# Diagnosabilities of Regular Networks 

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#### Abstract

In this paper, we study diagnosabilities of multiprocessor systems under two diagnosis models: the PMC model and the comparison model. In each model, we further consider two different diagnosis strategies: the precise diagnosis strategy proposed by Preparata et al. and the pessimistic diagnosis strategy proposed by Friedman. The main result of this paper is to determine diagnosabilities of regular networks with certain conditions, which include several widely used multiprocessor systems such as variants of hypercubes and many others.


Keywords. Diagnosis, diagnosis by comparison, hypercube, multiprocessor system, pessimistic diagnosis strategy, PMC model, precise diagnosis strategy.

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## 1 Introduction

Fault diagnosis is an important step in the design of multiprocessor systems and VLSI/WSIoriented computing systems. And automatic fault diagnosis has been considered an integral part of the process of achieving fault tolerance. A diagnosis strategy means a process to diagnose faults, and it is precise (respectively, pessimistic) if no fault-free processor is mistaken as a faulty one (respectively, a fault-free processor may be mistaken as a faulty one). In order to diagnose faults, a number of tests are performed among processors and the collection of all test results is referred to as a syndrome.

Suppose that $S$ is a system with at most $t$ faulty processors. Based on a precise diagnosis strategy, $S$ is $t$-diagnosable if given any syndrome, all faulty processors can be determined [28]. The maximum $t$ for which $S$ is $t$-diagnosable is called the diagnosability of $S$ [3]. On the other hand, based on a pessimistic diagnosis strategy, $S$ is $t / s$-diagnosable if given any syndrome, all faulty processors can be confined to a set of at most $s$ processors, where $t \leq s$ [18]. The maximum $t$ for which $S$ is $t / t$-diagnosable is also called the diagnosability of $S$ (23].

Preparata, Metzem, and Chien [28] first proposed a model, called the PMC model, for fault diagnosis in a multiprocessor system. Under the PMC model, all tests are performed between two adjacent processors, and it was assumed that a test result is reliable (respectively, unreliable) if the processor that initiates the test is fault-free (respectively, faulty). The PMC model was also adopted in [3], [16], [19], [20], [22], [23] and [35].

Malek [27] proposed another model, called the comparison model, under which each test is initiated by a unique arbitrator. The arbitrator feeds a pair of processors with the same task and input and then compares their outputs. It is assumed that the outputs are identical if they are fault-free, and distinct otherwise. Only a fault-free arbitrator can guarantee a reliable test result. Later, Maeng and Malek [25] modified Malek's model so that multiple arbitrators were allowed and each arbitrator can test any two of its adjacent processors. Maeng and Malek's model is referred to as the MM model. Sengupta and Dahbura 32]
further suggested a modification of the MM model, called the $M M^{*}$ model, in which any processor has to test another two processors if the former is adjacent to the later two. The $\mathrm{MM}^{*}$ model was also adopted in [2], 17] and 36].

Under the PMC model with a precise (respectively, pessimistic) strategy, an $n$-dimensional hypercube has diagnosability $n$ [3] (respectively, $2 n-2$ [23]); an $n$-dimensional enhanced hypercube has diagnosability $n+1$ (respectively, 2n) [35]; an $n$-dimensional Möbius cube has diagnosability $n$ (respectively, $2 n-2$ ) [16]; an $n$-dimensional star graph has diagnosability $n-1$ (respectively, $2 n-4$ ) [22]. On the other hand, under the $\mathrm{MM}^{*}$ model with a precise strategy, an $n$-dimensional hypercube has diagnosability $n$ [36] an $n$-dimensional enhanced hypercube has diagnosability $n+1$ [36]; an $n$-dimensional crossed cube has diagnosability $n$ [17]; a $k$-ary $n$-dimensional butterfly graph has diagnosability $2 k-2$ if $k \geq 3$ and $n \geq 3$ [2].

In this paper, we establish sufficient conditions for computing diagnosabilities of regular networks. Our results are valid for both the PMC and the MM* models with both the precise and the pessimistic strategies. As consequences, diagnosabilities of many well-known and unknown but potentially useful multiprocessor systems can be obtained. These include hypercubes, enhanced hypercubes, twisted cubes, crossed cubes, Möbius cubes, cube-connected cycles, tori, star graphs, and many others. Some of these are established in several papers as described in the previous paragraph, and many are new.

In the next section, we introduce definitions and notations which are used throughout this paper. We then derive in Section 3 the diagnosabilities of regular networks with certain conditions under different models and strategies. Consequently, the diagnosabilities of several widely used multiprocessor systems are determined in Section 4. Finally, in Section 5, we conclude the paper with some remarks.

## 2 Preliminaries

In the study of multiprocessor systems, the topology of a system is often adequately represented by a graph $G=(V, E)$, where each node $u \in V$ denotes a processor and each edge
$(u, v) \in E$ denotes a link between nodes $u$ and $v$. Previously, when the PMC model was adopted, a self-diagnosable system was often represented by a directed graph in which an arc directed from node $u$ to node $v$ means that $u$ can test $v$. On the other hand, when the MM* model was adopted, a self-diagnosable system was often represented by a multigraph in which an edge $(u, v)$ labeled with $w$ means that $w$ is an arbitrator for $u$ and $v$, i.e., $w$ can test both $u$ and $v$. Since multiple arbitrators for the same pair of nodes are allowed, the representing graph can be a multigraph.

Throughout this paper we use a graph $G=(V, E)$ to represent a self-diagnosable system. For a node $u$ of $G$, denote by $N(u)$ the set of all its neighboring nodes, i.e., $N(u)=$ $\{v \in V: v$ is adjacent to $u\}$. For a subset $S$ of $V$, let $N(S)=\cup_{v \in S} N(v)$.

Definition 1 Under the PMC model, a syndrome $\sigma$ for system $G$ is defined as follows. For any two distinct nodes $u$ and $v$ with $v \in N(u)$,

$$
\sigma(u, v)= \begin{cases}0, & \text { if } v \text { is tested by } u \text { to be fault-free; } \\ 1, & \text { if } v \text { is tested by } u \text { to be faulty. }\end{cases}
$$

Definition 2 Under the $M M^{*}$ model, a syndrome $\sigma$ for system $G$ is defined as follows. For any three distinct nodes $u$, $v$ and $w$ with $u, v \in N(w)$,

$$
\sigma(u, v ; w)= \begin{cases}0, & \text { if the test results of } u \text { and } v \text { by } w \text { are identical; } \\ 1, & \text { if the test results of } u \text { and } v \text { by } w \text { are distinct. }\end{cases}
$$

Notice that the test result initiated by a faulty processor is unreliable, and more than one syndrome may be produced for $G$ with faulty nodes. For each subset $F \subseteq V$, let $\Omega(F)$ represent the set of syndromes that can be produced if $F$ is the set of all faulty nodes. When $G$ has faulty nodes, a syndrome $\sigma$ is randomly generated for the purpose of fault diagnosis. We call $F$ an allowable fault set with respect to $\sigma$ under the PMC model (respectively, the MM ${ }^{*}$ model) if (1) and (2) hold (respectively, ( $1^{*}$ ) and ( $2^{*}$ ) hold).
(1) $\sigma(u, v)=0$ for $u \in V-F$ and $v \in V-F$.
(2) $\sigma(u, v)=1$ for $u \in V-F$ and $v \in F$.
(1*) $\sigma(u, v ; w)=0$ for $u \in V-F, v \in V-F$ and $w \in V-F$.
$\left(2^{*}\right) \sigma(u, v ; w)=1$ for $(u \in F$ or $v \in F)$ and $w \in V-F$.
It is easy to see that $F$ is an allowable fault set with respect to $\sigma$ if and only if $\sigma \in \Omega(F)$. Also, the set of all faulty nodes in $G$ is an allowable fault set with respect to $\sigma$.

