Algebraic connectivity: local and global maximizer graphs

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Abstract

Algebraic connectivity is one way to quantify graph connectivity, which in turn gauges robustness as a network. In this paper, we consider the problem of maximising algebraic connectivity both local and globally over all simple, undirected, unweighted graphs with a given number of vertices and edges. We pursue this optimization by equivalently minimizing the largest eigenvalue of the Laplacian of the 'complement graph'. We establish that the union of complete subgraphs are largest eigenvalue *local* minimizer graphs. Further, under sufficient conditions satisfied by the edge/vertex counts we prove that this union of complete components graphs are, in fact, Laplacian largest eigenvalue *global* maximizers; these results generalize the ones in the literature that are for just two components. These sufficient conditions can be viewed as quantifying situations where the component sizes are either 'quite homogeneous' or some of them are relatively 'negligibly small', and thus generalize known results of homogeneity of components. We finally relate this optimization with the Discrete Fourier Transform (DFT) and circulant graphs/matrices.

Keywords: Algebraic connectivity, Laplacian matrices, Circulant matrix, DFT AMS code: 05C50, 05C12, 15A42

1. Introduction

Graph connectivity finds application in networking, network security, transportation systems, multi-agent control and has been well studied in the literature. Connectivity of a graph G is also a measure of robustness as a network. Algebraic connectivity being one of measures of graph connectivity is defined as the second smallest eigenvalue

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 λ_{n-1} of the Laplacian matrix $L(G) \in \mathbb{R}^{n \times n}$ of the unweighted, undirected and simple graph G. In this paper, we consider only simple undirected, unweighted graphs with no self loops and no multiple edges between any pairs of the vertices. We study the problem of maximizing the algebraic connectivity of a graph for a given number of nodes and edges. We pursue this problem both: a *global* maximization across all graphs, and a *local* sense, in which we consider only one edge 'rearrangements' (defined precisely in Definition 3.1 below). Since algebraic connectivity and the problem of maximizing has received extensive attention and is well-understood, we quickly delve further into the problem formulation, and then touch other closely related work in the literature.

1.1. Notation

The notation we follow is standard and is included here for quick reference. The sets of real and complex numbers are denoted respectively by \mathbb{R} and \mathbb{C} . The largest eigenvalue of a symmetric matrix is denoted by λ_1 . Given an undirected graph G, the number of vertices |V(G)| is usually n, the number of edges |E(G)| is usually m, and the number of components of the graph is usually p. Further, the maximum degree across all vertices is denoted by Δ and d_{avg} is the average degree of vertices. The n eigenvalues of the Laplacian matrix L(G) are denoted by $\lambda_1(L(G)) \ge \lambda_2(L(G)) \ge \cdots \ge \lambda_{n-1}(L(G)) \ge \lambda_n(L(G)) = 0$. When the matrix L(G) and the graph G are clear from the context, we use just $\lambda_1, \ldots, \lambda_n$ to denote the eigenvalues, and when comparing the maximum eigenvalues of Laplacian matrices of different graphs, say G^m and G^c , we use $\lambda_1(G^m)$ and $\lambda_1(G^c)$. Note that, since L is symmetric, $\lambda_1(L(G)) = \max_{\|x\|_2=1} x^T L(G)x$.

We deal with integer-valued properties and their relation with other bounds, and in this context, we use the standard floor and the ceiling functions of x, denoted by $\lfloor x \rfloor$ and $\lceil x \rceil$, to mean the largest/smallest integer not greater-than/not-smaller-than the real number x respectively.

The complete graph in n vertices is denoted by K_n , and the complete bipartite graph with vertex sets having cardinalities p and q is denoted by $K_{p,q}$. Of course, our paper deals with complete multi-partite graphs, and in fact, with their complement graphs: which would then be union of complete graphs, denoted by $\bigcup K_i$.

The notion of complement graph G^c of a graph G^m is straightforward: it is a simple undirected graph with the same number (and indexing) of nodes and in which there is an edge in G^c between two nodes, by definition, if and only if there is no edge in G^m .

1.2. Problem formulation

The paper deals with the following mutually closely related problems.

Problem 1.1. The following sub-problems are inter-related for reasons clarified soon in the next section.

- (a) For a given number of vertices |V| = n and number of edges $|E| = m_1$, find an algebraic connectivity maximiser graph $G_1 = (V, E)$.
- (b) For given number of vertices |V| = n and number of edges $|E| = m_2$, minimise the largest Laplacian eigenvalue of the graph $G_2 = (V, E)$.

Further, each of the above optimizations can be pursued in one of two ways: globally and locally. For simplicity, we elaborate on just the second one, i.e. the largest eigenvalue minimization: we study the global case, and the 'local' case. More precisely,

- 1. finding a Largest Eigenvalue <u>Global</u> Minimizer (LEGM) graph that has the least largest eigenvalue possible for the given number of vertices and edges, and
- 2. finding a Largest Eigenvalue <u>Local</u> Minimizer (LELM), with 'local minima' in the sense that all one-edge reconnect graphs (see Definition 3.1) have either the same largest eigenvalue or higher.

Related work in the context of the above problem is pursued in the next section. The problem we consider in this paper also has a close link with circulant graphs (pursued further in Section 5) and DFT of time-symmetric vectors with entries from $\{0, 1\}$. The remark after the problem formulation below makes this precise.

Problem 1.2. DFT magnitude minimization: Given positive integers d and n with $1 \leq d \leq n-1$, consider a vector $x \in \{0,1\}^n$ with $x_1 = 0$ and $||x||_1 = d$, and further, x being 'time-symmetric', i.e. $x_i = x_{n+2-i}$ for i = 2, ..., n. Define $\bar{x} \in \mathbb{R}^n$ using x by $\bar{x}_1 = -d$, and $\bar{x}_i = x_i$ for all other i. Define the Discrete Fourier Transform $(DFT)^1$ of the vector \bar{x} by $X = DFT(\bar{x})$, and notice that $X \in \mathbb{R}^n$ due to the assumed time-symmetry. Consider the minimization problem: find x satisfying the conditions above such that $||X||_{\infty}$ is minimized.

Circulant matrices are pursued further in Section 5. The following remark motivates the assumptions within the problem formulations above.

Remark 1.3. The following points relate Problems 1.1 and 1.2 and Laplacian matrices of circulant graphs.

The condition x
₁ = −d means that the 'DC part' of x
 is zero and hence X₁ =
 0. Thus minimizing ||X||_∞ means that the focus is on the minimization of the
 maximum magnitude of all frequencies, except the DC.

