Inversion Symmetry of the Euclidean Group: Theory and Application in Robot Kinematics

Yuanqing Wu, Harald Löwe, Zexiang Li

***Abstract*—Just as a three dimensional (3-D) Euclidean space can be inverted through any of its points, the special Euclidean group $SE(3)$ admits an inversion symmetry through any of its group elements. In this paper, we show that the inversion symmetry of $SE(3)$ can be systematically exploited to study the kinematics of a variety of kinesiological and mechanical systems, and therefore has many potential applications in robot kinematics. The motion sets of these systems are inversion invariant (or symmetric) submanifolds of $SE(3)$ that resemble a 2-D plane in the 3-D Euclidean space. Symmetric submanifolds, unlike Lie subgroups of $SE(3)$, inherit unique geometric properties from inversion symmetry. They can be generated by kinematic chains with symmetric joint twists and joint variables. The main contribution of this paper is: (i) to give a complete classification of symmetric submanifolds of $SE(3)$; (ii) to investigate their geometric properties for robot applications; and (iii) to develop a generic method for synthesizing their kinematic chains.**

I. Introduction

The special Euclidean group $SE(3)$ refers to the *Lie group* of all proper rigid displacements of 3-D Euclidean space. It admits both a 6-D manifold structure and a compatible group structure under composition and inverse of displacements. Theory and application of Lie groups from a kinematics viewpoint is initiated by the work of Hervé ([2]) and Brockett ([3]), and is successfully applied to various aspects of robotics (kinematics [4], dynamics [5], control [6], localization [7], bio-inspired robotics [8], etc).

(Lie) subgroups of $SE(3)$ are subsets closed under group multiplication and inverse. A complete classification of ten classes of Lie subgroups of $SE(3)$ can be found in [2,9]. These subgroups can often be used to represent task spaces of mechanisms and robots with less than six DoFs: 1-DoF lower pairs or *primitive joints* such as helical, revolute or prismatic joint generate one-parameter subgroups of $SE(3)$; planar robots have a configuration space of $SE(2)$, the 3-D planar Euclidean group; orientation space of satellites and UAVs are represented by the 3-D special orthogonal group $SO(3)$; pick-and-place robots generate the 4-D Schönflies group.

The *Lie algebra* $se(3)$ of $SE(3)$ also plays an important role in robot kinematics. As infinitesimal transformations of $SE(3)$, elements of $se(3)$ (called *twists*) give rise to rigid displacements (elements of $SE(3)$) via the exponential map ([4]). $se(3)$ is often identified with the tangent space $T_I SE(3)$ at the identity $I \in SE(3)$, or *identity tangent space* for short ([10,11]). The exponential map restricted to a 1-D subspace of $se(3)$ generates a one-parameter subgroup of $SE(3)$. There is a one-to-one correspondence between Lie subalgebras (i.e., subspaces that are closed under the *Lie bracket*) of $se(3)$ and connected Lie subgroups of $SE(3)$ (see for example [10]-Ch IV, Theorem(8.7)). Besides, a twist also corresponds to the geometric notion of a screw; a subspace of $se(3)$ corresponds to a system of screws, or a *screw system* ([12]–[15]).

Recent advances in type synthesis of parallel robots ([16]–[24]), in particular, can be attributed to successful exploitation of the algebraic and geometric properties of $se(3)$. On the one hand, given a set of joint twists as a basis of $se(3)$, the corresponding kinematic chain or serial robot has the *motion pattern* of $SE(3)$ by the product of exponentials (POE) formula ([3,4], see also *canonical coordinates of the second kind* in [11]). The POE formula directly generalizes to Lie subgroups of $SE(3)$, such as in the Euler angle parametrization for $SO(3)$. These parameterizations are *natural* in the sense that a change of basis leaves motion pattern of the generated task space unaltered. In screw theory, the naturality of POE formula for $SE(3)$ and its Lie subgroups is also referred to as *full cycle mobility* ([13,23]). On the other hand, synthesis of parallel robots involves intersection of screw systems, or dually, sum of *reciprocal screw systems* ([12,13,21,23,24]).

Not all motion patterns can be modelled by Lie subgroups of $SE(3)$. The motion patterns of two DoF robot wrists or orientation devices ([25]–[27]), three to five DoF haptic devices ([28]), and five-axis machines ([29,30]) for example, can only be modelled by submanifolds of $SE(3)$. General submanifolds of $SE(3)$ lack group structure and Lie algebraic properties, and in general defy a systematic classification. The tangent spaces evolve in an unpredictable way, making it difficult to infer full cycle mobility from the screw system at any particular configuration ([21,23,24]). Notable exceptions include Hervé's dependent *product of* two Lie *subgroups* (POS, [31]), which subsequently found great success in type synthesis of novel

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Yuanqing Wu (corresponding author) and Zexiang Li are with Electronic and Computer Engineering, Hong Kong University of Science and Technology, Hong Kong, P.R. China (e-mail: troy.woo@gmail.com, eezxli@ust.hk). Dr.Wu is currently at the Interdepartmental Center for Health Sciences and Technologies of the University of Bologna, financially supported by the 2007-2013 ERDF Emilia-Romagna Regional Operational Programme.

Harald Löwe is with Institut Computational Mathematics, Technical University of Brauschweig, Deutschland (e-mail: h.loewe@tu-bs.de).

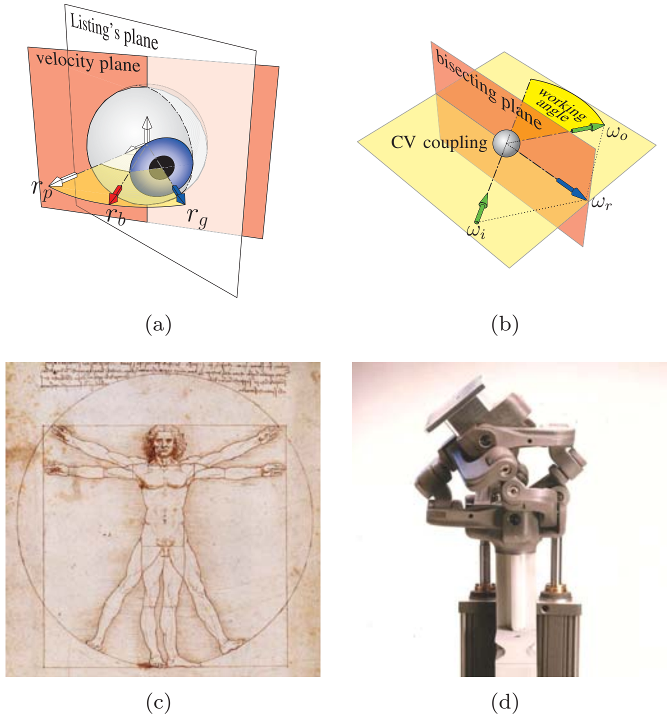

Fig. 1. (a): Listing's law of eye saccadic movement: r_p, r_g r_b denote the primary direction (perpendicular to the Listing's plane), the gaze direction and their angle bisector (perpendicular to the velocity plane); (b): principle of constant-velocity shaft coupling: ω_i, ω_o and $\omega_r = \omega_o - \omega_i$ are the input, output and CV coupling velocities respectively; (c): Leonardo Da Vinci's "Proportion of man"; (d): Mark Rosheim's Omni-wrist III (Courtesy of Mark Rosheim)

parallel robots ([24,32]). Meng et al. proved that POSs are well defined and classifiable submanifolds, and can be naturally represented by the POE formula too (see *Category II submanifolds* in [24]-Table II). Therefore POSs can be generated by kinematic chains of primitive joints. Carricato et al. [33,34] showed that the tangent spaces of a POS are all mutually congruent, thus defining what is called a *persistent screw system*. When a persistent screw system exists, the corresponding submanifold may be generated by the envelop of a tangent space smoothly moving in $SE(3)$ like a rigid body ([33]–[37]).

Although the POE formula unifies the study of Lie subgroup and POS motion patterns, it may fail to model the motion patterns of the following systems. First, consider human's eye saccadic movement. Donders (1848) noticed that human eyes only have 2 DoFs because its orientation is uniquely determined by the line of sight ([38]). This 2-DoF motion pattern, however, is not the same as that of the 2-axis Hooke's joint, which is a POS of two one-parameter subgroups. The instantaneous velocity of the latter violates the so called Listing's law [1] ([38], see Fig.1(a)). Similarly, consider a constant-velocity (CV) shaft coupling that allows a drive shaft to transmit revolute motion, through a variable angle, at a constant rotational speed ([16], see Fig. 1(b)). Such device has recently found applications in robotic wrists ([25,39,40]), robotic surgical tool ([41]), and hyper-redundant robot arms ([42]). A CV coupling does not have the same motion pattern as a Hooke's joint, since the latter is a well known non-CV joint. Unlike a Hooke's joint, the instantaneous velocity of a CV coupling (for intersecting shafts) always lies in the bisecting plane [2]. As the output shaft represented by ω_o turns away from the input shaft represented by ω_i, the bisecting plane turns in the same direction and half in magnitude. Third, consider the human shoulder complex movement. Rosheim ([43]) observed that the shoulder complex movement cannot be modeled as ball-in-socket motion, for otherwise it would not fit the geometrical proportions as depicted in Da Vinci's *Proportion of man* (see Fig. 1(c)). Instead, he claimed that the omni-wrists ([25]) give a better approximation of shoulder movement ([43]). An example of these wrists is given in Fig. 1(d), which employs a parallel kinematic structure with four identical $\mathcal{U}\cdot\mathcal{U}$ chains ($\mathcal{U}$ stands for universal joint) satisfying a CV coupling arrangement ([16,44]). Therefore, the shoulder complex motion pattern is neither a Lie subgroup nor a POS.

Progresses toward understanding the aforementioned motion patterns are very limited. Hunt [16] developed a general theory for analysis and synthesis of CV shaft couplings using screw theory. He observed that their joint screws obey a mirror or bilateral symmetry about the bisecting plane. Typical generators include $\mathcal{U}\cdot\mathcal{U}$, $\mathcal{R}\cdot\mathcal{S}\cdot\mathcal{R}$ ($\mathcal{R}$ stands for revolute joint and $\mathcal{S}$ for spherical or ball joint) and $\mathcal{R}\cdot\mathcal{PL}\cdot\mathcal{R}$ ($\mathcal{PL}$ denotes planar gliding joint) kinematic chains (see also [45]). But he did not specify the underlying submanifolds. Bonev approached several parallel CV couplings with the so called tilt and torsion angles (or modified Euler angles), a parametrization for $SO(3)$ different from the POE formula ([46]). He noticed that the torsion angles for these mechanisms are always zero (hence the name zero-torsion mechanisms). Carricato [45] used CV coupling chains connected in parallel to design orientational parallel manipulators with special decoupling properties. The submanifolds depicting their motion patterns have not been fully investigated yet.

Inspired by Hunt ([16]) and Bonev's ([47]) works, we intend to develop in this paper a holistic model for the aforementioned exceptional motion patterns. We shall show that the Listing's law of eye saccadic movement, the mirror symmetric arrangement of joint screws of a CV shaft coupling, and the vanishing of torsional motions can all be linked to a special property called the *inversion symmetry* of the special Euclidean group $SE(3)$. Mathematically, $SE(3)$ is called a *symmetric space* ([10]-Definition (8.1),[48,49]). We shall show that the exceptional motion patterns are inversion invariant (or symmetric) subsets

[1] The Listing's law about human eye movement, also called *the half-angle law*, states that: as the line of sight turns away from the normal of the Listing's plane (identity configuration), the instantaneous velocity plane turns away from the Listing's plane half in magnitude.

[2] The red plane shown in Fig.1(b): (i) it is perpendicular to the plane of input and output velocity ω_i and ω_o (the yellow plane in Fig.1(b)); and (ii) it bisects the complement of the working angle formed by ω_i and ω_o.

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of $SE(3)$, which we call the *symmetric submanifolds*. There is a similarity between Lie subgroups and symmetric submanifolds of $SE(3)$: a symmetric submanifold is generated by the exponential of its identity tangent space (compared with a Lie subgroup being the exponential image of its Lie subalgebra); symmetric submanifolds preserve the inversion symmetry (while Lie subgroups preserve group multiplication and inverse) and retain a natural coordinate known as the *canonical coordinates of the first kind* ([11], compared with the POE formula for Lie subgroups). The only revelation that comes close to our discovery is Selig's attempt to study full cycle mobility using totally geodesic submanifolds of $SE(3)$ (see [50]-Chapter 15.2). Both Lie subgroups and symmetric submanifolds are totally geodesic ([49]).

This paper is organized as follows. In Section II, we give a brief review of Lie group theory which is much needed in the introduction of inversion symmetry on $SE(3)$. In Section III, we introduce the notion of symmetric submanifolds, and investigate their geometric properties through the study of eye saccadic movement and constant-velocity coupling motion; a systematic classification of symmetric submanifolds is presented shortly after. In Section IV, we propose a systematic approach for synthesizing kinematic chains for symmetric submanifolds. Finally, we conclude our work in Section V.

II. Inversion symmetry on $SE(3)$

In this section, we shall first give a brief review of Lie group theory of $SE(3)$. Our presentation is similar to that in [24,30]. Then we shall give an introduction to inversion symmetry of $SE(3)$, which is an adaption from an elementary treatment of symmetric space in [48].

A. Lie group of rigid displacement: $SE(3)$

An element g of the *special Euclidean group* $SE(3)$ represents the displacement of a rigid body with respect to a reference configuration. It is convenient to introduce a reference frame a and attach its copy b (the body frame) to the rigid body. Then g corresponds to the homogeneous transformation matrix from b to a:

$$g = \begin{bmatrix} A & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4},\ A \in SO(3),\ p \in \mathbb{R}^3. \tag{1}$$

with the *proper rotation* A being an element of the special orthogonal group $SO(3)$:

$$SO(3) \triangleq \left\{ A \in \mathbb{R}^{3\times 3} | AA^T = I_{3\times 3},\ \det A = 1 \right\}.$$

Here $I_{3\times 3}$ denotes a three-by-three identity matrix.

Given the Lie group [3] structure of $SE(3)$, define the *left (right) translation* L_g (R_g), $g \in SE(3)$ by:

$$\begin{aligned} L_g &: SE(3) \to SE(3),\ h \mapsto gh, \\ (R_g &: SE(3) \to SE(3),\ h \mapsto hg.) \end{aligned} \tag{2}$$

[3] A Lie group is a differentiable manifold with a compatible group structure such that the group multiplication and inverse are differentiable mappings [10].

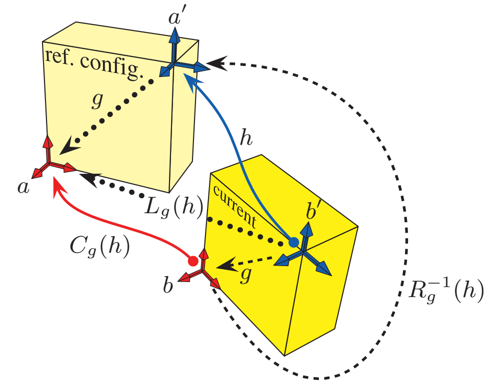


Fig. 2. Rigid body displacement and change of coordinate frames (a_i's: reference frame; b_i's: body frame; dashed lines: change of body frame; dotted lines: change of reference frame).

and also the *conjugation transformation* $C_g \triangleq L_g \circ R_{g^{-1}}$:

$$C_g : SE(3) \to SE(3),\ h \mapsto ghg^{-1}. \tag{3}$$

These are invertible differentiable maps (or *diffeomorphisms*) and satisfy the following properties:

$$\begin{aligned} &L_g \circ L_{g'} = L_{gg'}, \quad R_g \circ R_{g'} = R_{gg'}, \quad C_g \circ C_{g'} = C_{gg'}, \\ &(L_g)^{-1} = L_{g^{-1}}, \quad (R_g)^{-1} = R_{g^{-1}}, \quad (C_g)^{-1} = C_{g^{-1}}. \end{aligned}$$

where $g, g' \in SE(3)$. Physically speaking, a left (right) translation corresponds to a change of reference (body) frame. Changing the reference frame by $g \in SE(3)$ transforms a rigid motion h to its conjugation $C_g(h)$ (see Fig.2).

A subset H of $SE(3)$ is called a *(Lie) subgroup* if it is closed under group multiplication and inverse:

$$\forall g, h \in H \subset SE(3) \Rightarrow gh^{-1} \in H. \tag{4}$$

$SO(3)$ and the additive group $\mathbb{R}^3$ are Lie subgroups of $SE(3)$ under the homogeneous representation (1). The diffeomorphic image $C_g(H)$ of H retains group structure and is called a *conjugate subgroup*. Two subgroups are said to be equivalent if one is the conjugate of the other:

$$H \sim H' \text{ iff } H' = C_g(H),\ g \in SE(3).$$

Reference [24] gives a complete classification of Lie subgroups of $SE(3)$ up to conjugacy classes.

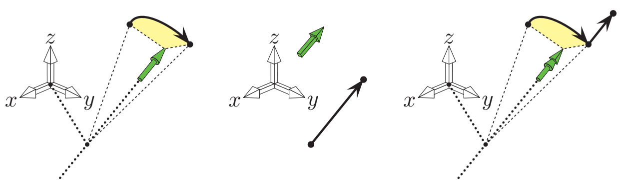
B. Differential analysis on $SE(3)$

The tangent space $T_g SE(3)$ of $SE(3)$ at a point $g \in SE(3)$ is defined as the space of all tangent vectors $\dot{g}(0)$ with $g(t) \in SE(3), t \in \mathbb{R}$ and $g(0) = g$. Define the *Lie algebra* $se(3)$ to be:

$$se(3) \triangleq T_I SE(3) = \left\{ \hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \middle| \omega,\ v \in \mathbb{R}^3 \right\} \tag{5}$$

where $\hat{\omega}$ is a skew symmetric matrix such that $\hat{\omega} u = \omega \times u, \forall u \in \mathbb{R}^3$. An element $\hat{\xi}$ of $se(3)$ is called a *twist*. It is identified with an element ξ in $\mathbb{R}^6$ via:

$$\wedge : \xi = \begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^6 \mapsto \hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in se(3).$$



(a) $\rho = 0$ (b) $\rho = \infty$ (c) $\rho \neq 0$

Fig. 3. Screw coordinate of a twist $\hat{\xi}$ representing rigid body motion, with solid curve indicating the motion of a point on the rigid body (graphical notation inspired by [23], courtesy of Dr.Xianwen Kong).

