arXiv:1712.08622v1 [math.OC] 22 Dec 2017

Analysis and Implementation of a Hourly Billing Mechanism for Demand Response Management

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Abstract-An important part of the Smart Grid literature on residential Demand Response deals with game-theoretic consumption models. Among those papers, the hourly billing model is of special interest as an intuitive and fair mechanism. We focus on this model and answer to several theoretical and practical questions. First, we prove the uniqueness of the consumption profile corresponding to the Nash equilibrium, and we analyze its efficiency by providing a bound on the Price of Anarchy. Next, we address the computational issue of the equilibrium profile by providing two algorithms: the cycling best response dynamics and a projected gradient descent method, and by giving an upper bound on their convergence rate to the equilibrium. Last, we simulate this demand response framework in a stochastic environment where the parameters depend on forecasts. We show numerically the relevance of an online demand response procedure, which reduces the impact of inaccurate forecasts.

Index Terms—Smart grid, Demand Response, Demand Side Management, Game Theory, Nash Equilibrium, Best Response.

I. INTRODUCTION

EMAND Response (DR) is a technique to exploit electricity consumers flexibilities by giving them particular incentives, in order to achieve some services to the grid e.g. reducing production and distribution costs or increasing renewable energy insertion [1]. In DR programs, the aggregated energy demand is a key metric and an aggregator interacts with active consumers, willing to minimize their electricity bill or maximize their utility, to optimize this demand profile. In such a framework, energy can be viewed as an asset demanded by customers, and which has a cost that depends on total demand and the time of demand-some congestion effects arise on the most demanded time periods and as producers have a limited quantity to offer. Game-theory offers a well adapted environment to model these congestion effects, and different game-theoretic framework have been proposed in the Smart Grid literature [2, 3, 4], following the seminal paper [5]. Among these, the hourly billing mechanism, introduced in [6, 7] and also adopted in [3, 8] emerges as a natural model-it has a structure of a routing congestion game [9]enjoying important fairness properties, although its theoretical analysis is harder than the mechanism proposed in [5]. In game-theoretic models, a major issue is to define an efficient procedure to compute and reach the consumption equilibrium associated with the game. Several papers [3, 8, 10] have investigated the complexity and algorithmic aspects associated to the notion of equilibrium. One can refer to [11] for a

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survey of classical optimization methods and their applicability depending on the DR framework. In [8], the authors consider a distributed generation environment and use an algorithm based on an iterative proximal best response. In [3], a hourly billing mechanism is used and the authors propose a proximal-point algorithm to compute the equilibrium.

In this paper, we investigate the theoretical properties and computational aspects of the hourly billing mechanism and discuss its practical implementation. The main contributions are the following. First, we prove the uniqueness of the equilibrium (Thm. 1) under a convexity assumption. The uniqueness of the equilibrium profiles was proved for the daily billing mechanism proposed in [5]. For the hourly billing mechanism, [3, Prop. 1] gives the uniqueness for a particular class of price functions. Our result applies to any convex and increasing price functions, and extends [9, Thm. 1] to a more general model where we consider bounds on the load at each time period. Next, we give a bound on the resulting Price of Anarchy (PoA), which shows the efficiency of the equilibrium (Thm. 2). This result should be compared with [5] where the model induces a PoA equal to one (optimality). Here, the PoA is numerically close to one but not one. Then, we bound the convergence rate of the best response algorithm in the case of affine prices (Thm. 3). In that case, convergence is known but to our knowledge, no bound on the rate has ever been given. The convergence has been conjectured more generally for any convex prices [12, 13]. We introduce a different algorithm based on a simultaneous projected gradient descent (Algo 2), and show its geometric convergence (Thm. 4) with a condition on the price functions only. To our knowledge, those results are also new. They can be compared with the algorithm 1 proposed in [3]. In particular, we allow a fix step-size and we do not need a proximal term. Last, we introduce an online DR procedure with receding horizons, to take into account updated forecasts in a stochastic environment. We show numerically, based on real consumption data, that this procedure can achieve significant savings compared to an offline procedure.

This paper reassembles and extends the main results on the hourly billing model announced in our conference papers [14, 15]: in Thm. 2 here, we give the upper bound on the Price of Anarchy [14, Thm. 2] and in Prop. 1 we use the same property than [14, Thm. 1]. We also use the potential property of the game noticed in [15, Thm. 2] and the *Best Response* algorithm presented in [14, Def. 3]. However there are several additional results in this paper: the theorem of uniqueness of the equilibrium presented here (Thm. 1) is stronger than [14, Thm. 1]. Also, the SIRD algorithm (Algo 2) and the convergence theorems (Thms. 3 and 4) were not presented in

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[14, 15]. We complete the simulation framework in [14, 15] by considering multiple forecasts on the nonflexible load and by introducing an online demand response procedure (Algo 3).

This paper is organized as follows: Sec. II gives the mathematical model of the DR framework and the associated billing mechanism, under the form of a game. In Sec. III, we introduce the notion of strong stability for general games and define two decentralized algorithms that enable to compute the equilibrium consumption profiles. We prove the convergence of those algorithms and give guarantees on their convergence rate. Finally, in Sec. IV, we define an *online* DR procedure and simulate it with historical consumption data of consumers with electric vehicles as flexible consumptions. We compare the performance of this online DR scheme to the *offline* version and other consumption scenarios.

NOTATION CONVENTION: through this paper, bold font ℓ is used to denote a vector as opposed to a scalar ℓ .

II. CONSUMPTION GAME WITH HOURLY BILLING

A. District of flexible consumers

We consider a network composed of a set $\mathcal{N} = \{1, \ldots, N\}$ of residential consumers linked to a local *aggregator*. Each household is equipped with a smart meter enabling two-way communication of information with the aggregator. We assume that each household electricity consumption can be divided into two parts: one which is *nonflexible* (lights, cooking appliances, TVs) and one which is *flexible* (Electric Vehicle charging, water heater, washing machine, etc). Moreover, each smart meter is linked to an *Electricity Consumption Scheduler* (ECS) that can automatically optimize and schedule the consumption profile of the flexible appliances of the user, given the constraints set by the consumer and physical constraints of each appliance.

B. From individual to aggregated consumption profiles

In the DR program, we determine a consumption profile for each consumer on a finite time horizon T. In this study, we take \mathcal{T} as a discrete set of time periods $\mathcal{T} = \{1, \ldots T\}$. In the simulations, \mathcal{T} will correspond to one day, and each time period t to one hour. We assume that each consumer has a Nonflexible (NF) consumption profile; we denote by $L_{\text{NF}} \in \mathbb{R}^T$, indexed by \mathcal{T} , the aggregated nonflexible load profile on the set of consumer \mathcal{N} . On top her nonflexible consumption, each consumer n has a flexible consumption profile that we denote by $\ell_n = (\ell_{n,1}, \cdots, \ell_{n,T}) \in \mathbb{R}^T$. The aggregated flexible load profile on the set of consumers is obtained as:

$$\boldsymbol{L} = (L_t)_{t \in \mathcal{T}} \in \mathbb{R}^T \text{ with } \forall t \in \mathcal{T}, L_t \stackrel{\Delta}{=} \sum_{n \in \mathcal{N}} \ell_{n,t} .$$
 (1)

C. Aggregator cost: a function of aggregated consumption

The aggregator is himself linked to electricity providers and we consider that he faces a per-unit (of energy) price function¹ $L_t \mapsto c_t(L_t)$ associated with each time period $t \in \mathcal{T}$ for the flexible electricity demand L_t given in (1), so that the cost of providing L_t at t is $L_t \times c_t(L_t)$. Price functions can depend on the nonflexible consumption profile L_{NF} because they model a global cost for the aggregator. In particular, we will assume in our model that the aggregator faces for each time period providing costs $C_t(.)$ that depend on the total load $L_{\text{NF},t} + L_t$. The price function $c_t(.)$ for the flexible part of consumption at t can be rewritten as:

$$c_t(L_t) \stackrel{\Delta}{=} \frac{1}{L_t} \left[C_t(L_{\text{NF},t} + L_t) - C_t(L_{\text{NF},t}) \right] \,. \tag{2}$$

Remark 1. Following (2), the price function $c_t(.)$ depends implicitly on the value of the nonflexible load $L_{NF,t}$ (even if the dependancy is not explicit in the notations used hereafter).

In our framework, we will consider the following assumptions on the price functions $(c_t)_t$:

Assumption 1. For each $t \in \mathcal{T}$, c_t is twice differentiable, strictly increasing and convex.

Assumption 2. For each $t \in \mathcal{T}$, c_t is twice differentiable, convex and strictly increasing. Moreover, there exists a > 0 s.t. for any t and admissible ℓ :

$$2c_t'(L_t)\left(1 - \left(\frac{c_t''(L_t)}{2c_t'(L_t)}\right)^2 \|\boldsymbol{\ell}_t\|_2^2\right) \ge a.$$
(3)

Assumption 3. For each $t \in \mathcal{T}$, c_t is affine, positive and increasing: $\forall t \in \mathcal{T}$, $c_t(\ell) = \alpha_t + \beta_t \ell$ with $\alpha_t, \beta_t \in (\mathbb{R}^*_+)^2$.

Remark 2. The latter three assumptions are more and more restrictive: Assumption 3 implies Assumption 2 with $a = 2 \min_t \beta_t$, and Assumption 2 obviously implies Assumption 1.

Remark 3. For a = 0, inequality (3) in Assumption 2 simplifies to the condition: $\|\boldsymbol{\ell}_t\|_2^{-1} \ge \left|\frac{c_t''(L_t)}{2c_t'(L_t)}\right|$. For each t, c_t'' has to be small relatively to c_t' .

