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Diagnosabilities of Regular Networks

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Abstract

In this paper, we study diagnosabilities of multiprocessor systems under two diagnosis models: the PMC model and the comparison model. In each model, we further consider two different diagnosis strategies: the precise diagnosis strategy proposed by Preparata *et al.* and the pessimistic diagnosis strategy proposed by Friedman. The main result of this paper is to determine diagnosabilities of regular networks with certain conditions, which include several widely used multiprocessor systems such as variants of hypercubes and many others.

Keywords. Diagnosis, diagnosis by comparison, hypercube, multiprocessor system, pessimistic diagnosis strategy, PMC model, precise diagnosis strategy.

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1 Introduction

Fault diagnosis is an important step in the design of multiprocessor systems and VLSI/WSIoriented computing systems. And automatic fault diagnosis has been considered an integral part of the process of achieving fault tolerance. A diagnosis strategy means a process to diagnose faults, and it is *precise* (respectively, *pessimistic*) if no fault-free processor is mistaken as a faulty one (respectively, a fault-free processor may be mistaken as a faulty one). In order to diagnose faults, a number of tests are performed among processors and the collection of all test results is referred to as a *syndrome*.

Suppose that S is a system with at most t faulty processors. Based on a precise diagnosis strategy, S is t-diagnosable if given any syndrome, all faulty processors can be determined [28]. The maximum t for which S is t-diagnosable is called the diagnosability of S [3]. On the other hand, based on a pessimistic diagnosis strategy, S is t/s-diagnosable if given any syndrome, all faulty processors can be confined to a set of at most s processors, where $t \leq s$ [18]. The maximum t for which S is t/t-diagnosable is also called the diagnosability of S [23].

Preparata, Metzem, and Chien [28] first proposed a model, called the *PMC model*, for fault diagnosis in a multiprocessor system. Under the PMC model, all tests are performed between two adjacent processors, and it was assumed that a test result is reliable (respectively, unreliable) if the processor that initiates the test is fault-free (respectively, faulty). The PMC model was also adopted in [3], [16], [19], [20], [22], [23] and [35].

Malek [27] proposed another model, called the comparison model, under which each test is initiated by a unique arbitrator. The arbitrator feeds a pair of processors with the same task and input and then compares their outputs. It is assumed that the outputs are identical if they are fault-free, and distinct otherwise. Only a fault-free arbitrator can guarantee a reliable test result. Later, Maeng and Malek [25] modified Malek's model so that multiple arbitrators were allowed and each arbitrator can test any two of its adjacent processors. Maeng and Malek's model is referred to as the *MM model*. Sengupta and Dahbura [32] further suggested a modification of the MM model, called the MM^* model, in which any processor has to test another two processors if the former is adjacent to the later two. The MM* model was also adopted in [2], [17] and [36].

Under the PMC model with a precise (respectively, pessimistic) strategy, an *n*-dimensional hypercube has diagnosability n [3] (respectively, 2n - 2 [23]); an *n*-dimensional enhanced hypercube has diagnosability n+1 (respectively, 2n) [35]; an *n*-dimensional Möbius cube has diagnosability n (respectively, 2n - 2) [16]; an *n*-dimensional star graph has diagnosability n - 1 (respectively, 2n - 4) [22]. On the other hand, under the MM* model with a precise strategy, an *n*-dimensional hypercube has diagnosability n [36]; an *n*-dimensional enhanced hypercube has diagnosability n + 1 [36]; an *n*-dimensional crossed cube has diagnosability n[17]; a *k*-ary *n*-dimensional butterfly graph has diagnosability 2k - 2 if $k \ge 3$ and $n \ge 3$ [2].

In this paper, we establish sufficient conditions for computing diagnosabilities of regular networks. Our results are valid for both the PMC and the MM* models with both the precise and the pessimistic strategies. As consequences, diagnosabilities of many well-known and unknown but potentially useful multiprocessor systems can be obtained. These include hypercubes, enhanced hypercubes, twisted cubes, crossed cubes, Möbius cubes, cube-connected cycles, tori, star graphs, and many others. Some of these are established in several papers as described in the previous paragraph, and many are new.

In the next section, we introduce definitions and notations which are used throughout this paper. We then derive in Section 3 the diagnosabilities of regular networks with certain conditions under different models and strategies. Consequently, the diagnosabilities of several widely used multiprocessor systems are determined in Section 4. Finally, in Section 5, we conclude the paper with some remarks.

2 Preliminaries

In the study of multiprocessor systems, the topology of a system is often adequately represented by a graph G = (V, E), where each node $u \in V$ denotes a processor and each edge $(u, v) \in E$ denotes a link between nodes u and v. Previously, when the PMC model was adopted, a self-diagnosable system was often represented by a directed graph in which an arc directed from node u to node v means that u can test v. On the other hand, when the MM* model was adopted, a self-diagnosable system was often represented by a multigraph in which an edge (u, v) labeled with w means that w is an arbitrator for u and v, i.e., wcan test both u and v. Since multiple arbitrators for the same pair of nodes are allowed, the representing graph can be a multigraph.

Throughout this paper we use a graph G = (V, E) to represent a self-diagnosable system. For a node u of G, denote by N(u) the set of all its neighboring nodes, i.e., $N(u) = \{v \in V : v \text{ is adjacent to } u\}$. For a subset S of V, let $N(S) = \bigcup_{v \in S} N(v)$.

Definition 1 Under the PMC model, a syndrome σ for system G is defined as follows. For any two distinct nodes u and v with $v \in N(u)$,

$$\sigma(u,v) = \begin{cases} 0, & \text{if } v \text{ is tested by } u \text{ to be fault-free;} \\ 1, & \text{if } v \text{ is tested by } u \text{ to be faulty.} \end{cases}$$

Definition 2 Under the MM* model, a syndrome σ for system G is defined as follows. For any three distinct nodes u, v and w with $u, v \in N(w)$,

$$\sigma(u, v; w) = \begin{cases} 0, & \text{if the test results of } u \text{ and } v \text{ by } w \text{ are identical;} \\ 1, & \text{if the test results of } u \text{ and } v \text{ by } w \text{ are distinct.} \end{cases}$$

Notice that the test result initiated by a faulty processor is unreliable, and more than one syndrome may be produced for G with faulty nodes. For each subset $F \subseteq V$, let $\Omega(F)$ represent the set of syndromes that can be produced if F is the set of all faulty nodes. When G has faulty nodes, a syndrome σ is randomly generated for the purpose of fault diagnosis. We call F an allowable fault set with respect to σ under the PMC model (respectively, the MM* model) if (1) and (2) hold (respectively, (1*) and (2*) hold).