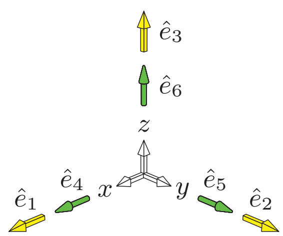


Fig. 4. Canonical basis of $se(3)$: $\hat{e}_1, \hat{e}_2, \hat{e}_3$: unit translation velocity along x, y and z axis; $\hat{e}_4, \hat{e}_5, \hat{e}_6$: unit rotation velocity about x, y and z axis.

ξ is referred to as the *screw coordinate* of $\hat{\xi}$. Therefore, $\hat{\xi}$ defines a screw passing through the base point $(\omega \times v)/\|\omega\|^2$, with unit direction $\omega/\|\omega\|$ and pitch $\rho \triangleq (\omega^T \cdot v)/\|\omega\|^2$. When $\omega = 0$, the pitch is said to be infinite (see Fig.3).

The isomorphic image of the canonical basis of $\mathbb{R}^6$ defines a basis of $se(3)$, which we denote by $\{\hat{e}_i\}_{i=1}^6$. They correspond to infinitesimal translations along and rotations about the x, y and z axes (see Fig.4).

The dual vector of a twist is called a *wrench* ([4]). Its screw coordinate is given by:

$$\xi^* = \begin{bmatrix} f \\ \tau \end{bmatrix} \triangleq \begin{bmatrix} \omega \\ v \end{bmatrix}.$$

where f is the linear force component and τ is the torque component. We shall denote the wrench space by $se(3)^*$. Given two screws $\xi_i = (v_i^T, \omega_i^T)^T, i = 1, 2$, the *natural pairing* $\langle \xi_1, \xi_2^* \rangle$ of a twist ξ_1 and a wrench ξ_2^* is given by the reciprocal product:

$$\langle \xi_1, \xi_2^* \rangle = \xi_1 \odot \xi_2 \triangleq v_1^T \cdot \omega_2 + v_2^T \cdot \omega_1.$$

Two screws ξ_1, ξ_2 are said to be reciprocal if $\xi_1 \odot \xi_2 = 0$. Physically speaking, the wrench about ξ_2 does no work on the twist about ξ_1.

$se(3)$ is equipped with a *Lie bracket* $[\cdot, \cdot]$, defined by:

$$\begin{aligned} [\cdot, \cdot] : se(3) \times se(3) &\to se(3), \\ (\hat{\xi}_1, \hat{\xi}_2) &\mapsto [\hat{\xi}_1, \hat{\xi}_2] \triangleq \hat{\xi}_1 \hat{\xi}_2 - \hat{\xi}_2 \hat{\xi}_1. \end{aligned} \tag{6}$$

It is bilinear, skew symmetric and satisfies the Jacobi identity ([10]):

$$[[\hat{\xi}_1, \hat{\xi}_2], \hat{\xi}_3] + [[\hat{\xi}_2, \hat{\xi}_3], \hat{\xi}_1] + [[\hat{\xi}_3, \hat{\xi}_1], \hat{\xi}_2] = 0. \tag{7}$$

A vector subspace $\mathfrak{h}$ of $se(3)$ is said to be a *Lie subalgebra* of $se(3)$ if it is closed under the Lie bracket:

$$\forall \hat{\xi}, \hat{\xi}' \in \mathfrak{h} \Rightarrow [\hat{\xi}, \hat{\xi}'] \in \mathfrak{h}. \tag{8}$$

The Lie algebra $\mathfrak{h} \triangleq T_I H$ of a Lie subgroup H of $SE(3)$ is automatically a Lie subalgebra of $se(3)$.

Left (right) translate of $se(3)$ gives the tangent space $T_g SE(3)$ at a generic configuration g:

$$\begin{aligned} T_g SE(3) = L_g(se(3)) &= \left\{ g\hat{\xi} | \hat{\xi} \in se(3) \right\}. \\ (= R_g(se(3)) &= \left\{ \hat{\xi} g | \hat{\xi} \in se(3) \right\}.) \end{aligned}$$

The *(spatial) velocity* $\hat{\xi}$ of a rigid motion $g(t), t \in \mathbb{R}$ is defined as the "pullback" of $\dot{g}(t)$ by $R_{g(t)}^{-1}$:

$$\hat{\xi} = R_{g(t)}^{-1} \dot{g}(t) = \dot{g} g^{-1} \in se(3).$$

We shall also refer to $se(3)$ as the *velocity space* of rigid motions on $SE(3)$.

Given $\hat{\xi} \in se(3)$, we define a left (right) invariant vector field $\hat{\xi}^l$ ($\hat{\xi}^r$) on $SE(3)$ by:

$$\begin{aligned} \hat{\xi}^l(g) &= L_g(\hat{\xi}) = g\hat{\xi}. \\ (\hat{\xi}^r(g) &= R_g(\hat{\xi}) = \hat{\xi} g.) \end{aligned} \tag{9}$$

Its integral curve passing through an initial point g is given by $g e^{t\hat{\xi}}, t \in \mathbb{R}$ ($e^{t\hat{\xi}} g, t \in \mathbb{R}$), with the exponential map $\exp : \hat{\xi} \mapsto e^{\hat{\xi}}$:

$$\exp : \hat{\xi} \mapsto e^{\hat{\xi}} = I + \hat{\xi} + \frac{\hat{\xi}^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{\hat{\xi}^k}{k!}.$$

being a local diffeomorphism from a neighborhood N_0 of $0 \in se(3)$ onto a neighborhood N_I of $I \in SE(3)$. We also denote $L_g \circ \exp$ by $\exp_g$, which maps onto $N_g \triangleq L_g(N_I)$. The integral curve of $\hat{\xi}^l$ ($\hat{\xi}^r$) passing through $I \in SE(3)$ is called a *one-parameter subgroup*, with group multiplication and inverse given by:

$$e^{s\hat{\xi}} e^{t\hat{\xi}} = e^{(s+t)\hat{\xi}}, \quad (e^{t\hat{\xi}})^{-1} = e^{-t\hat{\xi}}.$$

We shall denote the one-parameter subgroup by $\exp(\bar{\hat{\xi}})$, where $\bar{\hat{\xi}}$ denotes the subspace spanned by $\hat{\xi}$ (see Fig.5(a)). The Lie bracket of left and right invariant vector fields [4] admits the following rule ([48]):

$$\begin{cases} [\hat{\xi}^l, \hat{\zeta}^l] = [\hat{\xi}, \hat{\zeta}]^l, \\ [\hat{\xi}^r, \hat{\zeta}^r] = -[\hat{\xi}, \hat{\zeta}]^r, \quad \forall \hat{\xi}, \ \hat{\zeta} \in se(3). \\ [\hat{\xi}^l, \hat{\zeta}^r] = 0. \end{cases} \tag{10}$$

Given an ordered basis $(\hat{\xi}_1, \ldots, \hat{\xi}_6)$ of $se(3)$, the *canonical coordinates of the first kind* ([11]-Page 50):

$$\exp_{g,1} : (\alpha_1, \ldots, \alpha_6) \mapsto g \exp(\sum_{i=1}^{6} \alpha_i \hat{\xi}_i) \in N_g. \tag{11}$$

and the *canonical coordinates of the second kind* ([11]-Page 50):

$$\exp_{g,2} : (\beta_1, \ldots, \beta_6) \mapsto g \prod_{i=1}^{6} e^{\beta_i \hat{\xi}_i} \in N_g. \tag{12}$$

defines two local parametrization for $SE(3)$. In robotics community, (12) is also known as a *product of exponentials*

[4] See [10] for the definition of Lie bracket of vector fields.

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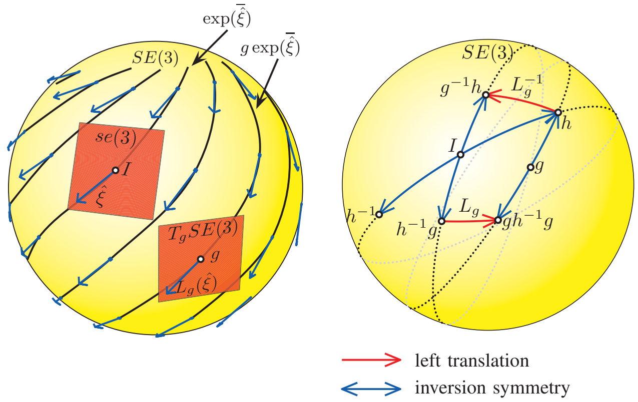


(a) (b)

Fig. 5. (a) Illustration of a left invariant vector field $\hat{\xi}^l(g) = L_g(\hat{\xi}) = g\hat{\xi}$ and its integral curves on $SE(3)$. The integral curve passing through I is the one-parameter subgroup $\exp(\bar{\hat{\xi}})$; (b) Illustration of inversion symmetry on $SE(3)$.

(POE) for the direct kinematics of a serial robot ([4]); the number of exponents can be less than six. For simplicity, we shall denote the *motion pattern* ([23], or motion type in [24]) generated from the POE of $(\hat{\xi}_1, \dots, \hat{\xi}_k)$ by $\{e^{\theta_1\hat{\xi}_1}\cdots e^{\theta_k\hat{\xi}_k}\}$, where the unspecified values θ_i's take an arbitrary real value.

The Baker-Campbell-Hausdorff (BCH) formula ([51]-Page 76) establishes a connection between the two types of canonical coordinates:

$$e^{\hat{\xi}}e^{\hat{\xi}'} = e^{\hat{\xi}+\hat{\xi}'+\frac{1}{2}[\hat{\xi},\hat{\xi}']+\frac{1}{12}([\hat{\xi},[\hat{\xi},\hat{\xi}']]+[\hat{\xi}',[\hat{\xi}',\hat{\xi}]])+\cdots}. \tag{13}$$

Consider the exponential map restricted to a Lie subalgebra $\mathfrak{h}$. $\exp(\mathfrak{h})$ is automatically a Lie subgroup by (4) and (13):

$$\begin{aligned}&\forall e^{\hat{\xi}}, e^{\hat{\xi}'} \in \exp(\mathfrak{h}) \Rightarrow \\ &e^{\hat{\xi}}(e^{\hat{\xi}'})^{-1} = \exp(\underbrace{\hat{\xi} - \hat{\xi}' - \frac{1}{2}[\hat{\xi},\hat{\xi}'] + \cdots}_{\in\mathfrak{h}}) \in \exp(\mathfrak{h}).\end{aligned} \tag{14}$$

The conjugation transformation C_g induces a linear isomorphism called the *Adjoint transformation* on $se(3)$:

$$Ad_g : se(3) \to se(3),\ \hat{\xi} \mapsto g\hat{\xi}g^{-1}.$$

with

$$(Ad_g)^{-1} = Ad_{g^{-1}}, \quad Ad_g \circ Ad_h = Ad_{gh}.$$

The *adjoint map* on $se(3)$ is defined as:

$$ad_{\hat{\xi}} \triangleq \frac{d}{dt}\left(Ad_{e^{t\hat{\xi}}}\right)\Big|_{t=0} : \hat{\xi}' \mapsto ad_{\hat{\xi}}(\hat{\xi}') = [\hat{\xi}, \hat{\xi}'].$$

The three maps are related by the exponential map ([51]-Page 48):

$$C_g(e^{\hat{\xi}}) = e^{Ad_g(\hat{\xi})}, \quad Ad_{e^{\hat{\xi}}} = e^{ad_{\hat{\xi}}}. \tag{15}$$

Physically speaking, Adjoint transformation represents change of reference frame for velocities, and the adjoint map is an infinitesimal Adjoint transformation.

C. $SE(3)$ as a symmetric space

We associate to each $g \in SE(3)$ a transformation S_g called *inversion symmetry* (see Fig.5(b)):

$$S_g : SE(3) \to SE(3),\ h \mapsto gh^{-1}g. \tag{16}$$

It is involutive, i.e. $S_g \circ S_g$ equals the identity map $\mathrm{id}_{SE(3)}$, and reverses the exponential map:

$$S_g(\exp_g(t\hat{\xi})) = g(ge^{t\hat{\xi}})^{-1}g = ge^{-t\hat{\xi}} = \exp_g(-t\hat{\xi}).$$

In particular, S_I coincides with the group inverse. Therefore we have $S_g = L_g \circ S_I \circ L_{g^{-1}}$. $SE(3)$ equipped with the inversion symmetry is called a *symmetric space*. A *quadratic displacement* Q_g with $g \in SE(3)$ is defined as:

$$Q_g \triangleq S_g \circ S_I : SE(3) \to SE(3),\ h \mapsto ghg. \tag{17}$$

A *symmetric submanifold* of $SE(3)$ is a submanifold M which is closed under inversion symmetry:

$$\forall g, h \in M \Rightarrow S_g(h) = gh^{-1}g \in M. \tag{18}$$

Lie subgroups of $SE(3)$ are obvious symmetric submanifolds, since they are closed under group multiplication and inverse and therefore also inversion symmetries.

D. Differential analysis of the symmetric space $SE(3)$

There are two special classes of vector fields pertaining to the inversion symmetry of $SE(3)$: the $(-)$-*derivations* $\mathfrak{D}_-$ and $(+)$-*derivations* $\mathfrak{D}_+$ ([48]-Page 81). Every twist $\hat{\xi} \in se(3)$ generates a $(-)$-*derivation* $\hat{\xi}_-$:

$$\hat{\xi}_-(g) \triangleq \frac{1}{2}(\hat{\xi}^l + \hat{\xi}^r)(g) = \frac{1}{2}(g\hat{\xi} + \hat{\xi}g). \tag{19}$$

It is readily verified that the integral curve of $\hat{\xi}_-$ passing through g is given by $Q_{\exp(\frac{t}{2}\hat{\xi})}(g) = e^{\frac{t}{2}\hat{\xi}}ge^{\frac{t}{2}\hat{\xi}}$ (see (17) for the definition of Q). Therefore $\mathfrak{D}_-$ consists of infinitesimal quadratic displacements.

The Lie bracket of two $(-)$-derivations $\hat{\xi}_-$ and $\hat{\zeta}_-$ defines a $(+)$-*derivation* $[\hat{\xi}, \hat{\zeta}]_+$:

$$\begin{aligned}[\hat{\xi},\hat{\zeta}]_+(g) &\triangleq [\hat{\xi}_-, \hat{\zeta}_-](g) = \frac{1}{2}([\hat{\xi},\hat{\zeta}]^l - [\hat{\xi},\hat{\zeta}]^r)(g). \\ &= \frac{1}{2}(g[\hat{\xi},\hat{\zeta}] - [\hat{\xi},\hat{\zeta}]g).\end{aligned} \tag{20}$$

where the second equality comes from (10). We say the $(+)$-derivation is generated by $[\hat{\xi}, \hat{\zeta}]$. Its integral curve passing through g is given by $C_{\exp(-\frac{t}{2}[\hat{\xi},\hat{\zeta}])}(g) = e^{-\frac{t}{2}[\hat{\xi},\hat{\zeta}]}ge^{\frac{t}{2}[\hat{\xi},\hat{\zeta}]}$. Taking Lie bracket of the derivations yields the following relations:

$$\begin{aligned}&[\mathfrak{D}_-, \mathfrak{D}_-] \subset \mathfrak{D}_+, \\ &[\mathfrak{D}_+, \mathfrak{D}_+] \subset \mathfrak{D}_+, \\ &[\mathfrak{D}_+, \mathfrak{D}_-] \subset \mathfrak{D}_-.\end{aligned} \tag{21}$$

It follows from (21) that: (i) $\mathfrak{D}_+$ is a Lie algebra; (ii) $\mathfrak{D}_-$ is closed under the *triple bracket* $[[\cdot,\cdot],\cdot]$:

$$[[\mathfrak{D}_-, \mathfrak{D}_-], \mathfrak{D}_-] \subset \mathfrak{D}_-.$$

For any $\hat{\xi}_-, \hat{\zeta}_-, \hat{\eta}_- \in \mathfrak{D}_-$, (10) gives:

$$\begin{aligned}[[\hat{\xi}_-, \hat{\zeta}_-], \hat{\eta}_-](g) &= \frac{1}{2}(g[[\hat{\xi}, \hat{\zeta}], \hat{\eta}] + [[\hat{\xi}, \hat{\zeta}], \hat{\eta}]g) \\ &= [[\hat{\xi}, \hat{\zeta}], \hat{\eta}]_-(g) \in \mathfrak{D}_-.\end{aligned} \tag{22}$$

It is clear from (22) that the triple bracket of three $(-)$-derivations is the $(-)$-derivation generated by a triple bracket of their generators in $se(3)$. We say that $se(3)$ with the triple bracket operation is a *Lie triple system* (LTS, see [48]-Page 80).

The similarity between Lie group theory and symmetric space theory we mentioned earlier is manifested by a correspondence of key vocabularies as shown in Table I.

TABLE I
KEY VOCABULARIES OF LIE GROUPS AND SYMMETRIC SPACES

Lie group structure	symmetric space structure
L_g $^{\text{Eq.(2)}}$, R_g $^{\text{Eq.(2)}}$, C_g $^{\text{Eq.(3)}}$	S_g $^{\text{Eq.(16)}}$, Q_g $^{\text{Eq.(17)}}$
Lie subgroup $^{\text{Eq.(4)}}$	symmetric submanifold $^{\text{Eq.(18)}}$
Lie algebra $^{\text{Eq.(5)}}$	Lie triple system $^{\text{Eq.(23)}}$
Lie bracket $[\cdot,\cdot]$ $^{\text{Eq.(6)}}$	triple product $[[\cdot,\cdot],\cdot]$ $^{\text{Eq.(22)}}$
$\hat{\xi}^l$ $^{\text{Eq.(9)}}$, $\hat{\xi}^r$ $^{\text{Eq.(9)}}$	$\hat{\xi}_-$ $^{\text{Eq.(19)}}$, $[\hat{\xi}, \hat{\zeta}]_+$ $^{\text{Eq.(20)}}$

III. SYMMETRIC SUBMANIFOLDS OF $SE(3)$: THE LISTING SPACE AND THE CV SPACE

In this section, we shall investigate properties of a symmetric submanifold of $SE(3)$ with two important examples, the Listing space of eye saccadic movement and the (3-D) CV space of a constant-velocity shaft coupling. Through the study of these submanifolds we introduce several important properties of general symmetric submanifolds. We also give a complete classification of symmetric submanifolds of $SE(3)$ and show that the Listing space and CV space are members of seven conjugacy classes of symmetric submanifolds.

