Intelligent Residential Air-Conditioning System with Smart-Grid Functionality

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Abstract—This paper sets forth a novel intelligent residential air-conditioning (A/C) system controller that has smart grid functionality. The qualifier "intelligent" means the A/C system has advanced computational capabilities and uses an array of environmental and occupancy parameters in order to provide optimal intertemporal comfort/cost trade-offs for the resident, conditional on anticipated retail energy prices. The term "smart-grid functionality" means that retail energy prices can depend on wholesale energy prices. Simulation studies are used to demonstrate the capabilities of the proposed A/C system controller.

Index Terms—Air conditioning, dynamic programming, intelligent control, smart grids.

Nomenclature

$\mathrm{BR_o}$	Nominal BTU rating (BTU/h) of the A/C system (at 35° C).
$\mathrm{COP_o}$	Nominal cooling coefficient-of-performance (unit-free) for the A/C system at 35° C.
C^a	Heat capacity (BTU/°F) of the internal air mass.
C^m	Heat capacity (BTU/°F) of the internal solid mass.
C_n	A/C electricity cost ($) during period n.
$E(\cdot)$	Expected value calculated using $f(\boldsymbol{\nu})$, the probability density function (pdf) for $\boldsymbol{\nu}$.
e_n	Electric energy consumption (kWh) of the A/C system during period n.
$E_n(\cdot)$	Expected value calculated using the marginal pdf for $\boldsymbol{\nu}_n$.
F	Pro-rated fixed cost ($) that the load-serving entity (LSE) charges R for A/C energy usage during each period n.
$f(\boldsymbol{\nu})$	Pdf for $\boldsymbol{\nu} = [\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_N]$.
$G_{\max}$	Maximum possible comfort level (Utils) achievable by house resident R during each period n from the thermal condition of his house.
G_n	Comfort (Utils) attained by R during period n from the thermal condition of his house.

h_1, h_2	Parameters appearing in R's comfort function that weigh R's thermal discomfort for the current and subsequent period, respectively.
I	R's targeted income expenditure level ($).
K	Conversion factor (3412.1 BTU $\approx$ 1 kWh).
k_{1n}, k_{2n}	Lower and upper temperature bounds for R's comfort function in period n.
m^i	Fraction of heat flow rate (unit-free) from internal heat flux to the internal solid mass.
m^l	Fraction of cooling load (unit-free) that indicates the latent cooling load inside the house, i.e., the unwanted moisture that needs to be removed.
m^s	Fraction of heat flow rate (unit-free) from solar radiation to the internal solid mass.
N	Number of successive time periods n comprising R's planning horizon, where period n is defined as the time interval $[(n-1)\Delta t, n\Delta t)$, for some fixed time step Δt.
N_a, N_m	Number of grid points corresponding to T_n^a, T_n^m, respectively.
NB_n	Net benefit (Utils) attained by R in period n (discounted to period 1).
$\mathbf{p}$	Vector of retail A/C energy prices for periods 1 through N, $\mathbf{p} = [p_1, p_2, \ldots, p_N]$.
p_n	Retail price ($/kWh) that the load-serving entity (LSE) charges R for A/C energy usage during period n.
$\mathbf{p}^y$	Vector of m consumption good prices, $\mathbf{p}^y = [p_1^y, p_2^y, \ldots, p_m^y]$.
p_j^y	Retail price paid by R per unit of good j.
$\dot{Q}_n$	Heat flow rate (BTU/h) from A/C system to inside air mass during period n.
$\dot{Q}_n^a$	Heat flow rate (BTU/h) from $\dot{Q}_n^s$ and $\dot{Q}_n^i$ to inside air mass during period n.
$\dot{Q}_n^i$	Heat flow rate (BTU/h) from internal appliances and occupants during period n.
$\dot{Q}_n^m$	Heat flow rate (BTU/h) from $\dot{Q}_n^s$ and $\dot{Q}_n^i$ to inside solid mass during period n.
$\dot{Q}_n^s$	Heat flow rate (BTU/h) from solar radiation during period n.
R	The resident of the house.
$\mathbf{r}_n$	Retail price-to-go sequence starting in period n, $\mathbf{r}_n = [p_n, p_{n+1}, \ldots, p_N]$.
TNB	Total net benefit (Utils) attained by R over the planning horizon (discounted to period 1).

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T_n^a	Internal air temperature (°F) at the beginning of period n.
T^b	R's bliss temperature, i.e., the inside air temperature (°F) at which R achieves his maximum comfort level.
T_n^m	Internal mass temperature (°F) at the beginning of period n (i.e., the equivalent temperature of the lumped solid mass).
T_n^o	Outside air temperature during period n (°F).
U^a	Thermal conductance (BTU/h/°F) between internal and external air mass defining the thermal envelope of the house.
U^m	Thermal conductance (BTU/h/°F) between the internal air mass and the solid mass.
$\mathbf{u}$	Sequence of A/C status conditions, $\mathbf{u} = [u_1, \ldots, u_N]$.
u_n	A/C system status (e.g., off or on) in period n.
$\mathbf{w}_n$	Vector of forcing terms in period n.
$\mathbf{x}$	Sequence of state vectors, $\mathbf{x} = [\mathbf{x}_1, \ldots, \mathbf{x}_N]$.
$\mathbf{x}_n$	State vector describing the condition of the house at the beginning of period n.
$\mathbf{y}$	Vector of consumption goods purchased by R during periods $1, \ldots, N$ in addition to A/C energy purchases, $\mathbf{y} = [y_1, \ldots, y_m]^T$.
$Z(\mathbf{y})$	Benefit obtained by R from consumption of $\mathbf{y}$.
α	Parameter (Utils/$) appearing in R's net benefit function that measures the benefit to R of a dollar of income.
β_n	Discount factor for R's net benefit in period n.
γ	Parameter appearing in R's comfort function that influences the shape of this function around a bliss temperature range.
$\boldsymbol{\nu}$	Sequence of stochastic environmental conditions, $\boldsymbol{\nu} = [\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_N]$.
$\boldsymbol{\nu}_n$	Vector of stochastic (external and internal) environmental conditions during period n.
ρ_n^o	Outside relative humidity during period n.

I. INTRODUCTION

THIS STUDY considers the design of a residential air-conditioning (A/C) system capable of responding intelligently to price signals in order to achieve optimal intertemporal comfort/cost trade-offs for a house resident. A key motivation for this study is a 2010 report by the United States Federal Energy Regulatory Commission (FERC) on demand response and advanced metering technology. This report concludes:

> "The investments in devices, controls and software to implement demand response remain one of the greatest barriers to increased penetration." [1, p. 56]

In line with this conclusion, the current paper carefully considers the complex interplay between the comfort/cost preferences of a house resident and the structural conditions constraining his A/C choices arising both from the physics of energy flows and the engineering limitations of A/C system implementations.

Previous research on comfort and energy management issues has largely focused on large building environments with many occupants [2]–[5]. As detailed in the 2009 survey by Dounis and Caraiscos [6], these studies consider not only heating and cooling systems but also other building design features such as window placements, window shading, mechanical ventilation systems, and lighting systems. Occupant comfort in these studies is typically a complex multi-faceted concept encompassing thermal comfort, visual comfort, and indoor air quality, in keeping with ASHRAE standards [7]. Various control methods are explored in these studies, including fuzzy controllers [8], fuzzy adaptive controllers [4], [9], and neural network controllers [10].

Nevertheless, in recent years the increasing interest in advanced metering infrastructure for households has encouraged researchers to focus more carefully on the energy usage choices of residential homeowners [11]. For example, Rogers et al. [12] study an interesting residential demand model, although without consideration of price signals. Guttromson et al. [13] and Chassin et al. [14] focus on the modeling of price-responsive residential demands constrained by internal and external state conditions. The latter studies are anchored by an Olympic Penninsula pilot project [15]. However, residential energy demands in these studies are modeled by means of pre-specified behavioral rules rather than as the solutions to residential optimization problems. More recent research has set forth formulations of the residential A/C control problem as an optimization problem. In this work the objective is to minimize some combination of thermal discomfort and energy usage under varying electricity prices [16]–[20].