¹For uniformity with the rest of this paper, we use indices of $x, \bar{x} \in \mathbb{R}^n$ and $X \in \mathbb{C}^n$ to vary from 1 to n, notwithstanding the typical DFT convention of using indices from 0 to n-1 for x, \bar{x} and X.

- 2. Entries in X are nothing but the negative of the eigenvalues of the Laplacian of the graph G_C constructed from x, and G_C is regular (of degree d) and is circulant; i.e., the Laplacian matrix is a circulant matrix.
- 3. The operation of defining $\bar{x} \in \mathbb{R}^n$ from $x \in \{0,1\}^n$ is one of adding an appropriately scaled discrete time impulse δ ; the impulse has equal amount of all frequencies. The DFT operation being linear on the signal space, this thus keeps the optimization focus on the non-DC part in the signal x.
- The operation of defining x̄ from x is like studying the eigenvalues of A − D (i.e. −L) instead of the adjacency matrix A, and note that the diagonal matrix D (the degree matrix) is merely d · I for this regular and circulant graph.

1.3. Organization of the paper

The rest of this paper is organized as follows. The next section relates the problem we pursue with other work in the literature and in what way our work generalizes existing results. Section 3 contains the main results of this paper, about *locally* optimal graphs. Further, in the context of *globally* optimal graphs, our main results that improve upon results in the literature and also formulate for the case of many components are contained in Section 4. In Section 5, we relate our work to the Discrete Fourier Transform and circulant matrices/graphs. We consider some examples in Section 6. We conclude the paper in Section 7, where we also summarize the contribution in this paper.

2. Background and other work in this area

Algebraic connectivity maximization of graphs has received much attention. The survey papers [11], [4], [3], [8] and [5] contain a wealth of results about upper/lower bounds on the algebraic connectivity, many of which we use crucially in our paper too. In particular, given that we pursue maximum eigenvalue minimization on the complement graph instead of directly algebraic connectivity (second-smallest eigenvalue) maximization, it would help the reader to quickly review Proposition 2.1 below to see why this approach of focussing on the complement is equivalent.

In the context of weighted graphs, [6] proposes an algorithm to find an edge to add to the graph to maximise algebraic connectivity, however, the edge weight here is a function of distance between the vertices. Closely aligned with our paper, [10] pursues both Algebraic Connectivity 'Local' Maximizers (ACLM) in the graph set of all one edge changes as in Definition 3.1 and also global maximizers, where for a given number of vertices and edges, conditions are formulated. Propositions 2.4 and 2.5 contain the exact statements from [10], since this work is relevant to the main results in our paper. Both local and global optima obtained in [10] pursue for the case when the complement has *two* components, while our paper generalizes to the case when the complement has any number of components, and also slightly improves the bounds for the case of two components.

Recall that for a graph G = (V, E), with V the vertex set and E the edge set, the Laplacian matrix is defined as L(G) = D(G) - A(G) where D(G) is the diagonal matrix with diagonal entries being degree of vertices and A(G) is the adjacency matrix of graph G. The second smallest eigenvalue of L is defined as the algebraic connectivity of the graph G: see [7]. This eigenvalue is also called the Fiedler value. The rest of this section contains results that we use and/or improve upon in this paper.

The following result crucially relates eigenvalues of the Laplacian matrices of a graph G^m and its complement G^c .

Proposition 2.1. ([11, page 148]) Let G^m be a simple undirected, unweighted graph and G^c be its complement. Then the largest eigenvalue of the graph $\lambda_1(G^m)$ satisfies, $\lambda_1(G^m) \leq n$. Further, the eigenvalues of the Laplacian matrices of G^m and G^c are related by $\lambda_i(G^c) = n - \lambda_{n-i}(G^m)$ for i=1, ..., n-1 and $\lambda_n(G^m) = \lambda_n(G^c) = 0$.

The next well-known result (from [13]) gives a lower bound for the maximum eigenvalue and also formulates the unique situation when the bound is tight.

Proposition 2.2. ([13, Theorem 3.19]) Consider a connected graph G with at least one edge, vertex set V(G) of cardinality n. Then the following hold.

- a) The maximum eigenvalue of the Laplacian matrix of the graph satisfies $\lambda_1(L(G)) \ge \Delta + 1$.
- b) $\lambda_1(L(G)) = \Delta + 1$ holds if and only if $\Delta = n 1$, i.e., there exists a 'star node' in G.

Of course, if a graph is not connected, then the above result can still be used by noting the obvious fact that the Laplacian matrix L_F of the full graph is a block diagonal matrix composed of that of the individual components, and hence the eigenvalues of L_F are the union of the individual Laplacian matrices' eigenvalues. The following result gives a different lower bound for the maximum eigenvalue and also the situation when this bound is tight.

Proposition 2.3. ([8, Theorem 3]) Let Graph G with $n \ge 2$ vertices and domination² number, denoted by γ . Then, $\lambda_1(G) \ge \lfloor \frac{n}{\gamma} \rfloor$ and, further, equality holds if and only if $G = G_a \bigcup G_b$ such that:

1. $|G_a| = \lfloor \frac{n}{\gamma} \rfloor$ and $\gamma(G_a) = 1$, and

 $^{^2}$ The domination number of a graph $\gamma({\rm G})$ is defined as the minimum size of the subset of vertices which are adjacent to every other vertex of the graph

2. $\gamma(G_b) = \gamma(G) - 1$ and $\lambda_1(G_b) \leq \lfloor \frac{n}{\gamma} \rfloor$.

The main results in our paper generalize the following results from [10] and we generalize these results to the case of more than two components (in the complement graph). For a specified number of vertices and edges, [10] studies the problem of Algebraic Connectivity Maximizer (ACM) graph and local algebraic maximizer graphs. The precise statements are below.

Proposition 2.4. [10, Theorem 3]: For integers $a \in \mathbb{Z}^+$, if $a \leq \lfloor \frac{n}{2} \rfloor$ and $a - \frac{2a^2}{n} < 1$, then for any $n \geq 3$, the complete bipartite graph $K_{a,n-a}$ is ACM in graphs with n vertices and a(n-a) edges.

Proposition 2.5. [10, Theorem 6]: For integers $a \in \mathbb{Z}^+$, if $a \leq \lfloor \frac{n}{2} \rfloor$, then the complete bipartite graph $K_{a,n-a}$ is ACLM in graphs with n vertices and a(n-a) edges.

