On Detection and Structural Reconstruction of Small-World Random Networks

T. Tony Cai¹, Tengyuan Liang¹ and Alexander Rakhlin¹

¹Department of Statistics, The Wharton School, University of Pennsylvania

Abstract

In this paper, we study detection and fast reconstruction of the celebrated Watts-Strogatz (WS) small-world random graph model (Watts and Strogatz, 1998) which aims to describe real-world complex networks that exhibit both high clustering and short average length properties. The WS model with neighborhood size k and rewiring probability probability β can be viewed as a continuous interpolation between a deterministic ring lattice graph and the Erdős-Rényi random graph. We study both the computational and statistical aspects of detecting the deterministic ring lattice structure (or local geographical links, strong ties) in the presence of random connections (or long range links, weak ties), and for its recovery. The phase diagram in terms of (k,β) is partitioned into several regions according to the difficulty of the problem. We propose distinct methods for the various regions.

1 Introduction

The "small-world" phenomenon aims to describe real-world complex networks that exhibit both high clustering and short average length properties. While most of the pairs of nodes are not friends, any node can be reached from another in a small number of hops. The Watts-Strogatz (WS) model, introduced in (Watts and Strogatz, 1998; Newman and Watts, 1999), is a popular generative model for networks that exhibit the small-world phenomenon. The WS model interpolates between the two extremes—the regular lattice graph on the one hand, and the random graph on the other. Considerable effort has been spent on studying the asymptotic statistical behavior (degree distribution, average path length, clustering coefficient, etc.) and the empirical performance of the WS model (Watts, 1999; Amaral et al., 2000; Barrat and Weigt, 2000; Latora and Marchiori, 2001; Van Der Hofstad, 2009). Successful applications of the WS model have been found in a range of disciplines, such as psychology (Milgram, 1967), epidemiology (Moore and Newman, 2000), medicine and health (Stam et al., 2007), to name a few. In one of the first algorithmic studies of the small-world networks, Kleinberg (2000) investigated the theoretical difficulty of finding the shortest path between any two nodes when one is restricted to use local algorithms, and further related the small-world notion to long range percolation on graphs (Benjamini and Berger, 2000; Coppersmith et al., 2002). The focus of the present paper is on statistical and computational aspects of the detection and recovery problems.

Given a network, the first statistical and computational challenge is to detect whether it enjoys the small world property, or whether the observation may be explained by the Erdős-Rényi random graph (the null hypothesis). The second question is concerned with the reconstruction of the neighborhood structure if the network does exhibit the small world phenomenon. In the language of social network analysis, the detection problem corresponds to detecting the existence of local geographical links (or

close friend connections, strong ties) in the presence of long range links (or random connections, weak ties). The reconstruction problem corresponds to distinguishing between these local links and long range links. The problem is statistically and computationally difficult due to the high-dimensional unobserved latent variable—the permutation matrix— which blurs the natural ordering of the ring structure.

Let us parametrize the WS model in the following way: the number of nodes is denoted by n, the neighborhood size by k, and the rewiring probability by β . Provided the adjacency matrix $A \in \mathbb{R}^{n \times n}$, we are interested in identifying the choices of (n, k, β) when detection and reconstruction of the small world random graph is possible. Specifically, we focus on the following two questions.

Detection Given the adjacency matrix A up to a permutation, when (in terms of n, k, β) and how (in terms of procedures) can one statistically distinguish whether it is a small world graph ($\beta < 1$), or a usual random graph with matching degree ($\beta = 1$). What if we restrict our attention to computationally efficient procedures?

Reconstruction Once the presence of the neighborhood structure is confirmed, when (in terms of n, k, β) and how (in terms of procedures) can one estimate the deterministic neighborhood structure? If one only aims to estimate the structure consistently (asymptotically correct), are there computationally efficient procedures, and what are their limitations?

We address the above questions by presenting a phase diagram in Figure 1. The phase diagram divides the parameter space into four disjoint regions according to the difficulty of the problem. We propose distinct methods for the regions where solutions are possible.

1.1 Why Small World Graph?

Finding and analyzing the appropriate statistical models for real-world complex networks is one of the main themes in network science. Many real empirical networks—for example, internet architecture, social networks, and biochemical pathways—exhibit two features simultaneously: high clustering among individual nodes and short distance between any two nodes. Consider the local tree rooted at a person. The high clustering property suggests prevalent existence of triadic closure, which significantly reduces the number of reachable people within a certain depth (in contrast to the regular tree case where this number grows exponentially with the depth), contradicting the short average length property. In a pathbreaking paper, Watts and Strogatz (1998) provided a mathematical model that resolves the above seemingly contradictory notions. The solution is surprisingly simple — interpolating between structural ring lattice graph and a random graph. The ring lattice provides the strong ties (i.e., homophily, connection to people who are similar to us) and triadic closure, while the random graph generates the weak ties (connection to people who are otherwise far-away), preserving the local-regular-branching-tree-like structure that induces short paths between pairs.

Given the small world model, it is natural to ask the statistical question of distinguishing the local links (geographical) and long range links (non-geographical) based on the observed graph.

1.2 Rewiring Model

Let us now define the WS model. Consider a ring lattice with n nodes, where each node is connected with its k nearest neighbors (k/2 on the left and k/2 on the right, k even for convenience). The rewiring process contains two procedures: erase and reconnect. First, erase each currently connected edge with probability β , independently. Next, reconnect each edge pair with probability β allowing multiplic-

ity. The observed symmetric adjacency matrix $A \in \{0,1\}^{n \times n}$ has the following structure under some unobserved permutation matrix $P_{\pi} \in \{0,1\}^{n \times n}$. For $1 \le i < j \le n$,

$$[P_{\pi}AP_{\pi}^{T}]_{ij} = 1 \quad \left\{ \begin{array}{ll} \text{w.p. } 1 - \beta(1 - \beta\frac{k}{n-1}), & \text{if } 0 < |i-j| \leq \frac{k}{2} \mod n - 1 - \frac{k}{2} \\ \text{w.p. } \beta\frac{k}{n-1}, & \text{otherwise} \end{array} \right.$$

and entries are independent of each other. Equivalently, we have for $1 \le i < j \le n$

$$A_{ij} = \kappa \left([P_{\pi} B P_{\pi}^T]_{ij} \right), \tag{1}$$

where $\kappa(\cdot)$ is the entry-wise i.i.d. Markov channel,

$$\kappa(0) \sim \mathsf{Bernoulli}\left(\beta \frac{k}{n-1}\right),$$

$$\kappa(1) \sim \mathsf{Bernoulli}\left(1 - \beta(1 - \beta \frac{k}{n-1})\right),$$

and $B \in \{0,1\}^{n \times n}$ indicates the support of the structural ring lattice

$$B_{ij} = \begin{cases} 1, & \text{if } 0 < |i-j| \le \frac{k}{2} \mod n - 1 - \frac{k}{2} \\ 0, & \text{otherwise} \end{cases}$$
 (2)

We denote by WS (n,k,β) the distribution of the random graph generated from the rewiring model, and denote by ER $(n,\frac{k}{n-1})$ the Erdős-Rényi random graph distribution (with matching average degree k). Remark that if $\beta=1$, the small world graph WS (n,k,β) reduces to ER $(n,\frac{k}{n-1})$, with no neighborhood structure. In contrast, if $\beta=0$, the small world graph WS (n,k,β) corresponds to the deterministic ring lattice, without random connections. We focus on the dependence of the gap $1-\beta=o(1)$ on n and k, such that distinguishing between WS (n,k,β) and ER $(n,\frac{k}{n-1})$ or reconstructing the ring lattice structure is statistically and computationally possible.

1.3 Summary of Results

The main theoretical and algorithmic results are summarized in this section. We first introduce several regions in terms of (n, k, β) , according to the difficulty of the problem instance, and then we present the results using the phase diagram in Figure 1. Except for the impossible region, we will introduce different algorithms with distinct computational properties.

Impossible region: $1-\beta < \sqrt{\frac{\log n}{n}} \lor \frac{\log n}{k}$. Inside this region no multiple testing procedure (regardless of computational budget) can succeed in distinguishing among the class of models including all of WS (n,k,β) and ER $(n,\frac{k}{n-1})$ with vanishing error.

Hard region: $\sqrt{\frac{\log n}{n}} \vee \frac{\log n}{k} \leq 1 - \beta < \sqrt{\frac{1}{k}} \vee \frac{\sqrt{\log n}}{k}$. It is possible to detect between WS (n, k, β) and ER $(n, \frac{k}{n-1})$ statistically with vanishing error; however the evaluation of the test statistic (5) (below) requires exponential time complexity (to the best of our knowledge).

Easy region: $\sqrt{\frac{1}{k}} \vee \frac{\sqrt{\log n}}{k} \leq 1 - \beta \leq \sqrt{\sqrt{\frac{\log n}{n}}} \vee \frac{\log n}{k}$. There exists an efficient spectral test that can distinguish between the small world random graph WS(n, k, β) and the Erdős-Rényi graph ER($n, \frac{k}{n-1}$) in near linear time (in the matrix size).