- (1) $\sigma(u, v) = 0$ for $u \in V F$ and $v \in V F$.
- (2) $\sigma(u, v) = 1$ for $u \in V F$ and $v \in F$.

- (1*) $\sigma(u, v; w) = 0$ for $u \in V F$, $v \in V F$ and $w \in V F$.
- (2^*) $\sigma(u, v; w) = 1$ for $(u \in F \text{ or } v \in F)$ and $w \in V F$.

It is easy to see that F is an allowable fault set with respect to σ if and only if $\sigma \in \Omega(F)$. Also, the set of all faulty nodes in G is an allowable fault set with respect to σ .

Two subsets F_1 and F_2 of V are distinguishable if $\Omega(F_1) \cap \Omega(F_2) = \emptyset$, and indistinguishable otherwise. When F_1 and F_2 are distinguishable, for each syndrome σ in $\Omega(F_1) \cup \Omega(F_2)$, exactly one of F_1 and F_2 is an allowable fault set with respect to σ . In this case, F_1 and F_2 are distinct. On the other hand, when F_1 and F_2 are indistinguishable, they are allowable fault sets with respect to each syndrome in $\Omega(F_1) \cap \Omega(F_2)$.

Definition 3 Under the precise diagnosis strategy, a system G = (V, E) is t-diagnosable if for any two subsets F_1 and F_2 of V such that $|F_1| \leq t$ and $|F_2| \leq t$, the sets F_1 and F_2 are distinguishable.

Definition 4 Under the pessimistic diagnosis strategy, a system G = (V, E) is t/t-diagnosable if for any two subsets F_1 and F_2 of V such that $|F_1| \le t$, $|F_2| \le t$ and $|F_1 \cup F_2| > t$, the sets F_1 and F_2 are distinguishable.

The following characterization is useful for the distinguishability of two sets under the MM* model. The symmetric difference of two sets A and B is defined as the set $A\Delta B = (A \cup B) - (A \cap B)$.

Lemma 1 ([32]) Suppose G = (V, E) is a system under the MM* model. Two distinct subsets F_1 and F_2 of V are distinguishable if and only if there is a node $v \in V - (F_1 \cup F_2)$ such that at least one of the following conditions holds.

- (1) $|N(v) \cap (F_1 F_2)| \ge 2.$
- (2) $|N(v) \cap (F_2 F_1)| \ge 2.$
- (3) $|N(v) (F_1 \cup F_2)| \ge 1$ and $|N(v) \cap (F_1 \Delta F_2)| \ge 1$.

3 Diagnosabilities of regular networks

This section determines diagnosabilities of regular networks with certain conditions. Our results are for systems under the PMC model and the MM* model each using both the precise and the pessimistic diagnosis strategies.

3.1 Precise diagnosis strategy

A graph is called *r*-regular if every node in this graph has the same degree r. A graph is triangle-free if it does not contain a complete graph of three nodes as a subgraph. All networks in this subsection are *r*-regular and triangle-free such that $N(u) \neq N(v)$ for every two adjacent nodes u and v. With these conditions, we prove the *r*-diagnosability of networks under the PMC model and the MM* model each using the precise diagnosis strategy, see Theorems 3 and 4 respectively. Our plan is as follows.

Suppose to the contrary that G is not r-diagnosable, in either model. Then, there are two indistinguishable and hence distinct sets F_1 and F_2 with $|F_1| \leq r$ and $|F_2| \leq r$. Using the conditions mentioned above for the networks, we first prove in Lemma 2 that there is a node $w \in F_1 \Delta F_2$ adjacent to some node $x \notin F_1 \cup F_2$. (For the purpose of discussion below, let F_3 denote the set of all such nodes x.) This is mainly because the conditions on the networks force that there are not too many edges between the nodes in $F_1 \cup F_2$. Having this lemma, the result for the PMC model then follows easily from the definition. For the result under the MM* model, a longer argument is needed. By the aid of Lemma 1 together with nodes in F_3 , we first establish that $|F_1 \cap F_2|$ is as large as to be either r - 1 or r - 2. Consequently, $F_1 - F_2$ and $F_2 - F_1$ both have at most two elements. These restrict the shape of G greatly. The rest of the proof is then separated into two cases depending on the size of $F_1 \cap F_2$.

We now start with the common lemma for the PMC model and the MM* model.

Lemma 2 Suppose $r \ge 2$ and G = (V, E) is an r-regular graph satisfying the following two conditions.

- (a) G is triangle-free.
- (b) $N(u) \neq N(v)$ for every two distinct nodes u and v of G.

Then, for any two distinct subsets F_1 and F_2 of V with $|F_1| \leq r$ and $|F_2| \leq r$, there exists a node $w \in F_1 \Delta F_2$ adjacent to some node $x \notin F_1 \cup F_2$.

Proof. Suppose to the contrary that $N(w) \subseteq F_1 \cup F_2$ for all nodes $w \in F_1 \Delta F_2$. As $F_1 \neq F_2$, we may choose $u \in F_1 \Delta F_2$. In this case, $N(u) \subseteq F_1 \cup F_2$. By the facts that |N(u)| = r and $|F_1 \cap F_2| < \max\{|F_1|, |F_2|\} \le r$, we know that u has a neighbor $v \in F_1 \Delta F_2$. Again, we have $N(v) \subseteq F_1 \cup F_2$. Since G is triangle-free, $N(u) \cap N(v) = \emptyset$. Therefore,

$$2r = |N(u)| + |N(v)| = |N(u) \cup N(v)| \le |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \le 2r.$$

Consequently, all inequalities are equalities and so $F_1 \cap F_2 = \emptyset$ and $F_1 \cup F_2$ is the disjoint union of N(u) and N(v). Therefore, $N(v) = (F_1 \cup F_2) - N(u)$. As $r \ge 2$, node u has another neighbor $v' \ne v$. Since $F_1 \cap F_2 = \emptyset$, we have $v' \in F_1 \Delta F_2$. By a similar argument as above, we have $N(v') = (F_1 \cup F_2) - N(u)$ and so N(v) = N(v'), a contradiction to condition (b). For the relation among these sets, see Figure 1.



Figure 1: Relation among the sets in the proof of Lemma 2.