Proposition 2.6. [12, Theorem 3.1] Consider Graph G with at least one edge and independence³ number $\alpha(G)$. Then, $\lambda_1(G) \ge \frac{n}{\alpha}$ and, further, equality holds if and only if α is factor of n and thus G then has α components each being $K_{\frac{n}{\alpha}}$.

We prove in this paper that the complement graph G^c made up of two complete components graph is LEGM under a very similar (and slightly relaxed) sufficient condition as compared to Proposition 2.4. We also extend the result of complete two components to multi-components and prove that the graph is LEGM under an appropriately generalized sufficient condition. This result (Theorem 4.2 below) generalizes Proposition 2.6 in a certain sense. The notion of Algebraic Connectivity Local Maximizers (ACLM) graph was introduced in [10]. The ACLM graph is the one in which if one edge is changed (i.e. one edge is either removed or reconnected to a different set of vertices), then its algebraic connectivity remains highest among all such 'one edge changed' graphs. ACLM graphs are thus not globally optimal, but at least locally optimal topologies and hence also usually globally suboptimal. In [10], it has been shown that the complete bipartite graph $K_{a,n-a}$ is an Algebraic Connectivity Local Maximizers (ACLM) in G for n vertices and a(n-a) edges graphs for $2 \leq a \leq \lfloor \frac{n}{2} \rfloor$; we generalize this result for the case that the complement graph has not just two components but in fact any number of components.

3. Main results: locally optimal graphs

In this section we present the main results of this paper which concern 'locally' optimal graphs. The notion of local is made precise in the definition below. This

³The independence number of graph $\alpha(G)$ is defined as the cardinality of the largest set of vertices of the graph with no edge connection between them

notion coincides with that of [10]. Local optimality is important when only simple rearrangements of the topology of a set of multi-agents, for example, is allowed and complicated rearrangements are disallowed. It helps to at least be locally optimal. Of course, globally optimal configurations would also need to satisfy this, and thus local optimality conditions are necessary conditions for global optimality too.

- **Definition 3.1.** (a) One edge reconnect of G_0 : Let $G_0(V, E_0)$ be a simple graph with |V| = n, and $|E_0| = m$. We define $G_1(V, E_1)$ be a <u>one-edge reconnect</u> of G_0 if G_1 is also a simple graph and one or both of nodes of exactly one edge differ from that of G_0 . Thus, we have one-edge reconnect if G_1 satisfies $|E_1| = m$ and $\operatorname{rank}(L_1 - L_0) = 2$.
 - (b) **One edge addition**: By one edge addition, we mean adding an edge to a graph while keeping the graph simple.

Using the above notion of one edge reconnects and one edge additions, we define a local minimizer graph; this is w.r.t. the largest eigenvalue of the Laplacian.

Definition 3.2. Largest Eigenvalue Local Minimizer graph: A graph G_0 is called a Largest Eigenvalue Local Minimizer (LELM) graph if G_0 has the least value of the Laplacian matrix's largest eigenvalue amongst all the simple graphs G obtained from G_0 by either a one edge reconnect or a one edge addition.

In the context of various possibilities of an edge reconnection or addition, it helps to visualize the case using a figure. We include various figures, and the proof techniques vary depending on these cases. In summary: when we have a union of complete subgraphs, then, an extra edge or an edge reconnection connects to complete components, and we make a distinction about whether the maximum degree increases or remains same, and whether the largest component (with vertex-size say n_1), or vertex-size slightly smaller than the largest (of size $n_1 - 1$), or further smaller was involved in the edge reconnection/addition. This distinction is needed to prove the local minimality of the graph proposed in Theorem 3.6.



Figure 1: Connection established by one edge addition between G_i and $G_j,$ where $|V(G_j)| \leqslant |V(G_i)| \leqslant |V(G_1)| - 2$

Lemma 3.3. Suppose a connection is established between complete graph components G_i and G_j by <u>adding</u> an edge to give G_{ij}^+ and let $L_{new} - L_{old} =: C_{add}$ is connection



Figure 2: Reconnection without increasing the maximum degree of G_i , where $|V(G_j)| \leq |V(G_i)| \leq |V(G_1)| - 2$



Figure 3: Reconnection with increasing the maximum degree of G_i , where $|V(G_j)| \leq |V(G_i)| \leq |V(G_1)| - 2$



Figure 4: Connection established by one edge addition to G_i , where $|V(G_j)| \leq |V(G_i)| = |V(G_1)| - 1$



Figure 5: Reconnection without increasing the maximum degree of G_i , where $|V(G_j)| \leq |V(G_i)| = |V(G_1)| - 1$

matrix. Then $\operatorname{rank}(L_{new} - L_{old}) = 1$ and the largest eigenvalue of C_{add} is 2 (refer to Figure 1).

Proof. Contribution to the Laplacian matrix of graph due to an edge addition has the structure:



Figure 6: Reconnection with increasing maximum degree of G_i , where $|V(G_j)| \leq |V(G_i)| = |V(G_1)| - 1$



Figure 7: Connection established between largest size component G_1 with any other size component by one edge addition



Figure 8: Reconnection of edge without increasing the maximum degree of largest size component G_1

$$C_{\rm add} \ = \left[\begin{array}{cccc} G_i & G_j & \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ \hline 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Clearly, the matrix $C_{\rm add}$ has rank one and the characteristic polynomial: $\chi_{C_{\rm add}}(s) = s^3(s-2)$. So, $\lambda_1(C_{\rm add}) = 2$.

For bigger or general size G_i and G_j with $|V(G_i)| + |V(G_j)| = a$, the structure of C_{add} remains same but with zeros padded appropriately. Thus, C_{add} has rank one in general also and the lemma is proved.

Lemma 3.4. Suppose a connection is established between complete graph components G_i and G_j by reconnecting an edge by removing one edge e and adding elsewhere such that both nodes of e change, to give G_{ij}^+ and let $L_{new} - L_{old} =: C_{\text{re-incr}}$ is connection matrix. Then $\text{rank}(L_{new} - L_{old}) = 2$ and the largest eigenvalue of $C_{\text{re-incr}}$ is 2 (refer to Figure 3).

Proof. Contribution to the Laplacian matrix of graph due to an edge reconnection as specified in the lemma has the following structure:

	Γ		G_i		G_j	1
		-1	1	0	0	
$C_{\text{re-incr}}$ =	=	1	-1	0	0	.
		0	0	1	-1	
	Ľ	0	0	-1	1	

Clearly, the matrix $C_{\text{re-incr}}$ has rank two and the characteristic polynomial: $\chi_{C_{\text{re-incr}}}(s) = s^2(s^2 - 2^2)$. Thus, $\lambda_1(C_{\text{re-incr}}) = 2$. Again, for the general case, zeros get padded appropriately and the lemma is thus proved.

