# Manipulating decision making of typical agents 

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#### Abstract

We investigate how the choice of decision makers can be varied under the presence of risk and uncertainty. Our analysis is based on the approach we have previously applied to individual decision makers, which we now generalize to the case of decision makers that are members of a society. The approach employs the mathematical techniques that are common in quantum theory, justifying our naming as Quantum Decision Theory. However, we do not assume that decision makers are quantum objects. The techniques of quantum theory are needed only for defining the prospect probabilities taking into account such hidden variables as behavioral biases and other subconscious feelings. The approach describes an agent's choice as a probabilistic event occurring with a probability that is the sum of a utility factor and of an attraction factor. The attraction factor embodies subjective and unconscious dimensions in the mind of the decision maker. We show that the typical aggregate amplitude of the attraction factor is $1 / 4$, and it can be either positive or negative depending on the relative attraction of the competing choices. The most efficient way of varying the decision makers choice is realized by influencing the attraction factor. This can be done in two ways. One method is to arrange in a special manner the payoff weights, which induces the required changes of the values of attraction factors. We show that a slight variation of the payoff weights can invert the sign of the attraction factors and reverse the decision preferences, even when the prospect utilities remain unchanged. The second method of influencing the decision makers choice is by providing information to decision makers. The methods of influencing decision making are illustrated by several experiments, whose outcomes are compared quantitatively with the predictions of our approach.


Index Terms-Decision theory, Decision making under risk and uncertainty, Group consultations, Social interactions, Information and knowledge

## I. InTRODUCTION

How to influence decision choices made by separate decision makers as well as by societies of many agents is an important and widely studied problem in psychology [1][6]. This problem is important for a variety of practical applications ranging from medicine [7], [8] to politics [9][13]. A number of articles are devoted to the effects of influencing decision making in economics, studying the role of different framing effects on product evaluation [14]-[16], consumer response to price [17]-[20], evaluation of retail outlets [21], market advertising [22]-[24], buying decisions [25], [26], perceptions of control and efficacy [27], distributive justice [28], performance feedbacks [29], and so on.

[^0]The principal possibility of influencing the choices of decision makers is based on the fact that decision makers do not exactly follow the prescriptions of expected utility theory, as formulated by von Neumann and Morgenstern [30]. Really, if each decision maker were to make decisions following the strict rules of utility theory, then it would be difficult, if possible at all, to influence his/her decisions without essentially varying the utility of the related lottery. However, it is well known that the choices of decision makers are not based solely on utility, but also are strongly influenced by emotions, prejudices, biases, and other subconscious feelings. There have been numerous attempts to modify utility theory by taking into account such subconscious degrees of freedom. For this purpose, a number of non-additive nonlinear probability models have been developed to account for the deviations from objective to subjective probabilities observed in human agents [31]-[40], trying to take into account satisfaction [41], anxiety [42], subjective perception [43], subjective utility [44], aspiration adaptation [45], [46], and so on. The necessity of taking into account the subconscious behavioral biases is emphasized in behavioral economics [47]. The variety of the approaches, deviating from the expected utility theory, are commonly named "non-expected utility theories" [48].

The non-expected utility theories have been thoroughly analyzed in several reviews [48]-[51]. The conclusion is that such theories are in the best case only descriptive, hence, do not have predictive power and do not explain numerous paradoxes existing in classical decision making. Moreover, their use ends up creating more paradoxes and inconsistencies than it resolves [50].
To overcome the problem, we have developed an approach based on the mathematical techniques of quantum theory [52][58], which explains our choice of its name, Quantum Decision Theory (QDT). We do not assume that decision makers are quantum objects. But the mathematical quantum techniques serve as the most convenient tool for taking into account the subconscious degrees of freedom of decision makers, similarly to how quantum theory avoids the explicit use of hidden variables, at the same time taking into account their possible existence resulting in the probabilistic formulation of the theory. We have shown that, in the frame of QDT, all paradoxes of classical decision making find simple and natural explanations [54], [57], [58]. QDT provides the expressions for discount functions, employed in the theory of time discounting [59], [60] and explains dynamical inconsistences [53]. Within QDT, behavioral biases result from interference and entanglement caused by decision makers deliberations [56]. While QDT has been developed to describe the behavior of human decision makers, it can also be used as a guide for creating artificial quantum intelligence [55].
In our previous publications [52]-[58], we have considered
a single decision maker. In the present paper, we generalize QDT, developed earlier for individual decision makers, to the case of decision makers interacting within a society. This generalization is formulated in Sec. II. Our main concern is the formulation of a mathematical model describing how decision makers can be influenced and how it would be possible to quantitatively evaluate the consequences of this influence. In experiments, one usually deals with large groups of decision makers with different preferences. In order to compare theoretical predictions with experimental results, we introduce and characterize, in Sec. III, the notion of typical social agents. Here and in what follows, by the term 'experiments', we mean empirical observations derived from the behavior of human subjects. In Sec. IV, we explain how it is possible to influence the typical decision makers preference by varying the arrangement of prospects. We formulate a criterion for the inversion of the attraction factor leading to the inversion of preferences. In Sec. V, the results, predicted by our approach, are compared with several classical experiments, demonstrating good quantitative agreement. In Sec. VI, we show how decision makers can be influenced by providing them additional either correct or wrong information. Section VII concludes.

## II. QUANTUM DECISION MAKING IN SOCIETY

In this section, we generalize the QDT approach, whose detailed exposition can be found in our previous publications, developed for individual decision makers, to a society of many decision makers. We recall that the decision makers are not quantum objects, but are normal humans. The techniques of quantum theory are employed merely for taking into account the hidden variables, such as emotions and biases of decision makers [58]. The possibility of taking into account hidden variables is at the heart of the quantum-theory techniques that allow for their existence by modifying the rules of calculating the quantum probabilities. This is why the quantum techniques make it possible to characterize human decision making, incorporating in it the existence of such hidden variables as subconscious feelings and behavioral biases. The efficiency of quantum techniques for human decision making is not because humans are quantum objects, but because these techniques are mathematically designed to accommodate the existence of hidden variables, which can be of a very different nature.

The theory presented below requires the use of some mathematical techniques that are common in quantum theory. But, as is explained above, the reader does not need to know anything about quantum theory. Actually, what one needs is the basic knowledge of the functional analysis in Hilbert space and the definition of scalar products used for introducing the prospect probabilities. In order to make the presentation self-consistent and to justify the derived results, we describe the main mathematical steps of the derivation, at the same time, omitting intermediate calculations for not overloading the reader. However, we cannot omit all mathematical formulas, since then it would not be clear how we get the final important expressions and why they have the properties that are essentially used in the following applications.

Let us consider a society of $N$ agents who are decision makers. The agents are enumerated by $\alpha=1,2, \ldots, N$. Each agent is characterized by a set $\left\{e_{\alpha n}: n=1,2, \ldots, d\right\}$ of $d$ elementary prospects that are represented by vectors $|\alpha n\rangle$ in a Hilbert space.