¹The original rewiring process in Watts and Strogatz (1998) does not allow multiplicity; however, for the simplicity of technical analysis, we focus on reconnection allowing multiplicity. These two rewiring processes are asymptotically equivalent.

Reconstructable region: $\sqrt{\sqrt{\frac{\log n}{n}}} \vee \frac{\log n}{k} < 1 - \beta \le 1$. In this region, not only is it possible to detect the existence of the lattice structure in a small-world graph, but it is also possible computationally to consistently estimate/reconstruct the neighborhood structure via a novel correlation thresholding procedure.

The following phase diagram provides an intuitive illustration of the above theoretical results. If we parametrize $k = n^x$, 0 < x < 1 and $1 - \beta = n^{-y}$, 0 < y < 1, each point $(x,y) \in [0,1]^2$ corresponds to a particular problem instance with parameter bundle $(n,k=n^x,\beta=1-n^{-y})$. According to the location of (x,y), the difficulty of the problem changes; for instance, the larger the x and the smaller the y is, the easier the problem becomes. The various above regions (plotted in $[0,1]^2$) are: impossible region (red region I), hard region (blue region II), easy region (green region III), reconstructable region (cyan region IV).

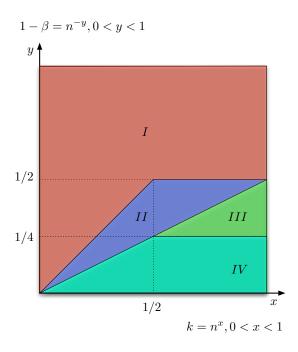


Figure 1: Phase diagram for small-world network: impossible region (red region I), hard region (blue region II), easy region (green region III), and reconstructable region (cyan region IV).

1.4 Notation

 $A,B,Z\in\mathbb{R}^{n\times n}$ denote symmetric matrices: A is the adjacency matrix, B is the structural signal matrix as in Equation (2), and $Z=A-\mathbb{E}A$ is the noise matrix. We denote the matrix of all ones by J. Notations \leq , \geq , \prec , \succ denote the asymptotic order: $a(n)\leq b(n)$ if and only if $\limsup_{n\to\infty}\frac{a(n)}{b(n)}\leq c$, with some constant c>0, a(n)< b(n) if and only if $\limsup_{n\to\infty}\frac{a(n)}{b(n)}=0$. C,C'>0 are universal constants that may change from line to line. For a symmetric matrix A, $\lambda_i(A)$, $1\leq i\leq n$ denotes the ranked eigenvalues, in a decreasing order. The inner-product $\langle A,B\rangle=\operatorname{tr}(A^TB)$ overloads the usual Euclidian inner-product and matrix inner-product. For any integer n, $[n]:=\{0,1,\ldots,n-1\}$ denotes the index set. Denote the permutation in symmetric group $\pi\in S_n$ and its associated matrix form as P_π .

For a graph G(V, E) generated from the Watts-Strogatz model $WS(n, k, \beta)$ with associated permuta-

tion π , for each node $v_i \in V$, $1 \le i \le |V|$, we denote

$$\mathcal{N}(v_i) := \left\{ v_j : 0 < |\pi^{-1}(i) - \pi^{-1}(j)| \le \frac{k}{2} \mod n - 1 - \frac{k}{2} \right\}$$

as the ring neighborhood nodes with respect to node v_i , before the permutation π applied.

1.5 Organization of the Paper

The following sections are dedicated to the theoretical justification of the various regions in Section 1.3. Specifically, Section 2 establishes the boundary for the impossible region I, where the problem is impossible to solve information theoretically. We contrast the hard region II with the regions III and IV in Section 3; here, the difference arises in statistical and computational aspects of detecting the strong tie structure inside random graph. Section 4 studies a correlation thresholding algorithm that reconstructs the neighborhood structure consistently when the parameters lie within the reconstructable region IV. We also study a spectral ordering algorithm which succeeds in reconstruction in a part of region III. Whether the remaining part of region III admits a recovery procedure is an open problem. Additional further directions are listed in Section 5.

2 The Impossible Region: Lower Bounds

We start with an information theoretic result that describes the difficulty of distinguishing among a class of models. The following Theorem 1 characterizes the impossible region, as in Section 1.3, in the language of minimax multiple testing error. The proof is postponed to Section 6.

Theorem 1 (Impossible Region). Consider the following statistical models: \mathscr{P}_0 denotes the probability measure of the Erdős-Rényi random graph $\mathsf{ER}(n,\frac{k}{n-1})$, and $\mathscr{P}_\pi,\pi\in S_{n-1}$ denote the probability measures of the Watts-Strogatz small-world graph $\mathsf{WS}(n,k,\beta)$ as in Equation (1) with different permutations π . Consider any selector $\phi:\{0,1\}^{n\times n}\to S_{n-1}\cup\{0\}$ that maps from the adjacency matrix $A\in\{0,1\}^{n\times n}$ to a decision in $S_{n-1}\cup\{0\}$. Then for any fixed $0<\alpha<1/8$, the following lower bound on multiple testing error holds:

$$\underline{\lim_{n\to\infty}} \min_{\phi} \max \left\{ \mathscr{P}_0(\phi \neq 0), \frac{1}{(n-1)!} \sum_{\pi \in S_{n-1}} \mathscr{P}_{\pi}(\phi \neq \pi) \right\} \geq 1 - 2\alpha,$$

when the parameters satisfy

$$1 - \beta \le C_{\alpha} \cdot \sqrt{\frac{\log n}{n}} \quad or \quad 1 - \beta \le C_{\alpha}' \cdot \frac{\log n}{k} \cdot \frac{1}{\log \frac{n \log n}{k^2}},$$

with constants C_{α} , C'_{α} that only depend on α . In other words, if

$$1 - \beta < \sqrt{\frac{\log n}{n}} \vee \frac{\log n}{k},$$

no multiple testing procedure can succeed in distinguishing, with vanishing error, the class of models containing all of WS (n, k, β) and ER $(n, \frac{k}{n-1})$.

The missing latent random variable, the permutation matrix P_{π} , is the object we are interested in recovering. A permutation matrix P_{π} induces a certain distribution on the adjacency matrix A. Thus the parameter space of interest, including models $\mathsf{WS}(n,k,\beta)$ and $\mathsf{ER}(n,\frac{k}{n-1})$, is of cardinality (n-1)!+1. Based on the observed adjacency matrix, distinguishing among the models including all of $\mathsf{WS}(n,k,\beta)$ and $\mathsf{ER}(n,\frac{k}{n-1})$ is equivalent to a multiple testing problem. The impossible region characterizes the information theoretic difficulty of this reconstruction problem by establishing the condition when minimax testing error does not vanish as $n,k(n)\to\infty$.

The "high dimensional" nature of this problem is mainly driven by the unknown permutation matrix, and this latent structure introduces difficulty both statistically and computationally. Statistically, via Le Cam's method, one can build a distance metric on permutation matrices using the distance between the corresponding measures (measures on adjacency matrices induced by the permutation structure). In order to characterize the intrinsic difficulty of estimating the permutation structure, one needs to understand the richness of the set of permutation matrices within certain distance to one particular element, a combinatorial task. Computationally, the combinatorial nature of the problem makes the "naive" approach computationally intensive.

3 Hard v.s. Easy Regions: Detection Statistics

This section studies the hard and easy regions in Section 1.3. First, we propose a near optimal test, the **maximum likelihood test**, that detects the ring structure above the information boundary derived in Theorem 1. However, the evaluation of the maximum likelihood test requires $\mathcal{O}(n^n)$ time complexity. The maximum likelihood test succeeds outside of region I, and, in particular, succeeds (statistically) in the hard region II. We then propose another efficient test, the **spectral test**, that detects the ring structure in time $\mathcal{O}^*(n^2)$ via the power method. The method succeeds in regions III and IV.

Theorem 2 below combines the results of Lemma 1 and Lemma 2.

Theorem 2 (Detection: Easy and Hard Boundaries). Consider the following statistical models: \mathcal{P}_0 denotes the distribution of the Erdős-Rényi random graph $\mathsf{ER}(n,\frac{k}{n-1})$, and $\mathcal{P}_\pi,\pi\in S_{n-1}$ denote distributions of the Watts-Strogatz small-world graph $\mathsf{WS}(n,k,\beta)$ with hidden permutation π . Consider any selector $\phi:\{0,1\}^{n\times n}\to\{0,1\}$ that maps an adjacency matrix to a binary decision (detection decision).

We say that minimax detection for the small-world random model is possible when

$$\lim_{n \to \infty} \min_{\phi} \max \left\{ \mathscr{P}_0(\phi \neq 0), \frac{1}{(n-1)!} \sum_{\pi \in S_{n-1}} \mathscr{P}_{\pi}(\phi \neq 1) \right\} = 0.$$
 (3)

If the parameter (n, k, β) satisfies

hard boundary:
$$1-\beta \ge \sqrt{\frac{\log n}{n}} \vee \frac{\log n}{k}$$
,

minimax detection is possible, and an exponential time **maximum likelihood test** (5) ensures (3). If, in addition, the parameter (n, k, β) satisfies

easy boundary:
$$1-\beta \ge \sqrt{\frac{1}{k}} \lor \frac{\sqrt{\log n}}{k}$$
,

then a near-linear time spectral test (7) ensures (3).