Lemma 3.5. Suppose a connection is established between complete graph components G_i and G_j by reconnecting an edge by removing one edge e and adding an edge such that only one node of e gets change, to give G_{ij}^+ , and let $L_{new} - L_{old} =: C_{\text{re-same}}$ be the connection matrix. Then $\operatorname{rank}(L_{new} - L_{old}) = 2$ and the largest eigenvalue of $C_{\text{re-same}}$ is $\sqrt{3}$ (refer to Figure 2).

Proof. Contribution to the Laplacian matrix of graph due to an edge reconnection as specified in the lemma has the following structure:

$$C_{\text{re-same}} = \begin{bmatrix} G_i & G_j \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ \hline 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly, the matrix $C_{\text{re-same}}$ has rank two and the characteristic polynomial: $\chi_{C_{\text{re-same}}}(s) = s^3(s^2 - 3)$. Thus, $\lambda_1(C_{\text{re-same}}) = \sqrt{3}$. Again, for the general case, zeros get padded appropriately and the lemma is thus proved.

With the above lemmas, we are ready to state and prove the first main result of this paper.

Theorem 3.6. : A graph G = (V, E) which is a union of complete components is a Largest Eigenvalue Local Minimizer (LELM). In other-words, for graph G of n number of vertices, m number of edges and p number of complete components of $|V(G_i)|$ size such that $\sum_{i=1}^{p} |V(G_i)| = n$ and $m = \sum_{i=1}^{p} |V(G_i)|C_2$, then $\lambda_1(G)$ is locally minimized, i.e. minimized w.r.t. one edge reconnects, one edge removals and one edge additions (as defined in Definition 3.1). Further, $\lambda_1(G) = \max_{i \in \{1, 2, \cdots, p\}} \{|V(G_i)|\}$. Proof. Let $G_1, G_2, ..., G_p$ be the components and the number of nodes involved in those components be $|V(G_i)| = n_i$ then $|E(G)| = \sum_{i \in \{1,2,..,p\}} |V(G_i)| C_2$. Without loss of generality, we assume that the components size have the following relation between them: $|V(G_1)| \ge |V(G_2)| \ge |V(G_3)| \ge ... |V(G_p)| > 0$. Thus the largest eigenvalue $\lambda_1(G)$ of the graph G is: $\lambda_1(G) = |V(G_1)|$, since $\lambda_1(L(K_{n_1})) = n_1$. In this setup if one edge is reconnected or one edge is added, it can be connected in the following 3 ways:

Case 1: Between components of smaller sizes G_i and G_j such that $|V(G_j)| \leq |V(G_i)| \leq |V(G_i)| < |V(G_i)| - 2$, i.e. both components G_i and G_j are at least two or more nodes smaller than the largest component's size (G_1) .

Case 2: Between component G_i and G_j with $|V(G_j)| \leq |V(G_i)| = |V(G_1)| - 1$.

Case 3: Between G_1 and any other component: same size as G_1 or smaller.

We now prove the theorem for each of the 3 cases. Note that for each case, we have three subcases: (a) Addition of an edge, (r_s) Removal and addition of an edge e such that only one vertex of e is changed, and (r_i) Removal and addition of an edge e such that both vertices of e are changed. We are not mentioning the one edge removal for local minimizer explicitly because by removing only one edge from complete component graphs, does not change its Laplacian largest eigenvalue. Hence, this proposed graph is trivially LELM.

Case 1: Between components of smaller sizes G_i and G_j (without loss of generality assuming $|V(G_i)| \ge |V(G_j)|$) such that $|V(G_i)| \le |V(G_1)| - 2$, i.e. both $G_i \& G_j$ are at least two nodes smaller than the largest component G_1 :

1a) By one edge addition (refer to Figure 1): If connection is established between G_i and G_j to give G_{ij}^+ by adding an edge, then the connection matrix C_{add} of Lemma 3.3, gets added to $L(G_i \oplus G_j)$.

Thus, due to the edge addition in between components we get, $L(G_{ij}^+)=L(G_i\oplus G_j)+C_{\rm add}$.

Also, $\lambda_1(L(G_{ij}^+)) = \max_{\|x\|_2 = 1} x^T L(G_{ij}^+) x = \max_{\|x\|_2 = 1} [x^T L(G_i \oplus G_j) x + x^T C_{\text{add}} x].$

Using Lemma 3.3, we have $\lambda_1(C_{\text{add}}) = 2$, which implies that $\lambda_1(L(G_{ij}^+)) \leq \lambda_1(L(G_i \oplus G_j)) + 2 = \lambda_1(L(G_i)) + 2 \leq \lambda_1(G_1)$.

Therefore, $\lambda_1(G) = \max\{\lambda_1(G_1), \lambda_1(G_2), ..., \lambda_1(G_{ij}^+)\} = \lambda_1(G_1)$. This proves that $\lambda_1(G)$ remains same and the proposed graph G is a $\lambda_1(G)$ local minimizer.

- 1r) One edge reconnect: If the connection established between G_i and G_j to give G_{ij}^+ by reconnecting one edge, then the following two different types of C connection matrix get added to $L(G_i \oplus G_j)$ depending upon how the reconnection of edge is done.
 - $1r_s$) Reconnection without increasing the maximum degree of G_i (refer to Figure 2):

Due to the reconnection, the connection matrix $C_{\text{re-same}}$ of Lemma 3.5 gets added and we get $L(G_{ij}^+) = L(G_i \oplus G_j) + C_{\text{re-same}}$.

Also,
$$\lambda_1(L(G_{ij}^+)) = \max_{\|x\|_2=1} x^T L(G_{ij}^+) x = \max_{\|x\|_2=1} [x^T L(G_i \oplus G_j) x + x^T C_{\text{re-same }} x]$$

Using Lemma 3.5, we have $\lambda_1(C_{\text{re-same}}) = \sqrt{3}$. $\implies \lambda_1(L(G_{ij}^+)) \leq \lambda_1(L(G_i \oplus G_j)) + \sqrt{3} = \lambda_1(L(G_i)) + \sqrt{3} < \lambda_1(L(G_i)) + 2 \leq \lambda_1(G_1)$. Therefore, $\lambda_1(G^+) = \max\{\lambda_1(G_1), \lambda_1(G_2), ..., \lambda_1(G_{ij}^+)\} = \lambda_1(G_1)$.