Two subsets $F_{1}$ and $F_{2}$ of $V$ are distinguishable if $\Omega\left(F_{1}\right) \cap \Omega\left(F_{2}\right)=\emptyset$, and indistinguishable otherwise. When $F_{1}$ and $F_{2}$ are distinguishable, for each syndrome $\sigma$ in $\Omega\left(F_{1}\right) \cup \Omega\left(F_{2}\right)$, exactly one of $F_{1}$ and $F_{2}$ is an allowable fault set with respect to $\sigma$. In this case, $F_{1}$ and $F_{2}$ are distinct. On the other hand, when $F_{1}$ and $F_{2}$ are indistinguishable, they are allowable fault sets with respect to each syndrome in $\Omega\left(F_{1}\right) \cap \Omega\left(F_{2}\right)$.

Definition 3 Under the precise diagnosis strategy, a system $G=(V, E)$ is $t$-diagnosable if for any two subsets $F_{1}$ and $F_{2}$ of $V$ such that $\left|F_{1}\right| \leq t$ and $\left|F_{2}\right| \leq t$, the sets $F_{1}$ and $F_{2}$ are distinguishable.

Definition 4 Under the pessimistic diagnosis strategy, a system $G=(V, E)$ is $t / t$-diagnosable if for any two subsets $F_{1}$ and $F_{2}$ of $V$ such that $\left|F_{1}\right| \leq t,\left|F_{2}\right| \leq t$ and $\left|F_{1} \cup F_{2}\right|>t$, the sets $F_{1}$ and $F_{2}$ are distinguishable.

The following characterization is useful for the distinguishability of two sets under the $\mathrm{MM}^{*}$ model. The symmetric difference of two sets $A$ and $B$ is defined as the set $A \Delta B=$ $(A \cup B)-(A \cap B)$.

Lemma 1 ([32]) Suppose $G=(V, E)$ is a system under the $M^{*}$ model. Two distinct subsets $F_{1}$ and $F_{2}$ of $V$ are distinguishable if and only if there is a node $v \in V-\left(F_{1} \cup F_{2}\right)$ such that at least one of the following conditions holds.
(1) $\left|N(v) \cap\left(F_{1}-F_{2}\right)\right| \geq 2$.
(2) $\left|N(v) \cap\left(F_{2}-F_{1}\right)\right| \geq 2$.
(3) $\left|N(v)-\left(F_{1} \cup F_{2}\right)\right| \geq 1$ and $\left|N(v) \cap\left(F_{1} \Delta F_{2}\right)\right| \geq 1$.

## 3 Diagnosabilities of regular networks

This section determines diagnosabilities of regular networks with certain conditions. Our results are for systems under the PMC model and the $\mathrm{MM}^{*}$ model each using both the precise and the pessimistic diagnosis strategies.

### 3.1 Precise diagnosis strategy

A graph is called r-regular if every node in this graph has the same degree $r$. A graph is triangle-free if it does not contain a complete graph of three nodes as a subgraph. All networks in this subsection are r-regular and triangle-free such that $N(u) \neq N(v)$ for every two adjacent nodes $u$ and $v$. With these conditions, we prove the $r$-diagnosability of networks under the PMC model and the MM* model each using the precise diagnosis strategy, see Theorems 3 and 4 respectively. Our plan is as follows.

Suppose to the contrary that $G$ is not $r$-diagnosable, in either model. Then, there are two indistinguishable and hence distinct sets $F_{1}$ and $F_{2}$ with $\left|F_{1}\right| \leq r$ and $\left|F_{2}\right| \leq r$. Using the conditions mentioned above for the networks, we first prove in Lemma 2 that there is a node $w \in F_{1} \Delta F_{2}$ adjacent to some node $x \notin F_{1} \cup F_{2}$. (For the purpose of discussion below, let $F_{3}$ denote the set of all such nodes $x$.) This is mainly because the conditions on the networks force that there are not too many edges between the nodes in $F_{1} \cup F_{2}$. Having this lemma, the result for the PMC model then follows easily from the definition. For the result under the $\mathrm{MM}^{*}$ model, a longer argument is needed. By the aid of Lemma together with nodes in $F_{3}$, we first establish that $\left|F_{1} \cap F_{2}\right|$ is as large as to be either $r-1$ or $r-2$. Consequently, $F_{1}-F_{2}$ and $F_{2}-F_{1}$ both have at most two elements. These restrict the shape of $G$ greatly. The rest of the proof is then separated into two cases depending on the size of $F_{1} \cap F_{2}$.

We now start with the common lemma for the PMC model and the MM* model.

Lemma 2 Suppose $r \geq 2$ and $G=(V, E)$ is an $r$-regular graph satisfying the following two conditions.
(a) $G$ is triangle-free.
(b) $N(u) \neq N(v)$ for every two distinct nodes $u$ and $v$ of $G$.

Then, for any two distinct subsets $F_{1}$ and $F_{2}$ of $V$ with $\left|F_{1}\right| \leq r$ and $\left|F_{2}\right| \leq r$, there exists a node $w \in F_{1} \Delta F_{2}$ adjacent to some node $x \notin F_{1} \cup F_{2}$.

Proof. Suppose to the contrary that $N(w) \subseteq F_{1} \cup F_{2}$ for all nodes $w \in F_{1} \Delta F_{2}$. As $F_{1} \neq F_{2}$, we may choose $u \in F_{1} \Delta F_{2}$. In this case, $N(u) \subseteq F_{1} \cup F_{2}$. By the facts that $|N(u)|=r$ and $\left|F_{1} \cap F_{2}\right|<\max \left\{\left|F_{1}\right|,\left|F_{2}\right|\right\} \leq r$, we know that $u$ has a neighbor $v \in F_{1} \Delta F_{2}$. Again, we have $N(v) \subseteq F_{1} \cup F_{2}$. Since $G$ is triangle-free, $N(u) \cap N(v)=\emptyset$. Therefore,

$$
2 r=|N(u)|+|N(v)|=|N(u) \cup N(v)| \leq\left|F_{1} \cup F_{2}\right|=\left|F_{1}\right|+\left|F_{2}\right|-\left|F_{1} \cap F_{2}\right| \leq 2 r .
$$

Consequently, all inequalities are equalities and so $F_{1} \cap F_{2}=\emptyset$ and $F_{1} \cup F_{2}$ is the disjoint union of $N(u)$ and $N(v)$. Therefore, $N(v)=\left(F_{1} \cup F_{2}\right)-N(u)$. As $r \geq 2$, node $u$ has another neighbor $v^{\prime} \neq v$. Since $F_{1} \cap F_{2}=\emptyset$, we have $v^{\prime} \in F_{1} \Delta F_{2}$. By a similar argument as above, we have $N\left(v^{\prime}\right)=\left(F_{1} \cup F_{2}\right)-N(u)$ and so $N(v)=N\left(v^{\prime}\right)$, a contradiction to condition (b). For the relation among these sets, see Figure 1


Figure 1: Relation among the sets in the proof of Lemma 2.

According to Lemma 2 and the definition of diagnosability of a system under the PMC model using the precise diagnosis strategy, we have

Theorem 3 If $r \geq 2$ and $G$ is an $r$-regular graph, then $G$ is $r$-diagnosable under the $P M C$ model using the precise diagnosis strategy if the following two conditions hold.
(a) $G$ is triangle-free.
(b) $N(u) \neq N(v)$ for every two distinct nodes $u$ and $v$ of $G$.

Proof. Suppose to the contrary that $G$ is not $r$-diagnosable. Then, by Definition 3, there exist two indistinguishable and hence distinct sets $F_{1}$ and $F_{2}$ with $\left|F_{1}\right| \leq r$ and $\left|F_{2}\right| \leq r$. By Lemma 2. there exists a node $w \in F_{1} \Delta F_{2}$ adjacent to some node $x \notin F_{1} \cup F_{2}$. Without loss of generality, we may assume that $w \in F_{1}-F_{2}$. Choose a syndrome $\sigma \in \Omega\left(F_{1}\right) \cap \Omega\left(F_{2}\right)$. If $\sigma(x, w)=0$ (respectively, $\sigma(x, w)=1$ ), then $F_{1}$ (respectively, $F_{2}$ ) is not an allowable fault set with respect to $\sigma$, a contradiction.