A. The Listing space and the CV space

Recall that a symmetric submanifold is a submanifold of $SE(3)$ that is closed under inversion symmetry (16). Consider a symmetric submanifold M with its identity tangent space $T_I M$ denoted by $\mathfrak{m}$.

Definition 1. *We say that a vector subspace $\mathfrak{m}$ of $se(3)$ is a* ***Lie triple system*** *(LTS) if it is closed under the triple bracket:*

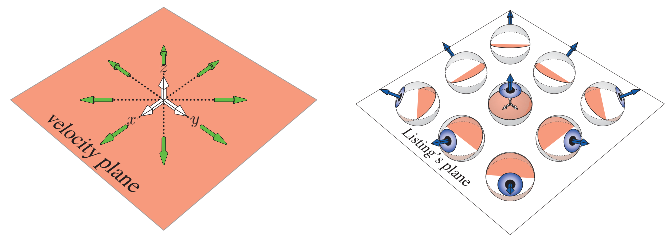
$$[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] \subset \mathfrak{m}. \tag{23}$$

Proposition 1. *If M is a symmetric submanifold of $SE(3)$ containing the identity, then $\mathfrak{m}$ is necessarily a Lie triple system of $se(3)$, i.e. $[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] \subset \mathfrak{m}$. Moreover, $\mathfrak{h}_\mathfrak{m} \triangleq [\mathfrak{m}, \mathfrak{m}]$ is a Lie subalgebra of $se(3)$, which we refer to as the* ***torsion algebra*** *of $\mathfrak{m}$.*

Proof. The proof of the first part is beyond the scope of this paper. The interested reader can refer to [48]-ChIII Thm.1.7. To prove the second part, notice that for any $\hat{u}, \hat{v}, \hat{s}, \hat{t} \in \mathfrak{m}$ and $[\hat{u}, \hat{v}], [\hat{s}, \hat{t}] \in \mathfrak{h}_\mathfrak{m}$, the Jacobi identity (7) gives:

$$[\underbrace{[\hat{u}, \hat{v}]}_{\in \mathfrak{h}_\mathfrak{m}}, \underbrace{[\hat{s}, \hat{t}]}_{\in \mathfrak{h}_\mathfrak{m}}] = -[\underbrace{[\hat{v}, [\hat{s}, \hat{t}]]}_{\in \mathfrak{m}}, \hat{u}] - [\underbrace{[[\hat{s}, \hat{t}], \hat{u}]}_{\in \mathfrak{m}}, \hat{v}] \in \mathfrak{h}_\mathfrak{m}.$$

since $\mathfrak{m}$ is closed under triple bracket. □



(a) (b)

Fig. 6. (a) Screw systems pertaining to the LTS $\mathfrak{m}_{2B}$ of the Listing space L: a pencil of zero-pitch screws; (b) Illustration of Listing's law of eye saccadic movement (with primary direction perpendicular to the Listing's plane): red planes indicate velocity planes; blue arrows indicate gaze direction.

As a restatement of the definition of LTS and its torsion algebra, we have the following corollary.

Corollary 1.1. *Given $\mathfrak{m}$ a LTS and its torsion algebra $\mathfrak{h}_\mathfrak{m} \triangleq [\mathfrak{m}, \mathfrak{m}]$,*

$$[\mathfrak{h}_\mathfrak{m}, \mathfrak{h}_\mathfrak{m}] \subset \mathfrak{h}_\mathfrak{m}, [\mathfrak{h}_\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}.$$

Example 1 (Listing space)**.** Polpitiya et al. [8] investigated the geometry and optimal control of the eye saccadic motion pattern. It is a 2-D submanifold of $SO(3)$ which they refer to as the *Listing space*:

$$L \triangleq \{A \in SO(3) | \exists \omega \in \mathbb{R}^2 \times \{0\} \text{ s.t. } A\omega = \omega\}.$$

It is straightforward to verify that L is the submanifold containing all one-parameter subgroups $\exp(\widehat{\omega})$ with $\omega \in \mathbb{R}^2 \times \{0\}$, namely with ω belonging to a plane pencil of zero-pitch screws (see Fig.6(a)).

We shall show that L is a symmetric submanifold of $SO(3)$. Consider first the identity tangent space of L. By identifying the Lie algebra of $SO(3)$ with $\mathbb{R}^3$ and denoting its canonical basis by (x, y, z), the Lie bracket coincides with vector cross product. The identity tangent space $\mathfrak{m}_{2B} \triangleq T_I L = \{\widehat{x}, \widehat{y}\}$ [5], which denotes the vector subspace spanned by $\hat{x}$ and $\hat{y}$, is not a Lie subalgebra since $[\hat{x}, \hat{y}] = \hat{z} \notin \mathfrak{m}_{2B}$. Instead, $\mathfrak{m}_{2B}$ is a LTS with triple bracket [6]:

$$[[\hat{x}, \hat{y}], \hat{x}] = \hat{y}, \quad [[\hat{x}, \hat{y}], \hat{y}] = -\hat{x}.$$

Next, it can be shown (see Appendix A) that $\hat{\omega}_-(A) = \frac{1}{2}(A\hat{\omega} + \hat{\omega}A), \hat{\omega} \in \mathfrak{m}_{2B}, A \in L$ defines a derivation on L;

[5] The subscript 2 in $\mathfrak{m}_{2B}$ denotes its dimension. The subscript B indicates that $\mathfrak{m}_{2B}$ is the second type 2-D LTS, which will be clear when we give a complete classification of LTSs in Sec.III-B.

[6] It is sufficient to check on the basis elements, since the Lie bracket $[\cdot, \cdot]$ is a bilinear operator.

the integral curves $Q_{\exp(\frac{t}{2}\hat{\omega})}(A)$ are completely contained in L. Therefore, for any $A = e^{\hat{\omega}}, A' = e^{\hat{\omega}'} \in L$:

$$S_A(A') = e^{\hat{\omega}} e^{-\hat{\omega}'} e^{\hat{\omega}} = Q_{\exp(\hat{\omega})}(e^{-\hat{\omega}'}) \in L.$$

and $L = \exp(\mathfrak{m}_{2B})$ is a symmetric submanifold of $SO(3)$ by (18). □

As a 2-D symmetric submanifold of $SO(3)$ (and therefore of $SE(3)$), the Listing space L admits the following properties.

First, its instantaneous velocity space at a general configuration is given by (see proof in Appendix A):

$$R^{-1}_{e^{\hat{\omega}}}(T_{e^{\hat{\omega}}} L) = Ad_{e^{\hat{\omega}/2}}(\mathfrak{m}_{2B}),\ \hat{\omega} \in \mathfrak{m}_{2B}. \tag{24}$$

Recall that Listing's law of eye saccadic movement (see Fig.1(a), Fig.6 and also [52]-Fig. 13) accurately predicts the eye angular velocity when the gaze direction rotates away from the primary direction about an arbitrary axis in the Listing plane and passing through the origin: the velocity plane also rotates away from the Listing's plane about the same axis, but with half magnitude. This characteristic is captured by (24) if we align the reference z-axis with the primary direction and the xy-plane with the Listing's plane. Therefore we shall refer to (24) as the *half-angle property.*

Second, the torsion algebra $\mathfrak{h}_{\mathfrak{m}_{2B}} = [\mathfrak{m}_{2B}, \mathfrak{m}_{2B}] = \{\overline{\hat{z}}\}$ is a Lie subalgebra. It defines a class of conjugations $C_{\exp(-\frac{t}{2}\hat{z})}$ (see the paragraph after (20)) that leaves L invariant:

$$C_{\exp(-\frac{t}{2}\hat{z})}(L) = L,\ \forall t \in \mathbb{R}. \tag{25a}$$

or infinitesimally,

$$Ad_{\exp(-\frac{t}{2}\hat{z})}(\mathfrak{m}_{2B}) = \mathfrak{m}_{2B}, \forall t \in \mathbb{R}. \tag{25b}$$

We shall refer to (25a) and (25b) as the *torsion invariance property.*

The Listing space L also reveals a subtle connection between Lie subgroups and symmetric submanifolds of $SE(3)$.

First, the exponential map takes a Lie subalgebra of $se(3)$ (locally) onto a connected Lie subgroup of $SE(3)$; it also takes the Lie triple system $\mathfrak{m}_{2B}$ onto the symmetric submanifold L. This turns out to be true in general.

Proposition 2. *Given a LTS $\mathfrak{m}$ of $se(3)$, the submanifold $\exp(\mathfrak{m})$ consisting of all one-parameter subgroups $\exp(\hat{\xi}), \hat{\xi} \in \mathfrak{m}$ is a symmetric submanifold of $SE(3)$. Conversely, any symmetric submanifold M of $SE(3)$ containing the identity I is necessarily of the form $\exp(\mathfrak{m})$ with $\mathfrak{m} = T_I M$ a LTS of $se(3)$.*

Proof. The proof is beyond the scope of this paper. The reader can refer to [48] and [49] for details. □

Second, notice that in Example 1 $\mathfrak{m}_{2B} \oplus \mathfrak{h}_{\mathfrak{m}_{2B}} = \{\overline{\hat{x}, \hat{y}, \hat{z}}\}$ is the Lie algebra of $SO(3)$. It is the minimal Lie subalgebra that contains $\mathfrak{m}_{2B}$, which we refer to as the *completion algebra* of $\mathfrak{m}_{2B}$; $SO(3)$ is the corresponding *completion group* ([50]) of L, i.e. the minimal Lie subgroup that contains L. Generalizing from these observations, we have:

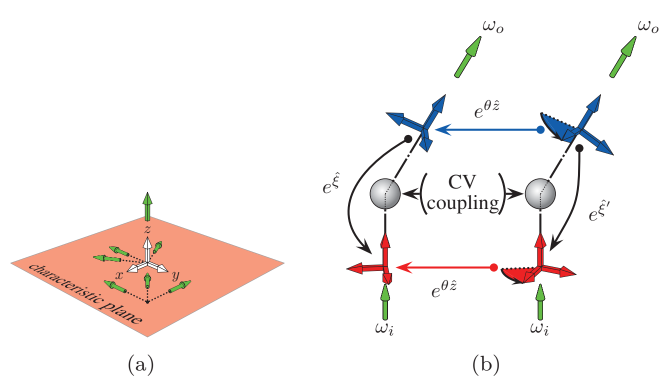


Fig. 7. (a) Screw systems pertaining to the LTS $\mathfrak{m}_{3B} = \{\overline{\hat{e}_3, \hat{e}_4, \hat{e}_5}\}$; (b) constant-velocity transmission by a 3-DoF CV shaft coupling.

Proposition 3. *The following statements are true about a symmetric submanifold M of $SE(3)$ and its LTS $\mathfrak{m}$:*

(1) Its velocity space at a generic point $g \in M$ is given by:

$$R_g^{-1}(T_g M) = Ad_{g^{\frac{1}{2}}}(\mathfrak{m}). \tag{26}$$

(2) $\mathfrak{g}_{\mathfrak{m}} \triangleq \mathfrak{m} + \mathfrak{h}_{\mathfrak{m}}$ is the completion algebra of $\mathfrak{m}$, and $G_M = \exp(\mathfrak{g}_{\mathfrak{m}})$ is the completion group of M;

*(3) $\mathfrak{m}$ is Adjoint invariant by elements of the **torsion group** $H_{\mathfrak{m}} \triangleq \exp(\mathfrak{h}_{\mathfrak{m}})$, and M is conjugation invariant by elements of $H_{\mathfrak{m}}$.*

Proof. (1) The proof is exactly the same as that of the Listing space (see Appendix A); (2) The vector subspace $\mathfrak{g}_{\mathfrak{m}}$ is closed under Lie bracket by Coro.1.1. It is minimal since any Lie subalgebra that contains $\mathfrak{m}$ necessarily contains $\mathfrak{h}_{\mathfrak{m}}$ too; (3) For any $[\hat{u}, \hat{v}] \in \mathfrak{h}_{\mathfrak{m}}$, $\hat{w} \in \mathfrak{m}$:

$$Ad_{e^{[\hat{u},\hat{v}]}}(\hat{w}) = e^{ad_{[\hat{u},\hat{v}]}}(\hat{w}) = \sum_{k=0}^{\infty} \frac{ad^k_{[\hat{u},\hat{v}]}}{k!} \hat{w} \in \mathfrak{m}. \tag{27}$$

from the fact that $ad_{[\hat{u},\hat{v}]}\hat{w} = [[\hat{u}, \hat{v}], \hat{w}] \in \mathfrak{m}$ (since $\mathfrak{m}$ is a LTS); the first equality follows from (15). Similarly:

$$C_{e^{[\hat{u},\hat{v}]}}(e^{\hat{w}}) = e^{Ad_{e^{[\hat{u},\hat{v}]}}(\hat{w})} \in \exp(\mathfrak{m}).$$

by (15) and (27). □

Example 2 (CV space)**.** Consider the screw system $\mathfrak{m}_{3B}$ of all zero-pitch coplanar screws [7], as depicted in Fig.7(a). It is used to model the instantaneous velocity space of a 3-DoF CV coupling for intersecting shafts ([16,45]). By aligning the xy-plane of the reference frame with the characteristic plane, $\mathfrak{m}$ corresponds to the 3-D subspace $\{\overline{\hat{e}_3, \hat{e}_4, \hat{e}_5}\}$ of $se(3)$ (see Fig.7(a)). It is straightforward to

[7] Note that infinite-pitch screws perpendicular to the characteristic plane are also included.

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verify that $\mathfrak{m}_{3B}$ is a LTS:

$$\begin{aligned} &[[\hat{e}_3,\hat{e}_4],\hat{e}_3]=0, && [[\hat{e}_3,\hat{e}_5],\hat{e}_3]=0,\\ &[[\hat{e}_3,\hat{e}_4],\hat{e}_4]=-\hat{e}_3, && [[\hat{e}_3,\hat{e}_5],\hat{e}_4]=0,\\ &[[\hat{e}_3,\hat{e}_4],\hat{e}_5]=0, && [[\hat{e}_3,\hat{e}_5],\hat{e}_5]=-\hat{e}_3,\\ &[[\hat{e}_4,\hat{e}_5],\hat{e}_3]=0, &&\\ &[[\hat{e}_4,\hat{e}_5],\hat{e}_4]=\hat{e}_5, &&\\ &[[\hat{e}_4,\hat{e}_5],\hat{e}_5]=-\hat{e}_4. && \end{aligned}$$

Therefore, $\exp(\mathfrak{m}_{3B})$ is a 3-D symmetric submanifold of $SE(3)$ by Prop.2.

We shall show that $\exp(\mathfrak{m}_{3B})$ is the motion pattern of 3-DoF (with plunging, see [16]) CV shaft couplings for intersecting shafts (which we refer to as the *3-D CV space*):

(a) Without loss of generality, we assume that at the starting configuration, both the input and output shafts coincide with the z axis. It can be verified computationally that $\exp(\mathfrak{m}_{3B})$ maintains intersection of input and output shafts [8]:

$$\underbrace{\begin{bmatrix}0\\z\end{bmatrix}}_{\substack{\text{input}\\\text{axis}}}\odot\underbrace{Ad_{e^{\hat{\xi}}}\left(\begin{bmatrix}0\\z\end{bmatrix}\right)}_{\substack{\text{output}\\\text{axis}}}=0,\ \forall\hat{\xi}\in\mathfrak{m}_{3B}.$$

(b) The condition for CV transmission can be depicted as follows: at a fixed working angle (see Fig.1(b)), when the input shaft sweeps through an angle of θ, the output shaft also sweeps through the same angle (see Fig.7(b)). This in turn requires the claimed CV motion pattern $\exp(\mathfrak{m}_{3B})$ to satisfy:

$$e^{\hat{\xi}'}=C_{e^{-\theta\hat{z}}}(e^{\hat{\xi}})\in\exp(\mathfrak{m}_{3B}),\ \forall\theta\in\mathbb{R},\hat{\xi}\in\mathfrak{m}_{3B}.$$

or, equivalently:

$$\hat{\xi}'=Ad_{e^{-\theta\hat{z}}}(\hat{\xi})\in\mathfrak{m}_{3B},\ \forall\theta\in\mathbb{R},\hat{\xi}\in\mathfrak{m}_{3B}.$$

which is satisfied by the torsion invariance property (25b). It is worth mentioning that the Listing space $L=\exp(\mathfrak{m}_{2B})$ is a 2-D submanifold of $\exp(\mathfrak{m}_{3B})$. It can be shown that L also satisfies (a) and (b), and is in fact the motion pattern for 2-DoF (non-plunging) CV shaft couplings (with intersecting shafts). □

Example 3 (tilt and torsion angles revisited)**.** In [46], Bonev proposed a set of modified Euler angles, called the *tilt and torsion angles* (see Fig.8), for design and control of parallel robots with two to three rotational DoFs. Several important implications of this notation can be extracted with the symmetric space theory. On the one hand, the axis angle ϕ and tilt angle θ define a parametrization for the Listing space $L=\exp(\mathfrak{m}_{2B})$:

$$(\phi,\theta)\mapsto\exp(\theta\cos\phi\hat{x}+\theta\sin\phi\hat{y})\in L. \quad (28)$$

[8]Two zero-pitch screws are concurrent if and only if their reciprocal product $\odot$ is zero ([13]).