 $\lambda_1(G)$ remains same and our graph is local minimizer.

 $1r_i$) Reconnection with increasing the maximum degree of G_i (refer to Figure 3):

Due to reconnection, the connection matrix $C_{\text{re-incr}}$ of Lemma 3.4 gets added and we get $L(G_{ij}^+) = L(G_i \oplus G_j) + C_{\text{re-incr}}$.

Also, $\lambda_1(L(G_{ij}^+)) = \max_{\|x\|_2=1} x^T L(G_{ij}^+) x = \max_{\|x\|_2=1} [x^T L(G_i \oplus G_j) x + x^T C_{\text{re-incr}} x]$ Using Lemma 3.4, $\lambda_1(C_{\text{re-incr}}) = 2$. $\implies \lambda_1(L(G_{ij}^+)) \leq \lambda_1(L(G_i \oplus G_j)) + 2 = \lambda_1(L(G_i)) + 2 \leq \lambda_1(G_1)$. Therefore, $\lambda_1(G^+) = \max\{\lambda_1(G_1), \lambda_1(G_2), ..., \lambda_1(G_{ij}^+)\} = \lambda_1(G_1)$. $\lambda_1(G)$ remains same and the proposed graph G graph is local minimizer.

This completes the proof of Case 1.

Case 2: Between component G_i of size $|V(G_i)| = |V(G_1)| - 1$ and any other component G_j of equal or smaller size than G_i i.e. $|V(G_j)| \leq |V(G_i)| = |V(G_1)| - 1$:

- 2a) By one edge addition (refer to Figure 4): Suppose connection is established between G_i and G_j to give G_{ij}^+ by adding an edge (using Proposition 2.2 b), $\lambda_1(G_{ij}^+) > |V(G_i)| + 1 = |V(G_1)| = \lambda_1(G_1).$ $\lambda_1(G^+) = \max\{\lambda_1(G_1), \lambda_1(G_2), ..., \lambda_1(G_{ij}^+)\} = \lambda_1(G_{ij}^+) > \lambda_1(G_1).$ Thus, proposed graph G is a local minimizer.
- 2r) One edge reconnect: Suppose connection is established between G_i and G_j to give G_{ij}^+ by relocating an edge, then following two different type of C connection matrix gets added to $L(G_i \oplus G_j)$ depending upon how reconnection of edge is done.
 - $2r_s$) Reconnection without increasing the maximum degree of G_i (refer to Figure 5):

Due to reconnection, we get $L(G_{ij}^+) = L(G_i \oplus G_j) + C_{\text{re-same}}$. Also, $\lambda_1(L(G_{ij}^+)) = \max_{\|x\|_2=1} x^T L(G_{ij}^+) x = \max_{\|x\|_2=1} [x^T L(G_i \oplus G_j) x + x^T C_{\text{re-same}} x]$ Using Lemma 3.5, $\lambda_1(C_{\text{re-same}}) = \sqrt{3}$.

 $\lambda_1(L(G_{ij}^+)) \leq \lambda_1(L(G_i \oplus G_j)) + \sqrt{3} = \lambda_1(L(G_i)) + \sqrt{3} = \lambda_1(L(G_1)) + \sqrt{3} - 1.$ So, in case of reconnecting without increasing maximum degree, we use the following relation:

 $\lambda_1(G_i) = \lambda_1(G_1) - 1 < \lambda_1(G_{ij}^+) \le \lambda_1(G_1) + \sqrt{3} - 1.$

Thus, $\lambda_1(G_1) \leq \lambda_1(G^+) \leq \lambda_1(G_1) + \sqrt{3} - 1$ implies $\lambda_1(G^+)$ either increases or remains same. Therefore again the proposed graph G graph is an LELM.

 $2r_i$) Reconnection with increasing the maximum degree of G_i (refer to Figure 6): Suppose connection is established between G_i and G_j to give G_{ij}^+ by

reconnecting an edge with increasing maximum degree of G_i , we get: (using Proposition 2.2), $\lambda_1(G_{ij}^+) > |V(G_i)| + 1 = |V(G_1)| = \lambda_1(G_1)$. $\lambda_1(G^+) = \max\{\lambda_1(G_1), \lambda_1(G_2), ..., \lambda_1(G_{ij}^+)\} = \lambda_1(G_{ij}^+)$. Hence, $\lambda_1(G)$ increases. Thus, the proposed graph G is an LELM.

This completes the proof of Case 2.

Cases 3: Between G_1 and any other component:

- 3a) By one edge addition (refer to Figure 7): Before addition of edge, we have $\lambda_1(G) = |V(G_1)|$. Then connection is established in two ways: between two largest size components and between largest and any other size components. Thus, addition of edge between components $K_{|V(G_1)|}$ and $K_{|V(G_i)|}$ leads to $\lambda_1(G) > |V(G_1)|$ (using Proposition 2.2)[13], [3]). Therefore, the proposed graph is a $\lambda_1(G)$ local minimizer (LELM).
- 3r) Reconnecting of edge with or without increasing the maximum degree of G_1 (refer to Figure 8): Here, the connection is established in cases with largest size component G_1 by re-connecting $K_{|V(G_1)|}$ and $K_{|V(G_i)|}$ either by increasing maximum degree of G_1 or not; similarly like addition of edge, the reconnection leads to $\lambda_1(G) > |V(G_1)|$ (using Proposition 2.2)[13], [3]). Hence the proposed graph G is again an LELM for this case also.

This completes the proof of Case 3 and also the proof of the theorem.

4. Main results: globally optimal graphs

In this section we obtain sufficient conditions for the union of complete graphs to be a global minimizer of the largest eigenvalue. The first main result of this section (Theorem 4.1) is a slight improvement (though claimed and proved on the *complement* graph using different proof techniques) to Proposition 2.4. The second main result of this section (Theorem 4.2) is a generalization to the case of more than two components and also gives the first one as a corollary, except the case of equality within the sufficient condition, Equation (1).

Theorem 4.1. Consider graph G = (V, E) of n number of vertices and m number of edges consisting of two complete components K_{ℓ} and $K_{n-\ell}$, i.e. $m = |E(G)| = {}^{\ell}C_2 + {}^{n-\ell}C_2$. Let without loss of generality $\ell \leq \frac{n}{2}$. Assume

$$\ell - \frac{2\ell^2}{n} \leqslant 1. \tag{1}$$

Then the graph $G = K_{\ell} \bigcup K_{n-\ell}$ is a Largest Eigenvalue Global Minimizer (LEGM).