For the discussion of the diagnosability under the $\mathrm{MM}^{*}$ model using the precise diagnosis strategy, we need the aid of Lemma 2 as well as Lemma The result is similar to that for the PMC model, except now there are two exceptional networks defined as follows.

The first graph is $G_{8}$ obtained from a 8-cycle joining the 4 pairs of the farest vertices. More precisely, $G_{8}$ is the graph with vertex set $V\left(G_{8}\right)=\left\{x_{1}, x_{2}, \ldots, x_{8}\right\}$ and edge set

$$
E\left(G_{8}\right)=\left\{\left(x_{i}, x_{i+1}\right): 1 \leq i \leq 7\right\} \cup\left\{\left(x_{8}, x_{1}\right)\right\} \cup\left\{\left(x_{j}, x_{j+4}\right): 1 \leq j \leq 4\right\}
$$

See Figure 2 for the graph $G_{8}$.


Figure 2: The graph $G_{8}$.

The second graph is $G_{n, n}$ obtained from the complete bipartite graph $K_{n, n}$ by removing a perfect matching. More formally, $G(n, n)$ is the graph with vertex set $V\left(G_{n, n}\right)=$
$\left\{x_{1}, x_{2}, \ldots, x_{n}, y_{1}, y_{2}, \ldots, y_{n}\right\}$ and edge set

$$
E\left(G_{n, n}\right)=\left\{\left(x_{i}, y_{j}\right): 1 \leq i \leq n, 1 \leq j \leq n \text { and } i \neq j\right\}
$$

See Figure 3 for the graph $G_{n, n}$.


Figure 3: The graph $G_{n, n}$.

We are now ready to establish diagnosabilities for regular networks under $\mathrm{MM}^{*}$ model using the precise diagnosis strategy.

Theorem 4 If $r \geq 3$ and $G$ is an $r$-regular graph, which is not isomorphic to $G_{8}$ or $G_{r+1, r+1}$, then $G$ is r-diagnosable under the $M M^{*}$ model using the precise diagnosis strategy if the following two conditions hold.
(a) $G$ is triangle-free.
(b) $N(u) \neq N(v)$ for every two distinct nodes $u$ and $v$ of $G$.

Proof. Suppose to the contrary that $G$ is not $r$-diagnosable. Then, by Definition 3, there exist two indistinguishable and hence distinct sets $F_{1}$ and $F_{2}$ with $\left|F_{1}\right| \leq r$ and $\left|F_{2}\right| \leq r$. According to Lemma [2, $F_{1} \Delta F_{2}$ has at least one node $w$ adjacent to some node $x \notin F_{1} \cup F_{2}$. Denote $F_{3}$ the set of all such nodes $x$. Since $F_{1}$ and $F_{2}$ are indistinguishable, none of the conditions in Lemma holds. It follows that for any node $v \in F_{3}$, we have
(i) $\left|N(v) \cap\left(F_{1}-F_{2}\right)\right| \leq 1$,
(ii) $\left|N(v) \cap\left(F_{2}-F_{1}\right)\right| \leq 1$,
(iii) $N(v) \subseteq F_{1} \cup F_{2}$.

Then, $F_{3}$ is an independent set with $N\left(F_{3}\right) \subseteq F_{1} \cup F_{2}$. Also, (i) and (ii) and $|N(v)|=r$ imply $\left|N(v) \cap F_{1} \cap F_{2}\right| \geq r-2$, which gives $\left|F_{1} \cap F_{2}\right| \geq r-2$ and so $\left|F_{1} \cap F_{2}\right|=r-1$ or $r-2$.

Choose a node $w \in F_{1} \Delta F_{2}$ which is adjacent to some node $x \in F_{3}$. Then, $\left|F_{1} \cup F_{2}\right| \geq$ $|N(x)| \geq r$. Suppose $\left|F_{1} \cup F_{2}\right|=r$. By (iii), $N(v)=F_{1} \cup F_{2}$ for all nodes $v \in F_{3}$. Condition (b) then implies that $F_{3}$ has just one node, which is $x$. In this case, $w$ must be adjacent to all other nodes in $F \cup F_{2}$. Thus a triangle forms, a contradiction. Hence, $\left|F_{1} \cup F_{2}\right| \geq r+1$.

Let $F_{3}=\left\{v_{1}, v_{2}, \ldots, v_{s}\right\}$ and consider the following two cases.
Case 1. $\left|F_{1} \cap F_{2}\right|=r-1$. In this case, $\left|F_{1} \cup F_{2}\right|=r+1$ and $\left|F_{1} \Delta F_{2}\right|=2$.
Let $F_{1} \cup F_{2}=\left\{w_{1}, w_{2}, \ldots, w_{r+1}\right\}$. As $G$ is $r$-regular, (iii) and condition (b) imply $s \leq r+1$ and, without loss of generality, $N\left(v_{i}\right)=\left(F_{1} \cup F_{2}\right)-\left\{w_{i}\right\}$ for $1 \leq i \leq s$. We claim that $F_{1} \cup F_{2}$ is independent. Suppose to the contrary that $w_{j}$ is adjacent to $w_{k}$ for some $j<k$. Since $G$ is triangle-free, any two neighbors of node $v_{i}$ are not adjacent. Hence, $w_{j} w_{k} \in E$ implies that $j=s=1$ or $j=1<k=s=2$. As $\left|F_{1} \Delta F_{2}\right|=2$, we may choose a vertex $w_{i} \neq w_{1}$ from $F_{1} \Delta F_{2}$. Then $N\left(w_{i}\right) \subseteq\left\{w_{1}\right\} \cup F_{3}$ and so $w_{i}$ has degree at most $1+s \leq 3$ and hence exactly 3 . Furthermore, $s=2$ and $w_{i}=w_{2}$, which is adjacent to $v_{1}$ and $v_{2}$, contradicting that $v_{2}$ is not adjacent to $w_{2}$. So, $F_{1} \cup F_{2}$ is an independent set. In this case, $N\left(w_{p}\right) \subseteq F_{3}$ and $N\left(w_{q}\right) \subseteq F_{3}$ for the two nodes $w_{p}, w_{q} \in F_{1} \Delta F_{2}$. Condition (b) and $s \leq r+1$ then imply that $s=r+1$ and so $G \cong G_{r+1, r+1}$, which is impossible.

Case 2. $\left|F_{1} \cap F_{2}\right|=r-2$. In this case, $\left|F_{1}-F_{2}\right| \leq 2$ and $\left|F_{2}-F_{1}\right| \leq 2$.
By (i)-(iii), $N\left(v_{i}\right)=\left(F_{1} \cap F_{2}\right) \cup\left\{v_{i}^{\prime}, v_{i}^{\prime \prime}\right\}$ for each $v_{i} \in F_{3}$, where $v_{i}^{\prime} \in F_{1}-F_{2}$ and $v_{i}^{\prime \prime} \in F_{2}-F_{1}$. Notice that the nodes $v_{i}^{\prime}$ (respectively, $v_{i}^{\prime \prime}$ ) are not necessarily distinct, but the sets $\left\{v_{i}^{\prime}, v_{i}^{\prime \prime}\right\}$ are distinct. Then, $\left|F_{1}-F_{2}\right| \leq 2$ and $\left|F_{2}-F_{1}\right| \leq 2$ imply $s \leq 4$. For the relation among these sets, see Figure 4

Since $G$ is triangle-free, neighbors of $v_{i}^{\prime}$ and $v_{i}^{\prime \prime}$ are in $F_{3}$ or in $\left(F_{1} \Delta F_{2}\right)-\left\{v_{i}^{\prime}, v_{i}^{\prime \prime}\right\}$. We first give four observations.
(1) If $N\left(v_{i}^{\prime}\right) \cap F_{3}=\left\{v_{i}\right\}$, then the other neighbors of $v_{i}^{\prime}$ are in $\left(F_{1} \Delta F_{2}\right)-\left\{v_{i}^{\prime}, v_{i}^{\prime \prime}\right\}$, which