Assumption 1 is standard in the congestion games literature and corresponds to "type-B" functions in the seminal paper [9]. This assumption is also made in most of the papers dealing with game-theoretic DR models as [7]. Indeed, it is justified by the fact that marginal costs of producing and providing electricity are increasing. Assumption 3 is also a standard assumption made in [16] because it enables fast computation of NE (see Sec. III), but it is very restrictive. Assumption 2 is not very explicit but is an in-between condition that comprises a larger set of functions than linear functions and for which our main results hold.

Whatever the assumption retained, the aggregator can influence the flexible consumption by sending incentives to consumers through a *billing mechanism*, that is, by defining what each consumer n will pay relatively to her flexible consumption profile ℓ_n , and in turn which quantity n's ECS will minimize.

D. Consumer's Optimization Problems

In this paper, following our studies in [14, 15], we will use an hourly proportional billing mechanism, where each

¹This price function can represent different objectives of the aggregator as the minimization of the distance to a targeted aggregated consumption profile L^* , the minimization of providing costs or the maximization of selfconsumption of (local) renewable production.

consumer $n \in \mathcal{N}$ minimizes her bill:

$$b_n(\boldsymbol{\ell}_n, \boldsymbol{\ell}_{-n}) \stackrel{\Delta}{=} \sum_{t \in \mathcal{T}} \ell_{n,t} c_t(L_t) , \qquad (4)$$

where $\ell_{-n} \stackrel{\Delta}{=} (\ell_m)_{m \neq n}$ denotes the consumption of all consumers but *n*. This billing mechanism was shown to have interesting fairness properties and is also adequate when considering consumers' utility functions (representing, e.g., temporal preferences for flexible consumption) [14, 15, 16]. It corresponds to a routing "atomic splittable" congestion game framework [9], well studied in the game theory literature, where we add the bounding constraints (5c) below.

Through her ECS, each consumer will adjust her flexible consumption profile $\ell_n \in \mathbb{R}^T$ to minimize her bill, which corresponds to the following optimization problem:

$$\min_{\boldsymbol{\ell}_n \in \mathbb{R}^T} b_n(\boldsymbol{\ell}_n, \boldsymbol{\ell}_{-n}) \tag{5a}$$

s.t.
$$\sum_{t \in \mathcal{T}} \ell_{n,t} = E_n$$
, (5b)

$$\underline{\ell}_{n,t} \leqslant \ell_{n,t} \leqslant \overline{\ell}_{n,t}, \forall t \in \mathcal{T} .$$
(5c)

Constraint (5b) ensures that the total energy given to n satisfies her daily flexible energy demand over \mathcal{T} , denoted by E_n , that we assume fixed and deterministic in our work². Constraint (5c) takes into account the physical power constraints and the personal scheduling constraints (supposed given by the user to her ECS). Note that taking $\underline{\ell}_{n,t} = \overline{\ell}_{n,t} = 0$ forces $\ell_{n,t} = 0$ so that constraint (5c) includes in particular unavailability during some time periods. We will denote by \mathcal{L}_n the feasible set of user n, given by constraints (5b-5c), and $\mathcal{L} \stackrel{\Delta}{=} \mathcal{L}_1 \times \cdots \times \mathcal{L}_N$ the Cartesian product of the feasible sets. As b_n depends both on ℓ_n and ℓ_{-n} through L_t , we get a N-person minimization game that we write in the standard form [17] as $\mathcal{G} \stackrel{\Delta}{=} (\mathcal{N}, \mathcal{L}, (b_n)_n)$.

In game-theoretic models, a desirable stability property is when each player n has no interest to deviate unilaterally from her current profile ℓ_n . This corresponds to the notion of Nash Equilibrium (NE), that is:

Def. 1 (Nash et al., 1950). *Nash Equilibrium (NE).* (ℓ_n^{NE}) is a NE of the minimization game: $\mathcal{G} = (\mathcal{N}, \mathcal{L}, (b_n)_n)$ iff for any player $n \in \mathcal{N}$:

$$b_n(\boldsymbol{\ell}_n^{\text{NE}}, \boldsymbol{\ell}_{-n}^{\text{NE}}) \leq b_n(\boldsymbol{\ell}_n, \boldsymbol{\ell}_{-n}^{\text{NE}}), \ \forall \boldsymbol{\ell}_n \in \mathcal{L}_n$$
$$\Leftrightarrow \boldsymbol{\ell}_n^{\text{NE}} \in \underset{\boldsymbol{\ell}_n \in \mathcal{L}_n}{\operatorname{argmin}} \ b_n(\boldsymbol{\ell}_n, \boldsymbol{\ell}_{-n}^{\text{NE}}) \ .$$

It is known that an NE may not exist or may not be unique, even in routing congestion games [9]. In our framework however, both properties are ensured, as stated in Thm. 1 below. This result extends the uniqueness theorem in [9] in presence of the constraint (5c) on power bounds.

Theorem 1. Under Assumption 1, G has a unique NE.

Proof: See Appendix A.

As said above, an NE is a very interesting situation in practice because of its stability: each player will only increase her bill if she changes her profile. However, an NE does not necessarily minimize the *social cost*:

$$\operatorname{SC}(\boldsymbol{\ell}) \stackrel{\Delta}{=} \sum_{n} b_n(\boldsymbol{\ell}) \ .$$
 (6)

Note that, with the billing equation (4), $SC(\ell)$ is equal to the total system $cost^3 \sum_t L_t c_t(L_t)$, so that this quantity should be minimized from the aggregator point of view. In general games, an NE can be sub-optimal in terms of SC. To measure the inefficiency of Nash Equilibria, a standard quantity is the Price of Anarchy:

Def. 2 (Koutsoupias *et al*, 1999). *Price of Anarchy (PoA)*. Given a N-player game $\mathcal{G} = (\mathcal{N}, \mathcal{L}, (b_n)_n)$ and \mathcal{L}_{NE} its set of Nash equilibria, the PoA is defined as the following ratio:

$$\operatorname{PoA}(\mathcal{G}) = \frac{\sup_{\boldsymbol{\ell} \in \mathcal{L}_{NE}} SC(\boldsymbol{\ell})}{\inf_{\boldsymbol{\ell} \in \mathcal{L}} SC(\boldsymbol{\ell})}$$

Note that, from the definition, the PoA is always greater than 1. Furthermore, finding an upper bound on the PoA ensures that the social cost at any NE will be relatively close to the minimal social cost. In general, bounding the PoA is a hard theoretical question in general congestion games [20, 21]. In [22], the authors give an upper bound if the price functions are polynomial with bounded degree and positive coefficients. With degree one (affine prices, Assumption 3) the bound is $PoA \leq 1.5$, that is, the NE profile can induce costs as much as 50% higher than the optimal costs: implementing such a framework would not be worthwhile for the aggregator, as uncoordinated consumers will probably perform better (in our simulations, the uncoordinated profiles induce costs 16% higher than the optimal costs, see Tab. I). However, the results in [22] are worst-case bounds and these bounds are only approached asymptotically⁴: in our simulations with affine prices, the PoA was always much lower than 1.5. One of the reasons is that in [22] the model does not consider the power constraints (5c), and a PoA of 1.5 might be reached in our case only if the constraints (5c) are coarse enough. To further explain the low PoA in our instances, we found the following theorem by precising the results of [22]:

Theorem 2. Under Assumption 3, define for any $t \in \mathcal{T}$, $\varphi_t = (1 + \frac{\alpha_t}{\beta_t \overline{L}_t})^2$, where $\overline{L}_t = \sum_n \overline{\ell}_{n,t}$ and $t_0 \stackrel{\Delta}{=} \arg\min_t \frac{\alpha_t}{\beta_t \overline{L}_t}$. Assuming that, for all $t \in \mathcal{T}$:

$$\varphi_t \leqslant \varphi_{t_0} + 2 + \sqrt{1 + \varphi_{t_0}} , \qquad (7)$$

the following inequality holds:

$$\operatorname{PoA}(\mathcal{G}) \leq \frac{1}{2} (1 + \sqrt{1 + \varphi_{t_0}^{-1}} + \frac{1}{2} \varphi_{t_0}^{-\frac{1}{2}}) .$$
 (8)

Proof: See Appendix B.

Remark 4. Using the inequality $\forall x \ge 0$, $\sqrt{1+x^2} \le 1+x$, (8) implies the following simplified—but coarser—bound:

$$\operatorname{PoA}(\mathcal{G}) \leq 1 + \frac{3}{4} \sup_{t \in \mathcal{T}} \left(1 + \frac{\alpha_t}{\beta_t \overline{L}_t} \right)^{-1}$$
 (9)

 3 In practice, the system costs can differ from the social cost of consumers, for instance if we consider that the aggregator makes a positive profit, or if we consider consumers utility functions as done in [15].

⁴Meaning that there exists a sequence of games $(\mathcal{G}_{\nu})_{\nu \geq 0}$, with parameters depending on ν , and affine price functions c_t such that $\operatorname{PoA}(\mathcal{G}_{\nu}) \xrightarrow[\nu \to \infty]{} 1.5$.

 $^{{}^{2}}E_{n}$ can be set by the consumer, induced by the physical parameters of her appliances (battery capacity), or computed by learning the consumer's habits.