According to Lemma 2 and the definition of diagnosability of a system under the PMC model using the precise diagnosis strategy, we have

Theorem 3 If $r \ge 2$ and G is an r-regular graph, then G is r-diagnosable under the PMC model using the precise diagnosis strategy if the following two conditions hold.

- (a) G is triangle-free.
- (b) $N(u) \neq N(v)$ for every two distinct nodes u and v of G.

Proof. Suppose to the contrary that G is not r-diagnosable. Then, by Definition 3, there exist two indistinguishable and hence distinct sets F_1 and F_2 with $|F_1| \leq r$ and $|F_2| \leq r$. By Lemma 2, there exists a node $w \in F_1 \Delta F_2$ adjacent to some node $x \notin F_1 \cup F_2$. Without loss of generality, we may assume that $w \in F_1 - F_2$. Choose a syndrome $\sigma \in \Omega(F_1) \cap \Omega(F_2)$. If $\sigma(x, w) = 0$ (respectively, $\sigma(x, w) = 1$), then F_1 (respectively, F_2) is not an allowable fault set with respect to σ , a contradiction.

For the discussion of the diagnosability under the MM^{*} model using the precise diagnosis strategy, we need the aid of Lemma 2 as well as Lemma 1. The result is similar to that for the PMC model, except now there are two exceptional networks defined as follows.

The first graph is G_8 obtained from a 8-cycle joining the 4 pairs of the farest vertices. More precisely, G_8 is the graph with vertex set $V(G_8) = \{x_1, x_2, \dots, x_8\}$ and edge set

 $E(G_8) = \{(x_i, x_{i+1}) : 1 \le i \le 7\} \cup \{(x_8, x_1)\} \cup \{(x_j, x_{j+4}) : 1 \le j \le 4\}.$

See Figure 2 for the graph G_8 .



Figure 2: The graph G_8 .

The second graph is $G_{n,n}$ obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching. More formally, G(n,n) is the graph with vertex set $V(G_{n,n}) =$ $\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\}$ and edge set

$$E(G_{n,n}) = \{ (x_i, y_j) : 1 \le i \le n, 1 \le j \le n \text{ and } i \ne j \}.$$

See Figure 3 for the graph $G_{n,n}$.



Figure 3: The graph $G_{n,n}$.

We are now ready to establish diagnosabilities for regular networks under MM^{*} model using the precise diagnosis strategy.

Theorem 4 If $r \ge 3$ and G is an r-regular graph, which is not isomorphic to G_8 or $G_{r+1,r+1}$, then G is r-diagnosable under the MM* model using the precise diagnosis strategy if the following two conditions hold.

- (a) G is triangle-free.
- (b) $N(u) \neq N(v)$ for every two distinct nodes u and v of G.

Proof. Suppose to the contrary that G is not r-diagnosable. Then, by Definition 3, there exist two indistinguishable and hence distinct sets F_1 and F_2 with $|F_1| \leq r$ and $|F_2| \leq r$. According to Lemma 2, $F_1 \Delta F_2$ has at least one node w adjacent to some node $x \notin F_1 \cup F_2$. Denote F_3 the set of all such nodes x. Since F_1 and F_2 are indistinguishable, none of the conditions in Lemma 1 holds. It follows that for any node $v \in F_3$, we have

- (i) $|N(v) \cap (F_1 F_2)| \le 1$,
- (ii) $|N(v) \cap (F_2 F_1)| \le 1$,
- (iii) $N(v) \subseteq F_1 \cup F_2$.

Then, F_3 is an independent set with $N(F_3) \subseteq F_1 \cup F_2$. Also, (i) and (ii) and |N(v)| = rimply $|N(v) \cap F_1 \cap F_2| \ge r-2$, which gives $|F_1 \cap F_2| \ge r-2$ and so $|F_1 \cap F_2| = r-1$ or r-2.

Choose a node $w \in F_1 \Delta F_2$ which is adjacent to some node $x \in F_3$. Then, $|F_1 \cup F_2| \ge |N(x)| \ge r$. Suppose $|F_1 \cup F_2| = r$. By (iii), $N(v) = F_1 \cup F_2$ for all nodes $v \in F_3$. Condition (b) then implies that F_3 has just one node, which is x. In this case, w must be adjacent to all other nodes in $F \cup F_2$. Thus a triangle forms, a contradiction. Hence, $|F_1 \cup F_2| \ge r + 1$.

Let $F_3 = \{v_1, v_2, \dots, v_s\}$ and consider the following two cases.

Case 1. $|F_1 \cap F_2| = r - 1$. In this case, $|F_1 \cup F_2| = r + 1$ and $|F_1 \Delta F_2| = 2$.

Let $F_1 \cup F_2 = \{w_1, w_2, \ldots, w_{r+1}\}$. As G is r-regular, (iii) and condition (b) imply $s \leq r+1$ and, without loss of generality, $N(v_i) = (F_1 \cup F_2) - \{w_i\}$ for $1 \leq i \leq s$. We claim that $F_1 \cup F_2$ is independent. Suppose to the contrary that w_j is adjacent to w_k for some j < k. Since G is triangle-free, any two neighbors of node v_i are not adjacent. Hence, $w_j w_k \in E$ implies that j = s = 1 or j = 1 < k = s = 2. As $|F_1 \Delta F_2| = 2$, we may choose a vertex $w_i \neq w_1$ from $F_1 \Delta F_2$. Then $N(w_i) \subseteq \{w_1\} \cup F_3$ and so w_i has degree at most $1 + s \leq 3$ and hence exactly 3. Furthermore, s = 2 and $w_i = w_2$, which is adjacent to v_1 and v_2 , contradicting that v_2 is not adjacent to w_2 . So, $F_1 \cup F_2$ is an independent set. In this case, $N(w_p) \subseteq F_3$ and $N(w_q) \subseteq F_3$ for the two nodes $w_p, w_q \in F_1 \Delta F_2$. Condition (b) and $s \leq r+1$ then imply that s = r+1 and so $G \cong G_{r+1,r+1}$, which is impossible.

Case 2. $|F_1 \cap F_2| = r - 2$. In this case, $|F_1 - F_2| \le 2$ and $|F_2 - F_1| \le 2$.