Figure 4: $\left|F_{1}-F_{2}\right| \leq 2$ and $\left|F_{2}-F_{1}\right| \leq 2$ imply $s \leq 4$.
has at most two nodes. Hence, $N\left(v_{i}^{\prime}\right)=\left\{v_{i}\right\} \cup\left(\left(F_{1} \Delta F_{2}\right)-\left\{v_{i}^{\prime}, v_{i}^{\prime \prime}\right\}\right)$ has exactly 3 nodes and $r=3$ and $\left|F_{1} \Delta F_{2}\right|=4$.
(2) If $N\left(v_{i}^{\prime}\right) \cap F_{3}=\left\{v_{i}, v_{j}\right\}$ with $i \neq j$, then the other neighbors of $v_{i}^{\prime}$ are in $\left(F_{1} \Delta F_{2}\right)-$ $\left\{v_{i}^{\prime}, v_{i}^{\prime \prime}, v_{j}^{\prime \prime}\right\}$, which has at most one node. Hence, $N\left(v_{i}^{\prime}\right)=\left\{v_{i}, v_{j}\right\} \cup\left(\left(F_{1} \Delta F_{2}\right)-\left\{v_{i}^{\prime}, v_{i}^{\prime \prime}, v_{j}^{\prime \prime}\right\}\right)$ has exactly 3 nodes and $r=3$ and $\left|F_{1} \Delta F_{2}\right|=4$.
(3) If there are at least 3 distinct nodes $v_{i}, v_{j}, v_{k} \in N\left(v_{i}^{\prime}\right) \cap F_{3}$, then $F_{2}-F_{1}$ contains at least three distinct nodes $v_{i}^{\prime \prime}, v_{j}^{\prime \prime}, v_{k}^{\prime \prime}$, which is impossible.
(4) Similarly, either $N\left(v_{i}^{\prime \prime}\right)=\left\{v_{i}\right\} \cup\left(\left(F_{1} \Delta F_{2}\right)-\left\{v_{i}^{\prime}, v_{i}^{\prime \prime}\right\}\right)$ or $N\left(v_{i}^{\prime \prime}\right)=\left\{v_{i}, v_{j}\right\} \cup\left(\left(F_{1} \Delta F_{2}\right)-\right.$ $\left.\left\{v_{i}^{\prime}, v_{j}^{\prime}, v_{i}^{\prime \prime}\right\}\right)$. In either case, $N\left(v_{i}^{\prime \prime}\right)$ has exactly $r=3$ nodes and $\left|F_{1} \Delta F_{2}\right|=4$.

Having the four observations, we now continue our proof. If $\left|N\left(v_{i}^{\prime}\right) \cap F_{3}\right|=\mid N\left(v_{i}^{\prime \prime}\right) \cap$ $F_{3} \mid=1$ for some $i$, then $N\left(v_{i}^{\prime}\right)=\left\{v_{i}\right\} \cup\left(\left(F_{1} \Delta F_{2}\right)-\left\{v_{i}^{\prime}, v_{i}^{\prime \prime}\right\}\right)=N\left(v_{i}^{\prime \prime}\right)$ by (1) and (4), contradicts condition (b).

Now, by symmetric, assume that $N\left(v_{1}^{\prime}\right) \cap F_{3}=\left\{v_{1}\right\}$ and $N\left(v_{1}^{\prime \prime}\right) \cap F_{3}=\left\{v_{1}, v_{2}\right\}$. By (1) and (4), the adjacency of the related nodes are shown as in the left of Figure 5. As $v_{2}^{\prime}$ is of degree 3 , it must be adjacent to one more node in $F_{3}$, say $v_{3}$. This implies that $G$ is in fact $G_{8}$ as in the right of Figure 5

The case of $\left|N\left(v_{1}^{\prime}\right) \cap F_{3}\right|=\left|N\left(v_{1}^{\prime \prime}\right) \cap F_{3}\right|=2$ is similar, except now $v_{1}^{\prime}$ is $x_{2}$.


Figure 5: $G$ is isomorphic to $G_{8}$.

### 3.2 Pessimistic diagnosis strategy

In parallel to the results of last subsection, in this subsection we establish $(2 r-2) /(2 r-$ 2)-diagnosability of networks under the PCM model and the MM* model each using the pessimistic diagnosis strategy, see Theorems 6 andrespectively. Arguments here are slightly more complicated than those in the previous subsection, and stronger conditions on the networks are necessary. More precisely, all networks considered in this subsection are rregular and triangle-free such that $|N(u) \cap N(v)| \leq 2$ for every two distinct nodes $u$ and $v$. Notice that the condition $|N(u) \cap N(v)| \leq 2$ is stronger than that $N(u) \neq N(v)$. In fact, when $G$ is $r$-regular, the former implies $|N(u) \cup N(v)| \geq 2 r-2$ while the later only implies $|N(u) \cup N(v)| \geq r+1$. For technical reason, we also have an exceptional graph $G_{5}$ which is the graph with vertex set $V_{5}=\left\{z, z_{1}, z_{2}, z_{3}, z_{4}, z_{5}\right\} \cup\left\{z_{I}: I \subseteq\{1,2,3,4,5\},|I|=2\right\}$ and edge set $E_{5}=\left\{z z_{i}, z_{i} z_{I}, z_{I} z_{J}: i \in\{1,2,3,4,5\}, i \in I, I, J \subseteq\{1,2,3,4,5\},|I|=|J|=2, I \cap J=\emptyset\right\}$.

Our plan is as follows. Suppose to the contrary that $G$ is not $(2 r-2) /(2 r-2)$ diagnosable, in either model. Then, there are two indistinguishable and hence distinct sets $F_{1}$ and $F_{2}$ with $\left|F_{1}\right| \leq 2 r-2$ and $\left|F_{2}\right| \leq 2 r-2$ but $\left|F_{1} \cup F_{2}\right|>2 r-2$. Using the conditions mentioned above for the networks, we first prove in Lemma国 that there is a node $w \in F_{1} \Delta F_{2}$ adjacent to some node $x \notin F_{1} \cup F_{2}$. (For the purpose of discussion below, let $F_{3}$ denote the set of all such nodes $x$.) Although the proof for Lemma 5 is longer than that for Lemma 2. the main reason is also that the conditions on the networks force that there are not too


Figure 6: The graph $G_{5}$, where $z_{i j}$ stands for $z_{\{i, j\}}$.
many edges between the nodes in $F_{1} \cup F_{2}$. Having this lemma, again, the result for the PMC model follows easily from the definition. For the result under the $\mathrm{MM}^{*}$ model, again, a longer argument is needed. By the aid of Lemma together with nodes in $F_{3}$, we first establish that $\left|F_{1} \cap F_{2}\right| \geq r-2$ and $\left|F_{1} \cup F_{2}\right| \leq 3 r-2$. It is then proved that $\left|N(w) \cap F_{3}\right| \leq 2$ for each node $w \in F_{1} \Delta F_{2}$. These restrict the connections between $F_{1} \Delta F_{2}$ and $F_{3}$. The rest of the proof is then separated into three cases depending on the sizes of $F_{3}$ and $N(p) \cap\left(F_{1} \Delta F_{2}\right)$ for $p \in F_{3}$.

We now start with the common lemma for the PMC model and the MM* model.

Lemma 5 Suppose $r \geq 5$ and $G$ is an r-regular graph, which is not isomorphic to $G_{5}$ and satisfies the following two conditions.
(a) $G$ is triangle-free.
(b) $|N(u) \cap N(v)| \leq 2$ for every two distinct nodes $u$ and $v$ of $G$.

Then, for any two distinct subsets $F_{1}$ and $F_{2}$ of $V$ with $\left|F_{1}\right| \leq 2 r-2$ and $\left|F_{2}\right| \leq 2 r-2$ but $\left|F_{1} \cup F_{2}\right|>2 r-2$, there exists a node $w \in F_{1} \Delta F_{2}$ adjacent to some node $x \notin F_{1} \cup F_{2}$.