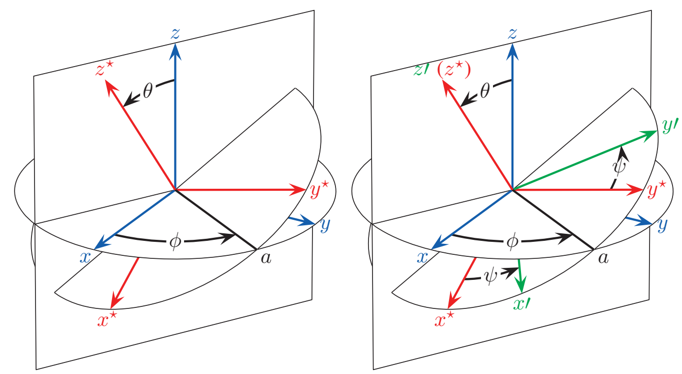


Fig. 8. Illustration of the tilt and torsion angles parametrization: on the left, tilt (about axis a); on the right, torsion (modified from Bonev's original illustration [46], courtesy of Prof.Ilian Bonev).

By substituting (ϕ,θ) with $(\alpha,\beta)=(\theta\cos\phi,\theta\sin\phi)$, (28) defines a canonical coordinate (of the first kind, (11)) for L with basis $(\hat{x},\hat{y})$:

$$\exp_1:(\alpha,\beta)\mapsto\exp(\alpha\hat{x}+\beta\hat{y})\in L.$$

On the other hand, the torsion angle ψ is the exponential coordinate for the torsion group $H_{\mathfrak{m}_{2B}}=\exp(\mathfrak{h}_{\mathfrak{m}_{2B}})=SO(2)$. Physically speaking, the torsion characterizes fluctuation in angular transmission by a Hooke joint, or the spindle rotation of a five-axes machine ([30]). Its existence obstructs constant-velocity transmission, and also leads to redundancy in representing spindle orientation space. The Listing space L is more preferable in these occasions.

Finally, by combining the two sets of parameters, (α,β,ψ) gives a parametrization of $SO(3)$, the completion group of L:

$$(\alpha,\beta,\psi)\mapsto\exp_1(\alpha,\beta)\cdot\exp_1(\psi)\in SO(3).$$

It also gives a relation between the symmetric space L, its torsion group $SO(2)$ and its completion group $SO(3)$. Its further implication to synthesis of motion generators will be clear to the reader in the next section. □

Generalizing from the above example, we have the following proposition.

Proposition 4. *If a LTS $\mathfrak{m}$ has trivial intersection with its torsion algebra $\mathfrak{h}_{\mathfrak{m}}\triangleq[\mathfrak{m},\mathfrak{m}]$:*

$$\mathfrak{m}\cap\mathfrak{h}_{\mathfrak{m}}=\mathfrak{m}\cap[\mathfrak{m},\mathfrak{m}]=\{0\}. \quad (29)$$

then the map:

$$\mathfrak{m}\times\mathfrak{h}_{\mathfrak{m}}\to G_M,(\hat{\xi},\hat{h})\mapsto e^{\hat{\xi}}e^{\hat{h}}. \quad (30)$$

is a local diffeomorphism. Consequently, by combining the canonical coordinates (of the first kind, (11)) of $M=\exp(\mathfrak{m})$ and $H_{\mathfrak{m}}=\exp(\mathfrak{h}_{\mathfrak{m}})$ for some basis $\{\hat{\xi}_i\}_{i=1}^k$ of $\mathfrak{m}$ and $\{\hat{h}_i\}_{i=1}^l$ of $\mathfrak{h}_{\mathfrak{m}}$:

$$\begin{cases}\exp_1:(\alpha_1,\dots,\alpha_k)\mapsto\exp\left(\sum_{i=1}^k\alpha_i\hat{\xi}_i\right)\in M,\\ \exp_1:(\beta_1,\dots,\beta_l)\mapsto\exp\left(\sum_{i=1}^l\beta_i\hat{h}_i\right)\in H_{\mathfrak{m}}.\end{cases}$$

TABLE II
COMPARISON OF PROPERTIES OF SYMMETRIC SUBMANIFOLDS AND LIE SUBGROUPS OF $SE(3)$.

geometric properties	symmetric submanifold M	Lie subgroup G
identity tangent space	LTS $\mathfrak{m}$ $^{\text{Prop.2}}$	Lie subalgebra $\mathfrak{g}$ $^{\text{Eq.(8)}}$
definition	$\exp(\mathfrak{m})$ $^{\text{Prop.1}}$	$\exp(\mathfrak{g})$ $^{\text{Eq.(14)}}$
Lie bracket	torsion algebra $\mathfrak{h}_\mathfrak{m} \triangleq [\mathfrak{m},\mathfrak{m}]$ $^{\text{Prop.1}}$	derived algebra $[\mathfrak{g},\mathfrak{g}]$
velocity space	half-angle property $Ad_{g^{\frac{1}{2}}}(\mathfrak{m})$ $^{\text{Prop.3-(1)}}$	$\mathfrak{g}$
conjugation invariance	by $H_\mathfrak{m}^{\text{Prop.3-(3)}}$	by G
completion algebra	$\mathfrak{g}_\mathfrak{m} \triangleq \mathfrak{m} + \mathfrak{h}_\mathfrak{m}^{\text{Prop.3-(2)}}$	$\mathfrak{g}$
parametrization	$\exp_1 \cdot \exp_1 : \mathfrak{m} \times \mathfrak{h}_\mathfrak{m} \to G_M^{\text{Prop.4}}$	$\exp_1, \exp_2 : \mathfrak{g} \to G$ $^{\text{Eq.(11),Eq.(12)}}$

gives a local coordinate system for its completion group G_M:

$$(\alpha_1,\ldots,\alpha_k;\beta_1,\ldots,\beta_l) \mapsto$$
$$\exp_1(\alpha_1,\ldots,\alpha_k)\cdot\exp_1(\beta_1,\ldots,\beta_l) \in G_M.$$

Proof. The proof is immediate from inverse function theorem (see for example [10]). □

The above proposition shows that for an arbitrary LTS $\mathfrak{m}$ satisfying (29), the completion group $G_M = \exp(\mathfrak{g}_\mathfrak{m})$ of the symmetric submanifold $M = \exp(\mathfrak{m})$ admits a "LTS-torsion" parametrization, which generalizes the tilt-torsion parametrization for $SO(3)$.

In light of Prop.4, we say that a rigid motion g in G_M is torsion-free (with respect to M) if its torsion coordinates $(\beta_1,\ldots,\beta_l)$ vanishes. A torsion-free motion g is contained in the symmetric submanifold $M = \exp(\mathfrak{m})$.

We end this subsection with Table II summarizing the aforementioned properties of a symmetric space, and comparing them to those of a Lie subgroup.

B. Systematic classification of symmetric submanifolds

We have so far introduced two instances of symmetric submanifolds of $SE(3)$, the Listing space (Example 1) and the (3-D) CV space (Example 2). They both admit exponential form $\exp(\mathfrak{m})$, with $\mathfrak{m}$ a LTS of $se(3)$ (see Def. 1). They also have similar geometric properties that distinguish them from the well known Lie subgroups.

According to Prop. 2 and Prop. 3, taking the exponential of a general LTS should produce a general symmetric submanifold having similar geometric properties. For a systematic classification of symmetric submanifolds of $SE(3)$ up to conjugation, it suffices to classify all LTSs of $se(3)$ up to Adjoint transformation (the equivalence is shown by (15)). Starting from a screw system of $se(3)$ (see for example [13]-Ch12 or [50]-Ch8), we can determine if it is a LTS by verifying triple bracket closure (23) for an arbitrarily chosen basis (the complete computation cannot be provided here due to space limitations and will be presented in [53].). A total of seven LTSs and their corresponding symmetric submanifolds are found and listed in Table III. The screw systems corresponding to the LTSs are depicted in Fig.6(a), Fig.7(a) and Fig.9. The *isotropy group* of a symmetric submanifold M is defined to be $\{g \in SE(3) | C_g(M) = M\}$. It contains or is equal to the torsion group H_M.

All the LTSs except $\mathfrak{m}_5 := \{\overline{\hat{e}_1,\hat{e}_2,\hat{e}_3,\hat{e}_4,\hat{e}_5}\}$ satisfy the condition $\mathfrak{m}\cap\mathfrak{h}_\mathfrak{m} = \{0\}$. According to Prop.4 , they admit a parameterization of the corresponding completion group G_M:

$$\mathfrak{m}\times\mathfrak{h}_\mathfrak{m} \to G_M, (\hat{\xi},\hat{h}) \mapsto e^{\hat{\xi}}e^{\hat{h}}.$$

On the other hand,

$$\mathfrak{m}_5 \cap \mathfrak{h}_{\mathfrak{m}_5} = \{\overline{\hat{e}_1,\hat{e}_2,\hat{e}_3}\}.$$

and a similar parametrization of its completion group $SE(3)$ does not exist. Consequently, it is difficult to give a unified treatment of $\mathfrak{m}_5$ with other LTSs when we synthesize kinematic chains of symmetric submanifolds in the next section. A separate treatment for $\mathfrak{m}_5$ can be found at the end of the next section.

Finally, we would like to point out that both the seven classes of symmetric submanifolds and the ten classes of Lie subgroups are *totally geodesic submanifolds* ([48,49]) of $SE(3)$. Such submanifolds are important for the study of full-cycle mobility ([50]-Ch15.2). But we will not delve into such a subject in this paper.

IV. KINEMATIC CHAINS OF SYMMETRIC SUBMANIFOLDS

We have shown that the concept of inversion symmetry and symmetric submanifolds of $SE(3)$ establish a connection between Listing's law of eye saccadic movement (half-angle property, Prop.3(1)), the motion pattern of CV couplings (torsion invariance property, Prop.3(3)), and also orientation representation of parallel robots (LTS-torsion parametrization, Prop.4). We shall now exploit the inversion symmetry to develop a systematic synthesis methodology for kinematic chains of symmetric submanifolds. We restrict ourselves to using the 1-DoF Reuleaux pairs, namely the $\mathcal{R},\mathcal{P}$ and $\mathcal{H}_\rho$ joints, with their corresponding twists shown in Fig.3.

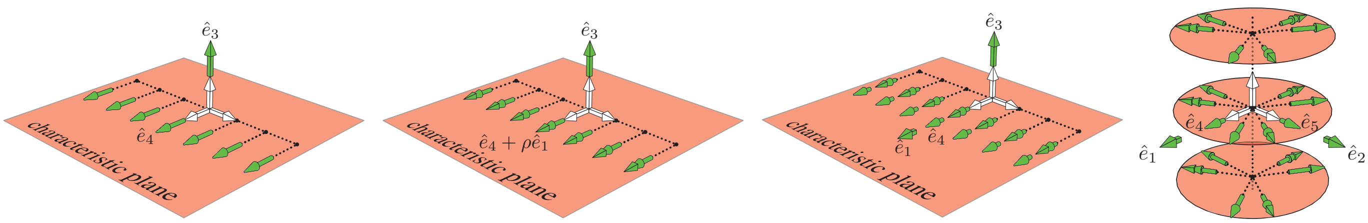
A. Symmetric chains of symmetric submanifolds

Consider a k-D LTS $\mathfrak{m}$ in Table III and a basis $\{\hat{\xi}_1,\ldots,\hat{\xi}_k\} \subset \mathfrak{m}$. In general, the POE $e^{\theta_1\hat{\xi}_1}\cdots e^{\theta_k\hat{\xi}_k}$ generated by the ordered basis $(\hat{\xi}_1,\ldots,\hat{\xi}_k)$ is not torsion-free. On the other hand, a POE with both symmetric joint twists and joint variables:

$$\begin{aligned} &e^{\theta_1\hat{\xi}_1}\cdots e^{\theta_k\hat{\xi}_k}\cdot e^{\theta_k\hat{\xi}_k}\cdots e^{\theta_1\hat{\xi}_1} \\ &= Q_{\exp(\theta_1\hat{\xi}_1)}\circ\cdots\circ Q_{\exp(\theta_{k-1}\hat{\xi}_{k-1})}(e^{2\theta_k\hat{\xi}_k}). \end{aligned} \tag{31}$$

TABLE III
CLASSIFICATION OF SYMMETRIC SUBMANIFOLDS OF $SE(3)$.

dim	LTS $\mathfrak{m}$	screw system [13]	torsion alg. $\mathfrak{h}_\mathfrak{m}$	completion alg. $\mathfrak{g}_\mathfrak{m}$	symmetric submanifold $M = \exp(\mathfrak{m})$	isotropy group
2	$\mathfrak{m}_{2A} \triangleq \{\hat{e}_3, \hat{e}_4\}$	2nd special 2-system	$\{\hat{e}_2\}$	$\{\hat{e}_2, \hat{e}_3, \hat{e}_4\}$	M_{2A}	$\exp(\hat{e}_1, \hat{e}_2)$
	$\mathfrak{m}^\rho_{2A} \triangleq \{\hat{e}_3, \rho\hat{e}_1 + \hat{e}_4\}$			$\{\hat{e}_2, \hat{e}_3, \rho\hat{e}_1 + \hat{e}_4\}$	M^ρ_{2A}	
	$\mathfrak{m}_{2B} \triangleq \{\hat{e}_4, \hat{e}_5\}$	1st special 2-system	$\{\hat{e}_6\}$	$\{\hat{e}_4, \hat{e}_5, \hat{e}_6\}$	M_{2B}: Listing space, 2D CV space	$\exp(\hat{e}_6)$
3	$\mathfrak{m}_{3A} \triangleq \{\hat{e}_1, \hat{e}_3, \hat{e}_4\}$	10th special 3-system	$\{\hat{e}_2\}$	$\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$	M_{3A}	$\exp(\hat{e}_1, \hat{e}_2)$
	$\mathfrak{m}_{3B} \triangleq \{\hat{e}_3, \hat{e}_4, \hat{e}_5\}$	4th special 3-system	$\{\hat{e}_1, \hat{e}_2, \hat{e}_6\}$		M_{3B}: 3-D CV space	$\exp(\hat{e}_1, \hat{e}_2, \hat{e}_6)$
4	$\mathfrak{m}_4 \triangleq \{\hat{e}_1, \hat{e}_2, \hat{e}_4, \hat{e}_5\}$	5th special 4-system	$\{\hat{e}_3, \hat{e}_6\}$	$se(3)$	M_4	$\exp(\hat{e}_3, \hat{e}_6)$
5	$\mathfrak{m}_5 \triangleq \{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4, \hat{e}_5\}$	special 5-system	$\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_6\}$		M_5: five-axes machining	$\exp(\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_6)$



(a) $\mathfrak{m}_{2A}$ (b) $\mathfrak{m}^\rho_{2A}$ (c) $\mathfrak{m}_{3A}$ (d) $\mathfrak{m}_4$

Fig. 9. Screw systems of Lie triple systems.

is necessarily torsion-free, since $\exp(\mathfrak{m})$ is closed under quadratic diplacements. We shall refer to the $2k$-tuple $(\hat{\xi}_1, \ldots, \hat{\xi}_k; \hat{\xi}_k, \ldots, \hat{\xi}_1)$ as a *symmetric chain* (SC). The symmetric arrangement of joint twists in a SC alone does not guarantee its motion being torsion-free: the following POE,

$$e^{\theta_1\hat{\xi}_1} \cdots e^{\theta_k\hat{\xi}_k} \cdot e^{\theta_{k+1}\hat{\xi}_k} \cdots e^{\theta_{2k}\hat{\xi}_1}.$$

is in general not torsion-free unless the joint variables are also symmetric:

$$\theta_i = \theta_{2k-i+1}, i = 1, \ldots, k. \tag{32}$$

In this case, we say the SC goes through a *symmetric motion*. The following proposition gives an alternative proof that the symmetric motion of a SC is necessarily torsion-free.

Proposition 5. *Given a LTS $\mathfrak{m}$ such that $\mathfrak{m} \cap \mathfrak{h}_\mathfrak{m} = \{0\}$, and a set $\{\hat{\xi}_1, \ldots, \hat{\xi}_r\} \subset \mathfrak{m}$ with r some positive integer ($\{\hat{\xi}_1, \ldots, \hat{\xi}_r\}$ not necessarily linearly independent), the POE $e^{\theta_1\hat{\xi}_1} \cdots e^{\theta_r\hat{\xi}_r}$ admits a unique representation:*

$$e^{\theta_1\hat{\xi}_1} \cdots e^{\theta_r\hat{\xi}_r} = e^{\hat{\xi}} e^{\hat{h}}, \ \hat{\xi} \in \mathfrak{m}, \ \hat{h} \in \mathfrak{h}_\mathfrak{m}. \tag{33}$$

Moreover,

$$e^{-\theta_1\hat{\xi}_1} \cdots e^{-\theta_r\hat{\xi}_r} = e^{-\hat{\xi}} e^{\hat{h}} \quad \textit{or} \quad e^{\theta_r\hat{\xi}_r} \cdots e^{\theta_1\hat{\xi}_1} = e^{-\hat{h}} e^{\hat{\xi}}.$$

Proof. See Appendix B. □

We emphasize that the condition $\mathfrak{m} \cap \mathfrak{h}_\mathfrak{m} = \{0\}$ cannot be omitted, because otherwise the exponents $\hat{\xi} \in \mathfrak{m}$ and $\hat{h} \in \mathfrak{h}_\mathfrak{m}$ are not uniquely defined.

Prop.5 states that negating the exponents in a POE of $\mathfrak{m}$ has the same effect of negating the exponent $\hat{\xi}$ in the unique representation $e^{\hat{\xi}} e^{\hat{h}}, \hat{\xi} \in \mathfrak{m}, \hat{h} \in \mathfrak{h}_\mathfrak{m}$. For this reason, we shall refer to $(\hat{\xi}_1, \ldots, \hat{\xi}_k)$ (and $(\hat{\xi}_k, \ldots, \hat{\xi}_1)$) in the SC as the *positive* (and *negative*) *subchain*, which we denote by SC^+ (and SC^-).