By (i)-(iii), $N(v_i) = (F_1 \cap F_2) \cup \{v'_i, v''_i\}$ for each $v_i \in F_3$, where $v'_i \in F_1 - F_2$ and $v''_i \in F_2 - F_1$. Notice that the nodes v'_i (respectively, v''_i) are not necessarily distinct, but the sets $\{v'_i, v''_i\}$ are distinct. Then, $|F_1 - F_2| \leq 2$ and $|F_2 - F_1| \leq 2$ imply $s \leq 4$. For the relation among these sets, see Figure 4.

Since G is triangle-free, neighbors of v'_i and v''_i are in F_3 or in $(F_1\Delta F_2) - \{v'_i, v''_i\}$. We first give four observations.

(1) If $N(v'_i) \cap F_3 = \{v_i\}$, then the other neighbors of v'_i are in $(F_1 \Delta F_2) - \{v'_i, v''_i\}$, which



Figure 4: $|F_1 - F_2| \le 2$ and $|F_2 - F_1| \le 2$ imply $s \le 4$.

has at most two nodes. Hence, $N(v'_i) = \{v_i\} \cup ((F_1 \Delta F_2) - \{v'_i, v''_i\})$ has exactly 3 nodes and r = 3 and $|F_1 \Delta F_2| = 4$.

(2) If $N(v'_i) \cap F_3 = \{v_i, v_j\}$ with $i \neq j$, then the other neighbors of v'_i are in $(F_1 \Delta F_2) - \{v'_i, v''_j, v''_j\}$, which has at most one node. Hence, $N(v'_i) = \{v_i, v_j\} \cup ((F_1 \Delta F_2) - \{v'_i, v''_j, v''_j\})$ has exactly 3 nodes and r = 3 and $|F_1 \Delta F_2| = 4$.

(3) If there are at least 3 distinct nodes $v_i, v_j, v_k \in N(v'_i) \cap F_3$, then $F_2 - F_1$ contains at least three distinct nodes v''_i, v''_j, v''_k , which is impossible.

(4) Similarly, either $N(v''_i) = \{v_i\} \cup ((F_1 \Delta F_2) - \{v'_i, v''_i\})$ or $N(v''_i) = \{v_i, v_j\} \cup ((F_1 \Delta F_2) - \{v'_i, v''_j\})$. In either case, $N(v''_i)$ has exactly r = 3 nodes and $|F_1 \Delta F_2| = 4$.

Having the four observations, we now continue our proof. If $|N(v'_i) \cap F_3| = |N(v''_i) \cap F_3| = 1$ for some *i*, then $N(v'_i) = \{v_i\} \cup ((F_1 \Delta F_2) - \{v'_i, v''_i\}) = N(v''_i)$ by (1) and (4), contradicts condition (b).

Now, by symmetric, assume that $N(v'_1) \cap F_3 = \{v_1\}$ and $N(v''_1) \cap F_3 = \{v_1, v_2\}$. By (1) and (4), the adjacency of the related nodes are shown as in the left of Figure 5. As v'_2 is of degree 3, it must be adjacent to one more node in F_3 , say v_3 . This implies that G is in fact G_8 as in the right of Figure 5.

The case of $|N(v_1') \cap F_3| = |N(v_1'') \cap F_3| = 2$ is similar, except now v_1' is x_2 . \Box



Figure 5: G is isomorphic to G_8 .

3.2 Pessimistic diagnosis strategy

In parallel to the results of last subsection, in this subsection we establish (2r-2)/(2r-2)-diagnosability of networks under the PCM model and the MM* model each using the pessimistic diagnosis strategy, see Theorems 6 and 7 respectively. Arguments here are slightly more complicated than those in the previous subsection, and stronger conditions on the networks are necessary. More precisely, all networks considered in this subsection are *r*-regular and triangle-free such that $|N(u) \cap N(v)| \le 2$ for every two distinct nodes *u* and *v*. Notice that the condition $|N(u) \cap N(v)| \le 2$ is stronger than that $N(u) \ne N(v)$. In fact, when *G* is *r*-regular, the former implies $|N(u) \cup N(v)| \ge 2r-2$ while the later only implies $|N(u) \cup N(v)| \ge r+1$. For technical reason, we also have an exceptional graph G_5 which is the graph with vertex set $V_5 = \{z, z_1, z_2, z_3, z_4, z_5\} \cup \{z_I : I \subseteq \{1, 2, 3, 4, 5\}, |I| = 2\}$ and edge set $E_5 = \{zz_i, z_iz_I, z_Iz_J : i \in \{1, 2, 3, 4, 5\}, i \in I, I, J \subseteq \{1, 2, 3, 4, 5\}, |I| = |J| = 2, I \cap J = \emptyset\}$.

Our plan is as follows. Suppose to the contrary that G is not (2r-2)/(2r-2)diagnosable, in either model. Then, there are two indistinguishable and hence distinct sets F_1 and F_2 with $|F_1| \leq 2r-2$ and $|F_2| \leq 2r-2$ but $|F_1 \cup F_2| > 2r-2$. Using the conditions mentioned above for the networks, we first prove in Lemma 5 that there is a node $w \in F_1 \Delta F_2$ adjacent to some node $x \notin F_1 \cup F_2$. (For the purpose of discussion below, let F_3 denote the set of all such nodes x.) Although the proof for Lemma 5 is longer than that for Lemma 2, the main reason is also that the conditions on the networks force that there are not too



Figure 6: The graph G_5 , where z_{ij} stands for $z_{\{i,j\}}$.

many edges between the nodes in $F_1 \cup F_2$. Having this lemma, again, the result for the PMC model follows easily from the definition. For the result under the MM* model, again, a longer argument is needed. By the aid of Lemma 1 together with nodes in F_3 , we first establish that $|F_1 \cap F_2| \ge r - 2$ and $|F_1 \cup F_2| \le 3r - 2$. It is then proved that $|N(w) \cap F_3| \le 2$ for each node $w \in F_1 \Delta F_2$. These restrict the connections between $F_1 \Delta F_2$ and F_3 . The rest of the proof is then separated into three cases depending on the sizes of F_3 and $N(p) \cap (F_1 \Delta F_2)$ for $p \in F_3$.

We now start with the common lemma for the PMC model and the MM* model.

Lemma 5 Suppose $r \ge 5$ and G is an r-regular graph, which is not isomorphic to G_5 and satisfies the following two conditions.

- (a) G is triangle-free.
- (b) $|N(u) \cap N(v)| \leq 2$ for every two distinct nodes u and v of G.