Proof of Theorem 2 consists of two parts, which will be addressed in the following two sections, respectively.

3.1 Maximum Likelihood Test

Consider the test statistic T_1 as the objective value of the following optimization

$$T_1(A) := \max_{P_{\pi}} \langle P_{\pi} B P_{\pi}^T, A \rangle, \tag{4}$$

where $P_{\pi} \in \{0,1\}^{n \times n}$ is taken over all permutation matrices and A is the observed adjacency matrix. The **maximum likelihood test** $\phi_1 : A \to \{0,1\}$ based on T_1 by

$$\phi_1(A) = \begin{cases} 1 & \text{if } T_1(A) \ge \frac{k}{n-1} nk + 2\sqrt{\frac{k}{n-1} nk \cdot \log n!} + \frac{2}{3} \cdot \log n! \\ 0 & \text{o.w.} \end{cases}$$
 (5)

The threshold is chosen as the rate $k^2 + \mathcal{O}\left(\sqrt{k^2 n \log \frac{n}{e}} \vee n \log \frac{n}{e}\right)$: if the objective value is of a greater order, then we believe the graph is generated from the small-world rewiring process with strong ties; otherwise we we cannot reject the null, the random graph model with only weak ties.

Lemma 1 (Guarantee for Maximum Likelihood Test). *The maximum likelihood test* ϕ_1 *in Equation* (5) *succeeds in detecting the small world random structure when*

$$1 - \beta \ge \sqrt{\frac{\log n}{n}} \vee \frac{\log n}{k},$$

in the sense that

$$\lim_{n,k(n)\to\infty} \max \left\{ \mathcal{P}_0(\phi_1\neq 0), \ \frac{1}{(n-1)!} \sum_{\pi\in S_{n-1}} \mathcal{P}_\pi(\phi_1\neq 1) \right\} = 0.$$

Remark 1. Lemma 1 can be viewed as the condition on the signal and noise separation. By solving the combinatorial optimization problem, the test statistics aggregates the signal that separates from the noise the most. An interesting open problem is, if we solve a relaxed version of the combinatorial optimization problem (4) within polynomial time complexity ϕ_1^{rel} , how much stronger the condition on $1-\beta$ needs to be to ensure power.

3.2 Spectral Test

For the spectral test, we calculate the second largest eigenvalue of the adjacency matrix A as the test statistic

$$T_2(A) := \lambda_2(A). \tag{6}$$

The **spectral test** $\phi_2: A \rightarrow \{0, 1\}$ is

$$\phi_2(A) = \begin{cases} 1 & \text{if } T_2(A) \ge \sqrt{k} \lor \sqrt{\log n} \\ 0 & \text{o.w.} \end{cases}$$
 (7)

Namely, if $\lambda_2(A)$ passes a certain threshold, we classify the graph as a small-world graph. Evaluation of (7) only requires near-linear time $\mathcal{O}^*(n^2)$.

Lemma 2 (Guarantee for Spectral Test). The second eigenvalue test ϕ_2 in Equation (7) satisfies

$$\lim_{n,k(n)\to\infty} \max \left\{ \mathscr{P}_0(\phi_2 \neq 0), \frac{1}{(n-1)!} \sum_{\pi \in S_{n-1}} \mathscr{P}_{\pi}(\phi_2 \neq 1) \right\} = 0$$

whenever

$$1 - \beta \ge \sqrt{\frac{1}{k}} \vee \frac{\sqrt{\log n}}{k}.$$

The main idea behind Lemma 2 is as follows. Let us look at the expectation of the adjacency matrix,

$$\mathbb{E}A = (1 - \beta)(1 - \beta \frac{k}{n - 1}) \cdot P_{\pi}^{T} B P_{\pi} + \beta \frac{k}{n - 1} \cdot (J - I),$$

where J is the matrix of all ones. The main structure matrix $P_{\pi}^{T}BP_{\pi}$ is a permuted version of the *circulant matrix* (see e.g. (Gray, 2006)). The spectrum of the circulant matrix B is highly structured, and is of distinct nature in comparison to the noise matrix $A - \mathbb{E}A$.

4 Reconstructable Region: Fast Structural Reconstruction

In this section, we discuss reconstruction of the ring structure in the Watts-Strogatz model. We show that in the reconstructable region (region IV in Figure 1), a **correlation thresholding procedure** succeed in reconstructing the ring neighborhood structure. As a by-product, once the neighborhood structure is known, one can distinguish between random edges and neighborhood edges for each node. A natural question is whether there is another algorithm that can work in a region (beyond region IV) where correlation thresholding fails. We show that in a certain regime with large k, a **spectral ordering procedure** outperforms the correlation thresholding procedure and succeeds in parts of regions III and IV (as depicted in Figure 2 below).

4.1 Correlation Thresholding

Consider the following **correlation thresholding procedure** for neighborhood reconstruction.

Algorithm 1: Correlation Thresholding for Neighborhood Reconstruction

Data: An adjacency matrix $A \in \mathbb{R}^{n \times n}$ for the graph G(V, E).

Result: For each node v_i , $1 \le i \le n$, an estimated set for neighborhood $\hat{\mathcal{N}}(v_i)$.

- 1. For each node v_i , calculate the correlation $\langle A_i, A_j \rangle$ for all $j \neq i$;
- 2. Sort the $\{\langle A_i, A_j \rangle, j \in [n] \setminus \{i\}\}$ in a decreasing order, select the largest k ones to form the estimated set $\hat{\mathcal{N}}(v_i)$;

Output: $\hat{\mathcal{N}}(v_i)$, for all $i \in [n]$

The following lemma proves consistency of the above Algorithm 1. Note the computational complexity is $\mathcal{O}(n \cdot \min\{\log n, k\})$ for each node using quick-sort, with a total runtime $\mathcal{O}^*(n^2)$.

Lemma 3 (Consistency of Correlation Thresholding). *Consider the Watts-Strogatz random graph* $WS(n, k, \beta)$. *Under the reconstructable regime IV (in Figure 1), that is,*

$$1 - \beta > \sqrt{\frac{\log n}{k}} \vee \left(\frac{\log n}{n}\right)^{1/4},\tag{8}$$

correlation thresholding provides a consistent estimate of the neighborhood set $\mathcal{N}(v_i)$ w.h.p in the sense that

$$\lim_{n,k(n)\to\infty}\max_{i\in[n]}\frac{|\hat{\mathcal{N}}(v_i)\triangle\mathcal{N}(v_i)|}{|\mathcal{N}(v_i)|}=0,$$

where \triangle denotes the symmetric set difference.

One interesting question in small-world networks is to distinguish between strong ties (structural edges induced by the ring lattice structure) and weak ties (edges due to random connections). The above lemma addresses this question by providing a consistent estimate of the neighborhood set for each node.

The condition under which consistency of correlation thresholding is ensured corresponds to the reconstructable region in Figure 1. One may ask if there is another algorithm that can provide a consistent estimate of the neighborhood set beyond region IV. The answer is yes, and we will show in the following section that under the regime when k is large (for instance, $k \ge n^{\frac{15}{16}}$), indeed it is possible to slightly improve on Algorithm 1.

4.2 Spectral Ordering

Consider the following **spectral ordering procedure**, which approximately reconstructs the ring lattice structure when k is large, i.e., $k > n^{\frac{7}{8}}$.

Algorithm 2: Spectral Reconstruction of Ring Structure

Data: An adjacency matrix $A \in \mathbb{R}^{n \times n}$ for the graph G(V, E).

Result: A ring embedding of the nodes *V*.

- 1. Calculate top 3 eigenvectors in the SVD $A = U\Sigma U^T$. Denote second and third eigenvectors as $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, respectively;
- 2. For each node i and the associated vector $A_{\cdot i} \in \mathbb{R}^n$, calculate the associated angle θ_i for vector $(u^T A_{\cdot i}, v^T A_{\cdot i})$;

Output: the sorted sequence $\{\theta_i\}_{i=1}^n$ and the corresponding ring embedding of the nodes. For each node v_i , $\hat{\mathcal{N}}(v_i)$ are the closest k nodes in the ring embedding.

The following Lemma 4 shows that when k is large, Algorithm 2 also provides consistent reconstruction of the ring lattice. Its computational complexity is $\mathcal{O}^*(n^2)$.