Applying Prop.5 to the symmetric motion (31):

$$e^{\theta_1\hat{\xi}_1} \cdots e^{\theta_k\hat{\xi}_k} \cdot e^{\theta_k\hat{\xi}_k} \cdots e^{\theta_1\hat{\xi}_1} = e^{\hat{\xi}} e^{\hat{h}} e^{-\hat{h}} e^{\hat{\xi}} = e^{2\hat{\xi}}.$$

The above equation serves as an explicit proof that the symmetric motion of a SC is torsion-free. Moreover, it shows that the magnitude of its exponent $2\hat{\xi} \in \mathfrak{m}$ is twice that of SC^+ or SC^-.

B. Symmetric twist pairs and general symmetric chains

More generally, consider the Adjoint transformation of a twist $\hat{\xi} \in \mathfrak{m}$ by a pair of POEs with equal and opposite exponents in $\mathfrak{m}$:

$$\begin{cases} \hat{\xi}^+ \triangleq Ad_{e^{\hat{\eta}_1} \cdots e^{\hat{\eta}_r}}(\hat{\xi}), \\ \hat{\xi}^- \triangleq Ad_{e^{-\hat{\eta}_1} \cdots e^{-\hat{\eta}_r}}(\hat{\xi}). \end{cases} \quad \hat{\eta}_1, \ldots, \hat{\eta}_r \in \mathfrak{m}. \tag{34}$$

We shall refer to $(\hat{\xi}^+; \hat{\xi}^-)$ as a *symmetric pair* (SP). The symmetric motion of a SP is necessarily torsion-free:

$$\begin{aligned} e^{\theta\hat{\xi}^+} e^{\theta\hat{\xi}^-} &= (e^{\hat{\eta}_1} \cdots e^{\hat{\eta}_r}) e^{\theta\hat{\xi}} (e^{\hat{\eta}_1} \cdots e^{\hat{\eta}_r})^{-1} \\ &\quad (e^{-\hat{\eta}_1} \cdots e^{-\hat{\eta}_r}) e^{\theta\hat{\xi}} (e^{-\hat{\eta}_1} \cdots e^{-\hat{\eta}_r})^{-1} \\ &= Q_{e^{\hat{\eta}_1}} \circ \cdots \circ Q_{e^{\hat{\eta}_r}} \circ Q_{e^{\theta\hat{\xi}}} \\ &\quad \circ Q_{e^{-\hat{\eta}_r}} \circ \cdots \circ Q_{e^{-\hat{\eta}_2}} (e^{-2\hat{\eta}_1}) \in \exp(\mathfrak{m}). \end{aligned}$$

It is not difficult to see from Prop.5 that the following corollary is true.

Corollary 5.1. *Let $\{(\hat{\xi}^+_i; \hat{\xi}^-_i)\}_{i=1}^k$ be a collection of SPs of a LTS $\mathfrak{m}$ such that $\mathfrak{m} \cap \mathfrak{h}_\mathfrak{m} = 0$, then we have:*

$$\begin{cases} e^{\theta_1\hat{\xi}^+_1} \cdots e^{\theta_k\hat{\xi}^+_k} = e^{\hat{\xi}} e^{\hat{h}}, \\ e^{-\theta_1\hat{\xi}^-_1} \cdots e^{-\theta_k\hat{\xi}^-_k} = e^{-\hat{\xi}} e^{\hat{h}}. \end{cases} \quad \hat{\xi} \in \mathfrak{m}, \ \hat{h} \in \mathfrak{h}_\mathfrak{m}.$$

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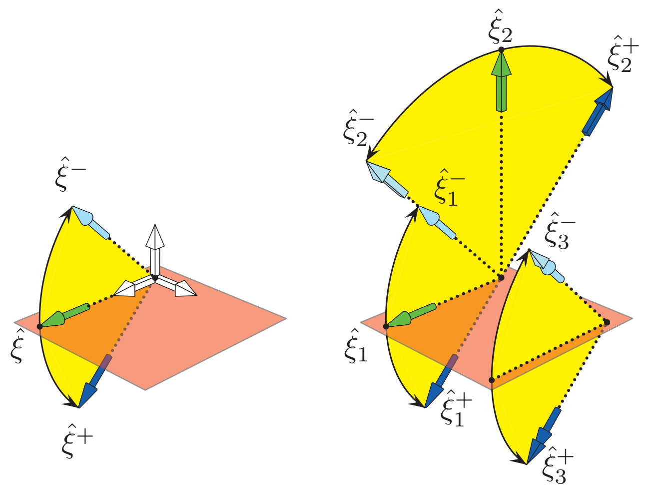


(a) M_{2B} (b) M_{3B}

Fig. 10. Symmetric pairs of CV spaces: (a) $\mathfrak{m}_{2B}$-SP: $(\hat{\xi}^+;\hat{\xi}^-) = (Ad_{e^{\hat{\eta}}}(\hat{\xi}); Ad_{e^{-\hat{\eta}}}(\hat{\xi}))$ with $\rho(\hat{\xi}^+) = \rho(\hat{\xi}^-) = 0$; (b): $\mathfrak{m}_{3B}$-SP: $(\hat{\xi}_1^+;\hat{\xi}_1^-)$ with $\rho(\hat{\xi}_1^+) = \rho(\hat{\xi}_1^-) = 0$, $(\hat{\xi}_2^+;\hat{\xi}_2^-)$ with $\rho(\hat{\xi}_2^+) = \rho(\hat{\xi}_2^-) = \infty$, and $(\hat{\xi}_3^+;\hat{\xi}_3^-)$ with $\rho(\hat{\xi}_3^+) = -\rho(\hat{\xi}_3^-)$.

Consequently,

$$e^{\theta_1\hat{\xi}_1^+}\cdots e^{\theta_k\hat{\xi}_k^+}\cdot e^{\theta_k\hat{\xi}_k^-}\cdots e^{\theta_1\hat{\xi}_1^-} = e^{2\hat{\xi}} \in \exp(\mathfrak{m}).$$

Therefore, a collection of SPs build up a kinematic chain $(\hat{\xi}_1^+,\ldots,\hat{\xi}_k^+;\hat{\xi}_k^-,\ldots,\hat{\xi}_1^-)$, which also generates a torsion-free motion under condition (32). On the one hand, it entails more design freedom than the special case where $\hat{\xi}_i^+ = \hat{\xi}_i^- = \hat{\xi}_i \in \mathfrak{m}, i = 1,\ldots,k$. In fact, it can be shown that the twists $\hat{\xi}_i^\pm$'s can take arbitrary values in the completion algebra $\mathfrak{g}_\mathfrak{m}$. On the other hand, the two twists in each SP $(\hat{\xi}_i^+;\hat{\xi}_i^-)$, although not equal, still satisfy certain symmetry conditions. Therefore, we shall refer to $(\hat{\xi}_1^+,\ldots,\hat{\xi}_k^+;\hat{\xi}_k^-\ldots,\hat{\xi}_1^-)$ as a *general* SC, or simply a SC.

To understand the symmetry involved in a SP, apply Prop.5 to (34) and we have:

$$\begin{cases} \hat{\xi}^+ = Ad_{e^{\hat{\eta}}}\circ Ad_{e^{\hat{h}}}(\hat{\xi}) = Ad_{e^{\hat{\eta}}}(\hat{\xi}'), \\ \hat{\xi}^- = Ad_{e^{-\hat{\eta}}}\circ Ad_{e^{\hat{h}}}(\hat{\xi}) = Ad_{e^{-\hat{\eta}}}(\hat{\xi}'). \end{cases} \tag{35}$$

where $\hat{\eta} \in \mathfrak{m}, \hat{h} \in \mathfrak{h}_\mathfrak{m}$, $e^{\hat{\eta}_1}\cdots e^{\hat{\eta}_r} = e^{\hat{\eta}}e^{\hat{h}}$ is the unique representation (33), and $\hat{\xi}' \triangleq Ad_{e^{\hat{h}}}(\hat{\xi}) \in \mathfrak{m}$ by the torsion invariance of $\mathfrak{m}$ (Prop.3-(3)). In other words,

Corollary 5.2. *Given a LTS $\mathfrak{m}$ such that $\mathfrak{m}\cap\mathfrak{h}_\mathfrak{m} = 0$, a symmetric pair $(\hat{\xi}^+;\hat{\xi}^-)$ as in (34) is generated by a pair of Adjoint transformations $(Ad_{e^{\hat{\eta}}}; Ad_{e^{-\hat{\eta}}})$, $\hat{\eta} \in \mathfrak{m}$ on a twist $\hat{\xi} \in \mathfrak{m}$.*

Moreover, $\{\hat{\xi}_i^+\}$'s determine $\{\hat{\xi}_i^-\}$'s in a unique way.

Corollary 5.3. *Given a LTS $\mathfrak{m}$ such that $\mathfrak{m}\cap\mathfrak{h}_\mathfrak{m} = 0$, and $\hat{\xi}^+ \in \mathfrak{g}_\mathfrak{m}$ in a symmetric pair $(\hat{\xi}^+;\hat{\xi}^-)$ of $\mathfrak{m}$, $\hat{\xi}^-$ is uniquely defined and independent of the choice of the Adjoint transformation $(Ad_{e^{\hat{\eta}}}; Ad_{e^{-\hat{\eta}}})$, $\hat{\eta} \in \mathfrak{m}$ as in Coro.5.2.*

Proof. See Appendix C. □

According to Coro.5.2 and Coro.5.3, we can synthesize a SP as follows. First, specify an arbitrary twist $\hat{\xi}^+$ in the completion algebra $\mathfrak{g}_\mathfrak{m}$; second, find an arbitrary twist $\hat{\eta} \in \mathfrak{m}$ such that $Ad_{e^{-\hat{\eta}}}$ brings $\hat{\xi}^+$ to lie in $\mathfrak{m}$:

$$\hat{\xi} \triangleq Ad_{e^{-\hat{\eta}}}(\hat{\xi}^+) \in \mathfrak{m} \quad \text{or} \quad \hat{\xi}^+ = Ad_{e^{\hat{\eta}}}(\hat{\xi}), \hat{\xi} \in \mathfrak{m}.$$

Then, $\hat{\xi}^- \triangleq Ad_{e^{-\hat{\eta}}}(\hat{\xi}) = Ad_{e^{-2\hat{\eta}}}(\hat{\xi}^+)$ is the unique twist that forms a SP with $\hat{\xi}^+$.

The symmetry of a SP $(\hat{\xi}^+;\hat{\xi}^-)$ also admits a more algebraic explanation.

Corollary 5.4. *Given a LTS $\mathfrak{m}$ such that $\mathfrak{m}\cap\mathfrak{h}_\mathfrak{m} = 0$, a symmetric pair $(\hat{\xi}^+;\hat{\xi}^-)$ of $\mathfrak{m}$ admits the following decomposition over the two subspaces $\mathfrak{m}$ and $\mathfrak{h}_\mathfrak{m}$ of the completion algebra $\mathfrak{g}_\mathfrak{m} = \mathfrak{m}\oplus\mathfrak{h}_\mathfrak{m}$:*

$$\begin{cases} \hat{\xi}^+ = \hat{\xi} + \hat{h}, \\ \hat{\xi}^- = \hat{\xi} - \hat{h} \end{cases}, \quad \hat{\xi} \in \mathfrak{m}, \hat{h} \in \mathfrak{h}_\mathfrak{m}. \tag{36}$$

Proof. The result is clear from the proof of Coro.5.3. □

Finally, the following corollary prescribes the condition on the choice of twists $\{\hat{\xi}_i^\pm\}_{i=1}^k$ such that the corresponding SC generates $M = \exp(\mathfrak{m})$ under condition (32).

Corollary 5.5. *Given a k-D LTS $\mathfrak{m}$ such that $\mathfrak{m}\cap\mathfrak{h}_\mathfrak{m} = 0$, a linearly independent set of twists $(\hat{\xi}_1^+,\ldots,\hat{\xi}_k^+), \hat{\xi}_i^+ \in \mathfrak{g}_\mathfrak{m}, i = 1,\ldots,k$ defines a symmetric chain of $\mathfrak{m}$ (by Coro.5.3 or Coro.5.4) if and only if one of the three following conditions are satisfied:*

1) $\{\overline{\hat{\xi}_1^+,\ldots,\hat{\xi}_k^+}\}\oplus\mathfrak{h}_\mathfrak{m} = \mathfrak{g}_\mathfrak{m}$.
2) $\{\overline{\hat{\xi}_1^-,\ldots,\hat{\xi}_k^-}\}\oplus\mathfrak{h}_\mathfrak{m} = \mathfrak{g}_\mathfrak{m}$.
3) $\{\overline{\hat{\xi}_1,\ldots,\hat{\xi}_k}\} = \mathfrak{m}$.

where $\hat{\xi}_i$ is the component of the unique decomposition of $\hat{\xi}_i^+ = \hat{\xi}_i + \hat{h}_i$ as in (36).

Proof. See Appendix D. □

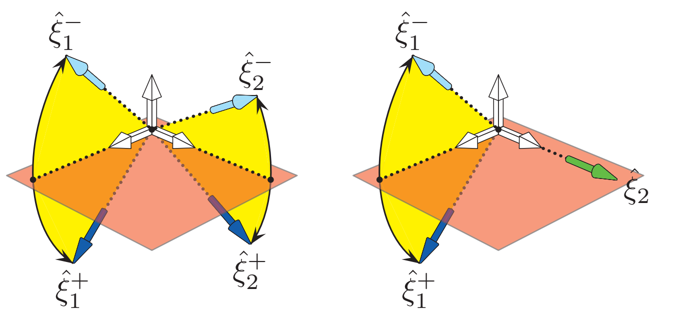
We emphasize that although in the above corollary, both $\{\hat{\xi}_i^+\}_{i=1}^k$ and $\{\hat{\xi}_i^-\}_{i=1}^k$ are linearly independent sets of twists, twists of the SC $(\hat{\xi}_1^+,\ldots,\hat{\xi}_k^+;\hat{\xi}_k^-,\ldots,\hat{\xi}_1^-)$ may become linearly dependent. Moreover, without the symmetric motion condition (32), the POE of the SC may generate an arbitrary subset of the completion group $G_M = \exp(\mathfrak{g}_\mathfrak{m})$.

C. Symmetry type of symmetric chains

We shall use both the geometric condition (35) and the algebraic condition (36) to study the particular symmetry type of each LTS.

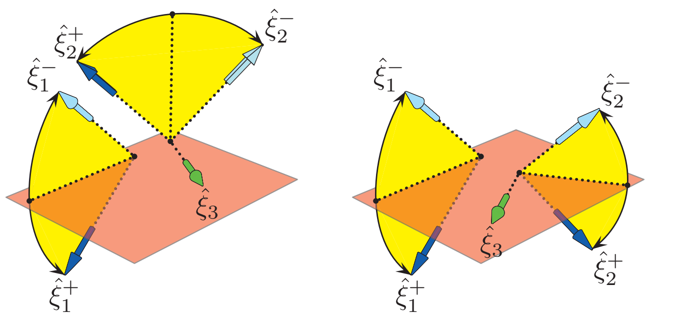
• **$\mathfrak{m}_{2B}$ symmetric chains:**

Consider the 2-*D* LTS $\mathfrak{m}_{2B} = \{\overline{\hat{e}_4,\hat{e}_5}\}$ of the 2-*D* CV space $\exp(\mathfrak{m}_{2B})$ as shown in Fig.6(a). It consists of a pencil of zero-pitch twists in the characteristic plane (Hunt's 1st special 2-system, [13]). A SP $(\hat{\xi}^+;\hat{\xi}^-) = (Ad_{e^{\hat{\eta}}}(\hat{\xi}); Ad_{e^{-\hat{\eta}}}(\hat{\xi})), \hat{\xi}, \hat{\eta} \in \mathfrak{m}_{2B}$ is generated by a pair of rotational displacements of $\hat{\xi}$ about the axis $\eta/\|\eta\|$ with magnitude $\pm\|\eta\|$. Pictorially, $\hat{\xi}^+$ and $\hat{\xi}^-$ are mirror or plane symmetric about the characteristic plane (the bisecting plane) of $\mathfrak{m}_{2B}$ (see Fig.10(a)).



(a) (b)

Fig. 11. SCs of $\mathfrak{m}_{2B}$: (a) even SC; (b) odd SC.



(a) (b)

Fig. 12. Typical SCs of $\mathfrak{m}_{3B}$: (a) $\mathcal{R}\cdot\mathcal{PL}\cdot\mathcal{R}$ chain ($(\hat{\xi}_2^+;\hat{\xi}_3;\hat{\xi}_2^-)$ forms a $\mathcal{PL}$ subchain); (b) $\mathcal{R}\cdot\mathcal{S}\cdot\mathcal{R}$ chain ($(\hat{\xi}_2^+;\hat{\xi}_3;\hat{\xi}_2^-)$ forms a $\mathcal{S}$ subchain).