Then, for any two distinct subsets F_1 and F_2 of V with $|F_1| \le 2r - 2$ and $|F_2| \le 2r - 2$ but $|F_1 \cup F_2| > 2r - 2$, there exists a node $w \in F_1 \Delta F_2$ adjacent to some node $x \notin F_1 \cup F_2$.

Proof. Suppose to the contrary that $N(w) \subseteq F_1 \cup F_2$ for all $w \in F_1 \Delta F_2$. By the assumptions, $F_1 \cap F_2$ is a proper subset of F_1 and F_2 , and so $|F_1 \Delta F_2| \ge 2$. We may choose two distinct vertices u and v from $F_1 \Delta F_2$. If N(u) and N(v) are subsets of $F_1 \cap F_2$, then

condition (b) implies that

$$|F_1 \cap F_2| \ge |N(u) \cup N(v)| = |N(u)| + |N(v)| - |N(u) \cap N(v)| \ge r + r - 2 = 2r - 2 \ge |F_1|,$$

contradicting to that fact that $F_1 \cap F_2$ is a proper subset of F_1 .

Therefore, either u or v is adjacent to a vertex in $F_1\Delta F_2$. So, we may choose two adjacent vertices x and y from $F_1\Delta F_2$. If $N(x) - \{y\}$ and $N(y) - \{x\}$ are subsets of $F_1 \cap F_2$, then condition (a) implies that $(N(x) - \{y\}) \cap (N(y) - \{x\}) = \emptyset$ and so

$$|F_1 \cap F_2| \ge |(N(x) - \{y\}) \cup (N(y) - \{x\})| = |N(x) - \{y\}| + |N(y) - \{x\}| = (r-1) + (r-1) \ge |F_1|,$$

again a contradiction.

This proves that $F_1 \Delta F_2$ has a vertex adjacent to at least two vertices in $F_1 \Delta F_2$. Now, choose a vertex $z \in F_1 \Delta F_2$ with a maximum number s of neighbors in $F_1 \Delta F_2$, where $2 \leq s \leq r$. Let these s neighbors of z be z_1, z_2, \ldots, z_s , and $A = \bigcup_{1 \leq i \leq s} (N(z_i) - \{z\})$. By condition (a), A does not contain z and its neighbors. Also, each z_i has r - 1 neighbors in A. By condition (b), each vertex in A has at most 2 neighbors in $\{z_1, z_2, \ldots, z_s\}$. Then $|A| \geq s(r-1)/2$. Therefore,

$$|F_1 \cup F_2| \ge 1 + |N(z)| + |A| \ge 1 + r + s(r-1)/2.$$

Also, by the choice of z, each node z_i has at most s neighbors in $F_1 \Delta F_2$ and hence at least r - s vertices in $F_1 \cap F_2$, which are not neighbors of z. This further implies that $|F_1 \cap F_2| \ge 2(r-s)$. Then,

$$4r - 4 \ge |F_1| + |F_2| = |F_1 \cup F_2| + |F_1 \cap F_2| \ge (1 + r + s(r-1)/2) + 2(r-s) = 3r + 1 + s(r-5)/2.$$

As $r \ge 5$ and $s \ge 2$, this inequality in fact is an equality and also r = 5 or s = 2. It is also the case that $|(F_1 \cap F_2) - N(z)| = r - s$, and each z_i is adjacent to any vertex in $(F_1 \cap F_2) - N(z)$. That is, $(F_1 \cap F_2) \cap A = (F_1 \cap F_2) - N(z)$.

If $2 \le s \le r-3$, then $|(F_1 \cap F_2) - N(z)| \ge 3$ and so $|N(z_1) \cap N(z_2)| \ge 3$, contradicting condition (b). If r = 5 and $2 = r-3 < s \le 4$, then $|(F_1 \cap F_2) \cap A| = |(F_1 \cap F_2) - N(z)| \ge 1$

and so $|N(z) \cap N(a)| \ge s > 2$ for any $a \in (F_1 \cap F_2) \cap A$, again impossible. Therefore, r = s = 5 and $F_1 \cap F_2 = \emptyset$. In this case, A has 10 vertices each adjacent to exactly two vertices in N(z). Also, by condition (b), two distinct vertices in A have distinct pair of neighbors in N(z). For $I = \{i, j\}$, we can use z_I to name the vertex of A adjacent to z_i and z_j . By condition (a), we also have that z_I is not adjacent to those z_J with $I \cap J \neq \emptyset$ and hence adjacent to those z_K with $I \cap K = \emptyset$. So, G is in fact G_5 , a contradiction.

According to Lemma 5 and the definition of diagnosability of a system under the PMC model using the pessimistic diagnosis strategy, we have

Theorem 6 If $r \ge 5$ and G is an r-regular graph, which is not isomorphic to G_5 , then G is (2r-2)/(2r-2)-diagnosable under the PMC model using the pessimistic strategy if the following two conditions hold.

- (a) G is triangle-free.
- (b) $|N(u) \cap N(v)| \leq 2$ for every two distinct nodes u and v of G.

Proof. Suppose to the contrary that G is not (2r-2)/(2r-2)-diagnosable. Then, by Definition 4, there exist two indistinguishable and hence distinct sets F_1 and F_2 with $|F_1| \leq 2r-2$ and $|F_2| \leq 2r-2$ but $|F_1 \cup F_2| > 2r-2$. According to Lemma 5, there exists a node $w \in F_1 \Delta F_2$ adjacent to some $x \notin F_1 \cup F_2$. Without loss of generality, we may assume that $w \in F_1 - F_2$. Choose a syndrome $\sigma \in \Omega(F_1) \cap \Omega(F_2)$. If $\sigma(x, w) = 0$ (respectively, $\sigma(x, w) = 1$), then F_1 (respectively, F_2) is not an allowable fault set with respect to σ , a contradiction.

Next, we establish diagnosabilities for regular networks under MM^{*} model using the precise diagnosis strategy.

Theorem 7 If $r \ge 6$ and G = (V, E) is an r-regular graph, then G is (2r - 2)/(2r - 2)diagnosable under the MM* model using the pessimistic strategy if the following two conditions hold.

- (a) G is triangle-free.
- (b) $|N(u) \cap N(v)| \leq 2$ for every two distinct nodes u and v of G.