Lemma 4 (Guarantee for Spectral Ordering). *Consider the Watts-Strogatz graph* $WS(n, k, \beta)$. *Assume k is large enough in the following sense:*

$$1 > \overline{\lim_{n,k(n) \to \infty}} \frac{\log k}{\log n} \ge \underline{\lim_{n,k(n) \to \infty}} \frac{\log k}{\log n} > \frac{7}{8}.$$

Under the regime

$$1 - \beta > \frac{n^{3.5}}{k^4},\tag{9}$$

the spectral ordering provides consistent estimate of the neighborhood set $\mathcal{N}(v_i)$ w.h.p. in the sense that

$$\lim_{n,k(n)\to\infty} \max_{i\in[n]} \frac{|\hat{\mathcal{N}}(v_i)\Delta\mathcal{N}(v_i)|}{|\mathcal{N}(v_i)|} = 0,$$

where \triangle denotes the symmetric set difference.

In Lemma 4, we can only prove consistency of spectral ordering under the technical condition that k is large. We do not believe this is due to an artifact of the proof. Even though the structural matrix (the signal) has large eigenvalues, the eigen-gap is not large enough. The spectral ordering succeeds when the spectral gap stands out over the noise level, which implies that k needs to be large enough.

Let us compare the region described in Equation (9) with the reconstructable region in Equation (8). We observe that spectral ordering pushes slightly beyond the reconstructable region when $k > n^{\frac{15}{16}}$, as shown in Figure 2.

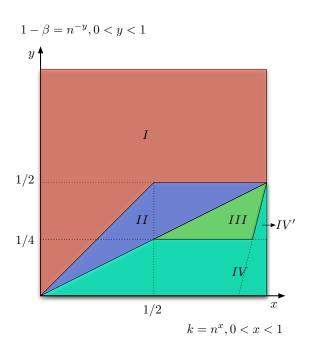


Figure 2: Phase diagram for small-world networks: impossible region (red region I), hard region (blue region II), easy region (green region III), and reconstructable region (cyan region IV and IV'). Compared to Figure 1, the spectral ordering procedure extends the reconstructable region (IV) when $k > n^{\frac{15}{16}}$ (IV').

5 Discussion

Reconstructable region We addressed the reconstruction problem via two distinct procedures, correlation thresholding and spectral ordering; however, whether there exists other computationally efficient algorithm that can significantly improve upon the current reconstructable region is still unknown. Designing new algorithms requires a deeper insight into the structure of the small-world model, and will probably shed light on better algorithms for mixed membership models.

Comparison to stochastic block model Recently, stochastic block models (SBM) have attracted considerable amount of attention from researchers in various fields. Community detection in stochastic block models focuses on recovering the hidden community information from the adjacency matrix that contains both noise and the latent permutation. The hidden community structure for classic SBM is illustrated in Figure 3 (the left one), as a block diagonal matrix. An interesting but theoretically more challenging extension to the classic SBM is the mixed membership SBM, where each node may simul-

taneously belong to several communities. The problem becomes more difficult when there are a growing number of communities and when each node belongs to several communities at the same time. Consider one easy case of the mix membership model, where the mix membership occurs only within neighborhood communities, as shown in the middle image of Figure 3. The small-world network we are investigating in this paper can be seen as an extreme case (shown on the right-most figure) of this easy mixed membership SBM, where each node falls in effectively k local clusters.

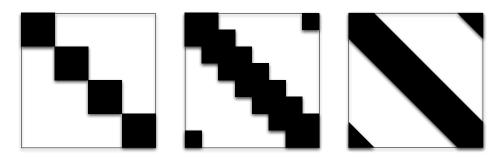


Figure 3: The structural matrices for stochastic block model (left), mixed membership SBM (middle), and small-world model (right). The black location denotes the support of the structural matrix.

In the small-world networks, identifying the structural links and random links becomes challenging since that are many local clusters (in constrast to relative small number of communities in SBM). This multitude of local clusters makes it difficult to analyze the effect of the hidden permutation on the structural matrix. We view the current paper as an initial attempt at attacking this problem.

6 Technical Proofs

Proof of Theorem 1. Denote the circulant matrix by B (it is B_{π} for any $\pi \in S_{n-1}$). The likelihood on $X \in \mathbb{R}^{n \times n}$ for WS model is

$$\begin{split} \mathcal{L}_{n,k,\beta}(X|B) &= \exp\left\{\log\frac{1-\beta(1-\beta\frac{k}{n-1})}{\beta(1-\beta\frac{k}{n-1})}\cdot\langle X,B\rangle + \log\frac{\beta\frac{k}{n-1}}{1-\beta\frac{k}{n-1}}\cdot\langle X,J-I-B\rangle \right. \\ &\qquad \qquad + nk\log(\beta(1-\beta\frac{k}{n-1})) + n(n-1-k)\log(1-\beta\frac{k}{n-1}) \right\} \\ &= \exp\left\{\left(\log\frac{1-\beta(1-\beta\frac{k}{n-1})}{\beta(1-\beta\frac{k}{n-1})} - \log\frac{\beta\frac{k}{n-1}}{1-\beta\frac{k}{n-1}}\right)\cdot\langle X,B\rangle + \log\frac{\beta\frac{k}{n-1}}{1-\beta\frac{k}{n-1}}\cdot\langle X,J-I\rangle \right. \\ &\qquad \qquad + nk\log(\beta(1-\beta\frac{k}{n-1})) + n(n-1-k)\log(1-\beta\frac{k}{n-1}) \right\}. \end{split}$$

For the Erdős-Rényi model, the likelihood is

$$\mathcal{L}_{n,k}(X) = \exp\left\{\log\frac{\frac{k}{n-1}}{1 - \frac{k}{n-1}} \cdot \langle X, J - I \rangle + n(n-1)\log(1 - \frac{k}{n-1})\right\}.$$

The Kullback-Leibler divergence between this two model is expressed in the following

$$\begin{split} \operatorname{KL}(P_{B}||P_{0}) &= \mathbb{E}_{X \sim P_{B}} \log \frac{P_{B}(X)}{P_{0}(X)} \\ &= \mathbb{E}_{X \sim P_{B}} \left\{ -\left(\log \frac{\frac{k}{n-1}}{1 - \frac{k}{n-1}} - \log \frac{\beta \frac{k}{n-1}}{1 - \beta \frac{k}{n-1}}\right) \cdot \langle X, J - I \rangle - n(n-1) \log (1 - \frac{k}{n-1}) \right. \\ &+ \left(\log \frac{1 - \beta (1 - \beta \frac{k}{n-1})}{\beta (1 - \beta \frac{k}{n-1})} - \log \frac{\beta \frac{k}{n-1}}{1 - \beta \frac{k}{n-1}}\right) \cdot \langle X, B \rangle + nk \log (\beta (1 - \beta \frac{k}{n-1})) + n(n-1-k) \log (1 - \beta \frac{k}{n-1})\right\} \\ &= -\left(\log \frac{\frac{k}{n-1}}{1 - \frac{k}{n-1}} - \log \frac{\beta \frac{k}{n-1}}{1 - \beta \frac{k}{n-1}}\right) \cdot \left\langle (1 - \beta) (1 - \beta \frac{k}{n-1})B + \beta \frac{k}{n-1} (J - I), J - I\right\rangle \\ &+ \left(\log \frac{1 - \beta (1 - \beta \frac{k}{n-1})}{\beta (1 - \beta \frac{k}{n-1})} - \log \frac{\beta \frac{k}{n-1}}{1 - \beta \frac{k}{n-1}}\right) \cdot \left\langle (1 - \beta) (1 - \beta \frac{k}{n-1})B + \beta \frac{k}{n-1} (J - I), B\right\rangle \\ &- n(n-1) \log (1 - \frac{k}{n-1}) + nk \log (\beta (1 - \beta \frac{k}{n-1})) + n(n-1-k) \log (1 - \beta \frac{k}{n-1}) \\ &= n(n-1) \log \frac{1 - \beta \frac{k}{n-1}}{1 - \frac{k}{n-1}} - nk \log \frac{1}{\beta} - \left[\log \frac{1}{\beta} + \log \frac{1 - \beta \frac{k}{n-1}}{1 - \frac{k}{n-1}}\right] nk \left[1 - (1 - \beta) \beta \frac{k}{n-1}\right] \\ &+ \left[\log \frac{1}{\beta} + \log \frac{1 - \beta (1 - \beta \frac{k}{n-1})}{\beta \frac{k}{n-1}}\right] + \log \frac{1 - \beta \frac{k}{n-1}}{1 - \frac{k}{n-1}}n \left[(n-1-k) + (1 - \beta) \beta \frac{k^{2}}{n-1}\right] \\ &+ \log \frac{1 - \beta (1 - \beta \frac{k}{n-1})}{\beta \frac{k}{n-1}} \cdot nk \left[1 - \beta (1 - \beta \frac{k}{n-1})\right]. \end{split}$$

Via the inequality $\log(1+x) < x$ for all x > -1, we can further simplify the above expression as

$$\begin{split} \mathrm{KL}(P_{B}||P_{0}) & \leq nk(1-\beta) \left[-\beta + \beta \frac{k}{n-1} + (1-\beta)\beta \frac{k^{2}}{n(n-1-k)} \right] + \frac{(1-\beta)(1-\beta \frac{k}{n-1})}{\beta \frac{k}{n-1}} nk \left[(1-\beta) + \beta^{2} \frac{k}{n-1} \right] \\ & \leq nk(1-\beta) \left[(1-\beta)\beta \frac{k}{n-1} + (1-\beta)\beta \frac{k^{2}}{n(n-1-k)} \right] + \frac{(1-\beta)^{2}(1-\beta \frac{k}{n-1})}{\beta} n(n-1) \leq C \cdot n^{2}(1-\beta)^{2}, \end{split}$$

where $0 < C < \frac{1}{2} \frac{k^2}{n(n-1)} + \frac{1}{\beta}$ is some universal constant (note we are interested in the case when β is close to 1).