The assumption (7) in Thm. 2 ensures that price functions (c_t) cannot differ too much from one time period to another. This would be the case if, for instance, the price functions are uniform over \mathcal{T} . One can see that, according to Thm. 2, the PoA converges to one when $\alpha_t/(\beta_t \overline{L}_t)$ converges to infinity: the PoA can be arbitrarily close to one if we choose the coefficients α_t large enough. This result is indeed intuitive: when the prices are constant ($\beta_t = 0$), they do not depend on the aggregates L and there is no congestion effect; the optimal profile is obtained by each consumer choosing the time periods with lowest prices, independently of ℓ_{-n} . Another interesting result is that the PoA also converges to one when the total load is low $(\overline{L}_t \to 0)$. Note that the left-hand-side of inequality (8) is decreasing with φ_0 and is equal to $(\frac{1+\sqrt{2}}{2})^2 \approx 1.457$ for $\varphi_0 = 1$ so our result is always tighter than the bound given in [22]. However, in our simulations with linear prices, the PoA was still lower than the bound (8), even when assumption (7) does not hold: the inequality (8) gives $PoA \leq 1.271$ (average on the simulated days), while the PoA on mean values from Tab. I is 1.017. In this regards, getting a tighter bound or generalizing our proof to more general price functions could be the subject of future work.

III. FAST COMPUTATION OF THE NASH EQUILIBRIUM

The computation of NE is a central problem in game theory [23]. For a practical implementation of a DR program, we need to be able to compute the NE consumption profile in small time. In this section, we provide two algorithms for computing the NE and analyze their convergence in our specific setting.

A. Two Decentralized Algorithms

Given a profile ℓ_{-n} of the others, consumer *n* is expected to choose the profile ℓ_n corresponding to a minimizer of (5), which is called its *Best Response*⁵. It is denoted by

$$BR_n: \boldsymbol{s}_n \mapsto \operatorname*{argmin}_{\boldsymbol{\ell}_n \in \mathcal{L}_n} \sum_t \ell_{n,t} c_t (\boldsymbol{s}_{n,t} + \ell_{n,t}) , \qquad (10)$$

which only depends on the sum of the load of the others $s_n \stackrel{\Delta}{=} \sum_{m \neq n} \ell_m \in \mathbb{R}^T$ because of the "aggregated" structure ⁶: from (4), we see that b_n only depends on ℓ_n and $L_n = s_n + \ell_n$. A natural algorithm for computing an NE is to iterate best responses and update the strategies, cycling over the set of users until convergence. This procedure, described as *Cycling Best-Response Dynamics* (CBRD) [24] is described by Algo 1.

Algo. 1	Cycling	Best Res	sponse D	ynamics ((CBRD)	1
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Require: $\ell^{(0)}$, k_{\max} , ε_{stop} 1: $k \leftarrow 0$, $\varepsilon^{(0)} \leftarrow \varepsilon_{stop}$ 2: while $\varepsilon^{(k)} \ge \varepsilon_{stop}$ & $k \leqslant k_{\max}$ do3: for n = 1 to N do4: $s_n^{(k)} = \sum_{m < n} \ell_m^{(k+1)} + \sum_{m > n} \ell_m^{(k)}$ 5: $\ell_n^{(k+1)} \leftarrow BR_n(s_n^{(k)})$ (using (10))6: end for7: $\varepsilon^{(k)} = \|\ell^{(k+1)} - \ell^{(k)}\|$ 8: $k \leftarrow k + 1$ 9: end while

Remark 5. In a general setting, $BR_n(s_n)$ can be multivalued. In that case, we can still use Algo 1 by arbitrarily choosing any element of $BR_n(s_n)$ at Line 5.

Notice that the **for** loop in Algo 1 (Line 3) implements sequential updates and cycles over the set of players in the arbitrary order⁷ 1, 2, ..., N. One could think of a *simultaneous* version of Algo 1 (without Line 4 and where Line 5 is executed by all players in parallel). However, we observed that doing so can prevent the convergence of Algo 1.

Another natural algorithm to compute the equilibrium is to emulate the projected gradient descent, well-known in convex optimization [25], by considering the gradient of each objective function of the players, as described in Algo 2.

Algo. 2 Simultaneous Improving Response Dynamics (SIRD)
Require: $\ell^{(0)}, k_{\max}, \varepsilon_{\text{stop}}, \gamma$
1: $k \leftarrow 0, \ \varepsilon^{(0)} \leftarrow \varepsilon_{\text{stop}}$
2: while $\varepsilon^{(k)} \ge \varepsilon_{\text{stop}}$ & $k \le k_{\text{max}}$ do
3: for $n = 1$ to N do
4: $\boldsymbol{\ell}_n^{(k+1)} \leftarrow \Pi_{\mathcal{L}_n} \left(\boldsymbol{\ell}_n^{(k)} - \gamma \nabla_n b_n(\boldsymbol{\ell}_n^{(k)}, \boldsymbol{\ell}_{-n}^{(k)}) \right)$
5: end for
6: $\varepsilon^{(k)} = \left\ \boldsymbol{\ell}^{(k+1)} - \boldsymbol{\ell}^{(k)} \right\ $
7: $k \leftarrow k+1$
8: end while

At Line 4 of Algo 2, $\Pi_{\mathcal{L}_n}$ denotes the projection on the feasibility set of consumer $n \mathcal{L}_n$. The chosen denomination *improving response* recall that, at each iteration of Algo 2, player *n improves* her profile ℓ_n by performing a projected gradient step (Line 4), but in general does not choose the best improvement as in Algo 1. Note that from Algo 1 to Algo 2, only the instructions within the **for** loop are changed: the update of s_n and computation of BR_n (Lines 4 and 5 of Algo 1) are replaced with the gradient step (Line 4 of Algo 2).

Remark 6. Both Algo 1 and Algo 2 can be implemented in a "decentralized" procedure: the instructions within the **for** loop (Line 4-5 for Algo 1 and Line 4 for Algo 2) can be performed locally by each consumer's ECS. In this way, the privacy of consumers is respected as they do not have to send information about their constraints (5b-5c) to the aggregator. On the other hand, they only need to receive information on the aggregated load $s_n^{(k)}$ and can hardly deduce the individual consumption of the other consumers.

The computational complexity of one iteration (within the **for** loop) of Algo 2 is equivalent to the complexity of the projection $\Pi_{\mathcal{L}_n}$, which can be computed with the Quadratic Program (QP) $\Pi_{\mathcal{L}_n}(\ell'_n) = \operatorname{argmin}_{\ell_n \in \mathcal{L}_n} \|\ell'_n - \ell_n\|_2^2$ so it would be of the same order of complexity (see [26, Lecture 4]) as one iteration (within the **for** loop) of algorithm CBRD with affine prices (Assumption 3). With the specific structure of the feasible set $\mathcal{L}_n \subset \mathbb{R}^T$, a QP can be solved very efficiently in $\mathcal{O}(T)$ [27], so that one iteration of Algo 2—as well as one BR with linear prices in Algo 1—will be very fast. Moreover,

⁵As player *n* gives its *best response* to the other players strategies. ⁶In a general setting, BR_n would be a function of ℓ_{-n} .

⁷Choosing a "good" order of the BR in the **for** loop might accelerate the convergence of the algorithm. This could be the subject of future work.

as we do not update sequentially the load of the others ℓ_{-n} in Algo 2, the projected gradient step within the **for** loop can be computed simultaneously and can be parallelized.

B. Game Stability and Convergence of Algos 1 and 2

To study the convergence to the unique NE of the two algorithms proposed in Sec. III-A, we use the notion of *stability*, and prove that the energy consumption game \mathcal{G} defined above is strongly stable under Assumption 2. The notion of stability was introduced in [28] in order to study different game dynamics in continuous time and their convergence to NE. We extend this property to a "strong" version (symmetrically to the concept of strong monotonicity for operators):

Def. 3 (Hofbauer and Sandholm, 2009). *Stable Game.* A minimization game $\mathcal{G} = (\mathcal{N}, \mathcal{L}, (b_n)_n)$ is stable iff

$$\forall \boldsymbol{\ell}, \boldsymbol{\ell}' \in \mathcal{L}, \ (\boldsymbol{\ell}' - \boldsymbol{\ell})^T (F(\boldsymbol{\ell}') - F(\boldsymbol{\ell})) \ge 0 , \qquad (11)$$

with $F(\boldsymbol{\ell}) \stackrel{\Delta}{=} (\nabla_n b_n(\boldsymbol{\ell}))_{n \in \mathcal{N}}$. Moreover, \mathcal{G} is a-strongly stable, with a constant a > 0, iff:

$$\forall \boldsymbol{\ell}, \boldsymbol{\ell}' \in \mathcal{L}, \ (\boldsymbol{\ell}' - \boldsymbol{\ell})^T. \left(F(\boldsymbol{\ell}') - F(\boldsymbol{\ell}) \right) \ge a \left\| \boldsymbol{\ell} - \boldsymbol{\ell}' \right\|^2 \ . \tag{12}$$

Remark 7. The condition of stability in (11) is equivalent to the condition of strict diagonal convexity in [29], which implies uniqueness of NE [29, Thm.2].

Def. 3 gives an abstract condition on an operator that depends on the objective functions of the players. In our case, players objectives (b_n) depend linearly on price functions $(c_t)_t$ through (4), so it is interesting to translate the condition of Def. 3 directly on the price functions, as stated in Prop. 1.

Proposition 1. Let a > 0 such that Assumption 2 holds. Then, the game G is a-strongly stable.

Proof: See Appendix Sec. C.

On the basis of this stability property, we first consider the question of the convergence of Algo 1. In general games, CBRD might not converge [30] or might take an exponential time to converge [31]. In atomic splittable congestion games on a parallel network, as in our case, the convergence and the speed of Algo 1 has been studied previously in [12] and [13], where the authors show by different methods that there is a geometric convergence in the case of N = 2 players and convex and strictly increasing price functions (Assumption 1). However, to the best of our knowledge, the convergence in this setting and for more players N > 2 is still an open question.