The current paper extends this prior work in three important directions. First, the A/C control strategy is formulated as a stochastic dynamic program in a manner that permits the controller to respond to both energy prices and randomly varying environmental conditions. We demonstrate that the underlying optimization problem can be solved in a reasonable amount of time using conventional computational resources by adopting a certainty equivalence approach, using weather forecast information that is nowadays readily available over the Internet. Moreover, this is done in a way that is minimally disruptive to the A/C system hardware, which is important for retrofitting existing residential A/C systems. Second, the thermal dynamics for the house and the A/C system are represented by means of physics-based models that are suitably realistic for residential A/C system control purposes (whereas previous work adopted simpler models to describe the plant dynamics). Third, the objective function expressing comfort/cost trade-offs for the household resident is rigorously motivated in terms of basic economic principles.

Section II sets out the stochastic optimal control problem in general terms: a residential A/C system determines energy usages over an N-period planning horizon to achieve optimal intertemporal comfort/cost trade-offs for the house resident, conditional on anticipated energy prices and on dynamically changing internal and external conditions. In this general formulation it is assumed that reliable state equations are available for determining the change in the thermal state of the resident's house from one period to the next as a function of the resident's A/C energy usage level and environmental parameters. It is also assumed that the resident's comfort level

is determined in each period by the indoor thermal state of his house at the beginning and end of the period.

Sections III and IV then address what might be done in the more practically relevant case in which the state equations for the resident's house must be approximated and the resident's achievable comfort levels are constrained by the mechanical requirements of the resident's A/C system. Illustrative findings from computer simulations demonstrating the capabilities of the resulting A/C system controller are reported in Section V. Concluding remarks are given in Section VI. Appendix A provides technical details regarding the use of a Luenberger observer to construct an estimate for mass temperature, and Appendix B provides additional motivation for the modeling of the resident's comfort/cost trade-offs.

II. GENERAL STOCHASTIC OPTIMAL CONTROL PROBLEM FOR A RESIDENTIAL A/C SYSTEM

A. Problem Formulation

For computational tractability, the planning horizon of the house resident is discretized into time periods $n = 1, \ldots, N$, and the continuous thermal dynamics of the house are correspondingly discretized into the discrete-time motion of a state vector $\mathbf{x}_n$. However, the dimension and content of the state vectors $\mathbf{x}_n$ are not restricted. Consequently, the state equation formulation in this section is generic and can be used to implement a wide variety of thermal models.[1]

The state $\mathbf{x}_{n+1}$ is assumed to be determined as a function S_n (time-varying for generality) of the previous state $\mathbf{x}_n$, the A/C status u_n, and a vector $\boldsymbol{\nu}_n$ of environmental parameters, as

$$\mathbf{x}_{n+1} = S_n(\mathbf{x}_n, u_n, \boldsymbol{\nu}_n), \quad n = 1, \ldots, N, \quad (1)$$

where the initial state at the beginning of period 1 is exogenously given as

$$\mathbf{x}_1 = \overline{\mathbf{x}}_1. \quad (2)$$

The A/C status u_n in (1) is assumed to be restricted to a domain U. For example, $U = \{-1, 0\}$ could represent a bang-bang control domain corresponding to cooling and off, whereas $U = [-1, 0]$ could represent a continuous control domain ranging from full cooling (-1) to off (0). Also, suppose the A/C heat flow rate $\dot{Q}_n$ is determined as a function

$$\dot{Q}_n = \dot{Q}(u_n, \boldsymbol{\nu}_n). \quad (3)$$

Finally, assume the electric energy usage e_n of the A/C system can be expressed as a function

$$e_n = e(\dot{Q}_n, \boldsymbol{\nu}_n) \equiv \tilde{e}(u_n, \boldsymbol{\nu}_n). \quad (4)$$

To model price-responsive demand for electricity, it is assumed that resident R has a retail contract with a load-serving entity (LSE) under which he pays a price p_n (\$/kWh) for his A/C energy usage e_n (kWh) plus a pro-rated fixed charge F (\$) to cover costs such as equipment purchases and

connection fees.[2] The total cost charged by the LSE to R during period n thus takes the form

$$C_n = C(p_n, e_n) = p_n e_n + F. \quad (5)$$

The sequence $\mathbf{p} = [p_1, p_2, \ldots, p_N]$ of retail A/C energy prices is assumed to be communicated by the LSE to R prior to the start of period 1. Although R has access to this price data, he does not need to act on a continual basis. Rather, it is the intelligent A/C controller that assumes this responsibility. Note that the time step of the A/C system model does not have to be the same as the time step of operations for the wholesale electric power market. For example, day-ahead market LMPs are determined on an hourly basis in the United States whereas an A/C system will typically run at a faster time rate. If hourly day-ahead market LMPs were to be charged to R as retail energy prices, the vector $\mathbf{p}$ would consist of 24 equal-length sub-vectors of constant-valued prices.

As in [12], the comfort level (Utils) attained by R in period n from the thermal condition of his house is measured as a time-varying function of the state vectors at the beginning and end of period n:

$$G_n = G(\mathbf{x}_n, \mathbf{x}_{n+1}, n). \quad (6)$$

As is standard in (power) economics, the comfort assessment (6) is assumed to be determined independently of any cost considerations.

From the viewpoint of period 1, the net benefit NB_n attained by R in period n is given by R's attained comfort level minus his energy purchase costs, weighted by a discount factor β_n, as follows:

$$NB_n = \beta_n[G(\mathbf{x}_n, \mathbf{x}_{n+1}, n) - \alpha C(p_n, \tilde{e}(u_n, \boldsymbol{\nu}_n))]. \quad (7)$$

The key parameter α (Utils/\$) appearing in (7) measures the benefit (utility) to R of an additional dollar of income. It permits costs measured in dollars to be expressed in benefit units (Utils), so that comfort/cost trade-offs can be calculated.

The precise sense in which α quantifies the trade-off between comfort satisfaction and energy cost for R is explained in some detail in Appendix B of our paper. Specifically, it is shown in Appendix B that α can be derived as the shadow price for R's budget constraint in a more fully articulated constrained benefit maximization problem: namely, the maximization of R's benefit from consumption of multiple goods/services (including thermal comfort) subject to a budget constraint. Thus, α measures R's "marginal benefit of income" at the optimization point, i.e., the drop in the maximized value of R's benefit that would result if R had one less dollar of income to spend (e.g., due to a higher energy price). For

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[1]In Section III, below, this generic formulation is concretely illustrated for a thermal model with two-dimensional state vectors $\mathbf{x}_n$, where the two state components are internal air temperature T_n^a and internal solid mass temperature T_n^m.

[2]In the general problem formulation presented in this section, the manner in which the LSE sets the A/C energy usage prices p_n is not restricted; hence, in particular, these prices do not need to bear any particular relationship to the prices paid by the LSE for its wholesale energy purchases. In reality, of course, an LSE that contracts with retail consumers having intelligent A/C system controllers as modeled in the current study will have to set its A/C energy usage prices in line with the prices it pays for energy at wholesale in order to remain profitable. For example, as illustrated below in Section V, p_n could be set equal to the day-ahead locational marginal price (LMP) paid by the LSE at wholesale plus a "mark-up" to cover additional types of operational costs.

simplicity, this section treats a reduced form of this more comprehensive benefit maximization problem in which R is in effect solving a first-order necessary condition for this more comprehensive problem, taking α as given.[3]

The total net benefit attained by R over the planning horizon from period 1 to period N, conditional on a given state sequence $\mathbf{x}$, A/C status sequence $\mathbf{u}$, environmental-term sequence $\boldsymbol{\nu}$, and price sequence $\mathbf{p}$ is calculated as the discounted sum of period-by-period net benefits:

$$\text{TNB}(\mathbf{x}, \mathbf{u}, \boldsymbol{\nu}, \mathbf{p}) = \Sigma_{n=1}^{N} \text{NB}_n. \qquad (8)$$

Let the expected value of (8), conditional on (1) through (7), be denoted by

$$E[\text{TNB}(\mathbf{x}, \mathbf{u}, \boldsymbol{\nu}, \mathbf{p})] = \int_V \text{TNB}(\mathbf{x}, \mathbf{u}, \boldsymbol{\nu}, \mathbf{p}) f(\boldsymbol{\nu}) d\boldsymbol{\nu} \qquad (9)$$

where V denotes the domain of possible environmental vectors $\boldsymbol{\nu}$ that could be realized during the planning horizon $\{1, \ldots, N\}$, and $f(\boldsymbol{\nu})$ denotes the joint probability density function (PDF) for $\boldsymbol{\nu}$.