Since $\mathfrak{g}_{\mathfrak{m}_{2B}} = \{\overline{\hat{e}_4, \hat{e}_5, \hat{e}_6}\}$, members of the $\mathfrak{m}_{2B}$-SC can be arbitrary twists not equal to scalar multiples of $\hat{e}_6$; a $\mathfrak{m}_{2B}$-SC consists of four twists (see Fig.11(a)):

$$\begin{cases} \hat{\xi}_1^+ = \hat{\xi}_1 + \hat{h}_1, \\ \hat{\xi}_1^- = \hat{\xi}_1 - \hat{h}_1. \end{cases} \quad \begin{cases} \hat{\xi}_2^+ = \hat{\xi}_2 + \hat{h}_2, \\ \hat{\xi}_2^- = \hat{\xi}_2 - \hat{h}_2. \end{cases}$$

where $\hat{\xi}_1, \hat{\xi}_2 \in \mathfrak{m}_{2B} = \{\overline{\hat{e}_4, \hat{e}_5}\}$ and $\hat{h}_1, \hat{h}_2 \in \mathfrak{h}_{\mathfrak{m}_{2B}} = \{\overline{\hat{e}_6}\}$, and such that:

$$\{\overline{\hat{\xi}_1, \hat{\xi}_2}\} = \mathfrak{m} = \{\overline{\hat{e}_4, \hat{e}_5}\}.$$

We shall refer to $(\hat{\xi}_1^+, \hat{\xi}_2^+; \hat{\xi}_2^-, \hat{\xi}_1^-)$ as an *even* SC. When $\hat{h}_2 = 0$, $\hat{\xi}_2^+ = \hat{\xi}_2^- = \hat{\xi}_2$. In this case, we can lump the two twists together and have a $\mathfrak{m}_{2B}$-SC with three twists $(\hat{\xi}_1^+; \hat{\xi}_2; \hat{\xi}_1^-)$ (see Fig.11(b)), which we refer to as an *odd* SC. Therefore, a $\mathfrak{m}_{2B}$-SC is a concentric $\mathcal{R}\cdot\mathcal{R}\cdot\mathcal{R}\cdot\mathcal{R}$ or $\mathcal{R}\cdot\mathcal{R}\cdot\mathcal{R}$ chain with bilateral symmetry about the characteristic plane. As we have pointed out earlier, without the symmetric motion condition (32), a $\mathfrak{m}$-SC may generate a general subset of the completion group $G_M = \exp(\mathfrak{g}_\mathfrak{m})$ instead of $\exp(\mathfrak{m})$. Twists of a SC need not be linearly independent (or non-redundant) either. Since $\mathfrak{g}_{\mathfrak{m}_{2B}}$ is the three dimensional spherical Lie algebra $\mathfrak{so}(3)$ of $SO(3)$, the concentric 4-$\mathcal{R}$ chain is necessarily redundant. When $\hat{h}_1 \neq 0$, the 3-$\mathcal{R}$ chain is a non-redundant $\mathfrak{so}(3)$-chain. When $\hat{h}_1 = \hat{h}_2 = 0$, $\hat{\xi}_i^+ = \hat{\xi}_i^- = \hat{\xi}_i \in \mathfrak{m}_{2B}, i = 1, 2$ and we have a singular $\mathfrak{so}(3)$-chain. Both the odd $\mathfrak{m}_{2B}$-SC and the even $\mathfrak{m}_{2B}$-SC can be found in the design of novel CV joints ([54,55]). □

• **$\mathfrak{m}_{3B}$ symmetric chains:**

Consider the 3-D LTS $\mathfrak{m}_{2B} = \{\overline{\hat{e}_3, \hat{e}_4, \hat{e}_5}\}$ of the 3-D CV space $\exp(\mathfrak{m}_{3B})$ as shown in Fig.7(a). It consists of a field of zero-pitch twists in, and infinite-pitch twists perpendicular to, the characterstic plane (Hunt's 4th special 3-system, [13]). Its SP $(\hat{\xi}^+; \hat{\xi}^-) = (Ad_{e^{\hat{\eta}}}(\hat{\xi}); Ad_{e^{-\hat{\eta}}}(\hat{\xi})), \hat{\xi}, \hat{\eta} \in \mathfrak{m}_{3B}$ can be one of the following cases:

1) $\rho(\hat{\eta}) = \rho(\hat{\xi}) = 0$. The SP corresponds to a pair of mirror symmetric revolute joints about the characteristic plane (see $(\hat{\xi}_1^+; \hat{\xi}_1^-)$ in Fig.10(b));
2) $\rho(\hat{\eta}) = \infty, \rho(\hat{\xi}) = 0$. Since the only infinite-pitch members of $\mathfrak{m}_{3B}$ are $\lambda\hat{e}_3, \lambda \in \mathbb{R}$, the SP is a parallel pair of mirror symmetric revolute joints parallel to the characteristic plane. This is a special case of 1) with $\hat{\eta}$ as a zero-pitch twist situated at infinity;
3) $\rho(\hat{\eta}) = 0, \rho(\hat{\xi}) = \infty$. The SP corresponds to a pair of prismatic joints that are symmetric about the z-axis in a plane containing them (see $(\hat{\xi}_2^+; \hat{\xi}_2^-)$ in Fig.10(b)). This is the same as a mirror symmetry about the characteristic plane if we flip the direction of $\hat{\xi}_2^-$. This will make no difference in type synthesis but will reverse the joint variable for $\hat{\xi}_2^-$ in the symmetric motion condition (32).

We can also acquire a pair of mirror symmetric helical joints by composing the algebraic and geometric condition. First, apply the algebraic condition (36):

$$\begin{cases} \hat{\xi}^+ = \hat{e}_4 + \rho\hat{e}_1, \\ \hat{\xi}^- = \hat{e}_4 - \rho\hat{e}_1. \end{cases} \quad \hat{e}_4 \in \mathfrak{m}_{3B}, \rho\hat{e}_1 \in \mathfrak{h}_{\mathfrak{m}_{3B}}.$$

where ρ is the desired pitch. Second, apply the geometric condition (35) to get a pair of mirror symmetric helical joints $(\hat{\xi}_3^+; \hat{\xi}_3^-) = (Ad_{e^{\hat{\eta}}}(\hat{\xi}^+); Ad_{e^{-\hat{\eta}}}(\hat{\xi}^-))$ (see Fig.10(b)). Note the algebraic condition implies that $\rho(\hat{\xi}_3^+) = -\rho(\hat{\xi}_3^-)$. This gives an explicit proof of Hunt's observation that mirror symmetric helical joints in a CV kinematic chain should have equal and opposite pitches ([16]).

A $\mathfrak{m}_{3B}$-SC consists of three SPs $\{\hat{\xi}_i^+; \hat{\xi}_i^-\}_{i=1}^3$, each being one of the three aforementioned cases. According to Coro.5.5, the three twists $\hat{\xi}_1^+, \hat{\xi}_2^+, \hat{\xi}_3^+ \in se(3)$ must be chosen such that:

$$\{\overline{\hat{\xi}_1^+, \hat{\xi}_2^+, \hat{\xi}_3^+}\} \oplus \{\overline{\hat{e}_1, \hat{e}_2, \hat{e}_6}\} = se(3).$$

The enumeration of eligible candidates is studied in our earlier work [30]. The two most commonly seen $\mathfrak{m}_{3B}$-SCs ([16]) are the mirror symmetric $\mathcal{R}\cdot\mathcal{PL}\cdot\mathcal{R}$ chain, which is equivalent to a mirror symmetric $\mathcal{R}\cdot\mathcal{R}\cdot\mathcal{P}\cdot\mathcal{R}\cdot\mathcal{R}$ chain (Fig.12(a)), and the mirror symmetric $\mathcal{R}\cdot\mathcal{S}\cdot\mathcal{R}$ chain, which is equivalent to a mirror symmetric 5-$\mathcal{R}$ chain (Fig.12(b)). When (32) is not enforced, these chains generate 5-D submanifolds of $SE(3)$. □

The above results corroborate Hunt's exhaustive classification of 3-D CV chains in [16]. It also confirms our earlier conclusion that the 3-D CV space is the 3-D symmetric submanifold $M_{3B} = \exp(\mathfrak{m}_{3B})$. Moreover, the mirror symmetry is a manifestation of the inversion symmetry of the underlying symmetric submanifold M_{3B}.

• $\mathfrak{m}_{2A}$ **and** $\mathfrak{m}_{2A}^{\rho}$ **symmetric chains:**

Consider the 2-D LTS $\mathfrak{m}_{2A} = \overline{\{\hat{e}_3, \hat{e}_4\}}$ (or $\mathfrak{m}_{2A}^{\rho} = \overline{\{\hat{e}_3, \hat{e}_4 + \rho\hat{e}_1\}}$) as shown in Fig.9(a) (or Fig.9(b)). It consists of all zero-pitch (or finite non-zero-pitch) twists parallel to the x-axis in the characterstic plane (xy-plane), and also twists of infinite-pitch perpendicular to the xy-plane (Hunt's 2nd special 2-system, [13]). Note that $\mathfrak{m}_{2A}$ is a *Lie triple subsystem* (LT subsystem) of $\mathfrak{m}_{3B}$; its SPs can be synthesized in a similar manner and therefore have the same type of symmetry: as shown in Fig.13(a), $\mathfrak{m}_{2A}$-SPs are mirror symmetric about the characteristic plane. $\mathfrak{m}_{2A}^{\rho}$-SPs admit exactly the same symmetry type, with the zero-pitch SP replaced by a SP of pitch ρ (see $(\hat{\xi}_2^+; \hat{\xi}_2^-)$ in Fig.13(b)). Unlike the case of $\mathfrak{m}_{3B}$, the two finite-pitch twists in $(\hat{\xi}_2^+; \hat{\xi}_2^-)$ in Fig.13(b) have equal but not opposite pitches for the obvious reason that the LTS $\mathfrak{m}_{2A}^{\rho}$ itself admits finite-pitch twists.

Since $\mathfrak{h}_{\mathfrak{m}_{2A}} = \overline{\{\hat{e}_2\}}$ and $\mathfrak{g}_{\mathfrak{m}_{2A}} = \overline{\{\hat{e}_2, \hat{e}_3, \hat{e}_4\}}$ is the Lie algebra of the 3-D planar Euclidean group, a $\mathfrak{m}_{2A}$-SC $(\hat{\xi}_1^+, \hat{\xi}_2^+; \hat{\xi}_2^-, \hat{\xi}_1^-)$ should satisfy:

$$\overline{\{\hat{\xi}_1^+, \hat{\xi}_2^+\}} \oplus \overline{\{\hat{e}_2\}} = \overline{\{\hat{e}_2, \hat{e}_3, \hat{e}_4\}}.$$

A $\mathfrak{m}_{2A}$-SC can be one of the following:

1) A mirror symmetric $\mathcal{R}\cdot\mathcal{R}\cdot\mathcal{R}\cdot\mathcal{R}$ or $\mathcal{R}\cdot\mathcal{R}\cdot\mathcal{R}$ chain with parallel axes;
2) A mirror symmetric $\mathcal{P}\cdot\mathcal{R}\cdot\mathcal{R}\cdot\mathcal{P}$ or $\mathcal{P}\cdot\mathcal{R}\cdot\mathcal{P}$ chain with $\mathcal{R}$ perpendicular to the two $\mathcal{P}$'s;
3) A mirror symmetric $\mathcal{R}\cdot\mathcal{P}\cdot\mathcal{P}\cdot\mathcal{R}$ or $\mathcal{R}\cdot\mathcal{P}\cdot\mathcal{R}$ with parallel $\mathcal{R}$'s both perpendicular to the $\mathcal{P}$'s.

These are all planar motion generators if the symmetric motion condition (32) is not enforced. Synthesis of $\mathfrak{m}_{2A}^{\rho}$ follows exactly the same approach and have exactly the same result with all revolute joints replaced by helical joints with pitch ρ. Therefore $\mathfrak{m}_{2A}^{\rho}$-SCs are planar helical motion generators when (32) is not enforced. □

• $\mathfrak{m}_{3A}$ **symmetric chains:**

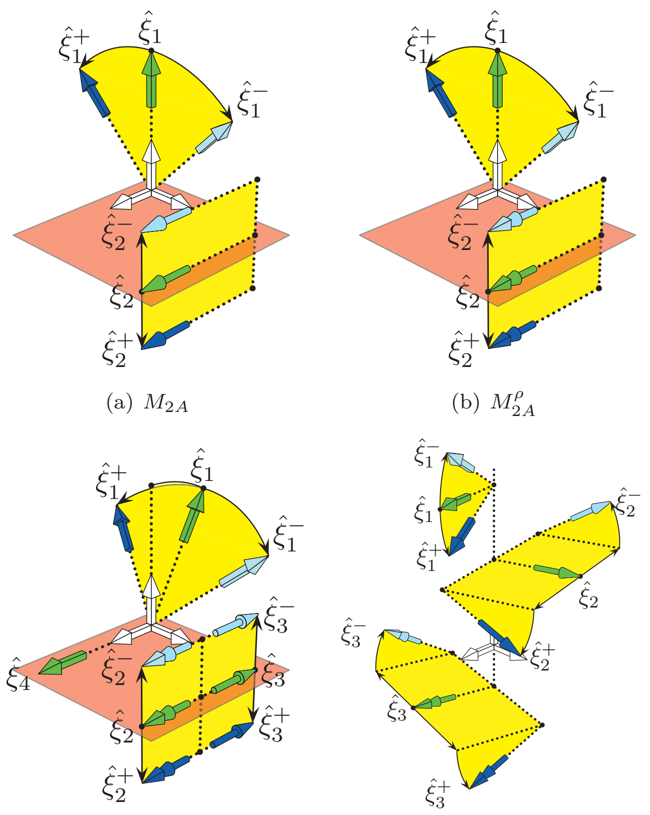
Consider the 3-D LTS $\mathfrak{m}_{3A} = \overline{\{\hat{e}_1, \hat{e}_3, \hat{e}_4\}}$ as shown in Fig.9(c). It consists of twists of all pitches on all lines parallel to the x-axis in the characteristic plane xy, and twists of infinite-pitch perpendicular to the xy-plane (Hunt's 10th special 3-system, [13]). From the fact that $\mathfrak{m}_{2A}^{(\rho)} \subset \mathfrak{m}_{3A}$, we see that any $\mathfrak{m}_{2A}$-SPs and $\mathfrak{m}_{2A}^{\rho}$-SPs are also $\mathfrak{m}_{3A}$-SPs (see $(\hat{\xi}_2^+; \hat{\xi}_2^-)$ and $(\hat{\xi}_3^+; \hat{\xi}_3^-)$ in Fig.13(c)). Besides, since $\mathfrak{h}_{\mathfrak{m}_{3A}} = \overline{\{\hat{e}_2\}}$, $\mathfrak{m}_{3A}$ admits the following SP by the algebraic condition (36):

$$\begin{cases} \hat{\xi}^+ = (\hat{e}_3 + \lambda\hat{e}_1) + \mu\hat{e}_2, \\ \hat{\xi}^- = (\hat{e}_3 + \lambda\hat{e}_1) - \mu\hat{e}_2. \end{cases} \quad \hat{e}_3 + \lambda\hat{e}_1 \in \mathfrak{m}_{3A}, \mu\hat{e}_2 \in \mathfrak{h}_{\mathfrak{m}_{3A}}.$$

for some real constants λ, μ (see $(\hat{\xi}_1^+; \hat{\xi}_1^-)$ in Fig.13(c)). The prismatic SP is no longer mirror symmetric about the characteristic plane xy, but instead becomes mirror symmetric about the xz-plane.

Since $\mathfrak{g}_{\mathfrak{m}_{3A}} = \overline{\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}}$ is the Lie algebra of the 4-D Schönflies group, a $\mathfrak{m}_{3A}$-SC $(\hat{\xi}_1^+, \hat{\xi}_2^+, \hat{\xi}_3^+; \hat{\xi}_3^-, \hat{\xi}_2^-, \hat{\xi}_1^-)$ should satisfy:

$$\overline{\{\hat{\xi}_1^+, \hat{\xi}_2^+, \hat{\xi}_3^+\}} \oplus \overline{\{\hat{e}_2\}} = \overline{\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}}.$$



(a) M_{2A} (b) M_{2A}^{ρ}

(c) M_{3A} (d) M_4

Fig. 13. Symmetric pairs of general symmetric submanifolds.

The enumeration of eligible candidates of $(\hat{\xi}_1^+, \hat{\xi}_2^+, \hat{\xi}_3^+)$ can be found in [30]. Since $\mathfrak{m}_{3A}$-SCs are 5 or 6-DoF chains (in comparison to the dimension of the Schönflies group being 4), they are redundant Schönflies motion generators in the absence of the symmetric motion condition (32). □

• $\mathfrak{m}_4$ **symmetric chains:**

Consider the 4-D LTS $\mathfrak{m}_4 = \overline{\{\hat{e}_1, \hat{e}_2, \hat{e}_4, \hat{e}_5\}}$ as shown in Fig.9(d). It consists of twists of all pitches along the lines of pencils in each plane normal to the z-axis, and that the centers of the pencils all lie on the z-axis (Hunt's 5th special 4-system, [13]). Since $\mathfrak{h}_{\mathfrak{m}_4} = \overline{\{\hat{e}_3, \hat{e}_6\}}$, a typical SP is given by the algebraic condition (36):

$$\begin{cases} \hat{\xi}^+ = (\lambda\hat{e}_4 + \mu\hat{e}_1) + (\sigma\hat{e}_6 + \tau\hat{e}_3) \\ \quad\;\; = (\lambda\hat{e}_4 + \sigma\hat{e}_6) + (\mu\hat{e}_1 + \tau\hat{e}_3), \\ \hat{\xi}^- = (\lambda\hat{e}_4 + \mu\hat{e}_1) - (\sigma\hat{e}_6 + \tau\hat{e}_3) \\ \quad\;\; = (\lambda\hat{e}_4 - \sigma\hat{e}_6) + (\mu\hat{e}_1 - \tau\hat{e}_3). \end{cases}$$
$$(\lambda\hat{e}_4 + \mu\hat{e}_1) \in \mathfrak{m}_4, (\sigma\hat{e}_6 + \tau\hat{e}_3) \in \mathfrak{h}_{\mathfrak{m}_4}.$$

for some arbitrary real constants $\lambda, \mu, \sigma, \tau$. The screw coordinates of $\hat{\xi}^+$ and $\hat{\xi}^-$ can be readily computed as follows:

- The direction vectors of ξ^+, ξ^- are given by:

$$\begin{cases} \omega^+ = \dfrac{1}{(\lambda^2 + \sigma^2)^{1/2}}(\lambda\hat{e}_4 + \sigma\hat{e}_6), \\ \omega^- = \dfrac{1}{(\lambda^2 + \sigma^2)^{1/2}}(\lambda\hat{e}_4 - \sigma\hat{e}_6). \end{cases}$$

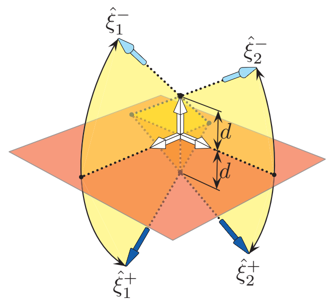


Fig. 14. A $\mathcal{U}\cdot\mathcal{U}$ SC as an incomplete $\mathfrak{m}_{3B}$-SC.

which are mirror symmetric about the xy-plane.

- The base points q^+, q^- of ξ^+, ξ^- are given by:

$$\begin{cases} q^+ = \dfrac{\sigma\mu - \lambda\tau}{\lambda^2+\sigma^2} e_2 \\ q^- = -\dfrac{\sigma\mu - \lambda\tau}{\lambda^2+\sigma^2} e_2. \end{cases}$$

which are mirror symmetric about the z-axis.