In our case, simulations show a geometric convergence rate for any instance of \mathcal{G} satisfying Assumption 3 and for any $N \in \mathbb{N}$, as illustrated in Fig. 1. In [13], it is conjectured that this geometric convergence may also holds under Assumption 1. Restricting ourselves to affine price functions, we notice that game \mathcal{G} is a potential game [15, 32] and we get the following guarantee on the rate of convergence of Algo 1: **Theorem 3.** Under Assumption 3, the sequence of iterates of Algorithm CBRD $(\ell^{(k)})_{k \ge 0}$ converges to the unique NE ℓ^{NE} of \mathcal{G} . Moreover, the convergence rate satisfies:

$$\left\|\boldsymbol{\ell}^{(k)} - \boldsymbol{\ell}^{\mathrm{NE}}\right\|_2 \leqslant C \frac{\sqrt{M}N}{\sqrt{a}} \times \frac{1}{\sqrt{k}},$$

where C depends on $\ell^{(0)}$ and the billing functions, $M = 2 \max_t \beta_t$ and $a = 2 \min_t \beta_t$.

Proof: See Appendix D. The result is implied by convergence of alternating block coordinate minimization method [33].

The proof of Thm. 3 uses the fact that $M = \max_n M_n$ where M_n is a Lipschitz constant of $\nabla_n b_n$, and a is a strong convexity (and *a*-strong stability) constant. To the best of our knowledge, the question to know if Thm. 3 holds for general price functions is open; it can be an avenue for future research.

It is easier to get a strong guarantee on the convergence rate of Algo 2 for general price functions, as stated in Thm. 4:

Theorem 4. Denote by M_n a Lipschitz constant of $\nabla_n b_n$ and $M \stackrel{\Delta}{=} \max_n M_n$. Under Assumption 2 (a- strong stability), for $\gamma \stackrel{\Delta}{=} a/(NM^2)$, SIRD converges. Moreover, we have:

$$\left\|\boldsymbol{\ell}^{\mathrm{NE}}-\boldsymbol{\ell}^{(k)}\right\|_{2} < \eta^{k} \left\|\boldsymbol{\ell}^{\mathrm{NE}}-\boldsymbol{\ell}^{(0)}\right\|_{2}$$

where $\eta = 1 - \frac{a^2}{NM^2}$.

Proof: See Appendix Sec. E.

Note that, under Assumption 3, as stated in Thm. 3 we have $M = 2 \max_t \beta_t$ and $a = 2 \min_t \beta_t$, which gives the explicit contraction ratio $\eta = 1 - \frac{\max_t \beta_t}{N \min_t \beta_t}$.

In practice, in spite of the weaker convergence result for CBRD, the convergence seems to also happen at a geometric rate, with a better ratio than the one found for Algorithm SIRD when the number of players N is small, as illustrated in Fig. 1. The convergence speed of both algorithm decreases with the number of users, as illustrated by the geometric coefficient η in Thm. 4. However, SIRD becomes faster than CBRD when the number of players is large enough ($N \ge 20$).



Fig. 1. Convergence of SIRD and CBRD with uniform affine price functions and T = 10. When the number of players N increases, the convergence rate of both algorithms decreases, but SIRD becomes faster than CBRD.

IV. SIMULATION OF ONLINE DEMAND RESPONSE

In this section, we simulate consumption under the DR framework described above. We propose a practical procedure to implement the DR framework: at each hour, the equilibrium profiles for the flexible consumption is re-computed for the hours ahead to the end of the optimization horizon T, using Algo 1 or Algo 2. Next, we detail this simulation framework.

A. Online Demand Response Procedure

The initial time horizon \mathcal{T} that we consider for the planning via DR starts each day at noon (t = 1) and stops at noon the day after (t = T). We describe an "online" procedure, computing the DR equilibrium flexible consumption profiles on time horizon $\{1, \ldots, T\}$ for each day. As the price functions c_t depend on the nonflexible load through (2), and as the accuracy of forecast of this load improves when approaching from real-time, we re-compute the equilibrium using updated forecasts at each time period.

Alg	o. 3 Online Demand Response Procedure
1:	Start at $t = 1$
2:	while $t \leq T$ do
3:	Set new horizon $\mathcal{T}^{(t)} = \{t, t+1, \dots, T\}$
4:	Get $\boldsymbol{L}_{\mathrm{NF}}$ forecast on $\mathcal{T}^{(t)}$: $\hat{\boldsymbol{L}}_{\mathrm{NF}}^{(t)} \stackrel{\Delta}{=} \left(\hat{L}_{\mathrm{NF}}^{(t)},s\right)_{t \leqslant s \leqslant T}$
5:	Re-compute prices $c_t(.)$ for $t \in \mathcal{T}^{(t)}$ from (2)
6:	Run Algo. SIRD or BRD to compute NE $\ell^{(t)}$ on $\mathcal{T}^{(t)}$
7:	for each user $n \in \mathcal{N}$ do
8:	Realize computed profile on time t, $\ell_{n,t}^{(t)}$
9:	Update energy demand $E_n \leftarrow E_n - \ell_{n,t}^{(t)}$
10:	end for
11:	Wait for $t+1$
12:	end while

Remark 8. In practice, the NE profile $\ell^{(t)}$ has to be computed before period t to begin consumption at time t (Line 8). If τ is an upper bound on the computation time of the NE profile (Line 6), then, as we want to use the latest available forecast, Lines 3-5 would be run just before $t - \tau$, Line 6 is run in the interval $[t - \tau, t]$ and Line 8 is executed through [t, t + 1].

Observe that in Algo 3, considering updated forecast at Line 4 leads to updated price functions $(c_t)_t$ (Line 5), according to equation (2). In turn, the updated price functions modify the objective function of user n, b_n , used in Line 6.

The difference of Algo 3 with an "offline" version is that we recompute the equilibrium consumption (Line 6) at each time for all the time periods ahead. In an offline DR, we would compute the equilibrium consumption for all the horizon $\mathcal{T} = \{1, \ldots, T\}$ only once, just before t = 1.

Proceeding with this "online" version has two main advantages. First, it enables to rely on updated forecast with new information acquired on the nonflexible load $L_{\rm NF}$ (Line 4) up to time $t-\tau$. Second, it also enables to cope with local issues as disconnection of an user or a communication bug: in that case, we do not follow lines 8 and 9 for the involved user, and this user will have the same energy demand for the next round at t + 1. **Theorem 5.** The online demand response procedure of Algo 3 is consistent: that is, if forecasts are perfect (i.e. $\forall t \in \mathcal{T}, \forall t' \in \mathcal{T}^{(t)}, \hat{L}_{NF,t'}^{(t)} = L_{NF,t'}$), then for any $t_2 > t_1$, the NE profile $\ell^{(t_1)}$ computed at t_1 with forecast $\hat{L}_{NF}^{(t_1)}$ is equal on $\{t_2, \ldots T\}$ to the NE profile $\ell^{(t_2)}$ computed at t_2 with forecast $\hat{L}_{NF}^{(t_2)}$.

Proof: See Appendix F.

Thm. 5 states a *dynamic programming principle* adapted to our game-theoretic framework. It ensures that, following Algo 3, the final realized profile ℓ will correspond to the NE under perfect forecasts. To quantify the value of this online procedure in the more realistic case of imperfect forecasts, we simulate it on the set of consumers and parameters taken from real data, defined below.

B. Consumers

We consider a set of N = 30 users owning an electric vehicle (EV) from the database of Texan residential consumers PecanStreet Inc. [34]. We consider that the charging of the EV is the only flexible appliance of the consumers managed through the DR program, while the remaining of the user's consumption is nonflexible and is taken as in the data. We denote by $\mathcal{D} \stackrel{\Delta}{=} \{16/01/01, \dots, 16/01/31\}$ the set of the 31 days of January 2016 for which we simulate the DR program and we index a parameter by $d \in \mathcal{D}$ when it is specific to day d. For constraints (5b-5c), we take, for each day $d \in D$, the total flexible demand of user n, $E_{n,d}$ as the total observed consumption for the EV of n on the time set $\mathcal{T} = \{1, \ldots, T\}$ (twenty-four hours of d from 12PM to 11AM on d + 1). The power lower bound is always taken to zero $\underline{\ell}_{n,d,t} = 0$. For the power upper bound $\overline{\ell}_{n,d,t}$, we consider two cases: if a positive power was given at d, t in the data, we take $\overline{\ell}_{n,d,t}$ as the maximum power given to n's EV in the data in the set \mathcal{D} . If the power given to the EV is 0 at d, t in the data, we take $\overline{\ell}_{n,d,t} = 0$ (*i.e.* we consider that the EV of n was not available to charge on period d, t).

C. Price Functions

Following [15], we consider that the aggregator has a providing cost for the global demand at time $t, D_t \stackrel{\Delta}{=} (L_{\text{NF},t} + L_t),$ that does not depend on the time, and given (in \$) by $C(D_t) \stackrel{\Delta}{=}$ $0.711 - 0.0417D_t + 0.00295D_t^2$ where the coefficients replicate the cost function of a real residential electricity provider. For this, we computed the average, minimum and maximum values of $L_{\text{NF},t}$ over all the hours of the 31 days of January 2016 on our set of 30 consumers and interpolate the three values (avg $L_{NF,t}$, min $L_{NF,t}$, max $L_{NF,t}$) to three respective prices proposed by the Texan distributor Coserv [35] so that $\tilde{c}(\arg L_{\rm NF} t) = 0.080$ kWh (price for "base" contracts), $\tilde{c}(\min L_{\mathrm{NF},t}) = 0.055$ kWh (price for Off-Peak hours in Time-of-Use contracts) and $\tilde{c}(\max L_{\text{NF},t}) = 0.14$ kWh (price for Peak hours). Following (2), the price for the flexible load is given by: $c_t(L_t) = (-4.17 + 0.590L_{\text{NF},t}) + 0.295L_t$, so that Assumption 3 holds.