Everywhere below, we employ the Dirac [61], [62] bracket notation, where a function $\psi_{n}(x)$ is represented as $|n\rangle$ and the scalar product of two functions is given by the formula

$$
\langle m \mid n\rangle \equiv \int \psi_{m}^{*}(x) \psi_{n}(x) d x
$$

Different elementary prospects are orthonormal to each other,

$$
\langle\alpha m \mid \alpha n\rangle=\delta_{m n},
$$

which symbolizes their mutual independence and incompatibility. The space of mind of an $\alpha$ - decision maker is the Hilbert space

$$
\begin{equation*}
\mathcal{H}_{\alpha} \equiv \operatorname{Span}_{n}\{|\alpha n\rangle\} \tag{1}
\end{equation*}
$$

The dimension of this space of mind is $d$. The space of mind of the whole society is the tensor product

$$
\begin{equation*}
\mathcal{H}=\bigotimes_{\alpha=1}^{N} \mathcal{H}_{\alpha} \tag{2}
\end{equation*}
$$

whose dimension is $N d$.
An $\alpha$ - agent deliberates on deciding between $L$ prospects forming a complete lattice

$$
\begin{equation*}
\mathcal{L}_{\alpha} \equiv\left\{\pi_{\alpha j}: j=1,2, \ldots, L\right\} \tag{3}
\end{equation*}
$$

Each prospect $\pi_{\alpha j}$ is represented by a vector $\left|\pi_{\alpha j}\right\rangle$ in the space of mind (1). The prospect vectors do not need to be orthonormal, which implies that they are not necessarily incompatible.

The prospect operator

$$
\begin{equation*}
\hat{P}\left(\pi_{\alpha j}\right) \equiv\left|\pi_{\alpha j}\right\rangle\left\langle\pi_{\alpha j}\right| \tag{4}
\end{equation*}
$$

acts on the space of mind (1). The set of all these operators is analogous to the algebra of local observables in quantum theory [62]. Respectively, the prospect probabilities are defined as the expectation values of the prospect operators. The expectation values for an individual decision maker are given by averaging the prospect operators over a strategic state of this decision maker [58], with such a strategic state being treated as a pure state represented by a single vector.

However, for the agents of a society, pure states of separate agents, generally, do not exist, since the society agents interact with each other by exchanging information. Moreover, the society as a whole may not be completely isolated from its surrounding. Therefore, the society state has to be characterized by a statistical operator $\hat{\rho}$ that is a non-negative normalized operator,

$$
\begin{equation*}
\operatorname{Tr}_{\mathcal{H}} \hat{\rho}=1 \tag{5}
\end{equation*}
$$

where the trace operation is over the society space (2). Then the expectation values of the prospect operators are given by the trace

$$
\begin{equation*}
p\left(\pi_{\alpha j}\right) \equiv \operatorname{Tr}_{\mathcal{H}} \hat{\rho} \hat{P}\left(\pi_{\alpha j}\right), \tag{6}
\end{equation*}
$$

defining the probabilities of the corresponding prospects. This definition makes the basic difference in the calculation of the prospect probabilities, as compared to the averaging over a single strategic state for individual decision makers [52]-[54].

Quantity (6), by its construction, is non-negative and defines the prospect probabilities under the normalization condition

$$
\begin{equation*}
\sum_{j=1}^{L} p\left(\pi_{\alpha j}\right)=1, \quad 0 \leq p\left(\pi_{\alpha j}\right) \leq 1 \tag{7}
\end{equation*}
$$

This definition of prospect probabilities is similar to the definition of quantum probabilities in the quantum theory of measurements [63].
Remembering that the prospect operator (4) acts on the space of mind (1) and introducing the reduced statistical operator

$$
\hat{\rho}_{\alpha} \equiv \operatorname{Tr}_{\mathcal{H} / \mathcal{H}_{\alpha}} \hat{\rho}
$$

in which the trace is over the partial factor space

$$
\mathcal{H} / \mathcal{H}_{\alpha} \equiv \bigotimes_{\beta(\neq \alpha)}^{N} \mathcal{H}_{\beta}
$$

makes it possible to rewrite the prospect probability (6) in the form

$$
\begin{equation*}
p\left(\pi_{\alpha j}\right)=\operatorname{Tr}_{\mathcal{H}_{\alpha}} \hat{\rho}_{\alpha} \hat{P}\left(\pi_{\alpha j}\right), \tag{8}
\end{equation*}
$$

with the trace over the space of mind (1).
Expanding the prospect vectors over the elementary prospect basis, and introducing the matrix elements

$$
\begin{align*}
\rho_{m n}^{\alpha} & \equiv\langle\alpha m| \hat{\rho}_{\alpha}|\alpha n\rangle \\
P_{m n}\left(\pi_{\alpha j}\right) & \equiv\langle\alpha m| \hat{P}\left(\pi_{\alpha j}\right)|\alpha n\rangle \tag{9}
\end{align*}
$$

it is straightforward to get the prospect probability

$$
\begin{equation*}
p\left(\pi_{\alpha j}\right)=f\left(\pi_{\alpha j}\right)+q\left(\pi_{\alpha j}\right), \tag{10}
\end{equation*}
$$

consisting of two terms. The first term, called the utility factor,

$$
\begin{equation*}
f\left(\pi_{\alpha j}\right)=\sum_{n} \rho_{n n}^{\alpha} P_{n n}\left(\pi_{\alpha j}\right), \tag{11}
\end{equation*}
$$

describes the classical objective probability, showing how the considered prospect is useful for the decision maker. While the second term, called the attraction factor,

$$
\begin{equation*}
q\left(\pi_{\alpha j}\right)=\sum_{m \neq n} \rho_{m n}^{\alpha} P_{n m}\left(\pi_{\alpha j}\right), \tag{12}
\end{equation*}
$$

characterizes the subjective influence of subconscious feelings, emotions, and biases and shows to what extent the prospect is attractive for the decision maker.

One could think that the form of probability (10) could be postulated, without deriving it from the preceding equations. Then, however, one would not know the properties of the terms $f(\pi)$ and $q(\pi)$. Hence, these properties should also be postulated, thus making the whole consideration overloaded by a number of postulates. Using the quantum techniques, we obtain the form of probability (10) automatically, which makes the approach self-consistent and well justified. In this way, the appearance of two terms in probability (10) is not an assumption, but it is the straightforward consequence of using
quantum techniques. The properties of these terms follow directly from Eqs. (1) to (8).

By its definition, the utility factor (11) is non-negative,

$$
\begin{equation*}
0 \leq f\left(\pi_{\alpha j}\right) \leq 1 \tag{13}
\end{equation*}
$$

and also it is normalized,

$$
\begin{equation*}
\sum_{j=1}^{L} f\left(\pi_{\alpha j}\right)=1 \tag{14}
\end{equation*}
$$

representing the classical objective probability. In the case when the prospect utilities $U\left(\pi_{\alpha j}\right)$ can be evaluated by means of classical utility theory, the utility factor takes the form

$$
\begin{equation*}
f\left(\pi_{\alpha j}\right)=\frac{U\left(\pi_{\alpha j}\right)}{\sum_{j} U\left(\pi_{\alpha j}\right)} . \tag{15}
\end{equation*}
$$

The attraction factor (12), by its definition, varies in the range

$$
\begin{equation*}
-1 \leq q\left(\pi_{\alpha j}\right) \leq 1 \tag{16}
\end{equation*}
$$

An important property of the attraction factor, following from conditions (7) and (14), is the alternation property

$$
\begin{equation*}
\sum_{j=1}^{L} q\left(\pi_{\alpha j}\right)=0 \tag{17}
\end{equation*}
$$