Proof. : This proof involves two cases depending on whether the inequality $\ell - \frac{2\ell^2}{n} \leq 1$ is strict (Case 1) or holds with equality (Case 2).

Case 1: $\ell - \frac{2\ell^2}{n} < 1$. First notice that when $\ell = \frac{n}{2}$, we get $\ell - \frac{2\ell^2}{n} = 0$ and $\ell < \frac{n}{2}$ is same as $0 < \ell - \frac{2\ell^2}{n}$. Hence, the assumption in the theorem gives $0 \le \ell - \frac{2\ell^2}{n} \le 1$. In order to prove the theorem, we obtain the average degree of the graph. Average degree (d_{avg}) of the graph $G = K_\ell \bigcup K_{n-\ell}$:

$$d_{avg} = \frac{2m}{n} = \frac{2}{n} \left\{ \frac{\ell^2 - \ell}{2} + \frac{(n - \ell)^2 - (n - \ell)}{2} \right\},\$$
$$= \frac{1}{n} \left\{ 2\ell^2 + n^2 - 2n\ell - n \right\},\$$
$$= n - \ell - 1 - \left(\ell - \frac{2\ell^2}{n}\right).$$

We use that the maximum degree of the graph, $\Delta \ge d_{avg}$. In fact, we also use that Δ should be an integer which implies $\Delta \ge \lfloor d_{avg} \rfloor$. If $0 \le \ell - \frac{2\ell^2}{n} < 1$, then the maximum degree, $\Delta \ge n - \ell - 1$.

Using Proposition 2.2a), for any graph that has as many edges as m, we get $\lambda_1(G) \ge \Delta + 1$ and thus $\lambda_1(G) \ge n - \ell$ for any graph having as many edges as $G = K_\ell \bigcup K_{n-\ell}$. For the proposed graph G, the largest eigenvalue of the graph, $\lambda_1(G) = \max\{\ell, n-\ell\} = n - \ell$.

Hence, the proposed graph G of theorem $K_{\ell} \bigcup K_{n-\ell}$ is an LEGM. **Case 2:** $\ell - \frac{2\ell^2}{n} = 1$. $\ell - \frac{2\ell^2}{n} = 1 \implies 2\ell^2 - n\ell + n = 0$ whose roots are: $\ell = \frac{n \pm \sqrt{n^2 - 8n}}{4}$.

Notice that for ℓ to be an integer the discriminant $n^2 - 8n$ needs to be a perfect square, i.e. $n^2 - 8n = p^2$, where $p \in \mathbb{Z}^+$.

It is easy to verify that a non-negative integer solution n exists only for n = 9 in which case $\ell = 3$.

For this case, i.e. $K_3 \bigcup K_6$, we have ${}^{3}C_2 + {}^{6}C_2 = 3 + 15 = 18$ edges, and $\lambda_1(G) = 6$. For this case, through a brute force exhaustive search for 18 edges, we conclude that $K_3 \bigcup K_6$ is an LEGM. (see also Example 6.1).

This completes the proof of Theorem 4.1.

We now generalize Theorem 4.1 and Proposition 2.6 to p, with p > 2, components.

Theorem 4.2. : Consider graph G of n number of vertices and m number of edges consisting of p complete components $K_{n_1}, K_{n_2}, \ldots, K_{n_p}$ such that $\sum_{i=1}^p n_i = n$ and $m = |E(G)| = \sum_{i=1}^p {}^{n_i}C_2$. Let without loss of generality $n_1 \ge n_2 \ge \ldots \ge n_p$. Assume $n_1^2 + n_2^2 + \ldots + n_p^2$

$$n_1 - \frac{n_1 + n_2 + \dots + n_p}{n} < 1.$$
⁽²⁾

Then the graph $G = K_{n_1} \bigcup K_{n_2} \ldots \bigcup K_{n_p}$ is a Largest Eigenvalue Global Minimizer (LEGM).

Proof. : For the graph $G = K_{n_1} \bigcup K_{n_2} \ldots \bigcup K_{n_p}$, first notice that $n_1 - \frac{n_1^2 + n_2^2 + \ldots + n_p^2}{n} \ge 0$.

This is because

$$\frac{n \times n_1 - (n_1^2 + n_2^2 + \dots + n_p^2)}{n} = \frac{n_2(n_1 - n_2) + n_3(n_1 - n_3) + \dots + n_p(n_1 - n_p)}{n}$$

and thus only when $\frac{n}{p} \in \mathbb{Z}$ (and hence $\frac{n}{p} = n_1 = n_2 = \ldots = n_p$, we have $n_1 - \frac{n_1^2 + n_2^2 + \ldots + n_p^2}{n} = 0$. For any other value of n and of n_i , we have $0 < n_1 - \frac{n_1^2 + n_2^2 + \ldots + n_p^2}{n}$ and thus $0 \leq n_1 - \frac{n_1^2 + n_2^2 + \ldots + n_p^2}{n}$ in general. The average degree (d_{avq}) of the graph:

$$d_{avg} = \frac{2m}{n} = 2\sum_{i=1}^{p} \frac{n_i C_2}{n},$$

= $\sum_{i=1}^{p} \frac{n_i^2 - n_i}{n} = \frac{n_1^2 + n_2^2 + \dots + n_p^2 - n}{n},$
= $n_1 - 1 - (n_1 - \frac{n_1^2 + n_2^2 \dots + n_p^2}{n}).$

We next use that the maximum degree of the graph, $\Delta \ge d_{avg}$. We also know that Δ should be an integer which implies $\Delta \ge \lceil d_{avg} \rceil$. If $0 \le n_1 - \frac{n_1^2 + n_2^2 \dots + n_p^2}{n} < 1$, then the maximum degree $\Delta \ge n_1 - 1$.

Using Proposition 2.2a), for any graph that has as many edges as m, we get: $\lambda_1(G) \ge \Delta + 1 \implies \lambda_1(G) \ge n_1$.

Finally, it remains to show that the proposed graph G satisfies $\lambda_1(G) = n_1$. Since, $n_1 \ge n_i$ and $\lambda_1(K_{n_i}) = n_i$, we conclude that graph proposed is LEGM. This completes the proof of Theorem 4.2.