D. Forecast of the Nonflexible Load

Here, as we assume that the prices depend on the nonflexible load, at each time t the aggregator has to compute a forecast $\hat{L}_{NF}^{(t)} \triangleq (\hat{L}_{NF,t}^{(t)}, \dots, \hat{L}_{NF,T}^{(t)})$ to be able to compute the equilibrium consumption for time periods $\{t, \dots, T\}$. To simulate the forecasts, we assume that the forecast made at time t for period $t' \ge t$, $\hat{L}_{NF,t'}^{(t)}$ has no bias, that is $\mathbb{E}[\hat{L}_{NF,t'}^{(t)}|\sigma(\mathcal{F}_t)] = L_{NF,t'}$ (where \mathcal{F}_t is the natural filtration over $(\hat{L}_{NF,t})_t$), and that we have perfect information at time t, that is: $\hat{L}_{NF,t}^{(t)} = L_{NF,t}$. Considering that $L_{NF,t} = P_t e^{X_t}$ where X_t follows an Ornstein-Uhlenbeck [36] process with mean reversing coefficient m and volatility σ , and P_t a seasonality factor that depends on the hour of the week (1st hour to 168th hour), we get for any $t \le t'$:

$$\hat{L}_{\text{NF},t'}^{(t)} = P_{t'} \left(\frac{L_{\text{NF},t}}{P_t}\right)^{e^{-m(t'-t)}} \exp\left(\frac{\sigma^2}{4m}(1 - e^{-2m(t'-t)})\right).$$

Using a least-squares regression on the observed data from years 2014 and 2015, we compute $m \simeq 0.198 \text{ h}^{-1}$ and $\sigma \simeq 0.117 \text{ h}^{-1/2}$. An example of the simulated forecasts made at four different time periods is given in Fig. 2.



Fig. 2. Forecasts of the nonflexible load $\hat{L}_{NF}^{(t)}$ evolving in time. We assume a perfect forecast at time t for t. Forecasting performance increases when approaching real time.

E. Gains with the Online DR Procedure

We run the DR Procedure as described in Sec. IV-A on each day of January 2016 to get the computed flexible consumption profile of each user ℓ_n , and compute the associated social cost on the DR horizon $\{1, \ldots, T\}$ for each day in \mathcal{D} . We compare the total social costs on all simulated days \mathcal{D} , obtained via the DR online procedure and the associated social costs with the four other consumption scenarios below:

1) *uncoordinated* case: no DR is implemented to control or incentivize consumers flexibility; the consumption profiles are taken as the observed value in the data;

2) offline DR: the equilibrium is computed only once at t = 1 and for the whole time horizon $\{1, \ldots T\}$ considering the first forecast $\hat{L}_{NF}^{(1)}$ available at t = 1,

3) perfect forecast DR: offline DR, where we take $\hat{L}_{NF}^{(1)} = L_{NF}$. With Thm. 5, it is useless to recompute the profiles at each time period,

4) *optimal* scenario: a centralized entity (with perfect forecasts) computes the flexible consumption profile ℓ that minimizes the system costs $\sum_t L_t c_t(L_t)$.

The NE is computed by implementing Algo 1 under Python 3.5, where each quadratic program (QP) minimization (corresponding to one *Best Response*, Line 5) is solved with the solver Cplex 12.6 through Pyomo API, run on a single thread on an *Intel i7@*2.6GHz. As for the stopping criterion of Algo 1, we take $\varepsilon_{\text{stop}} = 10^{-3}$. Under this configuration it takes on average around 80 seconds to compute each NE $(\ell_n^{\text{NE}} \in \mathbb{R}^{24}, n \in \{1...30\})$. The optimization problem to compute the *optimal* consumption profile satisfying all users constraints (5b-5c), for each simulated day (from 12PM to 11AM) in \mathcal{D} , is also a convex QP that is solved easily with the solver Cplex 12.6 in 0.31 seconds on average.

Cons. Scenario	Social Cost	Avg. Price	Gain			
Uncoordinated	\$ 1257.2	0.200 \$/kWh	_			
Offline DR	\$ 1231.6	0.195 \$/kWh	2.036%			
Online DR	\$ 1131.1	0.180 \$/kWh	10.03%			
Perfect forecast DR	\$ 1075.2	0.171 \$/kWh	14.47%			
Optimal scenario	\$ 1056.8	0.169 \$/kWh	15.94%			
TABLE I						

Social Costs, average prices and relative gain to the uncoordinated consumption scenario on January 2016.



Fig. 3. Consumption Profiles on a typical day, with the different scenarios listed in Sec. IV-E. *The optimal profile flattens the consumption. The online DR procedure of Algo 3 gets closer to the Perfect forecast (offline) DR profile.*

Tab. I summarizes numerical results, it gives the total costs on the 31 days of January 2016 and compares the gains of the different flexible consumption scenarios relatively to the uncoordinated one. We can see on this table that the online DR procedure achieves significant savings compared to the offline version for which the results are really low in average: using our offline DR decreases the system costs by 2%relatively to the uncoordinated profile, that is, when consumers behave without any incentives. Implementing this offline DR program might not be worthy as it still involves a sophisticated communication and automation structure and it adds more constraints for consumers. This low performance is directly linked to our simple and naive model for the nonflexible load forecasts, which results in inaccurate forecasts for the last hours, as seen in Fig. 2. If more advanced forecasting methods (see [37]) can improve this accuracy, we cannot get rid of the high variance due to the low scale of the small set of consumers (30 in our example, and several hundreds for an aggregator for a typical low-voltage station). The online DR procedure seems to bring a solution to this issue: even with our simple forecast model, we achieve more than 10% of savings, with a gap of only 5% from the scenario with perfect forecasts. Therefore, our results show that implementing the given online DR procedure, even without very accurate forecasts, will be worthwhile for the aggregator.

V. CONCLUSION

In this paper, we developed a game-theoretic model for a residential demand response program, and we have addressed several issues about its implementation. We gave several new theoretical results about the uniqueness and existence of a Nash equilibrium consumption profile, and we have shown that the two algorithms CBRD and SIRD provide approximations of the NE at an arbitrary accuracy in finite time. We have introduced and simulated an online procedure that recomputes the NE profiles at each time period to take into account new information. We have shown numerically that this online procedure achieves a small price of anarchy when the parameters are fixed but also when the demand is uncertain. The online procedure reduces the impact of inaccurate forecasts and will be interesting to implement for an aggregator.

Several extensions of this work can be undertaken. First of all, our online procedure can be directly applied in the presence of other sources of stochasticity such as interactions with market prices or local renewable production sources. The aggregator objective can also be generalized to take into account the distance to a reference load profile or to maximize consumption during renewable production peaks or could also reflect market prices. Also, two main theoretical questions are still open. First, the result on the PoA bound could be improved to be tighter to the numerical results, and generalized to a larger set of functions. Second, the convergence theorem for the Best Response Dynamics (CBRD) could also be improved, as the observed convergence rate is faster than the given bound, and the convergence is numerically observed for a larger set of prices than affine functions.

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APPENDIX A

Proof of Thm. 1: Uniqueness of NE in ${\cal G}$

The proof follows the one of [9], extending it to the constrained case with constraints of the form (5c).

We denote by λ_n the Lagrange multiplier associated to (5b), along with $\underline{\mu}_{n,t} \ge 0$ the multiplier associated to $\underline{\ell}_{n,t} \le \ell_{n,t}$ and $\overline{\mu}_{n,t} \ge 0$ the multiplier associated to $\ell_{n,t} \le \overline{\ell}_{n,t}$.

Note that the KKT conditions give that, at optimality:

$$\gamma_{n,t}(\ell_{n,t}, L_t) = \lambda_n + \underline{\mu}_{n,t} - \overline{\mu}_{n,t} , \qquad (13)$$

where $\gamma_{n,t}(\ell_{n,t}, L_t) \stackrel{\Delta}{=} c_t(L_t) + \ell_{n,t}c'_t(L_t)$ is the marginal cost of *n*. Let's consider ℓ and $\hat{\ell}$ two NE's. From (13), we get:

$$\begin{split} \ell_{n,t} < \ell_{n,t} \Rightarrow \overline{\mu}_{n,t} &= 0 \Rightarrow \gamma_{n,t}(\ell_{n,t},L_t) \geqslant \lambda_n \\ \text{and } \ell_{n,t} > \underline{\ell}_{n,t} \Rightarrow \underline{\mu}_{n,t} = 0 \Rightarrow \gamma_{n,t}(\ell_{n,t},L_t) \leqslant \lambda_n \end{split}$$

and the same inequalities hold for $\hat{\ell}$.