Putting this all together, the stochastic optimal control problem to be solved at the beginning of period 1 for determination of optimal A/C status choices $u_n^* \in U$ during periods $n = 1, \ldots, N$ can be expressed as follows:

$$\max E[\text{TNB}(\mathbf{x}, \mathbf{u}, \boldsymbol{\nu}, \mathbf{p})] \qquad (10)$$

with respect to $\mathbf{u} = [u_1, u_2, \ldots, u_N]^T$ subject to (1) and (2).

B. Closed-Loop Dynamic Programming Solution

Stochastic dynamic programming can be used to solve the control problem (10) in closed-loop form. That is, (10) can be solved in sequential form with the optimal A/C status value $u_n^*(\mathbf{x}_n; \mathbf{r}_n)$ in each period n expressed as a function of the current state $\mathbf{x}_n$, conditional on the price-to-go sequence $\mathbf{r}_n = [p_n, p_{n+1}, \ldots, p_N]^T$ consisting of the given retail energy prices from period n through the final planning period N. For any n satisfying $1 \leq n \leq N$, let $\text{Val}_n(\mathbf{x}_n; \mathbf{r}_n)$ denote the maximum expected total net benefits attainable by R starting from any feasible state $\mathbf{x}_n$, conditional on $\mathbf{r}_n$. That is, let $\text{Val}_n(\mathbf{x}_n; \mathbf{r}_n)$ denote R's price-conditioned *period-n value function*.[4]

From the developments in Section II-A, we can define

$$\text{Val}_N(\mathbf{x}_N; \mathbf{r}_N) = \max_{u_N} E_N[\text{NB}_N(\mathbf{x}_N, S_N(\mathbf{x}_N, u_N, \boldsymbol{\nu}_N), p_N, \tilde{e}(u_N, \boldsymbol{\nu}_N))] \qquad (11)$$

where $\mathbf{r}_N \equiv p_N$ denotes the retail energy price for period N, and the expectation is taken with respect to the randomly varying environmental conditions $\boldsymbol{\nu}_N$ for period N, conditional on $\mathbf{x}_N$ and exogenously given factors (such as $\mathbf{r}_N$). Note that the solution to (11) has the closed-loop form $u_N^* =$

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[3]As a practical matter, a household resident could experiment with different α values to find a value for this trade-off parameter that approximately reflects his true marginal benefit of income.

[4]The exogenously given price-to-go sequences $\mathbf{r}_n$ are explicitly included as conditioning factors in the optimal control and value functions in order to emphasize the price-responsive nature of the A/C system controller.

$u_N(\mathbf{x}_N; \mathbf{r}_N)$. It then follows, by definition, that resident R's value functions satisfy the following recursive relationship:[5] For $n = 1, \ldots, N-1$:

$$\text{Val}_n(\mathbf{x}_n; \mathbf{r}_n) = \max_{u_n} E_n[\text{NB}_n(\mathbf{x}_n, S_n(\mathbf{x}_n, u_n, \boldsymbol{\nu}_n), p_n, \tilde{e}(u_n, \boldsymbol{\nu}_n)) + \text{Val}_{n+1}(S_n(\mathbf{x}_n, u_n, \boldsymbol{\nu}_n); \mathbf{r}_{n+1})]. \qquad (12)$$

where $\mathbf{r}_n \equiv [p_n, \mathbf{r}_{n+1}]$.

Consequently, in principle, resident R at the beginning of period 1 can derive a closed-loop solution to his stochastic optimal control problem (10) as follows. He should first use (11) and (12) to derive his value functions $\text{Val}_n(\mathbf{x}_n; \mathbf{r}_n)$, starting at period $n = N$ and working backward to period $n = 1$. As a by-product of these calculations, for each period $n \geq 1$ the resident will obtain the optimal A/C status choice $u_n^*(\mathbf{x}_n; \mathbf{r}_n)$ as a function of $\mathbf{x}_n$, conditional on $\mathbf{r}_n$.

From the vantage point of the initial period, R does not yet know what state vectors $\mathbf{x}_n$ will be realized in subsequent periods due to the inherent uncertainty in the system. Nevertheless, he will know $\mathbf{x}_n$ at the beginning of each period n prior to his actual choice of an A/C status u_n. The closed-loop solutions $u_n^*(\mathbf{x}_n; \mathbf{r}_n)$ are thus complete contingency plans determining what A/C status choice should be optimally implemented at each future time, conditional on the state and price conditions at that time. Clearly, however, the state domain would have to be appropriately discretized to obtain a practically computable closed-loop solution. An example of such a discretization is provided in Section IV.

III. PHYSICS-BASED MODELING OF THE A/C SYSTEM

The A/C is a conventional residential system, such as the ones that may be typically found in United States middle-class residences. These are conventional systems, with an electrically powered central unit or a window/wall unit that cycles on and off to maintain the air temperature around a thermostat set point. This section provides explicit forms for the abstractly represented thermal state equation (1) and energy equation (4), as foundations for the proposed intelligent A/C system controller. The complexity of these forms arises because they are physically based. An important point here, however, is that house residents employing the proposed controller do not need to be exposed to this complexity; an interface can separate a resident from the internal workings of the controller. As will be clarified more carefully in Section IV below, all that a resident needs to be exposed to via this interface are "knobs" permitting him to adjust to his satisfaction the settings for his thermal comfort function parameters and his comfort/cost trade-off parameter α.

The thermal dynamics for a house are represented by means of an Equivalent Thermal Parameter (ETP) model [21], [22]. The ETP model supposes that the state of a house at time t consists of the inside air and mass temperatures, T^a and T^m, whose dynamics are defined by a system of two first-order

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[5]Equation (12) is a special case of Bellman's Principle of Optimality.

linear differential equations:

$$\frac{dT^a}{dt} = \frac{1}{C^a}\left[(T^o - T^a)U^a + (T^m - T^a)U^m + \dot{Q} + \dot{Q}^a\right] \tag{13}$$

$$\frac{dT^m}{dt} = \frac{1}{C^m}\left[(T^a - T^m)U^m + \dot{Q}^m\right]. \tag{14}$$

The parameters appearing above have been defined in the nomenclature; also

$$\dot{Q}^a = (1 - m^s)\dot{Q}^s + (1 - m^i)\dot{Q}^i \tag{15}$$

$$\dot{Q}^m = m^s\dot{Q}^s + m^i\dot{Q}^i. \tag{16}$$

For computational tractability, the above continuous-time system is transformed to a discrete-time system of the form

$$\mathbf{x}_{n+1} = \hat{\mathbf{A}}\mathbf{x}_n + \hat{\mathbf{B}}\mathbf{w}_n \tag{17}$$

under the assumption that all time-varying forcing terms are step functions that remain constant during each period n, with

$$\mathbf{w}_n = \begin{bmatrix} T^o_n & \dot{Q}^s_n & \dot{Q}^i_n & \dot{Q}_n \end{bmatrix}^T. \tag{18}$$

This discrete state equation is of the same form as (1). To see this, first note that the A/C heat flow rate $\dot{Q}_n$ depends on the A/C status u_n (cooling or off), which is represented by the following indicator function:

$$u_n = \begin{cases} 0 & \text{if A/C status = off} \\ -1 & \text{if A/C status = on} \end{cases} \tag{19}$$

The A/C heat flow rate $\dot{Q}_n$ represented by (3) is defined as

$$\dot{Q}_n = \dot{Q}(u_n, \boldsymbol{\nu}_n) = \frac{\text{BR}(T^o_n)}{m(\rho^o_n)}u_n \tag{20}$$

where the vector $\boldsymbol{\nu}_n$ contains all stochastic time-varying terms,

$$\boldsymbol{\nu}_n = \begin{bmatrix} T^o_n & \dot{Q}^s_n & \dot{Q}^i_n & \rho^o_n \end{bmatrix}^T. \tag{21}$$

In particular, the state function S_n in (1) reduces to a time-invariant function S of $(\mathbf{x}_n, u_n, \boldsymbol{\nu}_n)$ that is linear in $\mathbf{x}_n$.