Remark 4.3. Theorem 4.1 (and Theorem 4.2, for the number of components greater than two case) establish that when two components are of 'almost similar sizes' or the largest component is 'much larger than the smallest', we get a Largest Eigenvalue Global Minimizer graph. Both sufficient conditions, equations (1) and (2) are to be viewed as a relaxation on the condition ' α is factor of n' in Proposition 2.6. This is elaborated as follows. From Proposition 2.6, it is clear that for any integer n_1 , when we have $\bigcup_{i=1}^{p} K_{n_1}$, then this graph is an LEGM. Intuitively, by addition of a 'sufficiently small' component $K_{n_{p+1}}$, i.e. when $0 < n_{p+1} \ll n_1$, then LEGM would continue to hold. On the other hand, when n_p is 'slightly smaller' than n_1 , then too LEGM would continue to hold. In other words, not just when all components are of same size but when the components are 'quite homogeneous' or some of them are relatively 'negligibly small', then also LEGM property continues to hold: in that sense the sufficient condition $n_1 - \frac{n_1^2 + n_2^2 + \ldots + n_p^2}{n} < 1$ is a relaxation of the condition ' α is factor of n' of Proposition 2.6.

Theorem 4.4. Consider the algorithm below that takes n (number of vertices) and $m_{desired}$ (desired number of edges) as an input. Suppose the algorithm terminates with $m_{actual} = m_{desired}$, then the constructed graph is LELM. If the sufficient condition of Theorem 4.2 is met, then this proposed graph is also LEGM. Within the class of graphs which are LELM, this procedure gives the least λ_1 .

Proof. The claims in the theorem are straight forward and hence we summarize and dwell on only the key arguments. The algorithm constructs components: largest first and then smaller, etc. until all vertices are used up and the maximum number of edges (up to $m_{desired}$) are accommodated.

- By construction, the obtained graph is clearly LELM.
- Within the 'the while loop', the condition $x_i \leq n_i^{rem}$ ensures that the new components do not exceed the remaining number of vertices.
- Equation (4) ensures that ℓ_i is as large as possible for a given component size x_i .
- Equation (3) ensures that the ℓ_i components, each of x_i vertices, do not exceed the remaining number of vertices.

Thus, the construction procedure attempts to accommodates the desired number of edges with as small size components of complete graphs K_{x_i} as possible and hence is an LELM with least $\lambda_1 = x_1$.

Note that when the algorithm terminates, but with $m_{desired} > m_{actual}$, then the difference $m_{desired} - m_{actual} \leq n-2$: this can be seen easily and is hence not pursued. Obtaining a better upper bound is worth pursuing further.

Remark 4.5. Equations (3) and (4) within Algorithm 1 are to be understood as follows. It is understandable that to have λ_1 low, the complete components need to be individually of a small sizes. This is achieved by taking the min x_i satisfying equations (3) and (4). The condition $x_i \leq n_i^{rem}$ is about how many vertices are available for the next graph construction. For each x_i , the number of components K_{x_i} is ℓ_i . Each component of size x_i accommodates $x^i C_2$ edges and we try to have as many such components of size x_i as possible given the total remaining number of vertices n_i^{rem} , this is captured by $\ell_i \leq \frac{n_i^{rem}}{x_i}$. Finally, given a size x_i , it is required that the number of components ℓ_i of that size should be as large as possible to accommodate the desired (or yet to be accommodated) number of edges: this is ensured by Equation (4). Loosely speaking, increasing x_i helps in accommodating more edges, at the cost of a larger λ_1 and less number of components ℓ_i . On the other hand, smaller x_i aids in decreasing λ_1 , but would perhaps be unable to accommodate enough edges.

Algorithm 1: Edges inclusion in the graph

Input: Vertices count: *n*, number of edges desired: *m*_{desired}.

Output: Number of components p, their sizes and the number of edges actually accommodated m_{actual} .

Initialize i = 1, the number of vertices in the graph $n_1^{rem} := n$ and the desired number of edges to be accommodated in the graph $E_1^{rem} := E_{d_1} = m_{desired}$. while $E_{i+1}^{rem} > 0$ and $n_{i+1}^{rem} > 0$, do

Get x_i , ℓ_i by the following minimization: $\arg\min x_i \in \mathbb{Z}^+$, $x_i \leq n_i^{rem}$ such that there exists $l_i \in \mathbb{Z}^+$ satisfying (3) & (4): $\ell_i \leq \frac{n_i^{rem}}{2}$ (3)

$$\ell_i \leqslant \frac{1}{x_i}$$

$$\ell_i x_i C_2 \leqslant E_i^{rem} < (\ell_i + 1) x_i C_2$$

$$\tag{4}$$

 $E_{i+1}^{rem} := E_i^{rem} - \ell_i \, {}^{x_i}C_2 \text{ and } n_{i+1}^{rem} := n_i^{rem} - \ell_i x_i;$

 \mathbf{end}

Result: Suppose at i = s, one or both of the conditions, $E_{i+1}^{rem} > 0$ or

 $n_{i+1}^{rem} > 0$ gets violated, then the following are defined as the output.

- 1. The actual number of edges the constructed graph accommodates,
 - $m_{actual} := \sum_{i=1}^{s} \ell_i \, {}^{x_i}C_2 = |E(G)|.$
- 2. The number of components of the graph, $p = \sum_{i=1}^{s} \ell_i$.
- 3. The LELM graph $G := \underset{\ell_1 \text{ times}}{K_{x_1} \bigcup K_{x_2} \bigcup \cdots \bigcup \underset{\ell_s \text{ times}}{K_{x_s}}} K_{x_s}$, and $\lambda_1(G) = x_1$

5. Circulant matrices

Propositions 2.1 and 2.2 are about relations between the Laplacian eigenvalues for a graph and its complement, and about the max degree Δ providing a lower bound for the max eigenvalue. In particular, the lower bound $\Delta + 1$ is tight for the case when the graph contains a star node, i.e. the domination number (see Footnote 2) is 1. This naturally suggests that a relatively equitable distribution of edges that keeps the max-degree Δ low helps in keeping the maximum eigenvalue λ_1 also low.

Circulant matrices are such matrices: they are regular and contain a symmetry that indeed makes them LEGM for certain cases; we pursue this link in this section.