It is worth mentioning that the attraction factor comes into play only for composite prospects experiencing mutual interference [58]. For elementary prospects, it does not occur, being identically zero:

$$
q\left(e_{\alpha n}\right)=0 .
$$

Having defined the prospect probabilities, the prospects become naturally ordered. A prospect $\pi_{\alpha 1}$ is said to be preferred to a prospect $\pi_{\alpha 2}$ if and only if

$$
\begin{equation*}
p\left(\pi_{\alpha 1}\right)>p\left(\pi_{\alpha 2}\right) \quad\left(\pi_{\alpha 1}>\pi_{\alpha 2}\right) \tag{18}
\end{equation*}
$$

The prospects $\pi_{\alpha 1}$ and $\pi_{\alpha 2}$ are indifferent if and only if

$$
\begin{equation*}
p\left(\pi_{\alpha 1}\right)=p\left(\pi_{\alpha 2}\right) \quad\left(\pi_{\alpha 1}=\pi_{\alpha 2}\right) \tag{19}
\end{equation*}
$$

And the prospect $\pi_{\alpha 1}$ is preferred or indifferent to $\pi_{\alpha 2}$ if

$$
\begin{equation*}
p\left(\pi_{\alpha 1}\right) \geq p\left(\pi_{\alpha 2}\right) \quad\left(\pi_{\alpha 1} \geq \pi_{\alpha 2}\right) \tag{20}
\end{equation*}
$$

A prospect $\pi_{\alpha}^{*}$ that corresponds to the maximal probability

$$
p\left(\pi_{\alpha}^{*}\right)=\max _{j} p\left(\pi_{\alpha j}\right)
$$

is called optimal.
It is important to stress that the utility factor and attraction factor are principally different, having different mathematical properties, as described above. The term $f(\pi)$ contains only diagonal elements in sum (11), while term (12) contains only non-diagonal elements. In quantum theory, the non-diagonal terms characterize the existence of interference. When the latter is absent, the quantity $f(\pi)$ reduces to the classical probability. Similarly, in decision theory the term $f(\pi)$ is associated with the classical probability, while the second term $q(\pi)$ has no classical counterparts. The attraction factor in decision theory describes the interference of different prospect
modes, which is related to the deliberation of a decision maker choosing between several admissible possibilities.
Being principally different from both mathematical as well as decision-making points of view, the utility and attraction factors in no way could be combined into one quantity. Actually, the failure of the numerous "non-expected utility theories" [48] is due to the fact that in these approaches one has tried to construct a single quantity generalizing the expected utility, which has been shown to be impossible [48][51]. In the frame of the approach of the present paper, it is clear why such a combination of $f(\pi)$ and $q(\pi)$ is impossible, since they possess very different mathematical properties. And this impossibility is also easily understood in the frame of decision theory, where $f(\pi)$ describes an objective quantity that can be objectively measured, while $q(\pi)$ represents a subjective quantity that for a single decision maker can be found only empirically, though its aggregate value for a typical decision maker can be estimated as is explained in the following section.

The attraction factor in QDT is also basically different from the visceral factors considered in decision-making literature [64], where the visceral factors are assumed to be additional unknown variables entering the definition of utility functions. However, the explicit dependence of such utility functions on these visceral factors is also not known. Contrary to this, the properties of the attraction factor are prescribed by its derivation. In addition, including the visceral factors into utility functions leads to a redefinition of expected utility combining objective and subjective features, which is impossible, as discussed above.

## III. Typical behavior of social agents

Considering large societies, consisting of many agents $N \gg$ 1 and confronting numerous prospects, it is important to understand the typical behavior of such complex societies, corresponding to their behavior on average. The society is treated to be large, when $N$ is greater than 10 . Strictly speaking, the considered society has to contain so many members, for which it is admissible to collect reliable and representative statistical data, with a small standard error. For instance, measuring a quantity whose mean value is $M$, the typical statistical error for 10 agents is of the order of $0.3 M$ and for 100 agents, of order $0.1 M$.

## A. Definition of typical agent behavior

Let all agents in a society confront the same prospect lattice (3), with the same prospects $\pi_{j}=\pi_{\alpha j}$. The agents composing the society are different individuals and their decisions, even related to the same set of prospects, can vary, producing different probabilities $p\left(\pi_{\alpha j}\right)$.
The society as a whole can be characterized by the average probability

$$
\begin{equation*}
p\left(\pi_{j}\right) \equiv \frac{1}{N} \sum_{\alpha=1}^{N} p\left(\pi_{\alpha j}\right) \tag{21}
\end{equation*}
$$

averaged over all society members, which describes the typical behavior of agents. In view of expression (10), the typical
probability (21) reads as

$$
\begin{equation*}
p\left(\pi_{j}\right)=f\left(\pi_{j}\right)+q\left(\pi_{j}\right), \tag{22}
\end{equation*}
$$

with the typical utility factor

$$
\begin{equation*}
f\left(\pi_{j}\right) \equiv \frac{1}{N} \sum_{\alpha=1}^{N} f\left(\pi_{\alpha j}\right) \tag{23}
\end{equation*}
$$

and typical attraction factor

$$
\begin{equation*}
q\left(\pi_{j}\right) \equiv \frac{1}{N} \sum_{\alpha=1}^{N} q\left(\pi_{\alpha j}\right) \tag{24}
\end{equation*}
$$

Expression (22), with terms (23) and (24), appears here directly from using Eq. (10). That is, the occurrence of the attraction factor (24) is not an assumption, but the immediate consequence of the employed mathematical techniques, which themselves embody the entanglement of composite prospects. The appearance of such additional terms is typical of quantum theory, where they describe interference effects.
Because of Eqs. (13) and (14), the typical utility factor, describing the objective probability, satisfies the conditions

$$
\begin{equation*}
\sum_{j=1}^{L} f\left(\pi_{j}\right)=1, \quad 0 \leq f\left(\pi_{j}\right) \leq 1 \tag{25}
\end{equation*}
$$

In the case when it is defined by the prospect utilities according to Eq. (15), it reduces to the expression

$$
\begin{equation*}
f\left(\pi_{j}\right)=\frac{U\left(\pi_{j}\right)}{\sum_{j} U\left(\pi_{j}\right)}, \tag{26}
\end{equation*}
$$

since all agents have the same objective utilities: $U\left(\pi_{\alpha j}\right)=$ $U\left(\pi_{j}\right)$.

The attraction factor, generally, is not the same for different decision makers (it is not invariant with respect to a change of decision makers) but, owing to Eqs. (16) and (17), it preserves the alternation conditions

$$
\begin{equation*}
\sum_{j=1}^{L} q\left(\pi_{j}\right)=0, \quad-1 \leq q\left(\pi_{j}\right) \leq 1 \tag{27}
\end{equation*}
$$

In this way, each prospect is evaluated by the society with respect to two characteristics, its utility and its attractiveness. A prospect $\pi_{i}$ is more useful than $\pi_{j}$, if $f\left(\pi_{i}\right)>f\left(\pi_{j}\right)$. And a prospect $\pi_{i}$ is more attractive than $\pi_{j}$, if $q\left(\pi_{i}\right)>q\left(\pi_{j}\right)$. Therefore, a prospect can be more useful, but not preferred, being less attractive. As follows from expression (22), a prospect $\pi_{1}$ is preferred to a prospect $\pi_{2}$, in the sense of definition (18), when

$$
\begin{equation*}
p\left(\pi_{1}\right)>p\left(\pi_{2}\right) \quad\left(\pi_{1}>\pi_{2}\right) \tag{28}
\end{equation*}
$$

if and only if the inequality

$$
\begin{equation*}
f\left(\pi_{1}\right)-f\left(\pi_{2}\right)>q\left(\pi_{2}\right)-q\left(\pi_{1}\right) \tag{29}
\end{equation*}
$$

holds true.
Actually, the comparison of theory with experiment is meaningful only for a sufficiently large pool of decision makers, when the general typical features can be defined. In such a
large society, when the number of agents choosing a prospect $\pi_{j}$ is $N_{j}$, then the experimentally observed fraction

$$
\begin{equation*}
p_{\exp }\left(\pi_{j}\right) \equiv \frac{N_{j}}{N} \tag{30}
\end{equation*}
$$

provides the aggregate frequentist definition of probability that should be compared with the theoretical value (22).