- The pitches of ξ^+, ξ^- are equal and is given by:

$$\rho = \frac{\lambda\mu + \sigma\tau}{\lambda^2+\sigma^2}.$$

If we flip the direction of ω^-, $(\hat{\xi}^+;\hat{\xi}^-)$ admits a 2-fold rotational symmetry about the z-axis, with the two twists having equal pitch (see $(\hat{\xi}_2^+;\hat{\xi}_2^-)$ and $(\hat{\xi}_3^+;\hat{\xi}_3^-)$ in Fig.13(d)). When $\lambda = \sigma = 0$, we have an infinite-pitch $\mathfrak{m}_4$-SP which also admits 2-fold rotational symmetry about the z-axis if we flip the direction of ξ^- (see $(\hat{\xi}_1^+;\hat{\xi}_1^-)$ in Fig.13(d)).

Since $\mathfrak{g}_{\mathfrak{m}_4} = se(3)$, a $\mathfrak{m}_4$-SC $(\hat{\xi}_1^+,\dots,\hat{\xi}_4^+;\hat{\xi}_4^-,\dots,\hat{\xi}_1^-)$ should satisfy:

$$\{\overline{\hat{\xi}_1^+,\dots,\hat{\xi}_4^+}\} \oplus \{\overline{\hat{e}_3,\hat{e}_6}\} = se(3).$$

Eligible candidates of $(\hat{\xi}_1^+,\dots,\hat{\xi}_4^+)$ can be found in [30]. Since $\mathfrak{m}_4$-SCs are 7 or 8-DoF chains, they are redundant (and possibly, singular) $SE(3)$ motion generators in the absence of the symmetric motion condition (32). □

Finally, when a SC of a k-D LTS $\mathfrak{m}$ has less than k SPs, its symmetric motion generates a submanifold of the symmetric submanifold $M = \exp(\mathfrak{m})$. Such *incomplete SC*s are also prevalent in practice.

Example 4 ($\mathcal{U}\cdot\mathcal{U}$ SC)**.** A $\mathcal{U}\cdot\mathcal{U}$ SC (as shown in Fig.14) is briefly mentioned in [16] and studied in [45]. Its SPs are given by:

$$\begin{aligned} &\begin{cases} \hat{\xi}_1^+ = Ad_{e^{d\hat{e}_3}} \circ Ad_{e^{\hat{\omega}_1}} \hat{\xi}_1 \triangleq Ad_{e^{d\hat{e}_3}}(\hat{\eta}_1^-), \\ \hat{\xi}_1^- = Ad_{e^{-d\hat{e}_3}} \circ Ad_{e^{-\hat{\omega}_1}} \hat{\xi}_1 \triangleq Ad_{e^{-d\hat{e}_3}}(\hat{\eta}_1^+). \end{cases} \\ &\begin{cases} \hat{\xi}_2^+ = Ad_{e^{d\hat{e}_3}} \circ Ad_{e^{\hat{\omega}_2}} \hat{\xi}_2 \triangleq Ad_{e^{d\hat{e}_3}}(\hat{\eta}_2^-), \\ \hat{\xi}_2^- = Ad_{e^{-d\hat{e}_3}} \circ Ad_{e^{-\hat{\omega}_2}} \hat{\xi}_2 \triangleq Ad_{e^{-d\hat{e}_3}}(\hat{\eta}_2^+). \end{cases} \end{aligned} \tag{37}$$

where $d \in \mathbb{R}$, $\hat{\xi}_1, \hat{\xi}_2, \hat{\omega}_1, \hat{\omega}_2 \in \mathfrak{m}_{2B}$ and $(\hat{\eta}_1^+, \hat{\eta}_2^+; \hat{\eta}_2^-, \hat{\eta}_1^-)$ is a $\mathfrak{m}_{2B}$-SC. The SC generates a 2-D submanifold of M_{3B} under symmetric motion condition (32). It can be shown to also satisfy the CV transmission condition given in Example 2 (see [44] for more details).

When working under the symmetric motion condition (32), the $\mathcal{U}\cdot\mathcal{U}$ SC, in comparison to a $\mathfrak{m}_{3B}$-SC, has not a free but a dependent translational DoF ([44]). Mark Rosheim used this phenomenon to characterize the human shoulder complex movement ([43]). □

D. *Synthesis of* $\mathfrak{m}_5$ *kinematic chains*

We have shown that 5-D LTS $\mathfrak{m}_5 = \{\hat{e}_1,\dots,\hat{e}_5\}$ has a nontrivial intersection with its torsion algebra $\mathfrak{h}_{\mathfrak{m}_5}$. Consequently, the completion group $SE(3)$ of M_5 fails to admit a unique parametrization (31) by $\mathfrak{m}_5 \times \mathfrak{h}_{\mathfrak{m}_5}$. Therefore we cannot directly apply Prop.5 and its corollaries to synthesize SPs and SCs for $\mathfrak{m}_5$. We develop a type synthesis method for $\mathfrak{m}_5$-SCs directly from inversion symmetry (31). The following proposition gives an algebraic condition for synthesis of $\mathfrak{m}_5$-SPs and SCs.

Proposition 6. $\mathfrak{m}_5$ *admits a symmetric chain* $(\hat{\xi}_1^+,\dots,\hat{\xi}_5^+;\hat{\xi}_5^-,\dots,\hat{\xi}_1^-)$ *of the following form:*

$$\begin{cases} \hat{\xi}_i^+ = \hat{\xi}_i + \hat{h}_i, \\ \hat{\xi}_i^- = \hat{\xi}_i - \hat{h}_i. \end{cases} \quad \hat{\xi}_i \in \mathfrak{m}_5, \hat{h}_i \in \mathfrak{h}_{\mathfrak{m}_5}.$$

and such that:

$$\{\overline{\hat{\xi}_1,\dots,\hat{\xi}_5}\} = \mathfrak{m}_5.$$

The symmetric motion of $\mathfrak{m}_5$*-SCs generate* $M_5 = \exp(\mathfrak{m}_5)$.

Proof. See Appendix E. □

A closer look at Prop.6 shows that the SC$^+$ $(\hat{\xi}_1^+,\dots,\hat{\xi}_5^+)$ no longer determines the SC$^-$ $(\hat{\xi}_5^-,\dots,\hat{\xi}_1^-)$ in a unique way. The symmetry type of $\mathfrak{m}_5$-SPs therefore depend not only on the choice of ξ_i^+'s, but also on the choice of $\hat{h}_i$'s. Since all the aforementioned LTSs are LT subsystems of $\mathfrak{m}_5$, a $\mathfrak{m}_5$-SP may have any of the symmetry types of aforementioned LTSs.

V. Conclusion

In this paper, we have introduced inversion symmetry of the special Euclidean group $SE(3)$ and its symmetry invariant submanifolds arising from kinesiology and robot mechanical systems. They share many similarities with Lie subgroups of $SE(3)$, and therefore expand the known portfolio of motion patterns for the analysis and synthesis of many kinesiological joints or robot mechanical generators that defy a Lie group explanation.

The main contribution of our work is as follows. First, we have identified, for the first time, seven classes of symmetric submanifolds (Table III), which all admit the form of exponential of a Lie triple system of the Lie algebra $se(3)$. So far as the authors are aware of, this is also the first time the inversion symmetry of $SE(3)$ is studied. Second, we found that these symmetric submanifolds share both a list of geometric properties and also a universal type synthesis method for their kinematic chains:

- All symmetric submanifolds of $SE(3)$ are generated by the exponential of a corresponding Lie triple system (page 7, Prop.2);

- The tangent spaces of each symmetric submanifold admit the half-angle property (page 7, E.q.(26));
- Both the LTSs $\mathfrak{m}$ and the symmetric submanifolds $M = \exp(\mathfrak{m})$ are conjugation or Adjoint invariant by elements of the torsion group (page 7, Prop.3(3));
- All symmetric submanifolds except $M_5 = \exp(\mathfrak{m}_5)$ admits a LTS-torsion parametrization (page 8, Prop.4);
- All symmetric submanifolds admit a universal synthesis method for their kinematic chains (geometric approach on page 11, Coro.5.3; and algebraic approach on page 11, Coro.5.4).

On the one hand, the aforementioned geometric properties have potential applications in kinematics analysis of both kinesiological and robot mechanical systems. For example, we may use the half-angle property of M_{2A} to deduce that the coupler motion of an anti-parallel crank ([56]) (which is used to approximate the human knee joint model ([57])) has the motion pattern of a 1-D manifold of M_{2A} ([1]). The conjugation invariance property can be used to explain the fact that M_{2B} and M_{3B} type parallel robot wrists ([39,42]) can have identical and conjugate chains. Geometric properties of the Listing space (as the symmetric submanifold M_{2B}) may also offer new insight into the study of optimal control of human eye ([8]).

On the other hand, the complete classification of symmetric submanifolds and development of their geometric properties may have an impact on type synthesis of parallel robots. Making the connection of inversion symmetry of M_{3B} to the mirror symmetry of parallel CV couplings completes the story behind Hunt's observation some forty years ago ([16]). Besides, M_{3B} also happens to be the mixed freedom operation mode of the DYMO multi-mode parallel robot ([58]), and therefore offers new insight in type synthesis of multi-mode parallel robots. Finally, the inversion symmetry properties of symmetric submanifolds may offer a systematic approach to synthesizing interconnected parallel generators, as can be observed from several novel M_{2B}-CV joints ([54,55]). We will utilize the many geometric properties developed here in the type synthesis of parallel and interconnected generators for symmetric submanifolds in a separate treatment.

Acknowledgements

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Appendix A
$(-)$-derivation on the Listing space (Page 6)

We shall show that $\hat{\omega}^-(A) = \frac{1}{2}(A\hat{\omega} + \hat{\omega}A)$, $\hat{\omega} \in \mathfrak{m}_{2B}$, $A \in L$ defines a vector field on the Listing space $L = \exp(\mathfrak{m}_{2B})$. Recall that for $A = e^{\hat{v}}$, $\hat{v} \in \mathfrak{m}_{2B}$, the spatial velocity $\dot{A}A^{-1}$ is given by ([59]):

$$\begin{aligned}\dot{A}A^{-1} &= \frac{d}{dt}(e^{\hat{v}})e^{-\hat{v}} \\ &= \left(\int_0^1 e^{s\hat{v}}ds \cdot \omega\right)^{\wedge}, \omega = \dot{v} \in \mathbb{R}^2 \times \{0\}.\end{aligned} \tag{38}$$

Note that ω may be decomposed into components along and perpendicular to v, which we denote by $v_\parallel$ and $v_\perp$. Then (38) gives:

$$\dot{A}A^{-1} = \hat{v}_\parallel + \frac{\sin(\|v\|/2)}{\|v\|/2}(e^{\hat{v}/2}v_\perp)^\wedge \in Ad_{e^{\hat{v}/2}}\mathfrak{m}_{2B}. \tag{39}$$

This shows that $R_A^{-1}(T_A L) = Ad_{A^{1/2}}\mathfrak{m}_{2B}$.

On the other hand,

$$\begin{aligned}\hat{\omega}^-(A)A^{-1} &= \frac{1}{2}A^{\frac{1}{2}}(A^{\frac{1}{2}}\hat{\omega}A^{-\frac{1}{2}} + A^{-\frac{1}{2}}\hat{\omega}A^{\frac{1}{2}})A^{-\frac{1}{2}}. \\ &= \frac{1}{2}Ad_{e^{\hat{v}/2}}(Ad_{e^{\hat{v}/2}}\hat{\omega} + Ad_{e^{-\hat{v}/2}}\hat{\omega}) \\ &= \frac{1}{2}Ad_{e^{\hat{v}/2}}(e^{ad_{\hat{v}/2}}\hat{\omega} + e^{-ad_{\hat{v}/2}}\hat{\omega}) \\ &= Ad_{e^{\hat{v}/2}}\left(\sum_{k=0}^{\infty}\frac{ad^{2k}_{\hat{v}/2}}{(2k)!}\hat{\omega}\right) \\ &\in Ad_{e^{\hat{v}/2}}\mathfrak{m}_{2B} = R_A^{-1}(T_A L).\end{aligned} \tag{40}$$

The last inclusion relation is true since $\mathfrak{m}_{2B}$ is a LTS and therefore:

$$ad^2_{\hat{u}}\hat{\omega} = [\hat{u}, [\hat{u}, \hat{\omega}]] \in \mathfrak{m}_{2B},\ \forall \hat{u}, \hat{\omega} \in \mathfrak{m}_{2B}.$$

From (39) and (40), we see that $\hat{\omega}^-(A) \in T_A L$, and therefore defines a vector field on L. □

Appendix B
Proof of Prop.5 (Page 10)

The proof ot the first part is trivial: since $e^{\theta_1\hat{\xi}_1}\cdots e^{\theta_k\hat{\xi}_k}$ is an element of the complection group $G_M = \exp(\mathfrak{g}_\mathfrak{m})$, it admits a unique representation $e^{\hat{\xi}}e^{\hat{h}}, \hat{\xi} \in \mathfrak{m}, \hat{h} \in \mathfrak{h}_\mathfrak{m}$ by the diffeomorphism (30).

The second part of the proposition can be proved in two simpler steps. First, we shall show that for $\hat{\xi}_1, \hat{\xi}_2 \in \mathfrak{m}$, the following relation holds:

$$e^{\hat{\xi}_1}e^{\hat{\xi}_2} = e^{\hat{\xi}}e^{\hat{h}} \Rightarrow e^{-\hat{\xi}_1}e^{-\hat{\xi}_2} = e^{-\hat{\xi}}e^{\hat{h}}, \hat{\xi} \in \mathfrak{m}, \hat{h} \in \mathfrak{h}_\mathfrak{m}.$$

Apply the BCH formula (13) on both sides of the first equation:

$$\begin{aligned}&e^{\hat{\xi}_1+\hat{\xi}_2+\frac{1}{2}[\hat{\xi}_1,\hat{\xi}_2]+\frac{1}{12}([\hat{\xi}_1,[\hat{\xi}_1,\hat{\xi}_2]+[\hat{\xi}_2,[\hat{\xi}_2,\hat{\xi}_1])+\cdots} \\ &= e^{\hat{\xi}+\hat{h}+\frac{1}{2}[\hat{\xi},\hat{h}]+\frac{1}{12}([\hat{\xi},[\hat{\xi},\hat{h}]]+[\hat{h},[\hat{h},\hat{\xi}]])+\cdots}\end{aligned}$$

It is clear from Coro.1.1 that the terms in the exponents can be collected into $\mathfrak{m}$ and $\mathfrak{h}_\mathfrak{m}$:

$$\begin{aligned}&\hat{\xi}_1 + \hat{\xi}_2 + \frac{1}{12}([\hat{\xi}_1, [\hat{\xi}_1, \hat{\xi}_2] + [\hat{\xi}_2, [\hat{\xi}_2, \hat{\xi}_1]) + \cdots \\ &= \hat{\xi} + \frac{1}{2}[\hat{\xi}, \hat{h}] + \frac{1}{12}[\hat{h}, [\hat{h}, \hat{\xi}]] + \cdots \in \mathfrak{m}.\end{aligned} \tag{41a}$$

$$\frac{1}{2}[\hat{\xi}_1, \hat{\xi}_2] + \cdots = \hat{h} + \frac{1}{12}[\hat{\xi}, [\hat{\xi}, \hat{h}]] + \cdots \in \mathfrak{h}_{\mathfrak{m}}. \tag{41b}$$

It is clear that each term of (41a) is involved with an odd number of twists in $\mathfrak{m}$, and that of (41b) with an even number. We have $e^{-\hat{\xi}_1} e^{-\hat{\xi}_2} = e^{-\hat{\xi}} e^{\hat{h}}$ by changing the signs of $\hat{\xi}_1, \hat{\xi}_2$ and $\hat{\xi}$.