We start by showing the following implications:

$$\left(\hat{\lambda}_n \leqslant \lambda_n \text{ and } \hat{L}_t \geqslant L_t\right) \Rightarrow \hat{\ell}_{n,t} \leqslant \ell_{n,t} , \qquad (14)$$

$$\left(\hat{\lambda}_n \ge \lambda_n \text{ and } \hat{L}_t \le L_t\right) \Rightarrow \hat{\ell}_{n,t} \ge \ell_{n,t} .$$
 (15)

We show (14) as (15) is symmetric. If $\hat{\ell}_{n,t} = \underline{\ell}_{n,t}$ or $\ell_{n,t} = \overline{\ell}_{n,t}$, then $\hat{\ell}_{n,t} \leq \ell_{n,t}$ is clear. Else, $\hat{\ell}_{n,t} > \underline{\ell}_{n,t}$ and $\ell_{n,t} < \overline{\ell}_{n,t}$ so: $\gamma_{n,t}(\hat{\ell}_{n,t}, \hat{L}_t) \leq \hat{\lambda}_n \leq \lambda_n \leq \gamma_{n,t}(\ell_{n,t}, L_t) \leq \gamma_{n,t}(\ell_{n,t}, \hat{L}_t)$ (16)

where the last inequality holds as $\gamma_{n,t}$ is increasing in L_t . As $c'_t(\hat{L}_t) > 0$ from Assumption 1, we deduce that $\ell_{n,t} \ge \hat{\ell}_{n,t}$.

Now, let's consider $\mathcal{T}_1 = \left\{t : \hat{L}_t > L_t\right\}$ along with $\mathcal{T}_2 = \mathcal{T} \setminus \mathcal{T}_1 = \left\{t ; \hat{L}_t \leq L_t\right\}$ and $\mathcal{N}_a = \left\{n ; \hat{\lambda}_n > \lambda_n\right\}$. Suppose $\mathcal{T}_1 \neq \emptyset$. From constraint (5b) and from (15), we have:

$$\forall n \in \mathcal{N}_a, \sum_{t \in \mathcal{T}_1} \hat{\ell}_{n,t} = E_n - \sum_{t \in \mathcal{T}_2} \hat{\ell}_{n,t} \leqslant E_n - \sum_{t \in \mathcal{T}_2} \ell_{n,t} = \sum_{t \in \mathcal{T}_1} \ell_{n,t} .$$

On the other hand, considering for $t \in \mathcal{T}_1$ and $n \notin \mathcal{N}_a$, we have from (14) that $\hat{\ell}_{n,t} \leq \ell_{n,t}$, and thus:

$$\sum_{t \in \mathcal{T}_1} \hat{L}_t = \sum_{t \in \mathcal{T}_1} \sum_{n \in \mathcal{N}_a} \hat{\ell}_{n,t} + \sum_{t \in \mathcal{T}_1} \sum_{n \notin \mathcal{N}_a} \hat{\ell}_{n,t} \leqslant \sum_{t \in \mathcal{T}_1} L_t , \quad (17)$$

which is a contradiction. Thus $\mathcal{T}_1 = \emptyset$ and $\forall t$, $\dot{L}_t = L_t$. With this equality, we can precise (14) with:

$$\begin{bmatrix} \hat{\lambda}_n < \lambda_n \text{ and } \hat{L}_t = L_t \end{bmatrix} \Longrightarrow$$

$$\begin{bmatrix} \hat{\ell}_{n,t} < \ell_{n,t} \text{ or } \hat{\ell}_{n,t} = \ell_{n,t} = \underline{\ell}_{n,t} \text{ or } \hat{\ell}_{n,t} = \ell_{n,t} = \overline{\ell}_{n,t} \end{bmatrix}$$
(18)

and similarly for (15). Indeed, if $\hat{\ell}_{n,t} = \underline{\ell}_{n,t}$ or if $\ell_{n,t} = \overline{\ell}_{n,t}$ then the implication holds because $\ell_{n,t} \ge \underline{\ell}_{n,t}$ and $\ell_{n,t} \le \overline{\ell}_{n,t}$. Else, $\hat{\ell}_{n,t} > \underline{\ell}_{n,t}$ and $\ell_{n,t} < \overline{\ell}_{n,t}$, and the same sequence of inequalities as in (16) gives $\gamma_{n,t}(\hat{\ell}_{n,t}, L_t) < \gamma_{n,t}(x_{n,t}, L_t)$, implying that $\hat{\ell}_{n,t} < \ell_{n,t}$.

Suppose that there exists n s.t. $\hat{\lambda}_n < \lambda_n$. If only the two latter cases in (18) happen, then we have $\ell_{n,t} = \hat{\ell}_{n,t}$, $\forall t$. Else, there is at least one t for which $\hat{\ell}_{n,t} < \ell_{n,t}$, so $E_n = \sum_t \hat{\ell}_{n,t} < \sum_t \ell_{n,t} = E_n$ which can not happen. Thus, $\hat{\lambda}_n = \lambda_n$ for all n and (14) and (15) imply that $\ell_{n,t} = \hat{\ell}_{n,t}$ for all n and t.

Appendix B

PROOF OF THM. 2: POA UPPER BOUND

The proof relies on the notion of *local smoothness* introduced in [22]. The idea is to get a tighter bound than [22] by specifying the parameters of the affine price functions $(c_t)_t$ and by using the upper bound on L_t instead of looking at the worst possible cases as done in [22].

Let $\kappa_t \stackrel{\Delta}{=} \alpha_t / \beta_t$ so $c_t(x) = \beta_t(x + \kappa)$. From [22], we know that if there exist $\lambda, \mu > 0$ and a profile $\boldsymbol{y} \in \mathcal{L}$ satisfying for each $t \in \mathcal{T}$:

$$\forall \boldsymbol{x} \in \mathcal{L}, \ y_t(x_t + \kappa_t) + \frac{y_t^2}{4} \leqslant \lambda y_t(y_t + \kappa_t) + \mu x_t(x_t + \kappa_t), \ (19)$$

where $y_t = \sum_n y_{n,t}$ and $x_t = \sum_n x_{n,t}$ then \mathcal{G} is locally λ, μ -smooth for y, *i.e.* for any admissible profile $x \in \mathcal{L}$:

$$\sum_{n=1}^{N} b_n(\boldsymbol{x}) + \nabla_n b_n(\boldsymbol{x})^T (\boldsymbol{y}_n - \boldsymbol{x}_n) \leqslant \lambda \mathrm{SC}(\boldsymbol{y}) + \mu \mathrm{SC}(\boldsymbol{x}).$$

In that case, it follows from [22] that the PoA is bounded by $\lambda/(1-\mu)$. For the remaining of the proof, we fix t and omit subscript t in the notations. As done in [22], we introduce:

$$\phi_{xy}(\mu) \stackrel{\Delta}{=} \frac{y(x+\kappa) + \frac{y^2}{4} - \mu x(x+\kappa)}{y(y+\kappa)}$$

and $\lambda^*(\mu) \stackrel{\Delta}{=} \sup_{x,y \ge 0} \phi_{xy}(\mu)$.

 $\lambda^*(\mu)$ is the minimum value of $\lambda > 0$ such that (19) holds with values (λ, μ) . Let us compute an explicit expression of $\lambda^*(\mu)$. If x = 0, $\phi_{0,y}(\mu) = \frac{y+4b}{4(y+\kappa)}$ and $\frac{\partial \phi_{0,y}}{\partial y} < 0$ so $\sup_{x,y} \phi_{x,y}$ would be attained with y = 0 and is $\phi_{0,0} = 1$. Otherwise:

$$0 = \frac{\partial \phi}{\partial x} \Leftrightarrow \frac{1}{y(y+\kappa)}(y-2\mu x-\mu\kappa) \Rightarrow x = \frac{y-\kappa\mu}{2\mu}$$

but as $x \ge 0$, this supposes that $y \ge \mu \kappa$. We compute:

$$\phi_{\frac{y-\kappa\mu}{2\mu},y} = \frac{1}{y(y+\kappa)4\mu} \left((y+\kappa\mu)^2 + \mu y^2 \right) \stackrel{\Delta}{=} h(y) \; .$$

We can see that h' vanishes on \mathbb{R}_+ at $y_+ \stackrel{\Delta}{=} \frac{\kappa \mu^2 + \kappa \mu \sqrt{\mu^2 + 1 - \mu}}{1 - \mu}$ that gives a min of h so h is decreasing then increasing. At the lower bound $y = \kappa \mu$, we get $\phi = \frac{\kappa \mu + 4b}{4(\kappa \mu + \kappa)} = \frac{\mu + 4}{4(\mu + 1)} = \frac{1}{4} + \frac{3}{4(\mu + 1)} < 1$ which is not max as $\phi_{0,0} = 1$. At the upper bound $y = \overline{L}$, we have $h(\overline{L}) = \frac{(\overline{L} + \kappa \mu)^2 + \mu \overline{L}^2}{\overline{L}(\overline{L} + \kappa) 4\mu} = \lambda^*(\mu)$. Last, to compute the best bound $\inf_{\mu} \lambda^*(\mu)/(1-\mu)$, let us consider:

$$g(\mu) \stackrel{\Delta}{=} 4\overline{L}(\overline{L}+\kappa)\frac{\lambda^*(\mu)}{1-\mu} = \frac{(\overline{L}+\kappa\mu)^2 + \mu\overline{L}^2}{\mu(1-\mu)}$$

If we denote $\varphi \stackrel{\Delta}{=} (1+r)^2$ and $r \stackrel{\Delta}{=} \kappa/\overline{L}$, $g(\mu)$ is minimal at $\mu^* \stackrel{\Delta}{=} (-1+\sqrt{1+\varphi})/\varphi$. We finally get our PoA bound as:

$$\begin{split} &\frac{\lambda^*(\mu^*)}{1-\mu^*} = \frac{(3+2r)+2\sqrt{1+\varphi}}{4(1+r)} = \frac{1}{2} \left(1 + \sqrt{1+\frac{1}{\varphi}} + \frac{1}{2\sqrt{\varphi}} \right) \\ &= \frac{1}{2} \left(1 + \sqrt{1+(1+r)^{-2}} + (2(1+r))^{-1} \right) \leqslant 1 + \frac{3}{4(1+r)} \; . \end{split}$$

The last inequality gives a more explicit bound and is obtained from $\sqrt{a^2 + b^2} \leq a + b$ valid for any $a, b \geq 0$.