For comparing the empirical frequentist probability $p_{\text {exp }}(\pi)$ with the theoretical probability $p(\pi)$, we need to know how the latter can be calculated. The utility factor is explicitly defined in Eq. (26). The attraction factor, being a subjective quantity, essentially depends on the subjective state of a decision maker. Moreover, the same prospect, at different times, can be appreciated by a decision maker differently. Therefore, it seems that it is so much random that there is no way of finding its quantitative definition. However, as is explained above, the attraction factor possesses some general well defined and fixed properties. For instance, we know that it varies in the interval $[-1,1]$ and that it obeys the alternation condition (27). Being a random quantity does not preclude that it can enjoy some general typical properties. That is, an aggregate value of the attraction factor can be well defined. Under the aggregate value, we mean an average value, averaged either over many realizations of the same problem for a single decision maker or over the results for many decision makers deciding on a given problem. Such a typical value of the attraction factor can be found by accomplishing a series of experimental observations. Another way of theoretically defining the typical attraction factor is explained in the following section.

## B. Typical values of attraction factors

The attraction factors are subjective quantities that can be different for different decision makers. And for the same decision maker, attraction factors are different for different prospects, and even can be different for the same prospect at different times. This is equivalent to accepting that the attraction factor is a random quantity that can be characterized by a distribution $\varphi\left(q\left(\pi_{\alpha j}\right)\right)$. Since the attraction factor lies in the interval $[-1,1]$, its distribution is normalized as

$$
\begin{equation*}
\int_{-1}^{1} \varphi(q) d q=1 \tag{31}
\end{equation*}
$$

And, in view of the alternation condition (27), the mean value of the attraction factor is zero,

$$
\begin{equation*}
\int_{-1}^{1} \varphi(q) q d q=0 \tag{32}
\end{equation*}
$$

The exact attraction-factor distribution is unknown in general. In particular cases, it could be extracted from empirical observations. Moreover, even in the absence of any a priori empirical information, the typical values of the attraction factor, being a random quantity varying in the interval $[-1,1]$, can be estimated [58] in the following way.

Let us define the values

$$
\begin{equation*}
q_{+} \equiv \int_{0}^{1} \varphi(q) q d q, \quad q_{-} \equiv \int_{-1}^{0} \varphi(q) q d q \tag{33}
\end{equation*}
$$

which, according to the alternation condition (32), are related as

$$
\begin{equation*}
q_{+}+q_{-}=0 \tag{34}
\end{equation*}
$$

The absence of any a priori information implies that the distribution $\varphi(q)$ is uniform. This is evident from the generally accepted notion of no-a-priori information that implies the equiprobability of the variable in its whole domain. Also, as is well known, the equiprobable distribution provides the maximum of the Shannon entropy, which, in turn, characterizes the information measure [65].

In the case of the equiprobable distribution, the normalization condition yields $\varphi(q)=1 / 2$. As a result, the values (33) become

$$
\begin{equation*}
q_{+}=\frac{1}{4}, \quad q_{-}=-\frac{1}{4} . \tag{35}
\end{equation*}
$$

We have called the existence of such typical values of the attraction factor, corresponding to the non-informative priors, as the quarter law [58]. These values (35] can be used for estimating the influence of the attraction factors on the decision making of typical agents.
It has been proved in the earlier publications [53], [54], [57], [58] that the quarter law is in perfect agreement with a variety of empirical observations. Below we also show that the use of this typical value for the attraction factor is in good agreement with many other empirical data.

It is worth emphasizing that the value $1 / 4$ for the typical attraction factor is valid not only in the case of an equiprobable distribution, but also for a wide class of distributions. Let us take, for example, the symmetric beta-distribution

$$
\varphi(q)=\frac{\Gamma(2 \alpha)}{2 \Gamma^{2}(\alpha)}|q|^{\alpha-1}(1-|q|)^{\alpha-1}
$$

with the domain $[-1,1]$, often employed in many applications [66], where $\alpha$ is an arbitrary positive parameter. Then the typical values $q_{-}$and $q_{+}$are exactly $-1 / 4$ and $+1 / 4$, respectively, for any $\alpha>0$. The same quarter law follows from several other distributions normalized on the interval $[-1,1]$, for instance, from the symmetric quadratic distribution

$$
\varphi(q)=6\left(|q|-\frac{1}{2}\right)^{2}
$$

and from the symmetric triangular distribution

$$
\varphi(q)=\left\{\begin{array}{ll}
2|q|, & 0 \leq|q| \leq \frac{1}{2} \\
2(1-|q|), & \frac{1}{2}<|q| \leq 1
\end{array} .\right.
$$

In this way, the quarter law is not an ad hoc assumption, but it is a consequence of the theoretical evaluation for several distributions, which is in agreement with a number of empirical data.

## C. Quantitative resolution of classical paradoxes

The typical values of the attraction factor (35) make it possible to give quantitative predictions for decisions of typical decision makers. For instance, the disjunction effect, studied in different forms in a variety of experiments [67], was thoroughly analyzed [54], [58], and we found that the
empirically determined absolute value of the aggregate attraction factor $\left|q\left(\pi_{j}\right)\right|$ coincided with the value 0.25 predicted by expressions (35), within the typical statistical error of the order of $20 \%$ characterizing these experiments. The same quantitative agreement, between the QDT prediction for the absolute value of the attraction factors and empirical values, holds for experiments testing the conjunction fallacy [54], [58]. The planning paradox has also found a natural explanation within QDT [53]. Moreover, it has been shown [57] that QDT explains practically all typical paradoxes of classical decision making, arising when decisions are taken by typical decision makers.

