is noted that the structure of X has totally two ranks: only one M-clique is with the higher rank and it is a faction, and the remaining $n-1$ M-cliques have the same lower rank and they are followers, as shown in Fig. 15.

Among all macro-structures introduced in [27, Section 8.3], the three classes of macro-structures discussed in Lemmas 6.2, 6.3, and 6.4 are all potentially stable balanced structures under our coevolution. It is noted that the equilibrium appraisal structure also includes a ranked clusters of M-cliques structure specified as in Fig. 16. Different from the structure in Fig. 15, the structure in Fig. 16 has multiple factions and each follower may hold positive appraisals of more than one faction. One simple example for the equilibrium appraisal matrix in this case is that $X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ for any $\epsilon \geq 0$. However, this structure is not invariant under the coevolution system in general. Moreover, such an equilibrium is less-frequently observed in simulations with random initial conditions. In the example above, if the appraisal of node 1 held by the follower, node 3, increases for a sufficiently small amount, then the trajectory of the coevolution system leads to another equilibrium where node 3 is only directly connected to node 1.

6.2 Convergence of Invariant Macro-Structures

Now we integrate the invariant structure results in Lemmas 6.2, 6.3, and 6.4 with the convergence results in Section 5, which immediately implies the convergence properties of the stable macro-structures as in Corollary 6.1. In what follows we regard the classical balanced structure with two M-cliques as a special case of a clustering structure for the simplicity of presentation.

Corollary 6.1 (Convergence of generalized balanced structures). *For the coevolution system (4) and (5) with $\epsilon \geq 0$, each trajectory $X(t)$ converges exponentially fast to an equilibrium X^* in the following three scenarios:*

(i) (Convergence of a classical balanced structure with one cluster) *For a group of individuals with positive initial appraisals, $G^+(X(t))$ is a faction for all $t \geq 0$ and so is $G^+(X^*)$. Moreover, a positive appraisal consensus on each individual is achieved for the whole group in X^*.*

(ii) (Convergence of a clustering structure) *For a group of individuals with a clustering appraisal structure initially, the factions and outsiders of $G^+(X(0))$ remain unchanged in $G^+(X(t))$ for all $t \geq 0$ and in $G^+(X^*)$. An appraisal consensus of each individual of the group*

is achieved within each faction of $G^+(X^*)$: it is positive if the individual belongs to the faction and non-positive otherwise. An outsider occurs if and only if one cluster includes one individual and its self-appraisal is non-positive.

(iii) (Convergence of a ranked clusters of M-clique structure with form (8)) *For a group of individuals with an initial appraisal structure (8), the factions-followers structure with one faction remains unchanged in $G^+(X(t))$ for all $t \geq 0$ and in $G^+(X^*)$. The signs of all appraisals never change along the trajectory $X(t), t \geq 0$, and an appraisal consensus on each individual is achieved for the whole group in X^*.*

Different from Theorem 5.2 (iii), the first statement (i) of Corollary 6.1 assumes that all appraisals of the initial state are strictly positive, which implies the aperiodicity and irreducibility of all $X(t)$ and $W(t)$ along the trajectory. Therefore, ϵ could be equal to 0. Similarly, ϵ could be 0 for the second statement (ii). The third statement (iii) is a special case of Theorem 5.2 (ii), and therefore, the aperiodicity is satisfied implicitly and the statement holds for $\epsilon = 0$.

7 Conclusion

This article studies appraisal structure evolution among a group of individuals. Motivated by recent efforts on developing linkages between the major topics in sociological social psychology, we believe that it is interesting and meaningful to link social influence network theory with structural balance theory. As appraisals are subject to endogenous interpersonal influences, they may be influenced by others' appraisals. A network of such endogenous interpersonal influences is often formed in social groups. However, to the best of our knowledge, there are no dynamical models of appraisal structure which are directly evolved with the implications of such influence networks. It is not theoretically clear how the fundamental appraisals associated with persons' social identities are modified by the displayed influences of other group members, or how endogenous interpersonal influences in a group may generate equilibrium appraisals that are quite different from the initial array of appraisals.

We have presented novel results on the modeling and analysis of the coevolution of appraisal and influence networks. We derived a concise explicit dynamical model for the coevolution process and characterized completely its convergence and equilibrium structure properties. Our analysis also leads to several important implications to the study of signed social networks and structural balance theory. Specifically, our model shows that (i) for any initial appraisal matrix, the set of strongly connected components associated with the positive appraisal digraphs remains constant after finite time; (ii) for any initial appraisal matrix, the appraisal matrix trajectory converges asymptotically to an equilibrium, which has a factions-followers-outsiders structure: all individuals in a faction reach an appraisal consensus on each individual, all followers' appraisals are determined by the appraisals held by the individuals from the directly connected factions, and all outsiders have non-positive appraisals of each individual; and (iii) the appraisal structures according to the equilibria of the coevolution are balanced in sense that the two statements "my friend's enemy is my enemy" and "my friend's friend is my friend" are always satisfied in the associated social networks. The realizations of all possible equilibria of the coevolution fall into four distinct social structural classes. Meanwhile, three macro structural models are proved to be always stable subject to the proposed coevolution process. Overall, our model predicts a tendency of social appraisal structures to a set of structural equivalent bundles, i.e., a set of components where individuals have aligned interpersonal appraisals.

This paper presents only an introduction to appraisal evolution and structural balance models with implications of social influence networks, and much work remains to be done in order to understand the robustness of our formulation and its results. We assume here that the influence weights accorded by each individual are proportional to her positive appraisals on individuals of the social group. However, a large literature exists in social psychology on conditions that may affect individuals' influence network and its evolution (e.g., see our recent work [20]). We believe there are opportunities for a discussion on useful alternative mechanisms that adjust the relation between interpersonal appraisals and influences. Future research will be directed at validating our results with empirical data and identifying the qualitative roles of appraisal and influence coevolution mechanisms in the dynamics of signed social networks.

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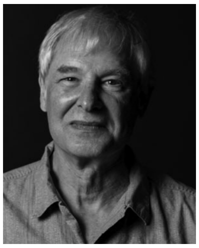

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