Remark that when $k \le n^{1/2}$, the above bound can be further strengthened, in the following sense (recall equation (10))

$$\begin{split} \text{KL}(P_B||P_0) & \leq nk(1-\beta) \left[-\beta + \beta \frac{k}{n-1} + (1-\beta)\beta \frac{k^2}{n(n-1-k)} \right] + \log \frac{1-\beta(1-\beta \frac{k}{n-1})}{\beta \frac{k}{n-1}} \cdot nk \left[1-\beta(1-\beta \frac{k}{n-1}) \right] \\ & \leq \left\{ \log \frac{1-\beta(1-\beta \frac{k}{n-1})}{\beta \frac{k}{n-1}} \cdot \frac{1-\beta(1-\beta \frac{k}{n-1})}{\beta \frac{k}{n-1}} \right\} \cdot k^2 \beta \frac{n}{n-1}. \end{split}$$

Denote $t := \frac{1 - \beta(1 - \beta \frac{k}{n-1})}{\beta \frac{k}{n-1}} = \frac{1 - \beta}{\beta} \frac{n-1}{k} + \beta$. Thus we have

$$KL(P_B||P_0) \le t \log t \cdot k^2 \beta \frac{n}{n-1}.$$
 (13)

Suppose for some constant $\alpha_* > 0$, and $\alpha = \alpha_* \cdot \frac{1}{\beta} (1 - \frac{1}{n})^2$, we have the following

$$t \le \alpha \frac{n \log \frac{n}{e}}{k^2} \cdot \frac{1}{\log \alpha \frac{n \log \frac{n}{e}}{k^2}} \tag{14}$$

and
$$t \log t \le \alpha \frac{n \log \frac{n}{e}}{k^2} \cdot \left(1 - \frac{\log \log \alpha \frac{n \log \frac{n}{e}}{k^2}}{\log \alpha \frac{n \log n}{k^2}}\right) < \alpha \frac{n \log \frac{n}{e}}{k^2}.$$
 (15)

Plug in the expression for t into (14), if

$$\frac{1-\beta}{\beta} \le \alpha (1+\frac{1}{n-1}) \cdot \frac{\log \frac{n}{e}}{k} \cdot \frac{1}{\log \alpha \frac{n\log \frac{n}{e}}{k^2}} - \frac{k}{n-1} \le \frac{\log n}{k} \frac{1}{\log \frac{n\log \frac{n}{e}}{k^2}}$$
 (16)

we have

$$t \leq \alpha \frac{n\log \frac{n}{e}}{k^2} \cdot \frac{1}{\log \alpha \frac{n\log \frac{n}{e}}{k^2}} \ \Rightarrow \ t\log t < \alpha \frac{n\log \frac{n}{e}}{k^2}$$

which further implies (via equation (13))

$$\frac{1}{(n-1)!} \sum_{\pi \in S_{n-1}} \text{KL}(P_{B_{\pi}} || P_0) \le t \log t \cdot k^2 \beta \frac{n}{n-1} \le \alpha_* \cdot \log(n-1)!.$$

Recalling Equation (10), if

$$1 - \beta \le \sqrt{\frac{\alpha_*}{C} \cdot \frac{(n-1)\log\frac{n}{e}}{n^2}} \approx \sqrt{\frac{\log n}{n}}$$
 (17)

we have

$$\frac{1}{(n-1)!} \sum_{\pi \in S_{n-1}} \mathrm{KL}(P_{B_{\pi}}||P_0) \le n^2 (1-\beta)^2 \le \alpha_* \cdot \log(n-1)!.$$

We invoke the following Lemma on minimax error through Kullbak-Leibler divergence.

Lemma 5 (Tsybakov (2009), Proposition 2.3). Let P_0, P_1, \ldots, P_M be probability measures on $(\mathcal{X}, \mathcal{A})$ satisfying

$$\frac{1}{M} \sum_{j=1}^{M} \text{KL}(P_j || P_0) \le \alpha \cdot \log M \tag{18}$$

with $0 < \alpha < \frac{1}{8}$. Then for any $\psi : \mathcal{X} \to [M+1]$

$$\max \left\{ P_0(\psi \neq 0), \frac{1}{M} \sum_{j=1}^M P_j(\psi \neq j) \right\} \geq \frac{\sqrt{M}}{\sqrt{M} + 1} \left(1 - 2\alpha - \sqrt{\frac{2\alpha}{\log M}} \right).$$

Collecting Equations (16) and (17), if either one of the conditions in Equations (16) and (17) holds, we have

$$\frac{1}{(n-1)!} \sum_{\pi \in S_{n-1}} \text{KL}(P_{B_{\pi}} || P_0) \le \alpha_* \cdot \log(n-1)!.$$
(19)

Putting things together, if

$$1 - \beta < \sqrt{\frac{\log n}{n}} \vee \frac{\log n}{k},$$

we have that Equation (19) hold. Applying Lemma 5, we complete the proof

$$\lim_{n \to \infty} \min_{\phi} \max \left\{ P_0(\phi \neq 0), \; \frac{1}{(n-1)!} \sum_{i=1}^{(n-1)!} P_i(\phi \neq i) \right\} \geq \lim_{n \to \infty} \frac{\sqrt{(n-1)!}}{1 + \sqrt{(n-1)!}} \left(1 - 2\alpha - \sqrt{\frac{2\alpha}{\log(n-1)!}} \right) = 1 - 2\alpha.$$

Proof of Lemma 1. Let us state the well-known Bernstein's inequality (Boucheron et al. (2013), Theorem 2.10), which will be used in the proof of this lemma.

Lemma 6 (Bernstein's inequality). Let $X_1, ..., X_n$ be independent bounded real-valued random variables. Assume that there exist positive numbers v and c such that

$$\begin{split} &\sum_{i=1}^n \mathbb{E}[X_i^2] \leq v, \\ &X_i \leq 3c, \forall 1 \leq i \leq n \ a.s. \end{split}$$

then we have, for all t > 0,

$$\mathbb{P}\left(\sum_{i=1}^{n} (X_i - \mathbb{E}X_i) \ge \sqrt{2vt} + ct\right) \le e^{-t}.$$
 (20)

First, let us consider the case when the adjacency matrix A is generated from the Erdős-Rényi random graph $\mathsf{ER}(n,\frac{k}{n-1})$. Recall Bernstein's inequality Lemma 6, for any P_π with $\pi \in S_{n-1}$, we know $\langle P_\pi B P_\pi^T, A \rangle$ has the same distribution as $\langle B, A \rangle$. Thus

$$\begin{split} \langle P_{\pi}BP_{\pi}^{T},A\rangle &\stackrel{\text{in law}}{=\!=\!=} \langle B,A\rangle = 2\sum_{i>j}A_{ij}B_{ij} \\ &= 2\sum_{i>j}\mathbb{E}[A_{ij}]B_{ij} + 2\sum_{i>j}(A_{ij} - \mathbb{E}[A_{ij}])B_{ij} \\ &\leq \frac{k}{n-1}nk + 2\sqrt{\frac{k}{n-1}nkt} + \frac{2}{3}t \end{split}$$

with probability at least $1 - \exp(-t)$. Here the last step is through Bernstein's inequality. There are nk/2 non-zero $B_{i,j}$, i > j, and it is clear that $A_{ij} \sim \text{Bernoulli}(\frac{k}{n-1})$, $2\sum_{i>j} \mathbb{E}[A_{ij}]B_{ij} = nk\frac{k}{n-1}$. Thus we can take $c = \frac{1}{3}$ and

$$\nu = \sum_{i < j} \mathbb{E}[A_{ij} B_{ij}]^2 = \sum_{i < j} \mathbb{E}[A_{ij}]^2 B_{ij} = \frac{nk}{2} \frac{k}{n - 1}$$

in Lemma 6. Via the union bound, take $t = \log n!$, we have

$$\max_{P_{\pi}} \langle P_{\pi}BP_{\pi}^{T}, A \rangle \leq \frac{k}{n-1}nk + 2\sqrt{\frac{k}{n-1}nk \cdot \log n!} + \frac{2}{3} \cdot \log n!$$

with probability at least $1 - (n-1)! \exp(-\log n!) = 1 - \frac{1}{n}$.