In order to illustrate how QDT resolves classical paradoxes, let us consider a typical paradox happening in decision making. In game theory, there is a series of games in which several subjects can choose either to cooperate with each other or to defect. Such setups have the general name of prisoner dilemma games. The cooperation paradox consists in the real behavior of game participants who often incline to cooperate despite the prescription of utility theory for defection. Below, we show that this paradox is easily resolved within QDT, which gives correct quantitative predictions.
The generic structure of the prisoner dilemma game is as follows. Two participants can either cooperate with each other or defect from cooperation. Let the cooperation action of one of them be denoted by $C_{1}$ and the defection by $D_{1}$. Similarly, the cooperation of the second subject is denoted by $C_{2}$ and the defection by $D_{2}$. Depending on their actions, the participants receive payoffs from the set

$$
\begin{equation*}
\mathbb{X}=\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\} \tag{36}
\end{equation*}
$$

whose values are arranged according to the inequality

$$
\begin{equation*}
x_{3}>x_{1}>x_{4}>x_{2} \tag{37}
\end{equation*}
$$

There are four admissible cases: both participants cooperate $\left(C_{1} C_{2}\right)$, one cooperates and the other defects $\left(C_{1} D_{2}\right)$, the first defects but the second cooperates $\left(D_{1} C_{2}\right)$, and both defect $\left(D_{1} D_{2}\right)$. The payoffs to each of them, depending on their actions, are given according to the rule

$$
\left[\begin{array}{ll}
C_{1} C_{2} & C_{1} D_{2}  \tag{38}\\
D_{1} C_{2} & D_{1} D_{2}
\end{array}\right] \rightarrow\left[\begin{array}{ll}
x_{1} x_{1} & x_{2} x_{3} \\
x_{3} x_{2} & x_{4} x_{4}
\end{array}\right]
$$

As is clear, the enumeration of the participants is arbitrary, so that it is possible to analyze the actions of any of them.

Each subject has to decide what to do, to cooperate or to defect, when he/she is not aware about the choice of the opponent. Then, for each of the participants, there are two prospects, either to cooperate,

$$
\begin{equation*}
\pi_{1}=C_{1}\left(C_{2}+D_{2}\right), \tag{39}
\end{equation*}
$$

or to defect,

$$
\begin{equation*}
\pi_{2}=D_{1}\left(C_{2}+D_{2}\right) \tag{40}
\end{equation*}
$$

The sum $\left(C_{2}+D_{2}\right)$ embodies the fact that the decision maker does not know the choice (cooperate or defect) of the second participant. In the absence of any information on the action chosen by the opponent, the probability for each of these actions is $1 / 2$. Assuming for simplicity a linear utility function
of the payoffs, the expected utility of cooperation for the first subject is

$$
\begin{equation*}
U\left(\pi_{1}\right)=\frac{1}{2} x_{1}+\frac{1}{2} x_{2}, \tag{41}
\end{equation*}
$$

while the utility of defection is

$$
\begin{equation*}
U\left(\pi_{2}\right)=\frac{1}{2} x_{3}+\frac{1}{2} x_{4} . \tag{42}
\end{equation*}
$$

The assumption of linear utility is not crucial, and can be removed by reinterpreting the payoff set (36) as a utility set. Because of condition (37), the utility of defection is always larger than that of cooperation, $U\left(\pi_{2}\right)>U\left(\pi_{1}\right)$. According to utility theory, this means that all subjects have always to prefer defection.
However, numerous empirical studies demonstrate that an essential fraction of participants choose to cooperate despite the prescription of utility theory. This contradiction between reality and the theoretical prescription constitutes the cooperation paradox [39], [68].
Considering the same game within the framework of QDT, we have the probabilities of the two prospects,

$$
\begin{equation*}
p\left(\pi_{1}\right)=f\left(\pi_{1}\right)+q\left(\pi_{1}\right), \quad p\left(\pi_{2}\right)=f\left(\pi_{2}\right)+q\left(\pi_{2}\right) \tag{43}
\end{equation*}
$$

Let us recall that humans possess the so-called propensity for cooperation, which is a well established empirical fact [69]-[71]. This propensity has developed during the history of humanity starting from the very beginning of human existence as hunters-gatherers. In the process of their development, humans noticed that cooperation is profitable for their survival and well-being. The propensity for cooperation has been the driving force for the creation of human societies, from tribes to states and country unions [69]-[72]. Without this feature, no social groups would be formed. The propensity to cooperation proposes that the attraction factor for cooperative prospect is larger than that for the defecting prospect, that is, $q\left(\pi_{1}\right)>q\left(\pi_{2}\right)$. In view of the alternation law 27, we have $q\left(\pi_{1}\right)=-q\left(\pi_{2}\right)$, which can be estimated by the typical value $1 / 4$, as in expressions (35). Hence, we can estimate the considered prospects by the equations

$$
\begin{equation*}
p\left(\pi_{1}\right)=f\left(\pi_{1}\right)+0.25, \quad p\left(\pi_{2}\right)=f\left(\pi_{2}\right)-0.25 \tag{44}
\end{equation*}
$$

From here, we see that, even if defection seems to be more useful than cooperation, so that $f\left(\pi_{2}\right)>f\left(\pi_{1}\right)$, the cooperative prospect can be preferred by some of the participants.

To illustrate numerically how this paradox is resolved, let us take the data from the experimental realization of the prisoner dilemma game by Tversky and Shafir [67]. Subjects played a series of prisoner dilemma games, without feedback. Three types of setups were used: first, when the subjects knew that the opponent had defected; second, when they knew that the opponent had cooperated; and third, when subjects did not know whether their opponent had cooperated or defected. The rate of cooperation was $3 \%$ when subjects knew that the opponent had defected, and $16 \%$ when they knew that the opponent had cooperated. However, when subjects did not know whether their opponent had cooperated or defected, the rate of cooperation was $37 \%$.

Treating the utility factors as classical probabilities, we have

$$
\begin{aligned}
& f\left(\pi_{1}\right)=\frac{1}{2} f\left(C_{1} \mid C_{2}\right)+\frac{1}{2} f\left(C_{1} \mid D_{2}\right) \\
& f\left(\pi_{2}\right)=\frac{1}{2} f\left(D_{1} \mid C_{2}\right)+\frac{1}{2} f\left(D_{1} \mid D_{2}\right)
\end{aligned}
$$

According to the Tversky-Shafir data,

$$
f\left(C_{1} \mid C_{2}\right)=0.16, \quad f\left(C_{1} \mid D_{2}\right)=0.03
$$

Therefore,

$$
\begin{equation*}
f\left(\pi_{1}\right)=0.10, \quad f\left(\pi_{2}\right)=0.90 \tag{45}
\end{equation*}
$$

Then, for the prospect probabilities (22), we get

$$
\begin{equation*}
p\left(\pi_{1}\right)=0.35, \quad p\left(\pi_{2}\right)=0.65 \tag{46}
\end{equation*}
$$

In this way, the fraction of subjects choosing cooperation is predicted to be $35 \%$. This is in remarkable agreement with the empirical data of $37 \%$ by [67]. Actually, within the statistical accuracy of the experiment, the predicted and empirical numbers are indistinguishable.

## IV. Influencing choice by reversing attraction FACTORS

In the prospect probability (22), the first term (23) is an objectively defined quantity characterizing, depending on the setup, either a classical probability or the prospect utility factor. It would, of course, be possible to change the society choice by varying the utility of prospects. This, however, would be just an objective shift of preferences caused by the varying prospect utilities.

More important is that it is possible to essentially change the decision makers choice merely by influencing the attractiveness of the considered prospects, without essentially varying their utilities. This means that the attraction factors are to be influenced.