Proof. Suppose to the contrary that G is not (2r-2)/(2r-2)-diagnosable. Then, by Definition 4, there exist two indistinguishable and hence distinct sets F_1 and F_2 with $|F_1| \leq 2r-2$ and $|F_2| \leq 2r-2$ but $|F_1 \cup F_2| > 2r-2$. According to Lemma 5, $F_1 \Delta F_2$ has at least one node w adjacent to some node $x \notin F_1 \cup F_2$. Denote F_3 the set of all such nodes x. Since F_1 and F_2 are indistinguishable, none of the conditions in Lemma 1 holds. It follows that for any node $v \in F_3$,

- (i) $|N(v) \cap (F_1 F_2)| \le 1$,
- (ii) $|N(v) \cap (F_2 F_1)| \le 1$,
- (iii) $N(v) \subseteq F_1 \cup F_2$.

Then, F_3 is an independent set with $N(F_3) \subseteq F_1 \cup F_2$. Also, (i) and (ii) and |N(v)| = rimply $|N(v) \cap F_1 \cap F_2| \ge r-2$, which gives $|F_1 \cap F_2| \ge r-2$ and so

$$|F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \le (2r - 2) + (2r - 2) - (r - 2) = 3r - 2.$$

We first claim that $|N(w) \cap F_3| \leq 2$ for each node $w \in F_1 \Delta F_2$. Assume to the contrary that $F_1 \Delta F_2$ has a node w adjacent to three distinct nodes p_1 , p_2 and p_3 in F_3 . Then, (i) to (iii) imply $|N(p_i) \cap (F_1 \cap F_2)| \geq r-2$ for $1 \leq i \leq 3$; and condition (b) implies $|N(p_i) \cap N(p_j) \cap (F_1 \cap F_2)| \leq 1$ for $i \neq j$. Thus, $|F_1 \cap F_2| \geq (r-2) + (r-3) + (r-4) = 3r-9$, and so $|F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq (2r-2) + (2r-2) - (3r-9) = r+5$. On the other hand, condition (b) implies $|F_1 \cup F_2| \geq |N(p_1) \cup N(p_2) \cup N(p_3)| \geq r + (r-2) + (r-4) \geq 3r-6 > r+5$ as $r \geq 6$, a contradiction.

Case 1. $|F_3| \ge 2$ and $|N(p) \cap (F_1 \Delta F_2)| = 1$ for each node $p \in F_3$.

Choose $p_1 \in F_3$ with $N(p_1) \cap (F_1 \Delta F_2) = \{w\}$. Also choose $p_2 \in (N(w) \cap F_3) - \{p_1\}$ if $|N(w) \cap F_3| = 2$, and any node $p_2 \in F_3 - \{p_1\}$ otherwise. By condition (b), $|F_1 \cap F_2| \ge |N(\{p_1, p_2\}) \cap F_1 \cap F_2| \ge (r-1) + (r-3) = 2r-4$. So, $|F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \le (2r-2) + (2r-2) - (2r-4) = 2r$. On the other hand, by condition (a), $N(w) \cap N(p_1) = \emptyset$. If w is (respectively, is not) adjacent to p_2 , by condition (a) (respectively, condition (b)), $N(w) \cap N(p_2) = \emptyset$ (respectively, $|N(w) \cap N(p_2)| \le 2$). In either case, $|N(w) - (N(\{p_1, p_2\}) \cup F_3)| \ge r - 3$. Hence $|F_1 \cup F_2| \ge$ $|N(\{p_1, p_2\})| + |N(w) - (N(\{p_1, p_2\}) \cup F_3)| \ge r + (r-2) + (r-3) > 2r$ as $r \ge 6$, a contradiction to $|F_1 \cup F_2| \le 2r$.

Case 2. $|F_3| \ge 2$ and $|N(p_1) \cap (F_1 \Delta F_2)| \ge 2$ for some node $p_1 \in F_3$.

Assume that p_1 is adjacent to two distinct nodes w_1 and w_2 in $F_1 \Delta F_2$. Furthermore, assume that $|N(w_1) \cap F_3| \ge |N(w_2) \cap F_3|$. Choose $p_2 \in (N(w_1) \cap F_3) - \{p_1\}$ if $|N(w_1) \cap F_3| = 2$, or $p_2 \in (N(w_2) \cap F_3) - \{p_1\}$ if $|N(w_2) \cap F_3| = 2$, or $p_2 \in F_3 - \{p_1\}$ otherwise. By (i)–(iii) and condition (b), $|F_1 \cap F_2| \ge |N(\{p_1, p_2\}) \cap (F_1 \cap F_2)| \ge (r-2) + (r-4) = 2r - 6$. Hence, $|F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \le (2r-2) + (2r-2) - (2r-6) = 2r + 2$.

On the other hand, condition (b) assures $|N(\{p_1, p_2\})| \ge r + (r-2) = 2r - 2$. If w_1 is adjacent to p_2 , then by conditions (a) and (b), $|N(w_1) - (N(\{p_1, p_2\}) \cup F_3)| \ge r - 2$ and $|N(w_2) - (N(w_1) \cup N(\{p_1, p_2\}) \cup F_3)| \ge r - 5$. See the left of Figure 7. Similarly, if w_1 is not adjacent to p_2 (as $N(w_1) \cap F_3 = 1$), then $|N(w_1) - (N(\{p_1, p_2\}) \cup F_3)| \ge r - 3$ and $|N(w_2) - (N(w_1) \cup N(\{p_1, p_2\}) \cup F_3)| \ge r - 4$. See the right of Figure 7. It follows that $|N(w_1) \cup N(w_2) - (N(\{p_1, p_2\}) \cup F_3)| \ge 2r - 7$. Hence, $|F_1 \cup F_2| \ge |N(\{p_1, p_2\})| + |N(w_1) \cup N(w_2) - (N(\{p_1, p_2\}) \cup F_3)| \ge 2r - 3$ as $r \ge 6$, a contradiction to $|F_1 \cup F_2| \le 2r + 2$.



Figure 7: Relation among $N(w_1)$, $N(w_2)$, $N(\{p_1, p_2\})$ and F_3 .

Case 3. $|F_3| = 1$, say $F_3 = \{p\}$.