Proof. Suppose to the contrary that $N(w) \subseteq F_{1} \cup F_{2}$ for all $w \in F_{1} \Delta F_{2}$. By the assumptions, $F_{1} \cap F_{2}$ is a proper subset of $F_{1}$ and $F_{2}$, and so $\left|F_{1} \Delta F_{2}\right| \geq 2$. We may choose two distinct vertices $u$ and $v$ from $F_{1} \Delta F_{2}$. If $N(u)$ and $N(v)$ are subsets of $F_{1} \cap F_{2}$, then
condition (b) implies that

$$
\left|F_{1} \cap F_{2}\right| \geq|N(u) \cup N(v)|=|N(u)|+|N(v)|-|N(u) \cap N(v)| \geq r+r-2=2 r-2 \geq\left|F_{1}\right|
$$

contradicting to that fact that $F_{1} \cap F_{2}$ is a proper subset of $F_{1}$.
Therefore, either $u$ or $v$ is adjacent to a vertex in $F_{1} \Delta F_{2}$. So, we may choose two adjacent vertices $x$ and $y$ from $F_{1} \Delta F_{2}$. If $N(x)-\{y\}$ and $N(y)-\{x\}$ are subsets of $F_{1} \cap F_{2}$, then condition (a) implies that $(N(x)-\{y\}) \cap(N(y)-\{x\})=\emptyset$ and so
$\left|F_{1} \cap F_{2}\right| \geq|(N(x)-\{y\}) \cup(N(y)-\{x\})|=|N(x)-\{y\}|+|N(y)-\{x\}|=(r-1)+(r-1) \geq\left|F_{1}\right|$, again a contradiction.

This proves that $F_{1} \Delta F_{2}$ has a vertex adjacent to at least two vertices in $F_{1} \Delta F_{2}$. Now, choose a vertex $z \in F_{1} \Delta F_{2}$ with a maximum number $s$ of neighbors in $F_{1} \Delta F_{2}$, where $2 \leq s \leq r$. Let these $s$ neighbors of $z$ be $z_{1}, z_{2}, \ldots, z_{s}$, and $A=\cup_{1 \leq i \leq s}\left(N\left(z_{i}\right)-\{z\}\right)$. By condition (a), $A$ does not contain $z$ and its neighbors. Also, each $z_{i}$ has $r-1$ neighbors in $A$. By condition (b), each vertex in $A$ has at most 2 neighbors in $\left\{z_{1}, z_{2}, \ldots, z_{s}\right\}$. Then $|A| \geq s(r-1) / 2$. Therefore,

$$
\left|F_{1} \cup F_{2}\right| \geq 1+|N(z)|+|A| \geq 1+r+s(r-1) / 2
$$

Also, by the choice of $z$, each node $z_{i}$ has at most $s$ neighbors in $F_{1} \Delta F_{2}$ and hence at least $r-s$ vertices in $F_{1} \cap F_{2}$, which are not neighbors of $z$. This further implies that $\left|F_{1} \cap F_{2}\right| \geq 2(r-s)$. Then,
$4 r-4 \geq\left|F_{1}\right|+\left|F_{2}\right|=\left|F_{1} \cup F_{2}\right|+\left|F_{1} \cap F_{2}\right| \geq(1+r+s(r-1) / 2)+2(r-s)=3 r+1+s(r-5) / 2$.

As $r \geq 5$ and $s \geq 2$, this inequality in fact is an equality and also $r=5$ or $s=2$. It is also the case that $\left|\left(F_{1} \cap F_{2}\right)-N(z)\right|=r-s$, and each $z_{i}$ is adjacent to any vertex in $\left(F_{1} \cap F_{2}\right)-N(z)$. That is, $\left(F_{1} \cap F_{2}\right) \cap A=\left(F_{1} \cap F_{2}\right)-N(z)$.

If $2 \leq s \leq r-3$, then $\left|\left(F_{1} \cap F_{2}\right)-N(z)\right| \geq 3$ and so $\left|N\left(z_{1}\right) \cap N\left(z_{2}\right)\right| \geq 3$, contradicting condition (b). If $r=5$ and $2=r-3<s \leq 4$, then $\left|\left(F_{1} \cap F_{2}\right) \cap A\right|=\left|\left(F_{1} \cap F_{2}\right)-N(z)\right| \geq 1$
and so $|N(z) \cap N(a)| \geq s>2$ for any $a \in\left(F_{1} \cap F_{2}\right) \cap A$, again impossible. Therefore, $r=s=5$ and $F_{1} \cap F_{2}=\emptyset$. In this case, $A$ has 10 vertices each adjacent to exactly two vertices in $N(z)$. Also, by condition (b), two distinct vertices in $A$ have distinct pair of neighbors in $N(z)$. For $I=\{i, j\}$, we can use $z_{I}$ to name the vertex of $A$ adjacent to $z_{i}$ and $z_{j}$. By condition (a), we also have that $z_{I}$ is not adjacent to those $z_{J}$ with $I \cap J \neq \emptyset$ and hence adjacent to those $z_{K}$ with $I \cap K=\emptyset$. So, $G$ is in fact $G_{5}$, a contradiction.

According to Lemma 5 and the definition of diagnosability of a system under the PMC model using the pessimistic diagnosis strategy, we have

Theorem 6 If $r \geq 5$ and $G$ is an r-regular graph, which is not isomorphic to $G_{5}$, then $G$ is $(2 r-2) /(2 r-2)$-diagnosable under the PMC model using the pessimistic strategy if the following two conditions hold.
(a) $G$ is triangle-free.
(b) $|N(u) \cap N(v)| \leq 2$ for every two distinct nodes $u$ and $v$ of $G$.

Proof. Suppose to the contrary that $G$ is not $(2 r-2) /(2 r-2)$-diagnosable. Then, by Definition 4, there exist two indistinguishable and hence distinct sets $F_{1}$ and $F_{2}$ with $\left|F_{1}\right| \leq 2 r-2$ and $\left|F_{2}\right| \leq 2 r-2$ but $\left|F_{1} \cup F_{2}\right|>2 r-2$. According to Lemma 囵, there exists a node $w \in F_{1} \Delta F_{2}$ adjacent to some $x \notin F_{1} \cup F_{2}$. Without loss of generality, we may assume that $w \in F_{1}-F_{2}$. Choose a syndrome $\sigma \in \Omega\left(F_{1}\right) \cap \Omega\left(F_{2}\right)$. If $\sigma(x, w)=0$ (respectively, $\sigma(x, w)=1$ ), then $F_{1}$ (respectively, $F_{2}$ ) is not an allowable fault set with respect to $\sigma$, a contradiction.

Next, we establish diagnosabilities for regular networks under MM* model using the precise diagnosis strategy.

Theorem 7 If $r \geq 6$ and $G=(V, E)$ is an r-regular graph, then $G$ is $(2 r-2) /(2 r-2)$ diagnosable under the $M M^{*}$ model using the pessimistic strategy if the following two conditions hold.
(a) $G$ is triangle-free.
(b) $|N(u) \cap N(v)| \leq 2$ for every two distinct nodes $u$ and $v$ of $G$.

Proof. Suppose to the contrary that $G$ is not $(2 r-2) /(2 r-2)$-diagnosable. Then, by Definition 4, there exist two indistinguishable and hence distinct sets $F_{1}$ and $F_{2}$ with $\left|F_{1}\right| \leq 2 r-2$ and $\left|F_{2}\right| \leq 2 r-2$ but $\left|F_{1} \cup F_{2}\right|>2 r-2$. According to Lemma $5 F_{1} \Delta F_{2}$ has at least one node $w$ adjacent to some node $x \notin F_{1} \cup F_{2}$. Denote $F_{3}$ the set of all such nodes $x$. Since $F_{1}$ and $F_{2}$ are indistinguishable, none of the conditions in Lemma holds. It follows that for any node $v \in F_{3}$,
(i) $\left|N(v) \cap\left(F_{1}-F_{2}\right)\right| \leq 1$,
(ii) $\left|N(v) \cap\left(F_{2}-F_{1}\right)\right| \leq 1$,
(iii) $N(v) \subseteq F_{1} \cup F_{2}$.