## A. Prospect probabilities for binary lattices

The most often and illustrative case is the choice between two prospects forming a binary lattice

$$
\begin{equation*}
\mathcal{L}=\left\{\pi_{1}, \pi_{2}\right\} . \tag{47}
\end{equation*}
$$

Suppose that the prospect $\pi_{1}$ is more attractive than $\pi_{2}$, which means that $q\left(\pi_{1}\right)>q\left(\pi_{2}\right)$. According to the alternation property (27), we have $q\left(\pi_{1}\right)=-q\left(\pi_{2}\right)$. Then, taking into account the quarter law (35), we can estimate the attraction factor $q\left(\pi_{1}\right)$ as $1 / 4$, while the attraction factor $q\left(\pi_{2}\right)$ as $-1 / 4$. Keeping in mind that a probability, by its meaning, lies in the interval $[0,1]$, the prospect probabilities can be evaluated by the formulas

$$
\begin{align*}
& p\left(\pi_{1}\right)=\operatorname{Ret}_{[0,1]}\left\{f\left(\pi_{1}\right)+\frac{1}{4}\right\}, \\
& p\left(\pi_{2}\right)=\operatorname{Ret}_{[0,1]}\left\{f\left(\pi_{2}\right)-\frac{1}{4}\right\}, \tag{48}
\end{align*}
$$

where the retract function

$$
\operatorname{Ret}_{[0,1]}\{z\} \equiv \begin{cases}0, & z<0 \\ z, & 0 \leq z \leq 1 \\ 1, & z>1\end{cases}
$$

is employed.

## B. Attraction factors and risk aversion

Formulas (48) can be used for evaluating the prospect probabilities in the case of the binary lattice (47). The classification of prospects onto more or less attractive is based on subjective feelings of decision makers. Among these, a very important role is played by the notion of aversion to uncertainty and risk, or ambiguity aversion [31], [32], [73]-[78]. It is possible to define as more attractive the prospect that provides more certain gain, hence, more uncertain loss [58].

It is worth recalling that the attraction factors are not fixed by the values $\pm 1 / 4$. These values have been obtained as noninformative priors allowing us to estimate the probability of selecting between the prospects. In particular cases, they can be different, since by their definition, they characterize subjective features of decision makers. Nevertheless, these noninformative priors provide a simple way for the probability estimation and lead to a very good agreement with empirical observations, as has been shown in our previous publications quantitatively resolving the classical paradoxes in decision making, such as disjunction effect and conjunction fallacy [54], [56], [58]. And in Sec. III C above, we have shown how the prisoner-dilemma paradox is quantitatively resolved within QDT. Below, we illustrate the applicability of the approach to several examples considered earlier by Kahneman and Tversky [31]. We stress that our theory has not been specially designed for explaining these examples, but the latter provide just one more illustration of the QDT approach that is general and can be applied to arbitrary cases as has been shown in our previous publications. We consider below different prospects with gains. The case of losses is also treatable by QDT. However, this case requires a separate consideration that is out of the scope of the present paper.

## C. Rule for defining attraction factor signs

The attraction factor sign is principally important, since it essentially influences the value of the prospect probability. The choice of this sign depends on the balance between the possible gain and related risk. Below, we describe how this choice can be done for the most often considered case of the binary prospect lattice.

Mathematically, the attraction factor, due to mode interference, arises only for the entangled composite prospects [63], which implies decisions under uncertainty. To the first glance, it seems that deciding between two simple lotteries does not explicitly involve uncertainty. However, it is necessary to stress that practically all decisions always deal with uncertainty, though it may be not explicitly formulated. Suppose, e.g., one has to decide between two lotteries $\pi_{1}$ and $\pi_{2}$. In the process of decision making, the uncertainty comes from two origins. One is related to the doubt about the objectivity of the setup suggesting the choice. The other, probably more important, is the uncertainty caused by the subjective hesitations of the decision maker with respect to his/her correct understanding of the problem and his/her knowledge of what would be the best criterion for making a particular choice. Therefore, even when one formally deals with two simple lotteries $\pi_{1}$ and $\pi_{2}$, one actually confronts the composite prospects $\pi_{1} \otimes B$
and $\pi_{2} \otimes B$, with $B=\left\{B_{1}, B_{2}\right\}$ being a set of two events. One of them, $B_{1}$, represents the confidence of the decision maker in the empirical setup as well as in the correctness of his/her decision. The other, $B_{2}$, corresponds to the disbelief of the decision maker in the suggested setup and/or in his/her understanding of the appropriate criteria for the choice. In what follows, we shall keep in mind the composite prospects $\pi_{i} \otimes B$, with $i=1,2$, while, for brevity, we shall write just $\pi_{i}$.

Let us consider two prospects

$$
\begin{align*}
& \pi_{1}=\left\{x_{i}, p_{1}\left(x_{i}\right): i=1,2, \ldots\right\}, \\
& \pi_{2}=\left\{y_{j}, p_{2}\left(y_{j}\right): j=1,2, \ldots\right\} . \tag{49}
\end{align*}
$$

The related maximal and minimal gains are denoted as

$$
\begin{array}{ll}
x_{\max } \equiv \sup _{i}\left\{x_{i}\right\}, & x_{\min } \equiv \inf _{i}\left\{x_{i}\right\}, \\
y_{\max } \equiv \sup _{j}\left\{y_{j}\right\}, & y_{\min } \equiv \inf _{j}\left\{y_{j}\right\} . \tag{50}
\end{array}
$$

The signs of the attraction factors for the binary prospect lattice, in view of the alternation condition (27), are connected with each other,

$$
\operatorname{sgn} q\left(\pi_{1}\right)=-\operatorname{sgn} q\left(\pi_{2}\right),
$$

because of which in what follows it is sufficient to analyze only one of them, say the sign of $q\left(\pi_{1}\right)$.

The first prospect gain factor is the ratio

$$
g\left(\pi_{1}\right) \equiv \frac{x_{\max }}{y_{\max }}
$$

showing how much the maximal gain of the first prospect is larger than that of the second one. On the other hand, the larger the probability of getting the minimal gain in the second prospect, the larger is the ratio

$$
r\left(\pi_{2}\right) \equiv \frac{p_{2}\left(y_{\min }\right)}{p_{1}\left(x_{\min }\right)},
$$

playing the role of the risk factor when choosing the second prospect. The combined influence of possible gain and risk is described by the product $g\left(\pi_{1}\right) r\left(\pi_{2}\right)$. The attractiveness of a prospect is characterized by how much the gain prevails over risk, that is, how the latter product $g\left(\pi_{1}\right) r\left(\pi_{2}\right)$ differs from one, hence by the sign of the value

$$
\begin{equation*}
\alpha\left(\pi_{1}\right) \equiv g\left(\pi_{1}\right) r\left(\pi_{2}\right)-1=\frac{x_{\max } p_{2}\left(y_{\min }\right)}{y_{\max } p_{1}\left(x_{\min }\right)}-1 . \tag{51}
\end{equation*}
$$

Then the sign of the first prospect attraction factor is defined by the rule

$$
\operatorname{sgn} q\left(\pi_{1}\right)=\left\{\begin{array}{ll}
+1, & \alpha\left(\pi_{1}\right)>0  \tag{52}\\
-1, & \alpha\left(\pi_{1}\right) \leq 0
\end{array} .\right.
$$

In the following section, we illustrate the practical application of this rule and show that it yields the results in good agreement with empirical observations. Let us stress that the formulated rule is designed for the case where the utilities of two prospects are close to each other and may be not applicable when these utilities are strongly different.

## V. Illustration of preference reversal by EXAMPLES

Here, we consider several examples of experiments described by Kahneman and Tversky [31]. In these experiments, the total number of decision makers was about equal to or smaller than one hundred, and the corresponding statistical errors were close to $\pm 0.1$. Decision makers had to choose between two prospects having the properties as those discussed above. Payoff were counted in monetary units, say in thousands of schekels, francs, or dollars. The kind of monetary units has no influence on the relative quantities, such as utility factors and prospect probabilities. Calculating the utility factors, we use for simplicity a linear utility function.