Alternatively, suppose *A* is from the small-world rewiring model WS(n, k, β), with permutation being the identity $\pi = e$. With probability at least $1 - \exp(-\log n) = 1 - \frac{1}{n}$,

$$\begin{split} \max_{P_{\pi}} \ \langle P_{\pi}BP_{\pi}^{T}, A \rangle &\geq \langle B, A \rangle \\ &= \langle B, \mathbb{E}[A] \rangle + \langle B, A - \mathbb{E}[A] \rangle \\ &\geq (1 - \beta + \beta^{2} \frac{k}{n - 1}) nk - \sqrt{nk \cdot \log n} \end{split}$$

where the last step is from Hoeffding's inequality: it is clear that for location (i, j) when $B_{ij} \neq 0$,

$$\mathbb{E}[A_{ij}] = 1 - \beta + \beta^2 \frac{k}{n-1},$$

and $0 \le A_{ij} \le 1$ almost surely.

Therefore if there exist a threshold T > 0 such that

$$(1 - \beta + \beta^2 \frac{k}{n - 1})nk - \sqrt{nk \cdot \log n} > T > \frac{k}{n - 1}nk + 2\sqrt{\frac{k}{n - 1}nk \cdot \log n!} + \frac{2}{3} \cdot \log n!$$
 (21)

we have that

$$\lim_{n,k(n)\to\infty} \; \max \; \left\{ P_0(\phi_1 \neq 0), \; \frac{1}{(n-1)!} \sum_{i=1}^{(n-1)!} P_i(\phi_1 \neq 1) \right\} \leq \lim_{n,k(n)\to\infty} \frac{1}{n} = 0.$$

The detailed calculation of Equation (21) yields that the test succeeds with high probability whenever

$$1 - \beta \ge \sqrt{\frac{\log n}{n}} \vee \frac{\log n}{k}.$$

Proof of Lemma 2. Under the rewiring model (Watts-Strogatz model) WS (n, k, β) with permutation P_{π}

$$P_{\pi}AP_{\pi}^{T} = (1 - \beta)(1 - \beta \frac{k}{n-1}) \cdot B + \beta \frac{k}{n-1} \cdot (J - I) + Z$$

where $J = 11^T \in \mathbb{R}^{n \times n}$, B is the ring structured signal matrix defined in Equation (2). We denote in short $A = \mathbb{E}A + Z$ as this signal and the noise part, and

$$B_{ij} = \begin{cases} 1 & \text{if } 0 < |i-j| \le \frac{k}{2} \mod n - 1 - \frac{k}{2} \\ 0 & \text{elsewhere} \end{cases}$$

B is a circulant matrix, whose spectrum is highly structured, and *Z* is a zero-mean noise random matrix. We first study the random fluctuation part, $Z = A - \mathbb{E}A$. Let us bound the expectation $\mathbb{E}\|A - \mathbb{E}A\|$ as a starting step, for any adjacency matrix $A \in \mathbb{R}^{n \times n}$ using the symmetrization trick. Denote $A' \sim A$

as the independent copy of A sharing the same distribution. Take $E,G \in \mathbb{R}^{n \times n}$ as random symmetric Rademacher and Gaussian matrices with entries E_{ij} , G_{ij} being, respectively, independent Rademacher and Gaussian. Denoting $A \circ B$ as matrix Hadamard product, we have

$$\begin{split} \mathbb{E}\|A - \mathbb{E}A\| &= \mathbb{E}\sup_{\|v\|_{\ell_2} = 1} \langle (A - \mathbb{E}A)\,v, v \rangle = \mathbb{E}\sup_{\|v\|_{\ell_2} = 1} \langle (A - \mathbb{E}_{A'}A')\,v, v \rangle \\ &\leq \mathbb{E}_A \mathbb{E}_{A'}\sup_{\|v\|_{\ell_2} = 1} \langle (A - A')\,v, v \rangle = \mathbb{E}_E \mathbb{E}_A \mathbb{E}_{A'}\sup_{\|v\|_{\ell_2} = 1} \langle [E \circ (A - A')]\,v, v \rangle \\ &\leq \mathbb{E}_A \mathbb{E}_E\sup_{\|v\|_{\ell_2} = 1} \langle [E \circ A]\,v, v \rangle + \mathbb{E}_{A'}\mathbb{E}_E\sup_{\|v\|_{\ell_2} = 1} \langle [-E \circ A']\,v, v \rangle \\ &= 2\mathbb{E}_A \mathbb{E}_E\sup_{\|v\|_{\ell_2} = 1} \langle [E \circ A]\,v, v \rangle \leq \frac{2}{\sqrt{2/\pi}} \cdot \mathbb{E}_A \mathbb{E}_E\sup_{\|v\|_{\ell_2} = 1} \langle [\mathbb{E}_G[|G|] \circ E \circ A]\,v, v \rangle \\ &\leq \sqrt{\frac{\pi}{2}} \cdot \mathbb{E}_A \mathbb{E}_E \mathbb{E}_G\sup_{\|v\|_{\ell_2} = 1} \langle [|G| \circ E \circ A]\,v, v \rangle = \sqrt{\frac{\pi}{2}} \cdot \mathbb{E}_A \mathbb{E}_G\sup_{\|v\|_{\ell_2} = 1} \langle [G \circ A]\,v, v \rangle \\ &= \sqrt{\frac{\pi}{2}} \cdot \mathbb{E}_A (\mathbb{E}_G \|G \circ A\|) \,. \end{split}$$

Recall the following Lemma from Bandeira and van Handel (2014).

Lemma 7 (Bandeira and van Handel (2014), Theorem 1.1). Let X be the $n \times n$ symmetric random matrix with $X = G \circ A$, where G_{ij} , i < j are i.i.d. N(0,1) and A_{ij} are given scalars. Then

$$\mathbb{E}_G \|X\| \leq \max_i \sqrt{\sum_j A_{ij}^2} + \max_{ij} |A_{ij}| \cdot \sqrt{\log n}.$$

Thus via Jensen's inequality and the above Lemma, we can continue

$$\begin{split} \mathbb{E}\|A - \mathbb{E}A\| &\leq \sqrt{\frac{\pi}{2}} \cdot \mathbb{E}_A \left(\mathbb{E}_G \|G \circ A\| \right) \\ &\lesssim \mathbb{E}_A \max_i \sqrt{\sum_j A_{ij}^2} + \max_{ij} |A_{ij}| \cdot \sqrt{\log n} \\ &\leq \sqrt{\mathbb{E}_A \max_i \sum_j A_{ij}^2} + \sqrt{\log n} \\ &\leq \sqrt{k + C_1 2 \sqrt{k \log n} + C_2 \log n} + \sqrt{\log n} \asymp \sqrt{k} \vee \sqrt{\log n}, \end{split}$$

where the last step uses Bernstein inequality Lemma 6. Moving from expectation $\mathbb{E}\|A - \mathbb{E}A\|$ to concentration on $\|A - \mathbb{E}A\|$ is through Talagrand's concentration inequality (see, Talagrand (1996) and Tao (2012) Theorem 2.1.13), since $\|\cdot\|$ is 1–Lipschitz convex function in our case (and the entries are bounded), thus with probability at least $1 - \frac{1}{n}$,

$$\|A - \mathbb{E}A\| \leq \mathbb{E}\|A - \mathbb{E}A\| + C \cdot \sqrt{\log n} \asymp \sqrt{k} \vee \sqrt{\log n}.$$

Now let us study the structural signal part. Matrix B is of the form circulant matrix, the associated polynomial is

$$f(x) = (x + x^{n-k/2}) \cdot \frac{x^{k/2} - 1}{x - 1}.$$

The eigen-structure is: collect for all j = 0, 1, ..., n/2

$$(\cos 0, \cos \frac{2\pi j}{n}, \cos \frac{2\pi 2j}{n}, \dots, \cos \frac{2\pi nj}{n})$$

and

$$(\sin 0, \cos \frac{2\pi j}{n}, \sin \frac{2\pi 2j}{n}, \dots, \sin \frac{2\pi nj}{n})$$

and the corresponding eigenvalue is

$$\lambda_j = f(w_j) = 2\sum_{i=1}^{k/2} \cos\left(i\frac{2\pi j}{n}\right).$$

Let us first assume $\frac{k}{n} \le \frac{1}{2}$, thus λ is the second largest eigenvalue

$$\lambda = 2\sum_{i=1}^{k/2} \cos\left(i\frac{2\pi}{n}\right) = \frac{2\sin\frac{k\pi}{2n}}{\sin\frac{\pi}{n}}\cos\frac{(k+2)\pi}{2n} \approx k.$$

Using Weyl's interlacing inequality, if there exist a T > 0 such that

$$\lambda_2(A_{WS}) \ge \lambda_2(\mathbb{E}[A_{WS}]) - \|Z\| > T > \|Z'\| > \lambda_2(A_{ER}),$$

where

$$\begin{split} \lambda_2(M) - \|Z\| &\geq (1-\beta)(1-\beta\frac{k}{n-1})\lambda - \sqrt{k} \vee \sqrt{\log n}, \\ \|Z'\| &\leq \sqrt{k} \vee \sqrt{\log n}, \end{split}$$

then we have

$$\lim_{n,k(n)\to\infty} \; \max \; \left\{ P_0(\phi_2 \neq 0), \; \frac{1}{(n-1)!} \sum_{i=1}^{(n-1)!} P_i(\phi_2 \neq 1) \right\} = 0.$$