Then, $F_{3}$ is an independent set with $N\left(F_{3}\right) \subseteq F_{1} \cup F_{2}$. Also, (i) and (ii) and $|N(v)|=r$ imply $\left|N(v) \cap F_{1} \cap F_{2}\right| \geq r-2$, which gives $\left|F_{1} \cap F_{2}\right| \geq r-2$ and so

$$
\left|F_{1} \cup F_{2}\right|=\left|F_{1}\right|+\left|F_{2}\right|-\left|F_{1} \cap F_{2}\right| \leq(2 r-2)+(2 r-2)-(r-2)=3 r-2
$$

We first claim that $\left|N(w) \cap F_{3}\right| \leq 2$ for each node $w \in F_{1} \Delta F_{2}$. Assume to the contrary that $F_{1} \Delta F_{2}$ has a node $w$ adjacent to three distinct nodes $p_{1}, p_{2}$ and $p_{3}$ in $F_{3}$. Then, (i) to (iii) imply $\left|N\left(p_{i}\right) \cap\left(F_{1} \cap F_{2}\right)\right| \geq r-2$ for $1 \leq i \leq 3$; and condition (b) implies $\left|N\left(p_{i}\right) \cap N\left(p_{j}\right) \cap\left(F_{1} \cap F_{2}\right)\right| \leq 1$ for $i \neq j$. Thus, $\left|F_{1} \cap F_{2}\right| \geq(r-2)+(r-3)+(r-4)=3 r-9$, and so $\left|F_{1} \cup F_{2}\right|=\left|F_{1}\right|+\left|F_{2}\right|-\left|F_{1} \cap F_{2}\right| \leq(2 r-2)+(2 r-2)-(3 r-9)=r+5$. On the other hand, condition (b) implies $\left|F_{1} \cup F_{2}\right| \geq\left|N\left(p_{1}\right) \cup N\left(p_{2}\right) \cup N\left(p_{3}\right)\right| \geq r+(r-2)+(r-4) \geq 3 r-6>r+5$ as $r \geq 6$, a contradiction.

Case 1. $\left|F_{3}\right| \geq 2$ and $\left|N(p) \cap\left(F_{1} \Delta F_{2}\right)\right|=1$ for each node $p \in F_{3}$.
Choose $p_{1} \in F_{3}$ with $N\left(p_{1}\right) \cap\left(F_{1} \Delta F_{2}\right)=\{w\}$. Also choose $p_{2} \in\left(N(w) \cap F_{3}\right)-\left\{p_{1}\right\}$ if $\left|N(w) \cap F_{3}\right|=2$, and any node $p_{2} \in F_{3}-\left\{p_{1}\right\}$ otherwise. By condition (b), $\left|F_{1} \cap F_{2}\right| \geq$ $\left|N\left(\left\{p_{1}, p_{2}\right\}\right) \cap F_{1} \cap F_{2}\right| \geq(r-1)+(r-3)=2 r-4$. So, $\left|F_{1} \cup F_{2}\right|=\left|F_{1}\right|+\left|F_{2}\right|-\left|F_{1} \cap F_{2}\right| \leq$ $(2 r-2)+(2 r-2)-(2 r-4)=2 r$.

On the other hand, by condition (a), $N(w) \cap N\left(p_{1}\right)=\emptyset$. If $w$ is (respectively, is not) adjacent to $p_{2}$, by condition (a) (respectively, condition (b)), $N(w) \cap N\left(p_{2}\right)=\emptyset$ (respectively, $\left.\left|N(w) \cap N\left(p_{2}\right)\right| \leq 2\right)$. In either case, $\left|N(w)-\left(N\left(\left\{p_{1}, p_{2}\right\}\right) \cup F_{3}\right)\right| \geq r-3$. Hence $\left|F_{1} \cup F_{2}\right| \geq$ $\left|N\left(\left\{p_{1}, p_{2}\right\}\right)\right|+\left|N(w)-\left(N\left(\left\{p_{1}, p_{2}\right\}\right) \cup F_{3}\right)\right| \geq r+(r-2)+(r-3)>2 r$ as $r \geq 6$, a contradiction to $\left|F_{1} \cup F_{2}\right| \leq 2 r$.

Case 2. $\left|F_{3}\right| \geq 2$ and $\left|N\left(p_{1}\right) \cap\left(F_{1} \Delta F_{2}\right)\right| \geq 2$ for some node $p_{1} \in F_{3}$.
Assume that $p_{1}$ is adjacent to two distinct nodes $w_{1}$ and $w_{2}$ in $F_{1} \Delta F_{2}$. Furthermore, assume that $\left|N\left(w_{1}\right) \cap F_{3}\right| \geq\left|N\left(w_{2}\right) \cap F_{3}\right|$. Choose $p_{2} \in\left(N\left(w_{1}\right) \cap F_{3}\right)-\left\{p_{1}\right\}$ if $\left|N\left(w_{1}\right) \cap F_{3}\right|=2$, or $p_{2} \in\left(N\left(w_{2}\right) \cap F_{3}\right)-\left\{p_{1}\right\}$ if $\left|N\left(w_{2}\right) \cap F_{3}\right|=2$, or $p_{2} \in F_{3}-\left\{p_{1}\right\}$ otherwise. By (i)-(iii) and condition (b), $\left|F_{1} \cap F_{2}\right| \geq\left|N\left(\left\{p_{1}, p_{2}\right\}\right) \cap\left(F_{1} \cap F_{2}\right)\right| \geq(r-2)+(r-4)=2 r-6$. Hence, $\left|F_{1} \cup F_{2}\right|=\left|F_{1}\right|+\left|F_{2}\right|-\left|F_{1} \cap F_{2}\right| \leq(2 r-2)+(2 r-2)-(2 r-6)=2 r+2$.

On the other hand, condition (b) assures $\left|N\left(\left\{p_{1}, p_{2}\right\}\right)\right| \geq r+(r-2)=2 r-2$. If $w_{1}$ is adjacent to $p_{2}$, then by conditions (a) and (b), $\left|N\left(w_{1}\right)-\left(N\left(\left\{p_{1}, p_{2}\right\}\right) \cup F_{3}\right)\right| \geq r-2$ and $\left|N\left(w_{2}\right)-\left(N\left(w_{1}\right) \cup N\left(\left\{p_{1}, p_{2}\right\}\right) \cup F_{3}\right)\right| \geq r-5$. See the left of Figure 7 Similarly, if $w_{1}$ is not adjacent to $p_{2}\left(\right.$ as $\left.N\left(w_{1}\right) \cap F_{3}=1\right)$, then $\left|N\left(w_{1}\right)-\left(N\left(\left\{p_{1}, p_{2}\right\}\right) \cup F_{3}\right)\right| \geq r-3$ and $\left|N\left(w_{2}\right)-\left(N\left(w_{1}\right) \cup N\left(\left\{p_{1}, p_{2}\right\}\right) \cup F_{3}\right)\right| \geq r-4$. See the right of Figure 7 It follows that $\left|N\left(w_{1}\right) \cup N\left(w_{2}\right)-\left(N\left(\left\{p_{1}, p_{2}\right\}\right) \cup F_{3}\right)\right| \geq 2 r-7$. Hence, $\left|F_{1} \cup F_{2}\right| \geq\left|N\left(\left\{p_{1}, p_{2}\right\}\right)\right|+\mid N\left(w_{1}\right) \cup$ $N\left(w_{2}\right)-\left(N\left(\left\{p_{1}, p_{2}\right\}\right) \cup F_{3}\right) \mid \geq 4 r-9 \geq 2 r+3$ as $r \geq 6$, a contradiction to $\left|F_{1} \cup F_{2}\right| \leq 2 r+2$.


Figure 7: Relation among $N\left(w_{1}\right), N\left(w_{2}\right), N\left(\left\{p_{1}, p_{2}\right\}\right)$ and $F_{3}$.

Case 3. $\left|F_{3}\right|=1$, say $F_{3}=\{p\}$.