The second step is to prove the general case using induction. Suppose the case with $r-1$ twists is true:

$$\begin{cases} e^{\hat{\xi}_2} \cdots e^{\hat{\xi}_r} = e^{\hat{\xi}} e^{\hat{h}}, \\ e^{-\hat{\xi}_2} \cdots e^{-\hat{\xi}_r} = e^{-\hat{\xi}} e^{\hat{h}}. \end{cases} \quad \hat{\xi} \in \mathfrak{m}, \hat{h} \in \mathfrak{h}.$$

Then the case with r twists reads:

$$e^{\hat{\xi}_1}(e^{\hat{\xi}_2} \cdots e^{\hat{\xi}_r}) = e^{\hat{\xi}_1} e^{\hat{\xi}} e^{\hat{h}} = e^{\hat{\xi}'} e^{\hat{h}'} e^{\hat{h}} = e^{\hat{\xi}'} e^{\hat{h}''}.$$

where $e^{\hat{\xi}_1} e^{\hat{\xi}} = e^{\hat{\xi}'} e^{\hat{h}'}, \hat{\xi}' \in \mathfrak{m}, \hat{h}' \in \mathfrak{h}_{\mathfrak{m}}$ is the unique representation (33) for $e^{\hat{\xi}_1} e^{\hat{\xi}}$, and $e^{\hat{h}'} e^{\hat{h}} = e^{\hat{h}''}$ for some $\hat{h}'' \in \mathfrak{h}_{\mathfrak{m}}$ since $\mathfrak{h}_{\mathfrak{m}}$ is a Lie algebra; and:

$$e^{-\hat{\xi}_1}(e^{-\hat{\xi}_2} \cdots e^{-\hat{\xi}_r}) = e^{-\hat{\xi}_1} e^{-\hat{\xi}} e^{\hat{h}} = e^{-\hat{\xi}'} e^{\hat{h}'} e^{\hat{h}} = e^{-\hat{\xi}'} e^{\hat{h}''}.$$

where $e^{-\hat{\xi}_1} e^{-\hat{\xi}} = e^{-\hat{\xi}'} e^{\hat{h}'}$ by the first step. □

Appendix C
Proof of Coro.5.3 (Page 11)

We shall show that two representations of $\hat{\xi}^+$ lead to the same twist $\hat{\xi}^-$, i.e.:

$$\hat{\xi}^+ = Ad_{e^{\hat{\eta}}}(\hat{\xi}) = Ad_{e^{\hat{\eta}'}}(\hat{\xi}') \Rightarrow \hat{\xi}^- = Ad_{e^{-\hat{\eta}}}(\hat{\xi}) = Ad_{e^{-\hat{\eta}'}}(\hat{\xi}').$$

for any $\hat{\eta}, \hat{\eta}', \hat{\xi}, \hat{\xi}' \in \mathfrak{m}$; or in light of (15), prove:

$$\hat{\xi}^+ = e^{ad_{\hat{\eta}}} \hat{\xi} = e^{ad_{\hat{\eta}'}} \hat{\xi}' \Rightarrow \hat{\xi}^- = e^{-ad_{\hat{\eta}}} \hat{\xi} = e^{-ad_{\hat{\eta}'}} \hat{\xi}'.$$

Collecting the even and odd terms of $e^{ad_{\hat{\eta}}} \hat{\xi} = (I + ad_{\hat{\eta}} + \frac{1}{2!} ad_{\hat{\eta}}^2 + \cdots)\hat{\xi}$ into $\mathfrak{m}$ and $\mathfrak{h}_{\mathfrak{m}}$, and equate them with those of $e^{ad_{\hat{\eta}'}} \hat{\xi}' = (I + ad_{\hat{\eta}'} + \frac{1}{2!} ad_{\hat{\eta}'}^2 + \cdots)\hat{\xi}'$:

$$\begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{ad_{\hat{\eta}}^{2k}}{(2k)!} \hat{\xi} = \sum_{k=0}^{\infty} \frac{ad_{\hat{\eta}'}^{2k}}{(2k)!} \hat{\xi}' \in \mathfrak{m}, \\ \displaystyle\sum_{k=0}^{\infty} \frac{ad_{\hat{\eta}}^{2k+1}}{(2k+1)!} \hat{\xi} = \sum_{k=0}^{\infty} \frac{ad_{\hat{\eta}'}^{2k+1}}{(2k+1)!} \hat{\xi}' \in \mathfrak{h}_{\mathfrak{m}}. \end{cases}$$

That $e^{-ad_{\hat{\eta}}} \hat{\xi} = e^{-ad_{\hat{\eta}'}} \hat{\xi}'$ follows from negating the odd terms in $\mathfrak{h}_{\mathfrak{m}}$. □

Appendix D
Proof of Coro.5.5 (Page 11)

Consider the SPs $\{(\hat{\xi}_i^+; \hat{\xi}_i^-)\}_{i=1}^k$ with:

$$\begin{cases} \hat{\xi}_i^+ = Ad_{e^{\hat{\eta}_i}} \hat{\xi}_i = e^{ad_{\hat{\eta}_i}} \hat{\xi}_i, \\ \hat{\xi}_i^- = Ad_{e^{-\hat{\eta}_i}} \hat{\xi}_i = e^{-ad_{\hat{\eta}_i}} \hat{\xi}_i. \end{cases} \quad \eta_i, \xi_i \in \mathfrak{m}.$$

By inverse function theorem [10], the POE $e^{\theta_1 \hat{\xi}_1^+} \cdots e^{\theta_k \hat{\xi}_k^+} \cdot e^{\theta_k \hat{\xi}_k^-} \cdots e^{\theta_1 \hat{\xi}_1^-}$ generates $\exp(\mathfrak{m})$ if and only if $\{\hat{\xi}_i^+ + \hat{\xi}_i^-\}_{i=1}^k$ is a basis of $\mathfrak{m}$. On the one hand,

$$\hat{\xi}_i^+ + \hat{\xi}_i^- = 2 \sum_{k=0}^{\infty} \frac{ad_{\hat{\eta}_i}^{2k}}{(2k)!} \hat{\xi}_i \in \mathfrak{m}.$$

On the other hand,

$$\hat{\xi}_i^+ = \underbrace{\sum_{k=0}^{\infty} \frac{ad_{\hat{\eta}_i}^{2k+1}}{(2k+1)!} \hat{\xi}_i}_{\in \mathfrak{h}_{\mathfrak{m}}} + \underbrace{\sum_{k=0}^{\infty} \frac{ad_{\hat{\eta}_i}^{2k}}{(2k)!} \hat{\xi}_i}_{= \frac{1}{2}(\hat{\xi}_i^+ + \hat{\xi}_i^-) \in \mathfrak{m}}.$$

Then,

$$\overline{\{\hat{\xi}_1^+, \ldots, \hat{\xi}_k^+\}} \oplus \mathfrak{h}_{\mathfrak{m}} = \mathfrak{g}_{\mathfrak{m}}.$$

if and only if

$$\overline{\{\hat{\xi}_1^+ + \hat{\xi}_1^-, \ldots, \hat{\xi}_k^+ + \hat{\xi}_k^-\}} = \mathfrak{m}.$$

The other two conditions can proved in a similar manner. □

Appendix E
Proof of Prop.6 (Page 14)

Consider first the symmetric motion of a $\mathfrak{m}_5$-SP $(\hat{\xi} + \hat{h}; \hat{\xi} - \hat{h}), \hat{\xi} \in \mathfrak{m}_5, \hat{h} \in \mathfrak{h}_{\mathfrak{m}_5}$. The function:

$$f(\hat{\xi}) \triangleq \log(e^{\hat{\xi}+\hat{h}} e^{\hat{\xi}-\hat{h}}).$$

is an odd function of $\hat{\xi}$ since:

$$f(-\hat{\xi}) = \log(e^{-\hat{\xi}+\hat{h}} e^{-\hat{\xi}-\hat{h}}) = -\log(e^{\hat{\xi}+\hat{h}} e^{\hat{\xi}-\hat{h}}).$$

By the BCH formula (13) and property of LTS (Coro.1.1), we see that $f(\hat{\xi}) \in \mathfrak{m}_5$ and therefore $e^{\hat{\xi}+\hat{h}} e^{\hat{\xi}-\hat{h}} = e^{f(\hat{\xi})} \in \exp(\mathfrak{m}_5)$. More generally,

$$f(\hat{\xi}, \hat{\xi}') \triangleq \log(e^{\hat{\xi}+\hat{h}} e^{\hat{\xi}'} e^{\hat{\xi}-\hat{h}}), \hat{\xi}, \hat{\xi}' \in \mathfrak{m}_5, \hat{h} \in \mathfrak{h}_{\mathfrak{m}_5}.$$

contains only terms with an odd number of $\hat{\xi}$ and $\hat{\xi}'$, since:

$$f(-\hat{\xi}, -\hat{\xi}') = \log(e^{-\hat{\xi}+\hat{h}} e^{-\hat{\xi}'} e^{-\hat{\xi}-\hat{h}}) = -\log(e^{\hat{\xi}+\hat{h}} e^{\hat{\xi}'} e^{\hat{\xi}-\hat{h}}).$$

By a similar argument as in the first case, $e^{\hat{\xi}+\hat{h}} e^{\hat{\xi}'} e^{\hat{\xi}-\hat{h}} \in \exp(\mathfrak{m}_5)$. Combining the two cases shows that the symmetric motion of a $\mathfrak{m}_5$-SC is indeed a motion in $\exp(\mathfrak{m}_5)$.

The last statement is a result of the inverse function theorem (see for example [10]). □

References

[1] Y. Wu, G. Liu, H. Löwe, and Z. Li, "Exponential submanifolds: A new kinematic model for mechanism analysis and synthesis," in *Robotics and Automation (ICRA), 2013 IEEE International Conference on*, May 2013, pp. 4177–4182.

[2] J. M. Herve, "Analyse structurelle des mécanismes par groupe des déplacements," *Mechanism and Machine Theory*, vol. 13, no. 4, pp. 437–450, 1978.

[3] R. Brockett, "Robotic manipulators and the product of exponentials formula," in *In P.A. Fuhrman, editor, Mathematical Theory of Networks and Systems.* Springer-Verlag, 1984, pp. 120–129.

[4] Z. Li, R. Murray, and S. Sastry, *A mathematical introduction to robotic manipulation.* CRC Press, 1994.

[5] F. C. Park, J. E. Bobrow, and S. R. Ploen, "A lie group formulation of robot dynamics," *The International journal of robotics research*, vol. 14, no. 6, pp. 609–618, 1995.

[6] F. Bullo and R. Murray, "Proportional derivative (pd) control on the euclidean group," in *European Control Conference*, vol. 2, 1995, pp. 1091–1097.

[7] J. Kwon, M. Choi, F. C. Park, and C. Chun, "Particle filtering on the euclidean group: framework and applications," *Robotica*, vol. 25, no. 06, pp. 725–737, 2007.

[8] A. D. Polpitiya, W. P. Dayawansa, C. F. Martin, and B. K. Ghosh, "Geometry and control of human eye movements," *Automatic Control, IEEE Transactions on*, vol. 52, no. 2, pp. 170–180, 2007.

[9] J. Beckers, J. Patera, M. Perroud, and P. Winternitz, "Subgroups of the euclidean group and symmetry breaking in nonrelativistic quantum mechanics," *Journal of mathematical physics*, vol. 18, no. 1, pp. 72–83, 1977.

[10] W. M. Boothby, *An introduction to differentiable manifolds and Riemannian geometry*, rev. 2nd ed. Amsterdam ; New York: Academic Press, 2003.

[11] A. Knapp, *Lie groups beyond an introduction*, 2nd ed. Boston: Birkhäuser, 2002.

[12] R. S. Ball, *A treatise on the theory of screws*. Cambridge Eng.: University Press, 1900.

[13] K. H. Hunt, *Kinematic geometry of mechanisms*. Oxford University Press, 1978.

[14] C. Gibson and K. Hunt, "Geometry of screw systems-1: Screws: Genesis and geometry," *Mechanism and Machine theory*, vol. 25, no. 1, pp. 1–10, 1990.

[15] ——, "Geometry of screw systems-2: classification of screw systems," *Mechanism and Machine Theory*, vol. 25, no. 1, pp. 11–27, 1990.

[16] K. Hunt, "Constant-velocity shaft couplings: a general theory," *ASME J. Eng. Ind*, vol. 95, no. 2, pp. 455–464, 1973.

[17] J. Hervé and F. Sparacino, "Structural synthesis of parallel robots generating spatial translation," in *Proc. 5th Int. Conf. Advanced Robotics*, vol. 1, 1991, pp. 808–813.

[18] A. Frisoli, D. Checcacci, F. Salsedo, and M. Bergamasco, "Synthesis by screw algebra of translating in-parallel actuated mechanisms," in *Advances in Robot Kinematics*. Springer, 2000, pp. 433–440.

[19] M. Carricato and V. Parenti-Castelli, "Singularity-free fully-isotropic translational parallel mechanisms," *The International Journal of Robotics Research*, vol. 21, no. 2, pp. 161–174, 2002.

[20] ——, "A family of 3-dof translational parallel manipulators," *Journal of mechanical design*, vol. 125, no. 2, pp. 302–307, 2003.

[21] Z. Huang and Q. Li, "Type synthesis of symmetrical lower-mobility parallel mechanisms using the constraint-synthesis method," *International Journal of Robotics Research*, vol. 22, no. 1, pp. 59–79, 2003.

[22] X. Kong and C. M. Gosselin, "Type synthesis of 3-dof spherical parallel manipulators based on screw theory," *Journal of Mechanical Design*, vol. 126, no. 1, pp. 101–108, 2004.

[23] X. Kong and C. Gosselin, *Type synthesis of parallel mechanisms*. Springer-Verlag, Berlin, Heidelberg, 2007.

[24] J. Meng, G. Liu, and Z. Li, "A geometric theory for analysis and synthesis of sub-6 dof parallel manipulators," *Robotics, IEEE Transactions on*, vol. 23, no. 4, pp. 625 –649, aug. 2007.

[25] M. Rosheim and G. Sauter, "New high-angulation omni-directional sensor mount," in *Proceedings of SPIE - The International Society for Optical Engineering*, vol. 4821, 2002, pp. 163–174.

[26] E. Samson, D. Laurendeau, M. Parizeau, S. Comtois, J. F. Allan, and C. Gosselin, "The agile stereo pair for active vision," *Machine Vision and Applications*, vol. 17, no. 1, pp. 32–50, 2006.

[27] J. M. Hervé, "Uncoupled actuation of pan-tilt wrists," *IEEE Transactions on Robotics*, vol. 22, no. 1, pp. 56–64, 2006.

[28] L. J. Stocco, S. E. Salcudean, and F. Sassani, "Optimal kinematic design of a haptic pen," *Mechatronics, IEEE/ASME Transactions on*, vol. 6, no. 3, pp. 210–220, 2001.

[29] E. L. Bohez, "Five-axis milling machine tool kinematic chain design and analysis," *International Journal of Machine Tools and Manufacture*, vol. 42, no. 4, pp. 505–520, 2002.

[30] Y. Wu, H. Wang, and Z. Li, "Quotient kinematics machines: Concept, analysis, and synthesis," *ASME Journal of Mechanisms and Robotics*, vol. 3, no. 4, pp. 041 004–1 – 041 004–1, Nov 2011.

[31] J. Hervé, "The planar spherical kinematic bond: Implementation in parallel mechanisms," pp. 1–19, 2003. [Online]. Available: http://www.parallemic.org/Reviews/Review013.html

[32] C. Lee and J. Herve, "Translational parallel manipulator with doubly planar limbs," *Mechanism and Machine Theory*, vol. 41, no. 4, pp. 359–486, 2006.

[33] M. Carricato and J. R. Martínez, "Persistent screw systems," in *Advances in Robot Kinematics: Motion in Man and Machine*. Springer, 2010, pp. 185–194.

[34] M. Carricato and D. Zlatanov, "Persistent screw systems," *Mechanism and Machine Theory*, vol. 73, pp. 296–313, 2014.

[35] M. Carricato and J. R. Martınez, "Persistent screw systems of dimension three," in *Proc. of 13th World Congress in Mechanism and Machine Science, Guanajuato, Mexico*, 2011, pp. 1–12.

[36] M. Carricato, "Persistent screw systems of dimension four," in *Latest Advances in Robot Kinematics*. Springer, 2012, pp. 147–156.

[37] ——, "Four-dimensional persistent screw systems of the general type," in *Computational Kinematics*, ser. Mechanisms and Machine Science. Springer Netherlands, 2014, vol. 15, pp. 299–306.

[38] V. Henn, *Three-dimensional kinematics of eye, head and limb movements*. Australia: Harwood Academic Publishers, 1997, ch. History of three-dimensional eye movement research, pp. 3–14.

[39] S. L. Canfield, A. J. Ganino, C. F. Reinholtz, and R. J. Salerno, "Spatial, parallel-architecture robotic carpal wrist," Dec. 23 1997, uS Patent 5,699,695.

[40] K. Sone, H. Isobe, and K. Yamada, "High angle active link," *NTN TECHNICAL REVIEW*, 2004.

[41] D. T. Wallace, S. C. Anderson, and S. Manzo, "Platform link wrist mechanism," Apr. 6 2010, uS Patent 7,691,098.

[42] R. Behrens, C. Kuchler, T. Forster, and N. Elkmann, "Kinematics analysis of a 3-dof joint for a novel hyper-redundant robot arm," in *Robotics and Automation (ICRA), 2011 IEEE International Conference on*. IEEE, 2011, pp. 3224–3229.

[43] M. E. Rosheim, *Leonardo's lost robots*. Springer, 2006.

[44] Y. Wu, Z. Li, and J. Shi, "Geometric properties of zero-torsion parallel kinematics machines," in *Intelligent Robots and Systems (IROS), 2010 IEEE/RSJ International Conference on*, pp. 2307–2312.

[45] M. Carricato, "Decoupled and homokinetic transmission of rotational motion via constant-velocity joints in closed-chain orientational manipulators," *Journal of Mechanisms and Robotics*, vol. 1, no. 4, pp. 1–14, 2009.

[46] I. A. Bonev, D. Zlatanov, and C. M. Gosselin, "Advantages of the modified euler angles in the design and control of pkms," *2002 Parallel Kinematic Machines International Conference*, pp. 171–188, 2002.

[47] I. Bonev, "Delta parallel robot-the story of success," *Newsletter*, 2001. [Online]. Available: http://www.parallemic.org/Reviews/Review002.html

[48] O. Loos, *Symmetric spaces*. WA Benjamin, 1969, vol. 1.

[49] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*. Academic press, 1978, vol. 80.

[50] J. Selig, *Geometrical Fundamental of Robotics*, ser. Monographs in Computer Science. New York ; Hong Kong: Springer, 2005.

[51] B. Hall, *Lie groups, Lie algebras, and representations: an elementary introduction*. Springer Verlag, 2003, vol. 222.

[52] D. Tweed and T. Vilis, "Geometric relations of eye position and velocity vectors during saccades," *Vision Research*, vol. 30, no. 1, pp. 111–127, 1990.

[53] Y. Wu and M. Carricato, "Classification of lie triple screw systems and application in kinematics," in *IFTOMM World Congress*, 2015.

[54] G. Thompson *et al.*, "Constant velocity coupling and control system therefor," Dec. 5 2006, uS Patent 7,144,326.

[55] H. Kocabas, "Design and analysis of a spherical constant velocity coupling mechanism," *Journal of Mechanical Design*, vol. 129, no. 9, pp. 991–998, 2007.

[56] F. Reuleaux, *The kinematics of machinery: outlines of a theory of machines*. Macmillan, 1876.

[57] V. Parenti-Castelli, A. Leardini, R. Di Gregorio, and J. O'Connor, "On the modeling of passive motion of the human knee joint by means of equivalent planar and spatial parallel mechanisms," *Autonomous Robots*, vol. 16, no. 2, pp. 219–232, 2004.

[58] D. Zlatanov, I. A. Bonev, and C. M. Gosselin, "Constraint singularities as c-space singularities," in *Advances in Robot Kinematics*. Springer, 2002, pp. 183–192.

[59] K. Okamura and F. Park, "Kinematic calibration using the product of exponentials formula," *Robotica*, vol. 14, no. 4, pp. 415–422, 1996.