Since $|F_1\Delta F_2| \ge 2$, we may choose two nodes u and v from $F_1\Delta F_2$ such that $p \in N(u)$. Let $A = \{u, v\}$. Notice that $A \cup (N(A) - F_3) \cup N(F_3) \subseteq F_1 \cup F_2$. By conditions (a) and (b), $|A \cup (N(A) - F_3)| \ge 2r - 1$ and $|N(F_3) - (A \cup N(A))| \ge r - 3$. Then, $|F_1 \cup F_2| \ge (2r - 1) + (r - 3) = 3r - 4$. We assume that $|F_1 \cup F_2| = 3r - 4 + s$, where $s \ge 0$. Hence, $|F_1 \cap F_2| = |F_1| + |F_2| - |F_1 \cup F_2| \le (2r - 2) + (2r - 2) - (3r - 4 + s) = r - s$. Also, $|N(F_3) \cap (F_1 \cap F_2)| \ge r - 2$. By condition (a), $|N(u) \cap (F_1\Delta F_2)| \ge |N(u)| - (|F_1 \cap F_2| - |N(F_3) \cap (F_1 \cap F_2)|) - |F_3| \ge r - ((r - s) - (r - 2)) - 1 = r - 3 + s$. Refer to Figure 8. Let $F_4 = N(u) \cap (F_1\Delta F_2)$ and $\alpha = |(N(F_4) - (A \cup N(A))) \cup N(F_3)|$. By conditions (a) and (b), for each node $x \in F_4$, $|N(x) - (A \cup N(A))| \ge r - 3$. Then, by condition (b), $\alpha \ge (|F_4|(r-3) - |N(F_3) - A|)/2 \ge ((r - 3 + s)(r - 3) - (r - 1))/2 \ge 2 + 3s/2$ as $r \ge 6$. Hence, $|F_1 \cup F_2| \ge |A \cup (N(A) - F_3) \cup N(F_3)| + \alpha \ge (2r - 1) + (r - 3) + (2 + 3s/2) = 3r - 2 + 3s/2$, a contradiction to $|F_1 \cup F_2| = 3r - 4 + s$.



Figure 8: Relation among N(u), $N(F_3) \cap F_1 \cap F_2$ and F_4 for Case 3. The two arrows represent that p and u are adjacent to all nodes in $N(F_3) \cap F_1 \cap F_2$ and F_4 , respectively.

4 Application to multiprocessor systems using regular networks

In this section we apply the four theorems in Section 3 to eight popular multiprocessor systems, while it is also possible to apply them to many other potentially useful ones not shown here. To introduce these systems, we need the following notations. Define $[m] = \{0, 1, \ldots, m-1\}$ and $[m]^n = \{x_{n-1}x_{n-2}\ldots x_0 : x_i \in [m] \text{ for } i \in [n]\}$, where m and n are positive integers. Let $x = x_{n-1}x_{n-2}\ldots x_0 \in [m]^n$ and $y = y_{n-1}y_{n-2}\ldots y_0 \in [m]^n$. The Hamming distance of x and y, denoted by H(x, y), is the number of indices i such that $x_i \neq y_i$.

Example 1 Hypercube Q_n [31]

A hypercube of n dimensions can be expressed by a graph $Q_n = (V, E)$ with $V = [2]^n$ and $E = \{(x, y) : H(x, y) = 1\}.$

Example 2 Enhanced hypercube $EQ_{n,s}$ [34]

An enhanced hypercube is just a hypercube augmented with certain extra links. More precisely, an (n, s)-enhanced hypercube can be expressed by a graph $EQ_{n,s} = (V, E)$ with $V = [2]^n$ and $E = \{(x, y) : H(x, y) = 1 \text{ or } y = x_{n-1}x_{n-2}\dots x_{s+1}\bar{x}_s\bar{x}_{s-1}\dots \bar{x}_0 \text{ for some}$ $0 \le s \le n-1\}$, where $\bar{x}_i = 1 - x_i$ for $0 \le i \le s$.

Example 3 Twisted cube TQ_n [15]

Assume that n is odd. Define $P_j(x) = (x_j + x_{j-1} + \ldots + x_0) \mod 2$, where $0 \le j \le n-1$. A twisted cube of n dimensions can be expressed by a graph $TQ_n = (V, E)$ with $V = [2]^n$ and E consisting of all (x, y)'s that satisfy the following two conditions for some $0 \le k \le (n-1)/2$:

- (1) $x_{2k}x_{2k-1} = \bar{y}_{2k}y_{2k-1}$ or $(x_{2k}x_{2k-1} = y_{2k}\bar{y}_{2k-1} \text{ and } P_{2k-2}(x) = 1)$ or $(x_{2k}x_{2k-1} = \bar{y}_{2k}\bar{y}_{2k-1} = \bar{y}_{2k}\bar{y}_{2k-1})$ and $P_{2k-2}(x) = 0$;
- (2) $x_{2j}x_{2j-1} = y_{2j}y_{2j-1}$ for all $j \neq k$,

where $x_0 x_{-1}$ is regarded as x_0 when k = 0.

Example 4 Möbius cube $MQ_n[12]$

A Möbius cube of n dimensions can be expressed by a graph $MQ_n = (V, E)$ with $V = [2]^n$ and E containing those (x, y)'s with $y = x_{n-1}x_{n-2}...x_{i+2} 0 \bar{x}_i x_{i-1}...x_0$ or $y = x_{n-1}x_{n-2}...x_{i+2} 1 \bar{x}_i \bar{x}_{i-1}...\bar{x}_0$ for some $0 \le i \le n-2$. Besides, E contains $(x, \bar{x}_{n-1}x_{n-2}...x_0)$ or $(x, \bar{x}_{n-1}\bar{x}_{n-2}...\bar{x}_0)$ but not both.

Example 5 Crossed cube CQ_n [14]

A crossed cube of n dimensions can be expressed by a graph $CQ_n = (V, E)$ with $V = [2]^n$ and E consisting of all (x, y)'s that satisfy the following conditions for some $1 \le m \le n$:

- (1) $x_{n-1}x_{n-2}\dots x_m x_{m-1} = y_{n-1}y_{n-2}\dots y_m \bar{y}_{m-1};$
- (2) $x_{m-2} = y_{m-2}$ if *m* is even;
- (3) $(x_{2i+1}x_{2i}, y_{2i+1}y_{2i}) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$ for $0 \le i \le \lfloor (m-1)/2 \rfloor 1$.