A matrix $C \in \mathbb{R}^{n \times n}$ is called circulant if each entry $c_{i,j}$, the entry in *i*-th row and *j*-th column satisfies: $c_{i,j} = c_{i+k,j+k}$, where the indices are considered to be modulo-*n* and - for this reason, and just for this sentence - indices i, j vary from 0 to n - 1. It is well-known (see [1]) that the set of circulant matrices form an *n*-dimensional subspace of $\mathbb{R}^{n \times n}$, and the entries of only the first row of *C* need to be specified for specifying *C*. A circulant graph is one whose Laplacian is a circulant matrix, after a permutation/re-ordering of the nodes, if needed. Define the matrix $J \in \mathbb{R}^{n \times n}$ such that $J_{ij} = 1$ for all $i, j \in \{1, 2, \dots, n\}$. Notice that nI - J is a circulant graph is same as the Laplacian of K_n , i.e. nI - J. This means that if G^m is a circulant graph, then so is its complement G^c .

We pursue further with Problem 1.2 and note that the DFT of the first row of a circulant matrix C are exactly the eigenvalues of C. Given integers n and m, the number of vertices and edges, due to the implicit regularity of a circulant graph, $2 \times m$ has to be divisible by n for a circulant graph G(V, E) to exist such that |V| = n and |E| = m.

Below is our first result in this context. We then come up with examples in the following section.

Theorem 5.1. Consider positive integers n and m satisfying the relation that n is a factor of 2m, and $\frac{2m}{n} + 1$ is a factor of n. Then, the following hold.

- 1. There exists a circulant graph G_c^m having n vertices and m edges.
- 2. G_c^m is an LEGM.
- 3. The first row of the adjacency matrix of G_c^m solves Problem 1.2.
- G^c_c, the complement of G^m_c, is also a circulant graph and has the highest algebraic connectivity, i.e. G^c_c is an ACM.

Proof. Notice that the condition on m is just that one can construct $G := \bigcup_{\substack{\ell \text{ times}}} K_i$, with $i := \frac{2m+n}{n}$ and $\ell := \frac{n}{i}$. By a careful renumbering of the vertices in G, it is

possible to obtain a circulant graph G_c . Note that renumbering of vertices is merely premultiplying and postmultiplying the Laplacian L by permutation matrices P and P^T , a unitary similarity transformation, does not change the eigenvalues of L.

Of course, the condition specified in the theorem is only a sufficient condition for a circulant matrix to be an LEGM. Examples 6.1 and 6.2 are included in the next section: The former (i.e. Example 6.1) is when the sufficient condition of Theorem 4.1 is met with equality, and, and the resulting union of complete components is an LEGM. Further, this case also admits a circulant matrix, which also is an LEGM, though it is not a union of complete components. The latter (Example 6.2) is a circulant matrix that has λ_1 significantly higher than the corresponding LELM constructed for n = 9and m = 18.

6. Examples

In this section we consider some examples. Table 1 contains many typical values of n and m (the total number of vertices and edges) and also lists which are LEGM (in addition to being LELM). Some more examples are elaborated here.

n	m	$\bigcup K_i$	λ_1	LEGM/LELM
9	10	4, 3, 2	4	LEGM
9	12	4, 4	4	LEGM
10	16	5, 4	5	LEGM
10	20	5, 5	5	LEGM
15	34	7, 5, 3	7	LELM
20	22	4, 4, 4, 2, 2, 2, 2, 2	4	LEGM
20	50	8,6,4,2	8	LELM
25	66	8,8,4,3,2	8	LELM
25	132	12, 12	12	LEGM
30	235	22, 2, 2, 2, 2, 2	22	LELM
32	136	10,10,10,2	10	LEGM

Table 1: λ_1 for complete components K_i having *n* vertices and *m* edges

Example 6.1. In this example, the sufficient condition Inequality (1) is satisfied with an equality and is not captured by Proposition 2.4, but handled by Theorem 4.1. Suppose the number of vertices, n = 9 and the number of edges, m = 18.

- The LELM graph is $K_6 \bigcup K_3$ with $\lambda_1(K_6 \bigcup K_3) = 6$, and by a simple exhaustive brute-force search, this also is an LEGM.
- Further, the circulant graph G_c with degree 4, represented by the circulant adjacency matrix having its first row as $[0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0]$ also has $\lambda_1(G_c) = 6$.

Thus $K_6 \bigcup K_3$ is not the unique LEGM and the circulant graph G_c has the same λ_1 value and is an LEGM too.

Example 6.2. Consider again the case when vertex/edge counts are n = 9 and m = 18, and we look for a circulant graph that maximizes the largest eigenvalue. As noted in the previous example, the LELM graph $K_6 \bigcup K_3$ gives $\lambda_1(LELM) = 6$. A different circulant graph G_c , obtained from the circulant adjacency matrix having its first row as $[0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0]$, has $\lambda_1(G_c) = 6.88$.

Example 6.3. Consider the case when the vertex/edge counts are n = 7 and m = 7.

- LELM graph is $K_4 \bigcup K_2 \bigcup K_1$ has $\lambda_1(LELM) = 4$.
- The circulant graph C_7 , the cycle graph on 7 nodes, represented by circulant adjacency matrix generated by $[0\ 1\ 0\ 0\ 0\ 1]$ has $\lambda_1(C_7) = 3.802$.

7. Concluding remarks

In this paper we showed how the graphs comprised of two or more complete components locally minimize the Laplacian's largest eigenvalue (LELM graphs): Theorems 3.6. This was achieved by a meticulous case by case analysis of various possibilities: see Figures 1 to 8, and Lemmas 3.3-3.5. Further, if the components sizes are either 'quite homogeneous' or some of them are relatively 'negligibly small' (as elaborated in Remark 4.5, which interpreted Equations (1) and (2) of Theorems 4.1 and 4.2), then this graph is not just local, but also a global minimizer of the largest eigenvalue for that many vertices and edges. This thus extends existing results in different and appropriate ways: Propositions 2.1, 2.2 and 2.3. We also proposed an algorithm to construct such a locally/globally optimum graph (Algorithm 1).

We also related our results to the well-studied class of graphs called circulant graphs: the significance being that due to the symmetry and fairly uniform distribution of edges across nodes within such graphs, they appear like the opposite of graphs that have a 'star node', and hence are potential candidates for minimization of the largest eigenvalue. The link between circulant graphs/matrices and the Discrete Fourier Transform is well-known, and the central problem considered in this paper thus translates to minimization of the maximum magnitude across all nonzero frequencies in a periodic discrete time signal (see Problem 1.2 and Remark 1.3).

Our method of maximizing the algebraic connectivity of a graph crucially uses Proposition 2.1. Thus all our results pertaining to minimization of the largest eigenvalue easily translate to the maximizing of the algebraic connectivity by noting that the main graph G^m and its complement G^c (on *n*-vertices) have the edge counts adding up to ${}^{n}C_2$, i.e. $\frac{n(n-1)}{2}$.

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