Next, following [22], for $\ell, \ell' \in \mathcal{L}$ (admissible solutions):

$$\sum_{n} b_{n}(\boldsymbol{\ell}) + \nabla_{n} b_{n}(\boldsymbol{\ell})^{T}(\boldsymbol{\ell}' - \boldsymbol{\ell})$$

$$= \sum_{n} \sum_{t \in \mathcal{T}} \ell_{n,t} \cdot c_{t}(L_{t}) + (\ell_{n,t}' - \ell_{n,t}) (c_{t}(L_{t}) + \ell_{n,t}c_{t}'(L_{t}))$$

$$= \sum_{t} L_{t}' \cdot c_{t}(L_{t}) + c_{t}'(L_{t}) \sum_{n} (\ell_{n,t}' \ell_{n,t} - \ell_{n,t}^{2})$$

$$\leqslant \sum_{t} L_{t}' \cdot c_{t}(L_{t}) + c_{t}'(L_{t}) \cdot \frac{L_{t}^{2}}{4} = \sum_{t} b_{t} \left[L_{t}'(L_{t} + \kappa_{t}) + \frac{L_{t}^{2}}{4} \right]$$

$$\leqslant \sum_{t} b_{t} \left[\lambda L_{t}'(L_{t}' + \kappa_{t}) + \mu L_{t}(L_{t} + \kappa_{t}) \right]$$
(20)
$$= \lambda \mathbf{SC}(\boldsymbol{\ell}') + \mu \mathbf{SC}(\boldsymbol{\ell})$$

where (20) is valid if (λ, μ) is chosen such that:

$$\forall t \in \mathcal{T}, \ \lambda \geqslant \frac{(\overline{L}_t + \kappa_t \mu)^2 + \mu \overline{L}_t^2}{\overline{L}_t (\overline{L}_t + \kappa_t) 4 \mu} \stackrel{\Delta}{=} \lambda_{\kappa_t}^*(\mu) \ .$$

Let us denote $t_0 \triangleq \operatorname{argmin} \kappa_t$ and choose $\mu^* \triangleq \mu_{t_0}^*$, $\lambda^* \triangleq \lambda_{\kappa_{t_0}}^*(\mu^*)$ (the optimal $(\overset{t}{\lambda}, \mu)$ for t_0), then we have to check that for all $t \in \mathcal{T}$, $\lambda^* \ge \lambda_{\kappa_t}^*(\mu^*)$. For that, it is sufficient to show that $r \mapsto \lambda_r^*(\mu^*)$ is decreasing on $[r_{t_0}, r_t]$, which is true if $r_t < -1 + \sqrt{1 + \frac{1-\mu^*}{\mu^{*2}}} \iff \varphi_{r_t} < \varphi_{r_{t_0}} + 2 + \sqrt{1 + \varphi_{r_{t_0}}}$ with $\varphi_r = (1+r)^2$, which gives condition (7) stated in Thm. 2.

APPENDIX C

Proof of Prop. 1: Strong Stability of ${\cal G}$

We denote by $G(\ell) \stackrel{\Delta}{=} J_F(\ell)$ the Jacobian of operator F. Since functions b_n are twice differentiable, condition (11) is equivalent to having the matrix $G(\ell) + G^T(\ell)$ positive definite for all $\ell \in \mathcal{L}$, that is, $G(\ell) + G(\ell)^T \succ 0$.

As $b_n = \sum_t b_{n,t}$, with $b_{n,t}(\ell_t) \stackrel{\Delta}{=} \ell_{n,t}c_t(L_t)$, is separable in t, we can re-index the matrix $G(\ell)$ to have a diagonal

block hourly matrix
$$G(\boldsymbol{\ell}) = diag(G_1, ..., G_H)$$
, with $G_t(\boldsymbol{\ell}_t) \triangleq \left(\frac{\partial^2 b_{n,t}}{\partial \ell_{n,t} \partial \ell_{m,t}}(\boldsymbol{\ell}_t)\right)_{n,m \in \mathcal{N}^2}$ and we get for all t :
 $G_t(\boldsymbol{\ell}_t) + G_t(\boldsymbol{\ell}_t)^T = \left(\frac{\partial^2 b_{n,t}(\boldsymbol{\ell}_t)}{\partial \ell_{m,t} \partial \ell_{n,t}} + \frac{\partial^2 b_{m,t}(\boldsymbol{\ell}_t)}{\partial \ell_{n,t} \partial \ell_{m,t}}\right)_{n,m}.$

Let $t \in \mathcal{T}$ and $\boldsymbol{x} \in \mathbb{R}^N \setminus \{0\}$. Furthermore, let $\sigma(\boldsymbol{x}, \boldsymbol{\ell}) \stackrel{\Delta}{=} \boldsymbol{x}^T \left(G_t(\boldsymbol{\ell}_t) + G_t^T(\boldsymbol{\ell}_t) \right) \boldsymbol{x}$. For notation simplicity, let us forget the index t and the argument (L) in functions c_t . We have:

$$\sigma(\boldsymbol{x}, \boldsymbol{\ell}) = \sum_{n=1}^{N} 2x_n^2 (\ell_n c'' + 2c') + 2\sum_{n < m} x_n x_m \left((\ell_n + \ell_m) c'' + 2c' \right) = \sum_{n=1}^{N} 2x_n^2 \left(r_n \gamma + (1 - r_n) a \right) + 2\sum_{n < m} \ell_n x_m \left((r_n + r_m) \gamma + (1 - r_n - r_m) a \right)$$

with $r_n \stackrel{\Delta}{=} \ell_n/L$, a = 2c' and $\gamma \stackrel{\Delta}{=} 2c' + Lc''$. Then we have:

$$\sigma = a \sum_{n} x_n^2 + a \left(\sum_{n} \left(1 - r_n \left(1 - \frac{\gamma}{a} \right) \right) \ell_n \right)^2 - \frac{(a - \gamma)^2}{a} \sum_{n, m} r_n r_m x_n x_m$$

which is the sum of three quadratic form: $q_1(\mathbf{x}) = 2a \sum x_n^2$ which has one eigenvalue 2a of multiplicity N, $q_2(\mathbf{x}) = 2a \left(\mathbf{x}^T \mathbf{v}^T \mathbf{v} \mathbf{x} \right)$ with $v_n \triangleq \sum_n 1 - \frac{\ell_n}{L} \left(1 - \frac{\gamma}{2a} \right)$ of rank one which has eigenvalues 0 of multiplicity N - 1 and $2a ||\mathbf{v}||_2^2$ of multiplicity 1, and a negative form of rank one $q_3(\mathbf{x}) = -\frac{1}{2a} \left(2a - \gamma \right)^2 \left(\sum_{n,m} \frac{\ell_n}{L} \frac{\ell_m}{L} \ell_n x_m \right)$ which nonzero eigenvalue is $-\frac{1}{2a} \left(2a - \gamma \right)^2 \sum_n \left(\frac{\ell_n}{L} \right)^2$. We deduce that the quadratic form $q_1 + q_2$ is positive

We deduce that the quadratic form $q_1 + q_2$ is positive definite, and that its eigenvalues are 2a with multiplicity N - 1 and $2a(1 + ||\boldsymbol{v}||_2^2)$ with multiplicity 1. Next we use the following result from perturbation theory:

Theorem 6 (Horn and Johnson, 2012, [38, p367]). If $A, E \in \mathcal{M}_n$ are two Hermitian matrices and if $\lambda_1 \leq ... \leq \lambda_n$ are the ordered eigenvalues of A and $\hat{\lambda}_1 \leq ... \leq \hat{\lambda}_n$ are the ordered values of A + E, and $\lambda_1^E \leq ... \leq \lambda_n^E$ are the ordered eigenvalues of E, then :

$$\langle k = 1..n, \ \lambda_1^E \leqslant \hat{\lambda}_k - \lambda_k \leqslant \lambda_n^E$$

$$and \ \left| \hat{\lambda}_k - \lambda_k \right| \leqslant \rho(E) = |||E|||_2$$

Applying this theorem with $A = q_1 + q_2$ and perturbation $E = q_3$ we get that the smallest eigenvalue $\hat{\lambda}_1$ of σ verifies:

$$\hat{\lambda}_1 \ge \min \left\{ \operatorname{Sp}(q_1 + q_2) \right\} - \frac{(a - \gamma)^2}{a} \sum_n r_n^2$$
$$= a \left(1 - \left(1 - \frac{\gamma}{a} \right)^2 \sum_n r_n^2 \right).$$

Replacing a and γ , we can get the condition of Assumption 2.