Example 6 Cube-connected cycles CCC_n [29]

Cube-connected cycles can be obtained by replacing each node of a hypercube with a cycle. More precisely, *cube-connected cycles* of n dimensions can be expressed by a graph $CCC_n = (V, E)$ with $V = \{[x, i] : x \in [2]^n \text{ and } i \in [n]\}$ and $E = \{([x, i], [x, j]) :$ $x \in [2]^n, i, j \in [n]$ and $j \equiv (i \pm 1) \mod n\} \cup \{([x, i], [y, i]) : x, y \in [2]^n, i \in [n] \text{ and}$ $y = x_{n-1}x_{n-2} \ldots x_{i+1}\bar{x}_i x_{i-1} \ldots x_0\}.$

Example 7 Torus $T_n(m)$ [6]

An *m*-sided torus of *n* dimensions can be expressed by a graph $T_n(m) = (V, E)$ with $V = [m]^n$ and $E = \{(x, y) : y_i \equiv (x_i \pm 1) \mod m \text{ for some } i \in [n] \text{ and } x_j = y_j \text{ for all } j \neq i \}.$

Example 8 Star graph S_n [1]

A star graph of n dimensions can be expressed by a graph $S_n = (V, E)$ with V being the set of all permutations of $\{1, 2, ..., n\}$, and E consisting of all (u, v)'s such that $u = u_1u_2...u_k...u_n$ and $v = u_ku_2...u_{k-1}u_1u_{k+1}...u_n$ (i.e., swap u_1 and u_k) for some $2 \le k \le n$. The diagnosabilities of these multiprocessor systems can be determined by the aid of Theorems 3, 4, 6 and 7. We first have to check if they satisfy the conditions in these theorems. As the checking is easy, we only summarize the results in Table I. Consequently, we have their diagnosabilities, as shown in Table II.

system	r-regular	triangle-free	$G_{r+1,r+1}$	G_8	G_5	$N(u) \neq N(v)$	$ N(u) \cap N(v) \le 2$
Q_n	r = n	yes	≇	≇	≇	yes if $n \ge 3$	yes if $n \ge 2$
$EQ_{n,s}$	r = n + 1	yes if $s \ge 2$	≇	≇	≇	yes if $n \ge 3$	yes if $n \ge 2, s \ne 2$
TQ_n	r = n	yes	≇	$\not\cong \text{if } n\neq 3$	≇	yes if $n \ge 3$	yes if $n \ge 2$
CQ_n	r = n	yes	≇	$\not\cong$ if $n \neq 3$	≇	yes if $n \ge 3$	yes if $n \ge 2$
MQ_n	r = n	yes	≇	$\not\cong \text{if } n\neq 3$	≇	yes if $n \ge 3$	yes if $n \ge 2$
CCC_n	$r = 3$ if $n \ge 3$	yes if $n \neq 3$	¥	¥	≇	yes	yes
$T_n(m)$	r = 2n	yes if $m \neq 3$	≇	≇	≇	yes if $n \ge 3$	yes if $n \ge 2$
S_n	r = n - 1	yes	¥	¥	≇	yes	yes

Table I: Properties of multiprocessor systems.

≇: not isomorphic.

 $N(u) \neq N(v)$: $N(u) \neq N(v)$ for any two distinct nodes u and v in V.

 $|N(u) \cap N(v)| \le 2$: $|N(u) \cap N(v)| \le 2$ for any two distinct nodes u and v in V.

Table II: Diagnosabilities of multiprocessor systems.

system		PMC	MM*		
	precise	pessimistic	precise	pessimistic	
Q_n	n [3]	2n - 2/2n - 2 [23]	n [36]	2n - 2/2n - 2	
$EQ_{n,s}$	n+1 [35]	2n/2n [35]	n+1 [36]	2n/2n	
TQ_n	n	2n - 2/2n - 2	n	2n - 2/2n - 2	
CQ_n	n	2n - 2/2n - 2	n [17]	2n - 2/2n - 2	
MQ_n	n [16]	2n - 2/2n - 2 [16]	n	2n - 2/2n - 2	
CCC_n	n+2	2n + 2/2n + 2	n+2	2n + 2/2n + 2	
$T_n(m)$	2n	4n - 2/4n - 2	2n	4n - 2/4n - 2	
S_n	n-1 [22]	2n-4/2n-4 [22]	n-1	2n - 4/2n - 4	

[i]: also obtained in [i]; all others are results of this paper.

5 Conclusion

Fault diagnosis of multiprocessor systems has received much attention since Preparata *et al.* [28] introduced the concepts of one-step diagnosis and sequential diagnosis. The *one-step diagnosis* requires that all faulty nodes are found out by decoding the syndrome, whereas the *sequential diagnosis* consists of several diagnosis and repair phases. In each phase, one or more faulty nodes will be determined and then repaired. The process is iterated until all faulty nodes are repaired.

The one-step diagnosability of a multiprocessor system S was defined to be the maximum number of faulty nodes allowed in S such that the one-step diagnosis of S can be performed. The sequential diagnosability of S was defined similarly. In [30], the problem of computing the sequential diagnosability for a general system was proved co-NP complete. In [24], lower bounds on sequential diagnosabilities of grids and hypercubes were suggested.

In [26], Maheshwari and Hakimi introduced a probabilistic model for fault diagnosis. A *p-probabilistically diagnosable* system requires that any set of faulty processors having a priori probability greater than or equal to p of occurring is uniquely diagnosable. In [33], the problem of determining whether a general system is *p*-probabilistically diagnosable or not was proved co-NP complete. A method of achieving an optimal diagnosis with maximum probability was presented in [8]. In [7], a probabilistic diagnosis algorithm was proposed whose probability of correct diagnosis could approach one if a slightly greater than linear number of tests were performed.

Another probabilistic diagnosis algorithm was proposed and evaluated in [10], on the basis of the concept that an aggregate of maximum cardinality is fault-free with probability approaching one if the cardinality of the actual fault set is smaller than the syndrome-dependent diagnosability. The *syndrome-dependent diagnosability* of a multiprocessor system is determined by evaluating the cardinality of the smallest consistent fault set that contains an aggregate of maximum cardinality. Lower bounds on syndrome-dependent diagnosabilities of toroidal grids and hypercubes were derived in [9].

In this paper, we have successfully computed one-step diagnosabilities of eight regular multiprocessor systems for two diagnosis models (i.e., the PMC and comparison models) and two diagnosis strategies (i.e., the precise and pessimistic diagnosis strategies). Our results were obtained as a consequence of four sufficient conditions. Compared with most of previous works which computed diagnosabilities only for individual systems, the four sufficient conditions can derive diagnosabilities for a class of regular systems. Our further research interests include computing sequential diagnosabilities and syndrome-dependent diagnosabilities of various systems for different diagnosis models and diagnosis strategies.

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