Finally, an explicit form for the energy consumption function (4) of the A/C is established as

$$e_n = e(\dot{Q}_n, \boldsymbol{\nu}_n) = K\frac{|\dot{Q}_n|}{\text{COP}(\boldsymbol{\nu}_n)}m(\rho^o_n)\Delta t. \tag{22}$$

Explicit numerical expressions for the functions that appear above are obtained from [22]:

$$\text{BR}(T^o_n) = \text{BR}_o\left(1.4892 - 0.0052T^o_n\right) \tag{23}$$

$$\text{COP}(\boldsymbol{\nu}_n) = \frac{\text{COP}_o}{-0.01364 + 0.01067T^o_n} \tag{24}$$

$$m(\rho^o_n) = 1.1 + \frac{m^l}{1 + \exp(4 - 0.1\rho^o_n)}. \tag{25}$$

IV. Controller Implementation

This section explains the envisioned practical implementation of the proposed intelligent A/C system controller, given the A/C system model described in Section III. This controller consists of two main parts, namely, the software running the scheduling algorithm and the wall control unit, as shown in Fig. 1.

Current residential A/C systems, whose logic is based on relatively simple thermostatic control, could be readily retrofitted by just upgrading their wall control units with the proposed intelligent unit (i.e., the A/C mechanical components would not need to be modified). The scheduling software could be programmed on the actual wall control unit; alternatively, in order to reduce hardware cost, it could run on a remote server as a cloud computing application, managed by an entity offering this service. The wall control unit also requires communications capability, for example, a wireless connection to the house's broadband internet.

At the time of installation, the four thermal parameters of the ETP model, namely, C^a, C^m, U^a, and U^m, would have to be programmed into the unit, since they are required for the model-based optimization process. These, together with m^i and m^s, may be determined using a standard spreadsheet-like calculation process based on the physical dimensions of the house such as the number of stories, the number and orientation of windows and doors, the floor area, and the level of thermal insulation [23]. The installer also would need to enter the BR and COP functions of the A/C unit.

For R's thermal comfort function, we adopt the following simple representation loosely based on the ANSI/ASHRAE 55-2010 standard [7] and similar to the one used in [12]:

$$\begin{aligned} G_n = G(\mathbf{x}_n, \mathbf{x}_{n+1}, n) = \\ = G_{\max} - h_1 f(\mathbf{x}_{n,1}, n) - h_2 f(\mathbf{x}_{n+1,1}, n+1), \end{aligned} \tag{26}$$

where the function f is defined as

$$f(x, n) = \begin{cases} [x - (T^b - k_{1n})]^\gamma & \text{if } x < T^b - k_{1n} \\ [x - (T^b + k_{2n})]^\gamma & \text{if } x > T^b + k_{2n} \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

The parameters h_1, h_2, k_{1n}, and k_{2n} are positive constants, whereas γ is a positive even integer. An increase in γ increases the magnitude of the slope of the discomfort function f when moving away from the bliss temperature range.[6]

This modeling of R's comfort function can be interpreted as follows. For periods n during which the house resident R is at home, he can set $k_{1n} = k_{2n} = 0$, so that he attains his maximum comfort level when the air temperature (the first element of $\mathbf{x}_n$) is maintained at his bliss temperature. When

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[6]The thermal comfort parameters h_1, h_2, and γ could be modeled as time varying without any technical difficulty. However, R's thermal comfort function (26) is meant to measure the true comfort (benefit) that R attains from the thermal state of his house under different thermal and occupancy conditions, independently of cost considerations. A change in the values of these parameters over time would therefore have to reflect some type of time variation in R's basic preferences for thermal comfort. This does not seem reasonable for the relatively short planning interval (one or two days) that we have in mind for the problem formulation.
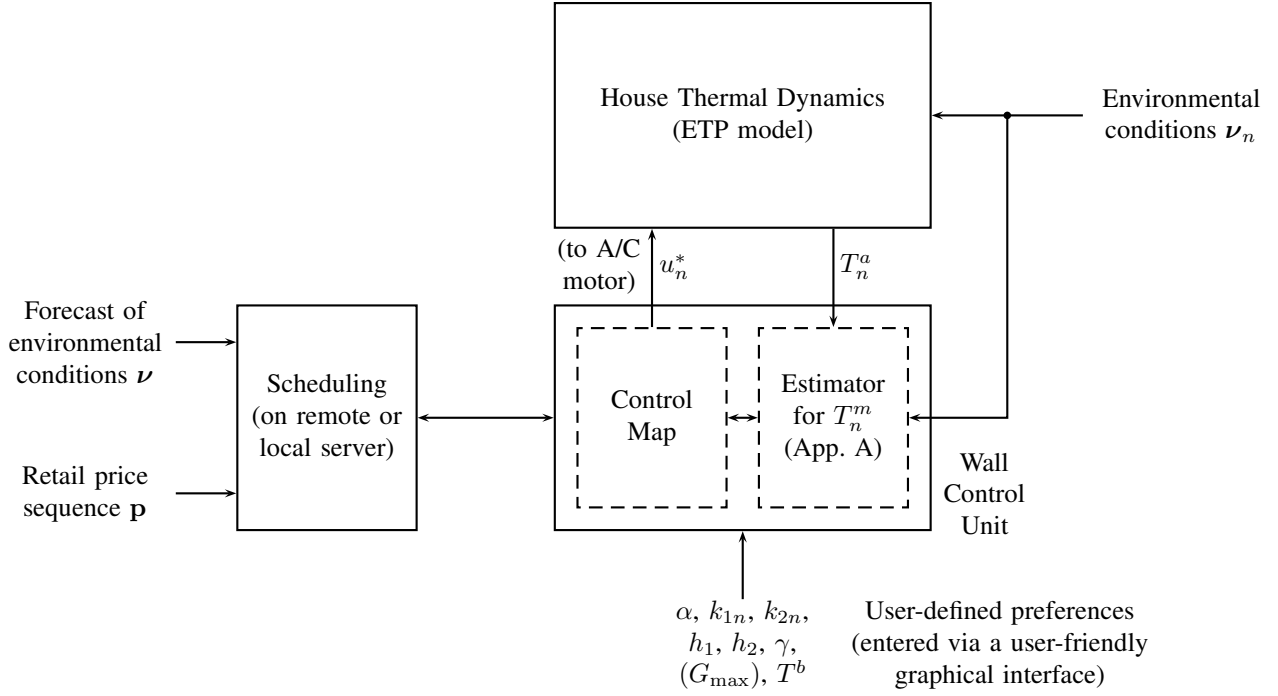
Fig. 1. Block-diagram schematic of the intelligent A/C system control.

the resident is not at home, nonzero values for k_{1n} and k_{2n} can be set, so that the same comfort level is attained within a range of temperatures, $T^b - k_{1n}$ and $T^b + k_{2n}$. In other words, the resident, while absent, is indifferent to the actual temperature inside the house, as long as it stays within the pre-specified range (for instance, to protect pets, foodstuff, or medicinal supplies). It should be noted that R could also decide to have nonzero k_{1n} and k_{2n} set-points even while at home, if this is his preference. The choice of constant representing the maximum comfort level attained ($G_{\max}$) is not of any practical significance, since it does not affect the result of the optimization. Its numerical value can be selected so that R's total net benefit has a positive value, measured in Utils, although this is not critical.