Example 1. One chooses between the prospects

$$
\pi_{1}=\{2.5,0.33|2.4,0.66| 0,0.01\}, \quad \pi_{2}=\{2.4,1\}
$$

Here, the first number of each pair corresponds to the payoff while the second number is the associated probability. Thus, the second prospect $\pi_{2}$ corresponds to the sure gain (probability 1 ) of 2.4 monetary units.

Using definition 26 of the utility factors

$$
f\left(\pi_{1}\right)=\frac{U\left(\pi_{1}\right)}{U\left(\pi_{1}\right)+U\left(\pi_{2}\right)}, \quad f\left(\pi_{2}\right)=\frac{U\left(\pi_{2}\right)}{U\left(\pi_{1}\right)+U\left(\pi_{2}\right)}
$$

with the utilities
$U\left(\pi_{1}\right)=2.5 \times 0.33+2.4 \times 0.66=2.409, \quad U\left(\pi_{2}\right)=2.4$,
gives the utility factors

$$
f\left(\pi_{1}\right)=0.501, \quad f\left(\pi_{2}\right)=0.499
$$

Following the rule described above, we get

$$
g\left(\pi_{1}\right)=1.042, \quad r\left(\pi_{2}\right)=0, \quad \alpha\left(\pi_{1}\right)=-1
$$

which tells us that the first prospect is less attractive. Then, again employing the non-informative priors, $q\left(\pi_{1}\right)$ can be estimated as $-1 / 4$, while $q\left(\pi_{2}\right)$, as $1 / 4$. Thus, we get the prospect probabilities

$$
p\left(\pi_{1}\right)=0.251, \quad p\left(\pi_{2}\right)=0.749
$$

In experiments, it was found that

$$
p_{\exp }\left(\pi_{1}\right)=0.18, \quad p_{\exp }\left(\pi_{2}\right)=0.82,
$$

which, within the experimental accuracy, coincides with the theoretical prediction.

Example 2. One considers the prospects

$$
\pi_{1}=\{2.5,0.33 \mid 0,0.67\}, \quad \pi_{2}=\{2.4,0.34 \mid 0,0.66\}
$$

The utility factors are practically the same as in the previous example:

$$
f\left(\pi_{1}\right)=0.503, \quad f\left(\pi_{2}\right)=0.497
$$

Now, the uncertainties of the two prospects are close to each other and the gain in the first prospect is a bit larger, which gives

$$
g\left(\pi_{1}\right)=1.042, \quad r\left(\pi_{2}\right)=0.985, \quad \alpha\left(\pi_{1}\right)=0.027
$$

which suggests that the first prospect is more attractive. This yields the prospect probabilities

$$
p\left(\pi_{1}\right)=0.753, \quad p\left(\pi_{2}\right)=0.247
$$

Again, this is in agreement with the experimental values

$$
p_{\exp }\left(\pi_{1}\right)=0.83, \quad p_{\exp }\left(\pi_{2}\right)=0.17
$$

being in the corridor of statistical errors.
Comparing the examples 1 and 2, we see that a change in the distribution of payoff weights, under the same payoffs, has lead to the reversal of the attraction factors and, as a result, to the preference reversal.

Example 3. The prospects are

$$
\pi_{1}=\{4,0.8 \mid 0,0.2\}, \quad \pi_{2}=\{3,1\}
$$

The utility factors (26) become

$$
f\left(\pi_{1}\right)=0.516, \quad f\left(\pi_{2}\right)=0.484
$$

In so far as

$$
g\left(\pi_{1}\right)=1.333, \quad r\left(\pi_{2}\right)=0, \quad \alpha\left(\pi_{1}\right)=-1
$$

the second prospect is more attractive. Then, we have the prospect probabilities

$$
p\left(\pi_{1}\right)=0.266, \quad p\left(\pi_{2}\right)=0.734
$$

which agree well with the empirical results

$$
p_{\exp }\left(\pi_{1}\right)=0.2, \quad p_{\exp }\left(\pi_{2}\right)=0.8
$$

Example 4. The prospects

$$
\pi_{1}=\{4,0.2 \mid 0,0.8\}, \quad \pi_{2}=\{3,0.25 \mid 0,0.75\}
$$

have the same payoffs and the same utility factors

$$
f\left(\pi_{1}\right)=0.516, \quad f\left(\pi_{2}\right)=0.484
$$

as in the previous case. Now we have

$$
g\left(\pi_{1}\right)=1.333, \quad r\left(\pi_{2}\right)=0.937, \quad \alpha\left(\pi_{1}\right)=0.249
$$

Hence the first prospect is more attractive. This gives the prospect probabilities

$$
p\left(\pi_{1}\right)=0.766, \quad p\left(\pi_{2}\right)=0.234
$$

with the reverse preference, as compared to Example 3. The experimental results

$$
p_{\text {exp }}\left(\pi_{1}\right)=0.65, \quad p_{\text {exp }}\left(\pi_{2}\right)=0.35
$$

are in agreement with the theoretical prediction.
Example 5. For the prospects

$$
\pi_{1}=\{6,0.45 \mid 0,0.55\}, \quad \pi_{2}=\{3,0.9 \mid 0,0.1\}
$$

the utility factors are equal,

$$
f\left(\pi_{1}\right)=0.5, \quad f\left(\pi_{2}\right)=0.5
$$

The second prospect is more attractive, since

$$
g\left(\pi_{1}\right)=2, \quad r\left(\pi_{2}\right)=0.182, \quad \alpha\left(\pi_{1}\right)=-0.636
$$

Then

$$
p\left(\pi_{1}\right)=0.25, \quad p\left(\pi_{2}\right)=0.75
$$

The experimental results

$$
p_{\exp }\left(\pi_{1}\right)=0.14, \quad p_{\exp }\left(\pi_{2}\right)=0.86
$$

within the statistical errors of $\pm 0.1$, agree with the theoretical prediction.

Example 6. The prospects

$$
\pi_{1}=\{6,0.001 \mid 0,0.999\}, \quad \pi_{2}=\{3,0.002 \mid 0,0.998\}
$$

lead to the same utility factors

$$
f\left(\pi_{1}\right)=0.5, \quad f\left(\pi_{2}\right)=0.5
$$

as in the previous example. The uncertainties of the two prospects are close to each other. However, the gain in the first prospect is essentially larger, which gives

$$
g\left(\pi_{1}\right)=2, \quad r\left(\pi_{2}\right)=0.999, \quad \alpha\left(\pi_{1}\right)=0.998
$$

This makes the second prospect less attractive. As a result, the prospect preference reverses, as compared to Example 5, with the prospect probabilities

$$
p\left(\pi_{1}\right)=0.75, \quad p\left(\pi_{2}\right)=0.25
$$

The experimental data

$$
p_{\exp }\left(\pi_{1}\right)=0.73, \quad p_{\exp }\left(\pi_{2}\right)=0.27
$$

practically coincide with the theoretical prediction, again demonstrating the preference reversal.