Therefore, we have the condition for which the second eigenvalue test succeeds:

$$(1-\beta)(1-\beta\frac{k}{n-1})\lambda > \sqrt{k}\vee\sqrt{\log n}$$

$$(1-\beta)(1-\beta\frac{k}{n-1}) > \frac{\sqrt{k\log n}\vee\log n}{\frac{2\sin\frac{k\pi}{2n}}{\sin\frac{\pi}{n}}\cos\frac{(k+2)\pi}{2n}} \asymp \sqrt{\frac{1}{k}}\vee\frac{\sqrt{\log n}}{k}.$$

Proof of Lemma 3. Take any two vectors A_i , A_j . that are two rows of the adjacency matrix. Denote the i, j-th rows have distance $|\pi^{-1}(i) - \pi^{-1}(j)|_{\text{ring}} = x$. This is equivalent to saying that the Hamming distance of the corresponding signal vectors satisfies $H(B_i, B_j) = 2x - 2$. Therefore the union of signal nodes for i, j-th row is of cardinality $|S_i \cup S_j| = k + x - 1$, common signal nodes is of cardinality $|S_i \cap S_j| = k - x + 1$,

unique signal is of cardinality $|S_i \triangle S_j| = 2x - 2$, and $|S_i^c \cap S_j^c| = n - k - x - 1$. The signal nodes is 1 with probability $p = 1 - \beta(1 - \beta \frac{k}{n-1})$, non signal is 1 with probability $q = \beta \frac{k}{n-1}$, and we have

$$\langle A_i.,A_j.\rangle = \sum_{l \in S_i \cap S_j} A_{il} A_{jl} + \sum_{l \in S_i \triangle S_j} A_{il} A_{jl} + \sum_{l \in S_i^c \cap S_j^c} A_{il} A_{jl}.$$

Observe as long as $l \neq i, j, A_{il}$ and A_{jl} are independent, and $\{A_{il}A_{jl}, l \in [n] \setminus \{i, j\}\}$ are independent of each other.

Let us bound each term via Bernstein's inequality Lemma 6,

$$\sum_{l \in S_i \cap S_j} A_{il} A_{jl} \in p^2 |S_i \cap S_j| \pm \left(\sqrt{2p^2 |S_i \cap S_j|t} + \frac{1}{3}t\right)$$

$$\sum_{l \in S_i \triangle S_j} A_{il} A_{jl} \in pq |S_i \triangle S_j| \pm \left(\sqrt{2pq |S_i \triangle S_j|t} + \frac{1}{3}t\right)$$

$$\sum_{l \in S_i^c \cap S_j^c} A_{il} A_{jl} \in q^2 |S_i^c \cap S_j^c| \pm \left(\sqrt{2q^2 |S_i^c \cap S_j^c|t} + \frac{1}{3}t\right)$$

with probability at least $1 - 6\exp(-t)$. We take $t = (2 + \epsilon)\log n$ for any $\epsilon > 0$, such that with probability at least $1 - Cn^{-\epsilon}$, the above bound holds for all pairs (i, j).

Thus for all $|\pi^{-1}(i) - \pi^{-1}(j)|_{\text{ring}} > k$ pairs,

$$\langle A_{i.}, A_{j.} \rangle \le 2kpq + (n-2k-2)q^2 + \left(\sqrt{4kpqt} + \sqrt{2(n-2k-2)q^2t} + t\right),$$

for $|\pi^{-1}(i) - \pi^{-1}(j)|_{\text{ring}} \le x$ pairs

$$\langle A_i, A_j, \rangle \geq (k-x+1)p^2 + (2x-2)pq + (n-k-x-1)q^2 - \left(\sqrt{2(k-x+1)p^2t} + \sqrt{2(2x-2)pqt} + \sqrt{2(n-k-x-1)q^2t} + t\right).$$

Thus, with $t = (2 + \epsilon) \log n$, $p = 1 - \beta(1 - \beta \frac{k}{n-1})$ and $q = \beta \frac{k}{n-1}$, if $x < x_0$ with

$$\frac{x_0}{k} := 1 - C_1 \sqrt{\frac{\log n}{k}} \frac{1}{1 - \beta} - C_2 \sqrt{\frac{\log n}{n}} \frac{1}{(1 - \beta)^2},$$

we have

$$\begin{split} (k-x+1)(p-q)^2 &\geq 2t + (2\sqrt{2}+1)\left(\sqrt{kp^2} + \sqrt{nq^2}\right)\sqrt{2t} \\ &\geq 2t + \left(\sqrt{2kpq} + \sqrt{(n-2k-2)q^2} + \sqrt{(k-x+1)p^2} + \sqrt{(2x-2)pq} + \sqrt{(n-k-x-1)q^2}\right)\sqrt{2t}, \end{split}$$

which further implies,

$$\begin{split} \min_{j:|\pi^{-1}(i)-\pi^{-1}(j)|_{\mathrm{ring}} \leq x_0} \ \langle A_i., A_j. \rangle &\geq \max_{j \notin \mathcal{N}(v_i)} \ \langle A_i., A_j. \rangle, \forall i \\ \max_{i \in [n]} \frac{|\hat{\mathcal{N}}(v_i) \triangle \mathcal{N}(v_i)|}{|\mathcal{N}(v_i)|} &\leq \frac{k-x_0}{k} = C_1 \sqrt{\frac{\log n}{k}} \frac{1}{1-\beta} + C_2 \sqrt{\frac{\log n}{n}} \frac{1}{(1-\beta)^2}. \end{split}$$

Therefore we can reconstruct the neighborhood consistently, under the condition

$$1-\beta > \sqrt{\frac{\log n}{k}} \vee \left(\frac{\log n}{n}\right)^{1/4}.$$

Proof of Lemma 4. Since eigen-structure is not affected by permutation, we will work under the case when the true permutation is identity. We work under a mild technical assumption that we have two independent observation of the adjacency matrix, one used for calculated the eigen-vector, the other used for projection. Note this is only a technical assumption for simplicity, and does not affect the theoretical result. Recall that A = M + Z, where $M = (1 - \beta)(1 - \beta \frac{k}{n-1}) \cdot B + \beta \frac{k}{n-1} \cdot (J - I)$ is the signal matrix. Denote the eigenvectors of M to be $U \in \mathbb{R}^{n \times n}$, and eigenvectors of A to be $\hat{U} \in \mathbb{R}^{n \times n}$. Classic Davis-Kahan perturbation bound informs us that

$$\|\hat{U}_{\cdot 2} - U_{\cdot 2}\| \le \frac{\|Z\|}{\Delta \lambda - \|Z\|}, \ \|\hat{U}_{\cdot 3} - U_{\cdot 3}\| \le \frac{\|Z\|}{\Delta \lambda - \|Z\|},$$

where the spectral gap $\Delta \lambda$ of *M* is

$$\begin{split} \Delta \lambda &:= (1-\beta)(1-\beta\frac{k}{n-1}) \cdot (\lambda_2 - \lambda_3) = (1-\beta)(1-\beta\frac{k}{n-1}) \cdot \left[2\sum_{i=1}^{k/2} \cos\left(i\frac{2\pi}{n}\right) - 2\sum_{i=1}^{k/2} \cos\left(i\frac{2\pi \cdot 2}{n}\right) \right] \\ &= (1-\beta)(1-\beta\frac{k}{n-1}) \left[\frac{2\sin\frac{k\pi}{2n}}{\sin\frac{\pi}{n}} \cos\frac{(k+2)\pi}{2n} - \frac{2\sin\frac{k\pi}{n}}{\sin\frac{2\pi}{n}} \cos\frac{(k+2)\pi}{n} \right] \\ &= (1-\beta)(1-\beta\frac{k}{n-1}) \frac{k^3}{n^2}. \end{split}$$

From the proof of Lemma 2, we know with high probability

$$||Z|| \le \sqrt{k} \vee \sqrt{\log n}.$$

We denote

$$\tan \hat{\theta}_i = \frac{\hat{U}_{\cdot 3}^T A_{\cdot i}}{\hat{U}_{\cdot 2}^T A_{\cdot i}},$$

and

$$\tan \theta_i = \frac{\langle U_{\cdot 3}, M_{\cdot i} \rangle}{\langle U_{\cdot 2}, M_{\cdot i} \rangle} = \frac{\frac{\lambda_2}{\sqrt{n}} \sin \frac{(i-1)2\pi}{n}}{\frac{\lambda_2}{\sqrt{n}} \cos \frac{(i-1)2\pi}{n}} = \tan \frac{(i-1)2\pi}{n}.$$