The resident R could program his comfort and cost preferences either directly on the wall control unit or (more realistically) via a user-friendly graphical user interface, which could run on R's personal computer, smart phone, or some other mobile computing device. The latter would allow R to program the device without directly entering numerical values; these would be determined internally by the software. The parameters reflecting R's preferences are communicated to the scheduling program. Whenever R decides to modify his bliss temperature or some other parameter, the updated parameter set would be re-sent, and the optimal scheduling would have to be recomputed. The scheduling algorithm also needs the day-ahead price sequence $\mathbf{p}$ and a forecast of future environmental conditions included in vector $\boldsymbol{\nu}$. In particular, it is quite challenging to obtain an accurate forecast of the internal heat flow rate $\dot{Q}_n^i$, which arises from various sources such as people, lights, and electrical appliances. Therefore, a typical variation of this term must be assumed, for example,

using the recommendations of [24]. Nevertheless, R might be willing to provide some additional information, such as the number of occupants and relevant details of their daily occupancy schedule, or whether visitors are expected on a certain date/time, which would help improve the scheduling.

Proper discretization of the state vector $\mathbf{x}_n$ is necessary for computational tractability when solving the scheduling problem. The internal air temperature is assumed to vary in the range $[T^b - 24, T^b + 24]$. To obtain reasonable accuracy, the range is discretized using $N_a = 481$ points, yielding an accuracy of 0.1 °F. The internal mass temperature is discretized with N_m points. Generally, the difference between T_n^a and T_n^m will be small. Herein, it is assumed that T_n^m lies in $[T_n^a - 4.8, T_n^a + 4.8]$, and that $N_m = 481$, yielding an accuracy of 0.02 °F for the difference $(T_n^a - T_n^m)$. A grid has thus been formed containing all allowable combinations of (T_n^a, T_n^m). When applying equation (17) during the dynamic programming algorithm, the states obtained are not guaranteed to lie on the grid, so they are moved to their nearest grid point, as illustrated in Fig. 2. This prevents the gradual increase of the grid size as dynamic programming proceeds backwards in time. Equations (11) and (12) are then used to develop the control map for the entire planning horizon. The control map is an $(N_a N_m) \times N$ matrix containing zeros or ones, where each element represents an on-or-off solution of (12). For instance, in this implementation, the computer memory required to store this map in binary format (using one bit for each element) is approximately 40 MB, or as low as 2 MB if sparse-matrix storage techniques are used. The dynamic programming algorithm was programmed in Matlab, and takes ca. 40 seconds to run on an Intel Core 2 Duo CPU E8400 3-GHz processor with 4 GB of RAM.
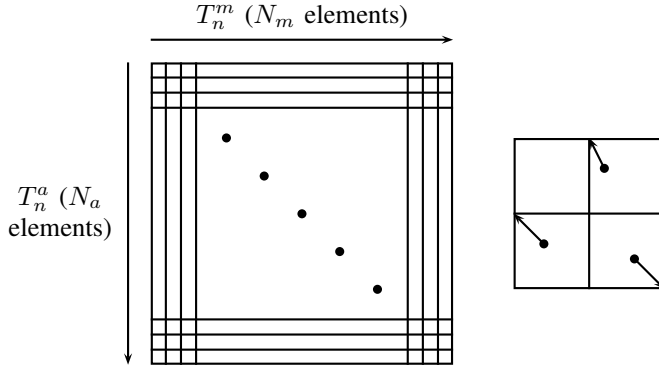
Fig. 2. Discretization of the state vector $\mathbf{x}_n$.

It should be noted that the internal mass temperature T_n^m cannot be obtained by direct measurement. However, as demonstrated in Appendix A, a Luenberger observer can be designed to estimate it using measurements of the environmental variables (and reasonable assumptions for the internal heat flow rate). These measurements could be obtained by actual temperature, solar irradiation, and humidity sensors installed at the house, or indirectly from weather monitoring websites.

V. SIMULATION RESULTS

This section reports simulation findings for the proposed intelligent A/C system controller. These simulation findings indicate that the controller works as expected to provide a flexible way for a house resident to optimally trade off thermal comfort against costs over time, conditional on his preferences for comfort, his anticipated occupancy times, and his A/C energy usage costs.

As discussed in Section II-A and Appendix B, the α parameter appearing in resident R's net benefit function (7) is an attribute of R reflecting his marginal benefit of income, not a control variable. Previous studies have not paid attention to the key role played by this attribute parameter in the determination of optimal comfort/cost trade-offs for household residents. Consequently, the simulations reported in this section explore outcomes for a range of possible α values for R.

Other parameter values are set as follows. The thermal comfort parameter values for R are set at $G_{\max} = 1.5$ Utils, $h_1 = h_2 = 0.04$ Utils/(°F)^2, $\gamma = 2$, and $T^b = 74$ °F. The thermal model parameter values for R's house are set at $C^a = 794.5$ BTU/°F, $C^m = 4726.4$ BTU/°F, $U^a = 444.3$ BTU/h/°F, $U^m = 7501$ BTU/h/°F, $m^s = m^i = 0.5$, BR$_o$ = 42000 BTU/h, COP$_o$ = 3.8, and $m^l = 0.3$. These were obtained for a hypothetical 1500 ft^2 single-story house with very good insulation.

Meteorological data are obtained from the typical meteorological year (TMY2) database [25], which contains records of a typical year for most of the regions in the United States. A relatively hot day is simulated based on the data corresponding to June 14th, 2009 in Detroit, Michigan. The data are smoothed to represent actual weather conditions and an offset is added to the temperature data. The day-ahead scheduling is carried out based on the outside temperature and relative humidity of the modified data as the forecast.
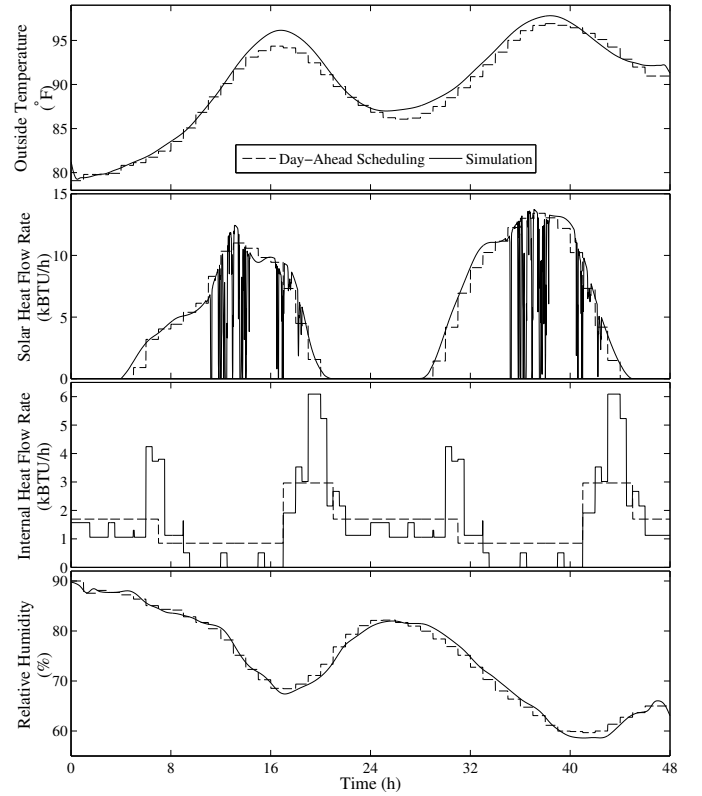

Fig. 3. Variation of environmental parameters for day-ahead scheduling and simulation.