Appendix D

PROOF OF THM. 3: CONVERGENCE OF CBRD

The key of the proof is that, under Assumption 3, the game is an exact potential game [32] with convex potential:

$$\Phi(\boldsymbol{\ell}) = \sum_{t \in \mathcal{T}} \alpha_t L_t + \frac{\beta_t}{2} (L_t^2 + \sum_n \ell_{n,t}^2) ,$$

that is, for any $\ell \in \mathcal{L}$ and any n, $\nabla_n \Phi(\ell) = \nabla_n b_n(\ell)$. Thus, the NE corresponds to the minimum of Φ and we have, for any $\ell \in \mathcal{L}$, $\underset{\ell_n \in \mathcal{L}_n}{\operatorname{argmin}} b_n(\ell_n, \ell_{-n}) = \underset{\ell_n \in \mathcal{L}_n}{\operatorname{argmin}} \Phi(\ell_n, \ell_{-n})$. Therefore, running Algo 1 is equivalent to performing an alternating block coordinate minimization on Φ . According to [33, Thm. 6.1], we get that:

$$\Phi(\boldsymbol{\ell}^{(k)}) - \Phi(\boldsymbol{\ell}^{\mathrm{NE}}) \leqslant \frac{2MN^2R^2c_5}{k}$$
(21)

with $M = \max_n M_n = 2 \max_t \beta_t$ (max of Lipschitz constants of $\nabla_n b_n = \nabla_n \Phi$), $R = \max_{\ell} \{ \| \ell - \ell^{\text{NE}} \| ; \Phi(x) \leq \Phi(\ell^{(0)}) \}$ and $c_5 = \max\{ \frac{2}{MN^2R^2} - 2, \Phi(\ell^{(1)}) - \Phi(\ell^{\text{NE}}), 2 \}$. But Φ is also strongly convex, that is, for any $\ell, \ell' \in \mathcal{L}$:

$$\Phi(\boldsymbol{\ell}) - \Phi(\boldsymbol{\ell}') \geqslant \langle \nabla \Phi(\boldsymbol{\ell}'), \boldsymbol{\ell} - \boldsymbol{\ell}' \rangle + \frac{a}{2} \left\| \boldsymbol{\ell} - \boldsymbol{\ell}' \right\|^2$$
(22)

with $a = 2 \min_t \beta_t$. Also, the minimality of ℓ^{NE} on the convex set \mathcal{L} implies that for any $\ell \in \mathcal{L}$:

$$\langle \nabla \Phi(\boldsymbol{\ell}^{\text{NE}}), \boldsymbol{\ell} - \boldsymbol{\ell}^{\text{NE}} \rangle \ge 0$$
 . (23)

Then from (22) and (23), we get for any $k \ge 0$:

$$\begin{split} \frac{a}{2} \left\| \boldsymbol{\ell}^{(k)} \!-\! \boldsymbol{\ell}^{\mathrm{NE}} \right\|^2 &\leqslant \Phi(\boldsymbol{\ell}^{(k)}) \!-\! \Phi(\boldsymbol{\ell}^{\mathrm{NE}}) + \langle \nabla \Phi(\boldsymbol{\ell}^{\mathrm{NE}}), \boldsymbol{\ell}^{\mathrm{NE}} \!-\! \boldsymbol{\ell}^{(k)} \rangle \\ &\leqslant \Phi(\boldsymbol{\ell}^{(k)}) - \Phi(\boldsymbol{\ell}^{\mathrm{NE}}) \ , \end{split}$$

and from (21) we get the convergence result of Thm. 3.

APPENDIX E Proof of Thm. 4: Convergence of SIRD

We analyze the convergence of the sequence $(T_{\gamma}^{k}(\boldsymbol{x}))_{k}$ where $[T_{\gamma}(\boldsymbol{\ell})]_{n} \stackrel{\Delta}{=} \Pi_{\mathcal{L}_{n}}(\boldsymbol{\ell} - \gamma \nabla_{n} f_{n}(\boldsymbol{\ell}_{n}, \boldsymbol{\ell}_{-n}))$. First notice that the unique NE of the game $\boldsymbol{\ell}^{\text{NE}}$ is the unique fixed point of T_{γ} i.e. $\boldsymbol{x}^{\text{NE}} = T_{\gamma}(\boldsymbol{x}^{\text{NE}})$. The idea is then to prove that T_{γ} is a η -contraction, for a given norm $\|\cdot\|$ which will imply the convergence rate

$$\|T^k(\boldsymbol{x}^{(0)}) - \boldsymbol{x}^{ ext{NE}}\| \leqslant \eta^k \| \boldsymbol{x}^{(0)} - \boldsymbol{x}^{ ext{NE}} \| \; ,$$

for any initial condition $\boldsymbol{x}^{(0)} \in \mathbb{R}^m$. Let $\|\cdot\|$ denote the Euclidean norm on \mathbb{R}^d for any positive integer d. As the projection on a convex set is nonexpansive [39, Corollary 12.20], we get for $\ell, \ell' \in \mathcal{L}$:

$$\|\mathbf{T}_{\gamma}(\boldsymbol{\ell}) - \mathbf{T}_{\gamma}(\boldsymbol{\ell}')\|^{2} = \sum_{n=1}^{N} \|T_{\gamma,n}(\boldsymbol{\ell}) - T_{\gamma,n}(\boldsymbol{\ell}')\|^{2}$$
$$= \sum_{n=1}^{N} \|\Pi_{\mathcal{L}_{n}}(\boldsymbol{\ell}_{n} - \gamma \nabla_{n} f_{n}(\boldsymbol{\ell})) - \Pi_{\mathcal{L}_{n}}(\boldsymbol{\ell}'_{n} - \gamma \nabla_{n} f_{n}(\boldsymbol{\ell}'))\|^{2}$$
$$\leqslant \sum_{n=1}^{N} \|\boldsymbol{\ell}_{n} - \boldsymbol{\ell}'_{n} + \gamma (\nabla_{n} f_{n}(\boldsymbol{\ell}') - \nabla_{n} f_{n}(\boldsymbol{\ell}))\|^{2}$$
$$= \sum_{n=1}^{N} \|\boldsymbol{\ell}_{n} - \boldsymbol{\ell}'_{n}\|^{2} + \gamma^{2} \|\nabla_{n} f_{n}(\boldsymbol{\ell}) - \nabla_{n} f_{n}^{h}(\boldsymbol{\ell}')\|^{2}$$
$$- 2\gamma \left\langle \nabla_{n} f_{n}(\boldsymbol{\ell}) - \nabla_{n} f_{n}(\boldsymbol{\ell}'), \boldsymbol{\ell}_{n} - \boldsymbol{\ell}'_{n} \right\rangle \right\rangle.$$

As we assume that for any n, $\nabla_n f_n$ is M_n -Lipschitz and let $M \stackrel{\Delta}{=} \max_n M_n$, then we have $\sum_{n=1}^N |\nabla_n f_n(\ell) -$ $\nabla_n f_n(\ell')|^2 \leq NM^2 ||\ell - \ell'||^2$. Besides, from *a*-strong stability Def. 3, we get :

$$\|\mathrm{T}_{\gamma}(\boldsymbol{\ell}) - \mathrm{T}_{\gamma}(\boldsymbol{\ell}')\|^{2} \leqslant \eta \|\boldsymbol{\ell} - \boldsymbol{\ell}'\|^{2}$$
,

with $\eta \stackrel{\Delta}{=} 1 + NM^2\gamma^2 - 2\gamma\alpha$. Minimizing on $\gamma > 0$ gives $\gamma = \frac{\alpha}{NM^2}$ and $\eta = 1 - \frac{\alpha^2}{NM^2} < 1$ and T_{γ} is a contraction.

APPENDIX F

PROOF OF THM. 5: CONSISTENCY OF DR PROCEDURE

Let $t_0 \in \{1, \ldots, T-1\}$ and let us denote by $\mathcal{G}^{(t_0)}$ the DR-game on hours $\{t_0, \ldots, T\}$ (considered at t_0 in the procedure) and $\mathcal{G}^{(t_0+1)}$ the DR-game on hours $\{t_0+1, \ldots, T\}$. Let $(\boldsymbol{x}, \boldsymbol{\lambda}, \underline{\mu}, \overline{\mu})$ be the unique NE of $\mathcal{G}^{(t_0)}$ associated with its dual variables (as defined in Sec. A) and \boldsymbol{y} the NE of $\mathcal{G}^{(t_0+1)}$. As we assumed perfect forecasts of the nonflexible load, the price functions $(c_t)_t$ considered in the BR of the players do not depend on the forecasts and are the same for the games $\mathcal{G}^{(t_0)}$ and $\mathcal{G}^{(t_0+1)}$. We want to show that $\boldsymbol{x}_t = \boldsymbol{y}_t$ for any $t \in \{t_0+1,\ldots,T\}$. As there is a unique NE of $\mathcal{G}^{(t_0+1)}$, it is sufficient to show that $(\boldsymbol{x}, \boldsymbol{\lambda}, \underline{\mu}, \overline{\mu})$ verifies the KKT conditions of each subproblem (5) of $\mathcal{G}^{(t_0+1)}$. As \boldsymbol{x} is a feasible solution of $\mathcal{G}^{(t_0)}$, it verifies the power bounds constraints (5c) as well as the total energy constraint (5b):

$$x_n^{t_0} + \sum_{t > t_0} x_{n,t} = E_n \iff \sum_{t > t_0} x_{n,t} = (E_n - x_n^{t_0}) ,$$
 (24)

where this last equality is exactly the total energy constraint (5b) for *n* in the game $\mathcal{G}^{(t_0+1)}$. Also, $\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\overline{\mu}}$ verify the complementarity constraints associated to constraint (5c) at each time $t \in \{t_0 + 1, \ldots, T\}$. Finally, the stationarity condition gives for any $t \in \{t_0 + 1, \ldots, T\}$, $\gamma_{n,t}(x_{n,t}, L_t) =$ $\lambda_n + \underline{\mu}_{n,t} - \overline{\mu}_{n,t}$ with $\gamma_{n,t}$ the marginal cost of *n* for time *t*. As each problem is convex, KKT conditions characterize the solution of each user's optimization problem (5), and thus \boldsymbol{x} is an NE of $\mathcal{G}^{(t_0+1)}$, and is therefore equal to \boldsymbol{y} .