Observe that

$$\tan \hat{\theta}_i = \frac{\langle \hat{U}_{\cdot 3}, A_{\cdot i} \rangle}{\langle \hat{U}_{\cdot 2}, A_{\cdot i} \rangle},$$
(22)

and for both the denominator and numerator, we have the bound

$$\begin{split} \langle \hat{U}_{.2}, A_{.i} \rangle &= \langle (\hat{U}_{.2} - U_{.2}) + U_{.2}, M_{.i} + Z_{.i} \rangle \\ &= \langle U_{.2}, M_{.i} \rangle + \langle \hat{U}_{.2} - U_{.2}, M_{.i} \rangle + \langle \hat{U}_{.2}, Z_{.i} \rangle \\ &\leq U_{.2}, M_{.i} \rangle + \|\hat{U}_{.2} - U_{.2}\| \|M_{.i}\| + |\langle \hat{U}_{.2}, Z_{.i} \rangle|. \end{split}$$

Thus we know

$$\max\{|\langle \hat{U}_{\cdot 2}, A_{\cdot i}\rangle - \langle U_{\cdot 2}, M_{\cdot i}\rangle|, |\langle \hat{U}_{\cdot 3}, A_{\cdot i}\rangle - \langle U_{\cdot 3}, M_{\cdot i}\rangle|\}$$
(23)

$$\leq \max\{\|\hat{U}_{\cdot 2} - U_{\cdot 2}\|\|M_{\cdot i}\| + |\langle \hat{U}_{\cdot 2}, Z_{\cdot i}\rangle|, \|\hat{U}_{\cdot 3} - U_{\cdot 3}\|\|M_{\cdot i}\| + |\langle \hat{U}_{\cdot 3}, Z_{\cdot i}\rangle|\}$$
(24)

$$\leq \frac{\sqrt{k} \vee \sqrt{\log n}}{\lambda_2 - \lambda_3 - \sqrt{k} \vee \sqrt{\log n}} \cdot \sqrt{k} (1 - \beta) + \sqrt{\log n}$$
(25)

where the last line follows from the definition of principal angle and Davis-Kahan bound and Hoeffding's inequality for $\langle \hat{U}_{\cdot 2}, Z_{\cdot i} \rangle$. Proceeding with Equation (22), without loss of generality, assume $0 \le \frac{(i-1)2\pi}{n} \le \frac{\pi}{2}$, for $0 \le \frac{(i-1)2\pi}{n} \le \frac{\pi}{4}$

$$\tan \hat{\theta}_{i} \leq \frac{\frac{\lambda_{2}}{\sqrt{n}} \sin \frac{(i-1)2\pi}{n} + \frac{\sqrt{k} \vee \sqrt{\log n}}{\lambda_{2} - \lambda_{3} - \sqrt{k} \vee \sqrt{\log n}} \cdot \sqrt{k} (1-\beta) + \sqrt{\log n}}{\frac{\lambda_{2}}{\sqrt{n}} \cos \frac{(i-1)2\pi}{n} - \frac{\sqrt{k} \vee \sqrt{\log n}}{\lambda_{2} - \lambda_{3} - \sqrt{k} \vee \sqrt{\log n}} \cdot \sqrt{k} (1-\beta) - \sqrt{\log n}} \tag{26}$$

$$= \frac{\sin\frac{(i-1)2\pi}{n} + \delta}{\cos\frac{(i-1)2\pi}{n} - \delta}, \quad \text{with } \delta = \frac{\sqrt{n}}{\lambda_2} \cdot \left\{ \frac{\sqrt{k} \vee \sqrt{\log n}}{\lambda_2 - \lambda_3 - \sqrt{k} \vee \sqrt{\log n}} \cdot \sqrt{k} (1 - \beta) + \sqrt{\log n} \right\}$$
(27)

$$\tan \hat{\theta}_i \ge \frac{\sin \frac{(i-1)2\pi}{n} - \delta}{\cos \frac{(i-1)2\pi}{n} + \delta},\tag{28}$$

with similar bounds for $\cot \hat{\theta}_i$ with $\frac{\pi}{4} \leq \frac{(i-1)2\pi}{n} \leq \frac{\pi}{2}$. Here the stochastic eror is bounded in the sense $\delta \approx \frac{n^{2.5}}{k^3} \frac{1}{1-\beta} \to 0$. From the above equation, we have

$$\begin{split} |\hat{\theta}_i - \theta_i| &\leq \min\{|\tan \hat{\theta}_i - \tan \theta_i|, |\cot \hat{\theta}_i - \cot \theta_i|\} \\ &\leq \min\left\{\frac{\delta(1 + \tan \theta_i)}{\cos \theta_i - \delta}, \frac{\delta(1 + \cot \theta_i)}{\sin \theta_i - \delta}\right\} \leq \frac{2\delta}{\frac{1}{\sqrt{2}} - \delta}. \end{split}$$

For any i, we have the bound on the stochastic error $|\hat{\theta}_i - \theta_i| < C \cdot \delta = \frac{n^{2.5}}{k^3} \cdot \frac{1}{1-\beta}$. And for all $j \in \mathcal{N}(i)$ in the neighborhood, the support is $|\theta_j - \theta_i| \le \frac{2\pi k}{n}$. Fix any i, for any $j \notin \mathcal{N}(v_i)$,

$$\min_{j \notin \mathcal{N}(v_i)} |\hat{\theta}_j - \hat{\theta}_i| \ge \frac{2\pi k}{n} - C\delta > (\frac{2\pi k}{n} - C'\delta) + C\delta \ge \max_{|j-i| < k - C'n\delta} |\hat{\theta}_j - \hat{\theta}_i|,$$

with $C \le 4\sqrt{2}$ and C' > 2C. Therefore, the following bound on symmetric set difference holds

$$\max_{i \in [n]} \frac{|\hat{\mathcal{N}}(v_i) \triangle \mathcal{N}(v_i)|}{|\mathcal{N}(v_i)|} \le \frac{C' \cdot n\delta}{k} \le \frac{C' \cdot n \cdot \frac{n^{2.5}}{k^3} \frac{1}{1-\beta}}{k}.$$

In summary under the condition

$$1-\beta > \frac{n^{3.5}}{k^4},$$

one can recover the neighborhood consistently w.h.p. in the sense

$$\lim_{n,k(n)\to\infty} \max_{i\in[n]} \frac{|\hat{\mathcal{N}}(v_i)\Delta\mathcal{N}(v_i)|}{|\mathcal{N}(v_i)|} = 0.$$

acknowledgement

The authors thank Elchanan Mossel for many helpful discussions and suggestions for improving the paper.

References

- Luis A Nunes Amaral, Antonio Scala, Marc Barthelemy, and H Eugene Stanley. Classes of small-world networks. *Proceedings of the national academy of sciences*, 97(21):11149–11152, 2000.
- Afonso S Bandeira and Ramon van Handel. Sharp nonasymptotic bounds on the norm of random matrices with independent entries. *arXiv preprint arXiv:1408.6185*, 2014.
- Alain Barrat and Martin Weigt. On the properties of small-world network models. *The European Physical Journal B-Condensed Matter and Complex Systems*, 13(3):547–560, 2000.
- Itai Benjamini and Noam Berger. The diameter of long-range percolation clusters on finite cycles. *arXiv* preprint math/0012070, 2000.
- Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration inequalities: A nonasymptotic theory of independence*. OUP Oxford, 2013.
- Don Coppersmith, David Gamarnik, and Maxim Sviridenko. The diameter of a long range percolation graph. In *Proceedings of the thirteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 329–337. Society for Industrial and Applied Mathematics, 2002.
- Robert M Gray. Toeplitz and circulant matrices: A review. now publishers inc, 2006.
- Jon Kleinberg. The small-world phenomenon: An algorithmic perspective. In *Proceedings of the thirty-second annual ACM symposium on Theory of computing*, pages 163–170. ACM, 2000.
- Vito Latora and Massimo Marchiori. Efficient behavior of small-world networks. *Physical review letters*, 87(19):198701, 2001.
- Stanley Milgram. The small world problem. Psychology today, 2(1):60-67, 1967.
- Cristopher Moore and Mark EJ Newman. Epidemics and percolation in small-world networks. *Physical Review E*, 61(5):5678, 2000.
- Mark EJ Newman and Duncan J Watts. Scaling and percolation in the small-world network model. *Physical Review E*, 60(6):7332, 1999.
- CJ Stam, BF Jones, G Nolte, M Breakspear, and Ph Scheltens. Small-world networks and functional connectivity in alzheimer's disease. *Cerebral cortex*, 17(1):92–99, 2007.
- Michel Talagrand. A new look at independence. The Annals of probability, pages 1–34, 1996.
- Terence Tao. *Topics in random matrix theory*, volume 132. American Mathematical Soc., 2012.
- Alexandre B Tsybakov. *Introduction to nonparametric estimation*, volume 11. Springer Series in Statistics, 2009.

- Remco Van Der Hofstad. Random graphs and complex networks. *Available on http://www. win. tue. nl/rhofstad/NotesRGCN. pdf*, page 11, 2009.
- Duncan J Watts. *Small worlds: the dynamics of networks between order and randomness.* Princeton university press, 1999.
- Duncan J Watts and Steven H Strogatz. Collective dynamics of 'small-world' networks. *nature*, 393(6684): 440–442, 1998.