For the simulation of the A/C system, artificial conditions are synthesized based on the modified TMY2 data. To this end, a small perturbation is superimposed on the modified data to simulate actual (different than forecasted) conditions. The solar radiation incident on the house is a function of the direct normal radiation and the diffuse horizontal radiation. The solar heat gain factor [22] is then used to calculate the heat flow rate from the solar radiation. Radiation data are obtained from the TMY2 file; however, since these are provided on an hourly time-scale, other higher-frequency recorded data from NREL [26] are used to simulate cloud movement in a more realistic fashion.

A crudely predetermined schedule of appliances (based on the design value of internal heat flow rate [22]) is used to construct the internal heat flow rate for the day-ahead scheduling. A finer variation of appliances and occupant activity is assumed to occur in the simulation. The variation of all environmental parameters used for day-ahead scheduling and in simulations is depicted in Fig. 3. For scheduling, the variables are represented by piecewise constant functions, changing every hour.

The retail price corresponding to the chosen region (Detroit, MI) is the day-ahead LMP obtained from an historical LMP report [27] for the Midwest ISO. The price p_n in (5) includes the LMP plus a mark-up of 5 cents/kWh, whereas $F = 0$. The retail price variation is shown in Fig. 4.

Simulations are run using a 2-day planning horizon, where each period Δt is 2 minutes long (implying $N = 1440$). The discount factors β_n in (7) are specified to be 1.0 for the first
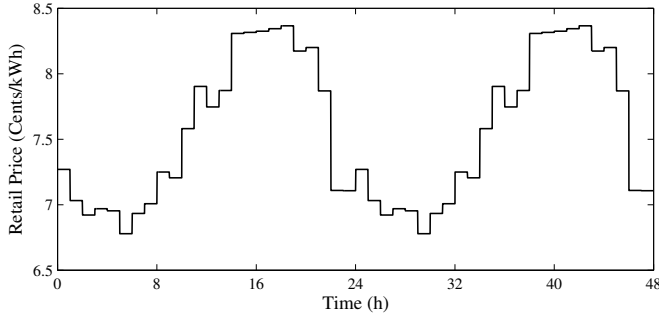
Fig. 4. Retail price variation.

TABLE I
RESULTS WITH $k_{1n} = k_{2n} = 0$

α	24-hour Electricity Cost (\$)	24-hour Thermal Comfort (Utils)	24-hour Net Benefit (Utils)	24-hour Energy (kWh)
N/A	2.14	1078.8	N/A	27.3
0	2.14	1072.4	1072.4	27.4
50	2.13	1070.7	964.1	27.3
200	2.11	1064.3	641.8	27.0
500	2.09	1052.7	9.7	26.7
1000	2.04	1016.3	-1026.6	26.2
2000	1.88	862.7	-2888.8	24.1
3000	1.77	675.0	-4624.6	22.7
4000	1.70	429.7	-6374.5	22.0

day of the planning horizon and 0.9 for the second day of the planning horizon.

The general A/C controller set out in Section II-A postulates the existence of a joint PDF for the environmental variables over the planning horizon. Nevertheless, for implementation purposes, it would generally be very difficult to obtain or estimate such a joint PDF. Here we make use of a "certainty equivalence" approach to derive an approximate solution for the optimal on/off A/C controls. This approach replaces the random environmental variables over the planning horizon by their expected values, reducing the problem to a deterministic dynamic programming problem. Since the application at hand involves only a short two-day planning horizon, the approximate solution should be reasonably close to the optimal solution.

A few simplifications are introduced to ease the presentation of results. First, because day-ahead LMPs cannot be known with certainty two days in advance, it is assumed that the price sequence for the second day of the planning horizon is forecasted to be the same as for the first day. Second, although a new optimization takes place at the end of each day for a two-day planning horizon, optimization outcomes are only shown for the first 24 hours of each two-day planning horizon.

Finally, it should be noted that the two-day rolling-horizon optimization implemented for the application at hand to generate updates to the A/C control map could instead be undertaken at shorter intervals (e.g., hourly). A shorter rolling-horizon specification would presumably permit a greater forecast accuracy for the environmental variables and improved comfort/cost optimization outcomes, but at the cost of increased computational time.

A. Resident Stays at Home Throughout the Day

As a first case study, the resident is assumed to remain at home throughout the day, maintaining a constant bliss temperature, and $k_{1n} = k_{2n} = 0$. For comparison purposes, a simulation was first run over a 24-hour horizon using a classical A/C thermostat operating with simple hysteresis control, with a deadband of ± 0.25 °F. The thermal comfort obtained was 1078.8 Utils. (The ideal daily thermal comfort is $(N/2)G_{\max} = 1080$ Utils.) The energy consumption of the A/C system was 27.3 kWh and the electricity cost was \$2.14. These results are listed as the first row of Table I for the reader's convenience.
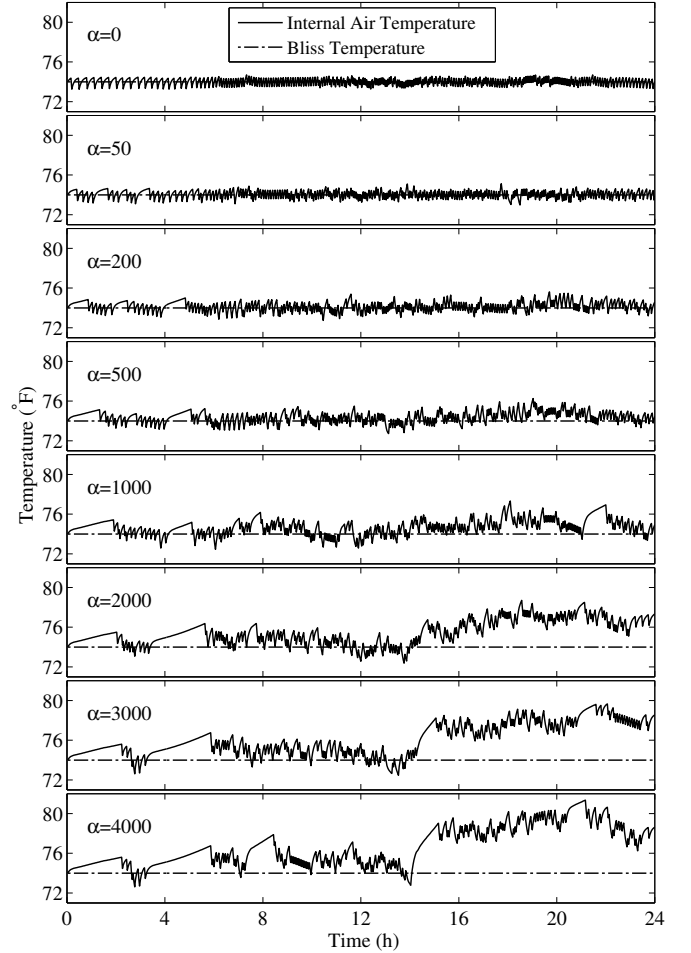

Fig. 5. Variation of internal air temperature (T_n^a) for various α values.

Fig. 5 shows the variation of the indoor air temperature inside R's house for a range of α values. It is obvious that, as α is increased, the deviations of T_n^a from T^b become increasingly prominent. Table I summarizes the results. As expected, an increase in α results in increased electric energy savings but lower thermal comfort.