Since $\left|F_{1} \Delta F_{2}\right| \geq 2$, we may choose two nodes $u$ and $v$ from $F_{1} \Delta F_{2}$ such that $p \in N(u)$. Let $A=\{u, v\}$. Notice that $A \cup\left(N(A)-F_{3}\right) \cup N\left(F_{3}\right) \subseteq F_{1} \cup F_{2}$. By conditions (a) and (b), $\left|A \cup\left(N(A)-F_{3}\right)\right| \geq 2 r-1$ and $\left|N\left(F_{3}\right)-(A \cup N(A))\right| \geq r-3$. Then, $\left|F_{1} \cup F_{2}\right| \geq$ $(2 r-1)+(r-3)=3 r-4$. We assume that $\left|F_{1} \cup F_{2}\right|=3 r-4+s$, where $s \geq 0$. Hence, $\left|F_{1} \cap F_{2}\right|=\left|F_{1}\right|+\left|F_{2}\right|-\left|F_{1} \cup F_{2}\right| \leq(2 r-2)+(2 r-2)-(3 r-4+s)=r-s$. Also, $\left|N\left(F_{3}\right) \cap\left(F_{1} \cap F_{2}\right)\right| \geq r-2$. By condition (a), $\left|N(u) \cap\left(F_{1} \Delta F_{2}\right)\right| \geq|N(u)|-\left(\left|F_{1} \cap F_{2}\right|-\right.$ $\left.\left|N\left(F_{3}\right) \cap\left(F_{1} \cap F_{2}\right)\right|\right)-\left|F_{3}\right| \geq r-((r-s)-(r-2))-1=r-3+s$. Refer to Figure 8. Let $F_{4}=N(u) \cap\left(F_{1} \Delta F_{2}\right)$ and $\alpha=\left|\left(N\left(F_{4}\right)-(A \cup N(A))\right) \cup N\left(F_{3}\right)\right|$. By conditions (a) and (b), for each node $x \in F_{4},|N(x)-(A \cup N(A))| \geq r-3$. Then, by condition (b), $\alpha \geq\left(\left|F_{4}\right|(r-3)-\left|N\left(F_{3}\right)-A\right|\right) / 2 \geq((r-3+s)(r-3)-(r-1)) / 2 \geq 2+3 s / 2$ as $r \geq 6$. Hence, $\left|F_{1} \cup F_{2}\right| \geq\left|A \cup\left(N(A)-F_{3}\right) \cup N\left(F_{3}\right)\right|+\alpha \geq(2 r-1)+(r-3)+(2+3 s / 2)=3 r-2+3 s / 2$, a contradiction to $\left|F_{1} \cup F_{2}\right|=3 r-4+s$.


Figure 8: Relation among $N(u), N\left(F_{3}\right) \cap F_{1} \cap F_{2}$ and $F_{4}$ for Case 3. The two arrows represent that $p$ and $u$ are adjacent to all nodes in $N\left(F_{3}\right) \cap F_{1} \cap F_{2}$ and $F_{4}$, respectively.

## 4 Application to multiprocessor systems using regular networks

In this section we apply the four theorems in Section 3 to eight popular multiprocessor systems, while it is also possible to apply them to many other potentially useful ones not
shown here. To introduce these systems, we need the following notations. Define $[m]=$ $\{0,1, \ldots, m-1\}$ and $[m]^{n}=\left\{x_{n-1} x_{n-2} \ldots x_{0}: x_{i} \in[m]\right.$ for $\left.i \in[n]\right\}$, where $m$ and $n$ are positive integers. Let $x=x_{n-1} x_{n-2} \ldots x_{0} \in[m]^{n}$ and $y=y_{n-1} y_{n-2} \ldots y_{0} \in[m]^{n}$. The Hamming distance of $x$ and $y$, denoted by $H(x, y)$, is the number of indices $i$ such that $x_{i} \neq y_{i}$.

## Example 1 Hypercube $Q_{n}$ [31]

A hypercube of $n$ dimensions can be expressed by a graph $Q_{n}=(V, E)$ with $V=[2]^{n}$ and $E=\{(x, y): H(x, y)=1\}$.

## Example 2 Enhanced hypercube $E Q_{n, s}$ [34]

An enhanced hypercube is just a hypercube augmented with certain extra links. More precisely, an $(n, s)$-enhanced hypercube can be expressed by a graph $E Q_{n, s}=(V, E)$ with $V=[2]^{n}$ and $E=\left\{(x, y): H(x, y)=1\right.$ or $y=x_{n-1} x_{n-2} \ldots x_{s+1} \bar{x}_{s} \bar{x}_{s-1} \ldots \bar{x}_{0}$ for some $0 \leq s \leq n-1\}$, where $\bar{x}_{i}=1-x_{i}$ for $0 \leq i \leq s$.

## Example 3 Twisted cube $T Q_{n}$ [15]

Assume that $n$ is odd. Define $P_{j}(x)=\left(x_{j}+x_{j-1}+\ldots+x_{0}\right) \bmod 2$, where $0 \leq j \leq n-1$. A twisted cube of $n$ dimensions can be expressed by a graph $T Q_{n}=(V, E)$ with $V=[2]^{n}$ and $E$ consisting of all $(x, y)$ 's that satisfy the following two conditions for some $0 \leq k \leq(n-1) / 2$ :
(1) $x_{2 k} x_{2 k-1}=\bar{y}_{2 k} y_{2 k-1}$ or $\left(x_{2 k} x_{2 k-1}=y_{2 k} \bar{y}_{2 k-1}\right.$ and $\left.P_{2 k-2}(x)=1\right)$ or $\left(x_{2 k} x_{2 k-1}=\bar{y}_{2 k} \bar{y}_{2 k-1}\right.$ and $\left.P_{2 k-2}(x)=0\right)$;
(2) $x_{2 j} x_{2 j-1}=y_{2 j} y_{2 j-1}$ for all $j \neq k$,
where $x_{0} x_{-1}$ is regarded as $x_{0}$ when $k=0$.

## Example 4 Möbius cube $M Q_{n}[12]$

A Möbius cube of $n$ dimensions can be expressed by a graph $M Q_{n}=(V, E)$ with $V=[2]^{n}$ and $E$ containing those $(x, y)$ 's with $y=x_{n-1} x_{n-2} \ldots x_{i+2} 0 \bar{x}_{i} x_{i-1} \ldots x_{0}$ or $y=x_{n-1} x_{n-2} \ldots x_{i+2} 1 \bar{x}_{i} \bar{x}_{i-1} \ldots \bar{x}_{0}$ for some $0 \leq i \leq n-2$. Besides, $E$ contains $(x$, $\left.\bar{x}_{n-1} x_{n-2} \ldots x_{0}\right)$ or $\left(x, \bar{x}_{n-1} \bar{x}_{n-2} \ldots \bar{x}_{0}\right)$ but not both.

## Example 5 Crossed cube $C Q_{n}$ [14]

A crossed cube of $n$ dimensions can be expressed by a graph $C Q_{n}=(V, E)$ with $V=[2]^{n}$ and $E$ consisting of all $(x, y)$ 's that satisfy the following conditions for some $1 \leq m \leq n$ :
(1) $x_{n-1} x_{n-2} \ldots x_{m} x_{m-1}=y_{n-1} y_{n-2} \ldots y_{m} \bar{y}_{m-1}$;
(2) $x_{m-2}=y_{m-2}$ if $m$ is even;
(3) $\left(x_{2 i+1} x_{2 i}, y_{2 i+1} y_{2 i}\right) \in\{(00,00),(10,10),(01,11),(11,01)\}$ for $0 \leq i \leq\lfloor(m-1) / 2\rfloor-1$.

## Example 6 Cube-connected cycles $C C C_{n}$ [29]

Cube-connected cycles can be obtained by replacing each node of a hypercube with a cycle. More precisely, cube-connected cycles of $n$ dimensions can be expressed by a graph $C C C_{n}=(V, E)$ with $V=\left\{[x, i]: x \in[2]^{n}\right.$ and $\left.i \in[n]\right\}$ and $E=\{([x, i],[x, j])$ : $x \in[2]^{n}, i, j \in[n]$ and $\left.j \equiv(i \pm 1) \bmod n\right\} \cup\left\{([x, i],[y, i]): x, y \in[2]^{n}, i \in[n]\right.$ and $\left.y=x_{n-1} x_{n-2} \ldots x_{i+1} \overline{x_{i}} x_{i-1} \ldots x_{0}\right\}$.

## Example 7 Torus $\boldsymbol{T}_{\boldsymbol{n}}(\boldsymbol{m})$ [6]

An $m$-sided torus of $n$ dimensions can be expressed by a graph $T_{n}(m)=(V, E)$ with $V=[m]^{n}$ and $E=\left\{(x, y): y_{i} \equiv\left(x_{i} \pm 1\right) \bmod m\right.$ for some $i \in[n]$ and $x_{j}=y_{j}$ for all $\left.j \neq i\right\}$.

## Example 8 Star graph $S_{n}$ [1]

A star graph of $n$ dimensions can be expressed by a graph $S_{n}=(V, E)$ with $V$ being the set of all permutations of $\{1,2, \ldots, n\}$, and $E$ consisting of all $(u, v)$ 's such that $u=$ $u_{1} u_{2} \ldots u_{k} \ldots u_{n}$ and $v=u_{k} u_{2} \ldots u_{k-1} u_{1} u_{k+1} \ldots u_{n}$ (i.e., swap $u_{1}$ and $u_{k}$ ) for some $2 \leq k \leq n$.

The diagnosabilities of these multiprocessor systems can be determined by the aid of Theorems 3, 4, 6] and 7. We first have to check if they satisfy the conditions in these theorems. As the checking is easy, we only summarize the results in Table I. Consequently, we have their diagnosabilities, as shown in Table II.

Table I: Properties of multiprocessor systems.

| system | $r$-regular | triangle-free | $G_{r+1, r+1}$ | $G_{8}$ | $G_{5}$ | $N(u) \neq N(v)$ | $\|N(u) \cap N(v)\| \leq 2$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $Q_{n}$ | $r=n$ | yes | $\neq$ | $\neq$ | $\not \approx$ | yes if $n \geq 3$ | yes if $n \geq 2$ |
| $E Q_{n, s}$ | $r=n+1$ | yes if $s \geq 2$ | $\not \approx$ | $\not \approx$ | $\nsupseteq$ | yes if $n \geq 3$ | yes if $n \geq 2, s \neq 2$ |
| $T Q_{n}$ | $r=n$ | yes | $\neq$ | $\not \approx$ if $n \neq 3$ | $\neq$ | yes if $n \geq 3$ | yes if $n \geq 2$ |
| $C Q_{n}$ | $r=n$ | yes | $\not \approx$ | $\not \approx$ if $n \neq 3$ | $\nsupseteq$ | yes if $n \geq 3$ | yes if $n \geq 2$ |
| $M Q_{n}$ | $r=n$ | yes | $\not \approx$ | $\nsupseteq$ if $n \neq 3$ | $\neq$ | yes if $n \geq 3$ | yes if $n \geq 2$ |
| $C C C_{n}$ | $r=3$ if $n \geq 3$ | yes if $n \neq 3$ | $\not \approx$ | $\neq$ | $\not \approx$ | yes | yes |
| $T_{n}(m)$ | $r=2 n$ | yes if $m \neq 3$ | $\nsupseteq$ | $\not \approx$ | $\nsupseteq$ | yes if $n \geq 3$ | yes if $n \geq 2$ |
| $S_{n}$ | $r=n-1$ | yes | $\not \approx$ | $\not \approx$ | $\neq$ | yes | yes |

$\neq$ : not isomorphic.
$N(u) \neq N(v): N(u) \neq N(v)$ for any two distinct nodes $u$ and $v$ in $V$.
$|N(u) \cap N(v)| \leq 2:|N(u) \cap N(v)| \leq 2$ for any two distinct nodes $u$ and $v$ in $V$.

Table II: Diagnosabilities of multiprocessor systems.

| system | PMC |  | MM $^{*}$ |  |
| :--- | :---: | :---: | :---: | :---: |
|  | precise | pessimistic | precise | pessimistic |
| $Q_{n}$ | $n[\mathbf{3}]$ | $2 n-2 / 2 n-2[\mathbf{2 3}]$ | $n[\mathbf{3 6}]$ | $2 n-2 / 2 n-2$ |
| $E Q_{n, s}$ | $n+1[\mathbf{3 5}]$ | $2 n / 2 n[\mathbf{3 5}]$ | $n+1[\mathbf{3 6}]$ | $2 n / 2 n$ |
| $T Q_{n}$ | $n$ | $2 n-2 / 2 n-2$ | $n$ | $2 n-2 / 2 n-2$ |
| $C Q_{n}$ | $n$ | $2 n-2 / 2 n-2$ | $n[\mathbf{1 7}]$ | $2 n-2 / 2 n-2$ |
| $M Q_{n}$ | $n[\mathbf{1 6}]$ | $2 n-2 / 2 n-2[\mathbf{1 6}]$ | $n$ | $2 n-2 / 2 n-2$ |
| $C C C_{n}$ | $n+2$ | $2 n+2 / 2 n+2$ | $n+2$ | $2 n+2 / 2 n+2$ |
| $T_{n}(m)$ | $2 n$ | $4 n-2 / 4 n-2$ | $2 n$ | $4 n-2 / 4 n-2$ |
| $S_{n}$ | $n-1[\mathbf{2 2}]$ | $2 n-4 / 2 n-4[\mathbf{2 2}]$ | $n-1$ | $2 n-4 / 2 n-4$ |

[i]: also obtained in [i]; all others are results of this paper.

## 5 Conclusion

Fault diagnosis of multiprocessor systems has received much attention since Preparata et al. [28] introduced the concepts of one-step diagnosis and sequential diagnosis. The one-step diagnosis requires that all faulty nodes are found out by decoding the syndrome, whereas the sequential diagnosis consists of several diagnosis and repair phases. In each phase, one or more faulty nodes will be determined and then repaired. The process is iterated until all faulty nodes are repaired.

The one-step diagnosability of a multiprocessor system $S$ was defined to be the maximum number of faulty nodes allowed in $S$ such that the one-step diagnosis of $S$ can be performed. The sequential diagnosability of $S$ was defined similarly. In [30, the problem of computing the sequential diagnosability for a general system was proved co-NP complete. In [24], lower bounds on sequential diagnosabilities of grids and hypercubes were suggested.

In [26, Maheshwari and Hakimi introduced a probabilistic model for fault diagnosis. A p-probabilistically diagnosable system requires that any set of faulty processors having a priori probability greater than or equal to $p$ of occurring is uniquely diagnosable. In [33], the problem of determining whether a general system is p-probabilistically diagnosable or not was proved co-NP complete. A method of achieving an optimal diagnosis with maximum probability was presented in [8]. In [7], a probabilistic diagnosis algorithm was proposed whose probability of correct diagnosis could approach one if a slightly greater than linear number of tests were performed.

Another probabilistic diagnosis algorithm was proposed and evaluated in [10], on the basis of the concept that an aggregate of maximum cardinality is fault-free with probability approaching one if the cardinality of the actual fault set is smaller than the syndromedependent diagnosability. The syndrome-dependent diagnosability of a multiprocessor system is determined by evaluating the cardinality of the smallest consistent fault set that contains an aggregate of maximum cardinality. Lower bounds on syndrome-dependent diagnosabilities of toroidal grids and hypercubes were derived in (9].

In this paper, we have successfully computed one-step diagnosabilities of eight regular multiprocessor systems for two diagnosis models (i.e., the PMC and comparison models) and two diagnosis strategies (i.e., the precise and pessimistic diagnosis strategies). Our results were obtained as a consequence of four sufficient conditions. Compared with most of previous works which computed diagnosabilities only for individual systems, the four sufficient conditions can derive diagnosabilities for a class of regular systems. Our further research interests include computing sequential diagnosabilities and syndrome-dependent diagnosabilities of various systems for different diagnosis models and diagnosis strategies.

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