B. Resident Leaves Home During the Day

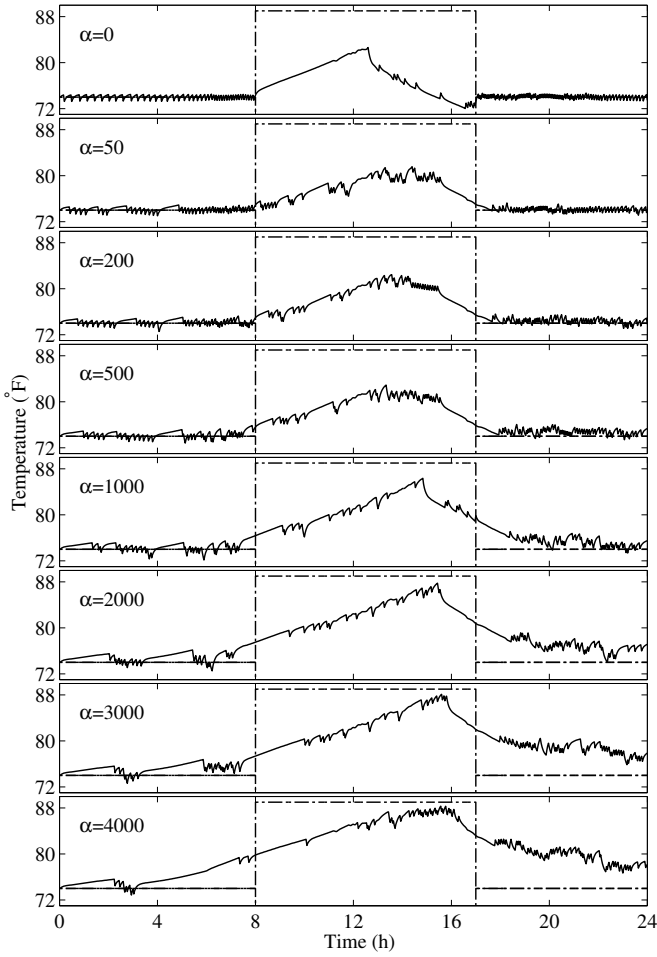For the second case study, the resident is assumed to leave the house from 8 am to 5 pm. During this time, k_{1n} and

Fig. 6. Variation of internal air temperature (T_n^a) for various α values and $k_{1n} = k_{2n} = 15$ °F while the resident is not at home.

TABLE II
RESULTS WITH NONZERO k_{1n}, k_{2n}

α	24-hour Electricity Cost ($)	24-hour Thermal Comfort (Utils)	24-hour Net Benefit (Utils)	24-hour Energy (kWh)
0	2.09	1075.2	1075.2	26.4
50	2.05	1073.1	970.8	25.8
200	2.02	1067.6	664.3	25.4
500	1.96	1055.9	73.6	24.7
1000	1.88	990.6	-891.6	23.7
2000	1.71	826.1	-2594.8	21.4
3000	1.60	582.8	-4214.3	20.1
4000	1.49	287.9	-5670.8	18.5

The critical parameter α appearing in the household resident's net benefit function (7) plays a key role in the determination of the resident's optimal comfort/cost trade-offs. As detailed in Appendix B, α reflects an attribute of the household resident—namely, his marginal benefit of income—that depends on his preferences and on his opportunities for the purchase of alternative goods; α is not a "control variable." As seen in Section V, we envision our A/C controller as having an α "knob" that each household resident can fine tune to match his own particular preferences and choice environment. n Section V we provide numerical examples to show how different settings for this alpha "knob" for different residents would affect the A/C energy usages resulting from the optimal on-off A/C control settings generated by the A/C controller, all else equal.

In a possible future smart-grid scenario, dynamically varying price signals can be communicated to households, thereby achieving active demand response. Our thermal comfort model can form a basis for studying the aggregation of price-sensitive demand emanating from a residential area, since A/C systems constitute a substantial component of residential energy consumption during the summer. The methodology can also be adopted by LSEs to forecast price-sensitive load from their retail customers. Furthermore, there is an interesting feedback loop connecting wholesale load to wholesale prices to retail prices to retail load and back up to wholesale load. In fact, this feedback loop is currently being explored by means of systematic simulation studies [28].

As stressed in Section II, our study is agnostic regarding the exact method by which the LSE servicing the A/C energy needs of retail consumers determines the A/C energy prices. Clearly, however, the ability to offer retail energy contracts under which the prices charged vary dynamically with changing conditions could open up strategic opportunities for profit-seeking LSEs. This important topic is part of our ongoing research.

The general discrete-time SDP problem set out in Section II for the household resident does not assume a finite domain either for the control actions u_n or for the state vectors $\mathbf{x}_n$. For practical application, however, finite discretizations are introduced in Section III for the control and state domains that render our SDP formulation equivalent to a finite-horizon discrete-time Markov Decision Process (MDP). In future studies it would be of interest to compare and contrast our SDP

k_{2n} are set to 15 °F. The simulation results shown in Fig. 6 exhibit a markedly different pattern from the previous case study. Most notably, the A/C controller makes a decision to switch off during the morning hours. For the extreme case of $\alpha = 0$, this switch occurs as soon as the resident leaves home. However, as α is increased, the A/C turns off earlier than that. It is also interesting to observe how the controller decides to cool down the house in anticipation of the resident's arrival at home at 5 p.m., and how this decision varies with different α values.

Table II summarizes the results, which follow a similar trend as for the previous experiment. Comparing Tables I and II, we find that the cost of electricity and the energy consumption have decreased considerably. This is because the A/C is mostly turned off during the time R is not at home.

VI. CONCLUSION

The purpose of this paper is to present the control of an A/C system by stochastic dynamic programming (SDP) to achieve optimal intertemporal trade-offs between thermal comfort and A/C energy costs for a household resident conditional on retail A/C energy prices and environmental conditions. A thermal comfort model is used to capture the thermal preferences of the resident.

solution approach to approaches that have been introduced in the literature for the approximate solution of MDP problems.

It is also of great interest to design a similar controller for inverter-based systems, which are rapidly gaining market share worldwide, because they offer increased efficiency and energy savings (albeit with increased capital cost). However, this is not the case in the United States, where most residential A/C systems are still commonly based on simple on/off control. Therefore, one important advantage of our simple "bang-bang" proposed control is that it lends itself to the retrofitting of existing systems (at least in the USA) with minimal intervention required on the mechanical A/C components. Nevertheless, the general mathematical formulation outlined in Section II certainly permits the formulation of a continuous problem, which would be appropriate for an inverter-based A/C system. This is an important topic for future work.

The general formulation (6) for the household resident's thermal comfort function set out in Section II permits thermal comfort to depend on the initial and final state vectors during period n as well as directly on n. For concrete illustration, however, Section IV uses a simplified thermal comfort function (26) that depends only on the initial and final air temperature of the resident's house for any period n as well as on period-specific lower and upper bounds for the resident's comfort function reflecting whether the resident is actually at home during period n. In future studies it would be of great interest to explore more carefully the implications of alternative thermal comfort function specifications for the welfare of household residents and for system performance more generally. Moreover, it would be important to refine further the physical model of the A/C system, in order to study the impact of improved modeling on the optimization results.

Finally, in future work we intend to implement the proposed intelligent A/C system controller in practice, and to conduct experiments to test its performance. The current study provides the theoretical underpinnings for this experimental validation.

APPENDIX A
LUENBERGER OBSERVER TO ESTIMATE T^m

The ETP model (13)–(14) can be written as

$$\begin{bmatrix} \dot{T}^a \\ \dot{T}^m \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} T^a \\ T^m \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1^T \\ \mathbf{b}_2^T \end{bmatrix} \mathbf{w} \quad (28)$$

where A_{11}, A_{12}, A_{21} and A_{22} are the scalar elements of the matrix $\mathbf{A}$, and $\mathbf{b}_1^T$ and $\mathbf{b}_2^T$ are the two rows of the matrix $\mathbf{B}$. An estimate for the mass temperature can be constructed as

$$\dot{\hat{T}}^m = (A_{22} - \tilde{K}A_{12})\hat{T}^m + A_{21}T^a + \mathbf{b}_2^T\mathbf{w}$$
$$+ \tilde{K}(\dot{T}^a - A_{11}T^a - \mathbf{b}_1^T\mathbf{w}) . \quad (29)$$

The gain $\tilde{K}$ is chosen such that $A_{22} - \tilde{K}A_{12} < 0$, in which case it can be shown that the error $T^m - \hat{T}^m$ asymptotically approaches zero as $t \to \infty$ [29]. However, this estimator requires knowledge of $\dot{T}^a$, which is unknown. To eliminate $\dot{T}^a$, we let $z \equiv \hat{T}^m - \tilde{K}T^a$, and (29) leads to a modified estimator

in terms of z, given by

$$\dot{z} = (A_{22} - \tilde{K}A_{12})z + [(A_{22} - \tilde{K}A_{12})\tilde{K} + A_{21} - \tilde{K}A_{11}]T^a$$
$$+ (\mathbf{b}_2^T - \tilde{K}\mathbf{b}_1^T)\mathbf{w} . \quad (30)$$

The mass temperature is estimated from $\hat{T}^m = z + \tilde{K}T^a$. This observer logic could be readily programmed in the wall unit, in discrete-time form. For the simulation studies of Section V, the gain was set to $\tilde{K} = -7$.

APPENDIX B
EXTENDED MOTIVATION FOR THE COMFORT/COST TRADE-OFF MODEL

Here we present additional motivation for the form of resident R's comfort/cost trade-off problem (10) set out in Section II-A. In particular, we show that, for an appropriate choice of α, the solution of this problem can be viewed as a necessary condition for the solution of a more comprehensive problem involving the budget-constrained maximization of the benefit attained by R over periods $1, \ldots, N$ from the consumption of multiple goods in addition to thermal comfort.

As is standard in microeconomic treatments of multi-good optimization problems, suppose the multi-good benefit obtained by R over periods $1, \ldots, N$ is given by the function

$$W(\mathbf{u}, \mathbf{y}) = \sum_{n=1}^{N} \beta_n G(\mathbf{x}_n, S_n(\mathbf{x}_n, u_n, \boldsymbol{\nu}_n), n) + Z(\mathbf{y}) , \quad (31)$$

where the state vectors $\mathbf{x}_n$ satisfy the state equations (1) and (2), and the dependence of W on the exogenously given terms $\boldsymbol{\nu}$ and $\bar{\mathbf{x}}_1$ has been supressed from the notation. As in Section II-A, the summation term measures the benefit (comfort) attained by R from the thermal conditions inside his house during periods $1, \ldots, N$. Now, however, there is also a second term, $Z(\mathbf{y})$, measuring the benefit (satisfaction) attained by R from the consumption of a vector $\mathbf{y} = [y_1, \ldots, y_m]^T$ of m additional types of goods during periods $1, \ldots, N$. Assume that R strictly prefers more of each of these goods to less, all else equal, implying that $Z(\mathbf{y})$ is a strictly increasing function of y_j for each $j = 1, \ldots, m$.

Let $\mathbf{p}^y = [p_1^y, \ldots, p_m^y]$, where p_j^y denotes the dollar amount paid by R per unit of consumption of good j. Also, assume that the A/C electric energy prices $\mathbf{p} = [p_1, \ldots, p_N]$, the goods prices $\mathbf{p}^y$, and the environmental conditions $\boldsymbol{\nu} = [\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_N]$ are known by R prior to the start of period 1. Let I (\$) denote R's target total income expenditure level for periods $1, \ldots, N$, and let $\mathbf{u} = [u_1, \ldots, u_N]$ and $\mathbf{y}$ denote the choice vectors for R.

Now consider the following optimization problem for R involving the maximization of his multi-good benefit function (31) subject to a budget constraint:[7]

$$\max W(\mathbf{u}, \mathbf{y}) \quad (32)$$

[7]For expositional simplicity, the restriction of u_n to some admissible domain U and the restriction of $\mathbf{y}$ to the nonnegative orthant in Euclidean m-space are ignored below. Also, the assumed nonsatiation of R with respect to consumption of $\mathbf{y}$ guarantees that R will satisfy his budget constraint as a strict equality.

with respect to choice of $\mathbf{u}$ and $\mathbf{y}$, subject to

$$\sum_{n=1}^{N} \beta_n C(p_n, \tilde{e}_n(u_n, \boldsymbol{\nu}_n)) + \mathbf{p}^y \cdot \mathbf{y} = I . \quad (33)$$

Let α denote the Lagrange multiplier corresponding to the budget constraint (33), and form the Lagrangian function $\mathcal{L}$ as follows:

$$\mathcal{L}(\mathbf{u}, \mathbf{y}, \alpha, I) = W(\mathbf{u}, \mathbf{y})$$
$$+ \alpha \left[I - \sum_{n=1}^{N} \beta_n C(p_n, \tilde{e}_n(u_n, \boldsymbol{\nu}_n)) - \mathbf{p}^y \cdot \mathbf{y} \right] \quad (34)$$

Suppose the usual Karush-Kuhn-Tucker (KKT) first-order necessary conditions expressed in terms of the Lagrangian function $\mathcal{L}$ result in unique solutions $(\mathbf{u}^*, \mathbf{y}^*, \alpha^*)$ for $\mathbf{u}$, $\mathbf{y}$, and α. Let these solutions be expressed in the form

$$(\mathbf{u}^*, \mathbf{y}^*, \alpha^*) = (\mathbf{u}(I), \mathbf{y}(I), \alpha(I)), \quad (35)$$

where dependence on all exogenous variables except income I has been suppressed from the notation. Given certain regularity conditions, it follows by the envelope theorem[8] that $\alpha(I)$ measures R's marginal benefit of income[9] in the sense that

$$\alpha(I) = \frac{dW(\mathbf{u}(I), \mathbf{y}(I))}{dI} . \quad (36)$$

That is, $\alpha(I)$ measures the change in R's optimized multi-good benefits with respect to a change in his income I, evaluated at the solution point.

Finally, here is the interesting observation that motivates this appendix discussion. If α is pre-set at the level $\alpha(I)$ in the Lagrangian function $\mathcal{L}$ in (34), this function separates into two parts, one involving only $\mathbf{u}$ and the other involving only $\mathbf{y}$, as follows:

$$\sum_{n=1}^{N} \beta_n \left[G(\mathbf{x}_n, \mathbf{x}_{n+1}, n) - \alpha(I) C(p_n, \tilde{e}_n(u_n, \boldsymbol{\nu}_n)) \right] \quad (37)$$

and

$$Z(\mathbf{y}) + \alpha(I)[I - \mathbf{p}^y \cdot \mathbf{y}] . \quad (38)$$

The optimal setting of $\alpha(I)$ in (37) and (38) guarantees that R's income I is optimally split between expenditures on electric energy for A/C and expenditures on the consumption goods $\mathbf{y}$. Consequently, the two parts can be separately treated as individual optimization problems.

In particular, the maximization of (37) with respect to $\mathbf{u}$, the approach taken in Section II, results in the satisfaction of the KKT necessary first-order conditions for the choice of $\mathbf{u}$ corresponding to the more comprehensive budget-constrained multi-good benefit maximization problem handled in this appendix that involves a simultaneous choice of both $\mathbf{u}$ and $\mathbf{y}$. Thus, by appropriate trial-and-error experimentation, resident R could arrive at a setting for the comfort/cost trade-off factor α in (7) that approximately achieves his optimal A/C energy usage solution for this more comprehensive problem.

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[8]Applied to the problem at hand, the envelope theorem [30, Chap. 1, Thm. 1.F.1] guarantees that:

$$dW(\mathbf{u}(I), \mathbf{y}(I))/dI = d\mathcal{L}(\mathbf{u}(I), \mathbf{y}(I), \alpha(I), I)/dI =$$
$$= \partial\mathcal{L}(\mathbf{u}(I), \mathbf{y}(I), \alpha(I), I)/\partial I = \alpha(I)$$

where d denotes total differentiation and ∂ denotes partial differentiation.

[9]In the economics literature, in which consumer benefits are assumed to be measured by "utility functions," it is standard to refer to α as a marginal utility of income measure.

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