Example 7. Consider the prospects

$$
\pi_{1}=\{6,0.25 \mid 0,0.75\}, \quad \pi_{2}=\{4,0.25|2,0.25| 0,0.5\}
$$

The utility factors are

$$
f\left(\pi_{1}\right)=0.5, \quad f\left(\pi_{2}\right)=0.5
$$

Now we have

$$
g\left(\pi_{1}\right)=1.5, \quad r\left(\pi_{2}\right)=0.667, \quad \alpha\left(\pi_{1}\right)=0
$$

Hence, the first prospect is less attractive, leading to the probabilities

$$
p\left(\pi_{1}\right)=0.25, \quad p\left(\pi_{2}\right)=0.75
$$

The empirical data

$$
p_{\exp }\left(\pi_{1}\right)=0.18, \quad p_{\exp }\left(\pi_{2}\right)=0.82
$$

within the experimental accuracy, are in agreement with the theoretical prediction.

The results are summarized in the Table where, in the last column, the error

$$
\Delta\left(\pi_{1}\right)=\left|p\left(\pi_{1}\right)-p_{\exp }\left(\pi_{1}\right)\right|
$$

is shown. For all cases, this error is about 0.1 , which is the same as the standard error 0.1 for the corresponding experiments.

In the above examples, we have considered prospects that are characterized by different gains. The treatment of

TABLE I
UTILITY FACTOR $f\left(\pi_{1}\right)$, PROSPECT PROBABILITY $p\left(\pi_{1}\right)$, EMPIRICAL FREQUENCY $p_{\exp }\left(\pi_{1}\right)$, AND THE DEVIATION $\Delta\left(\pi_{1}\right)$ FOR THE CONSIDERED EXAMPLES OF DECISION-MAKING MANIPULATION.

|  | $f\left(\pi_{1}\right)$ | $p\left(\pi_{1}\right)$ | $p_{\exp }\left(\pi_{1}\right)$ | $\Delta\left(\pi_{1}\right)$ |
| :--- | :--- | :--- | :--- | :--- |
| 1 | 0.501 | 0.251 | 0.18 | 0.07 |
| 2 | 0.503 | 0.753 | 0.83 | 0.08 |
| 3 | 0.516 | 0.266 | 0.20 | 0.07 |
| 4 | 0.516 | 0.766 | 0.65 | 0.12 |
| 5 | 0.5 | 0.25 | 0.14 | 0.11 |
| 6 | 0.5 | 0.75 | 0.73 | 0.02 |
| 7 | 0.5 | 0.25 | 0.18 | 0.07 |

prospects, involving losses, is a separate problem. Strictly speaking, in real life, in order to lose something, it is in general the case that one possesses a wealth no less than the loss. However, there are also examples of negative wealth, associated with debts that are larger than present equity. For a firm, this leads in general to bankruptcy. Rationally, agents should also default on their debts, if they can, a situation that often but not always occurs, as for instance exemplified by the many cases of negative equity of homeowners in the USA [79] and Great Britain [80] following the real estate price collapse and financial crisis. Thus, in general, we should expect that the prospect probabilities depend on the initial richness of decision makers. But in the laboratory experiments, one usually considers artificial situations, with imaginary or unrealistic small losses, when the starting assets are not important. The real and imaginary losses are rather different things and are to be treated differently. These delicate problems are out of the scope of the present paper and will be treated in a separate publication.

Our aim has been to demonstrate the fact that, under practically the same utility, by appropriately arranging the payoff weights, it is possible to realize the reversal of the attraction factors and, as a result, the reversal of decision preferences.

## VI. INFLUENCE BY VARYING AVAILABLE INFORMATION

The standard setup of studying decision making in the laboratory is when decision makers are assumed to give responses without consulting each other and without looking for additional information. However, in a number of experimental studies, it has been found that decisions can essentially change when the agents are allowed to consult with each other, increasing by this their mutual information [81]-[87], or when they can get additional information by learning from their own experience [88].

When the objective parts of the prospect probabilities are assumed to remain invariant, the influence on decision making of the obtained information can be realized by varying the attraction factors. Therefore, we have to understand how the latter vary with respect to the change of information available to decision makers.

Let us denote by $\mu$ the measure of information available to a decision maker. This measure can be defined according to one
of the known ways of measuring information [89]. Decision making depends on the amount of information and varies when it changes [90].

Generalizing the consideration of Sec. II, we take into account that the society state, represented by the statistical operator $\hat{\rho}(\mu)$, depends on the available information $\mu$. This means that the society state $\hat{\rho}$ is influenced by the received additional information $\mu$, which transforms $\hat{\rho}$ into $\hat{\rho}(\mu)$. The transformation law can be represented by a unitary evolution operator, as is described below. In simple language, this implies that the society state depends on the available information and varies when the level of information changes [91]. In mathematical terms, the amount of the received information can be quantified by the Kullback-Leibler [92] information

$$
I_{K L}(\mu)=\operatorname{Tr} \hat{\rho}(\mu) \ln \frac{\hat{\rho}(\mu)}{\hat{\rho}} .
$$

The prospect probabilities of an $\alpha$ - agent take the form

$$
\begin{equation*}
p\left(\pi_{\alpha j}, \mu\right)=\operatorname{Tr}_{\mathcal{H}} \hat{\rho}(\mu) \hat{P}\left(\pi_{\alpha j}\right) \tag{53}
\end{equation*}
$$

where all notations are the same as in Sec. II.
The variation of the society state with information can be described by the information evolution operator $\hat{U}(\mu)$, so that

$$
\begin{equation*}
\hat{\rho}(\mu)=\hat{U}(\mu) \hat{\rho} \hat{U}^{+}(\mu) \tag{54}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{\rho}(0)=\hat{\rho} \tag{55}
\end{equation*}
$$

As before, the society state is normalized, such that

$$
\begin{equation*}
\operatorname{Tr}_{\mathcal{H}} \hat{\rho}(\mu)=1 \tag{56}
\end{equation*}
$$

The initial condition 55) yields

$$
\begin{equation*}
\hat{U}(0)=\hat{1}_{\mathcal{H}}, \tag{57}
\end{equation*}
$$

with $\hat{1}_{\mathcal{H}}$ being the unity operator on space (2). And the normalization condition (56) requires that the evolution operator be a unitary operator:

$$
\begin{equation*}
\hat{U}^{+}(\mu) \hat{U}(\mu)=\hat{1}_{\mathcal{H}} \tag{58}
\end{equation*}
$$

These properties make it possible to represent the evolution operator as

$$
\begin{equation*}
\hat{U}(\mu)=e^{-i \hat{H} \mu} \tag{59}
\end{equation*}
$$

where $\hat{H}$, acting on space (2), is called the evolution generator.
The general form of the evolution generator can be written as the sum of the terms acting on each of the decision makers in the society and the term characterizing the interactions between these decision makers:

$$
\begin{equation*}
\hat{H}=\bigoplus_{\alpha=1}^{N} \hat{H}_{\alpha}+\hat{H}_{i n t} \tag{60}
\end{equation*}
$$

where $\hat{H}_{\alpha}$ acts on space (1) and $\hat{H}_{i n t}$, on space (2).
The agents of the society are considered as separate individuals who, though interacting with each other, do not loose their personal identities and are able to take individual decisions. In mathematical language, this means that agents
are quasi-isolated [93]. The mathematical formulation of the quasi-isolated state reads as the commutation condition

$$
\begin{equation*}
\left[\hat{H}_{\alpha} \bigotimes \hat{1}_{\mathcal{H}}, \hat{H}_{\text {int }}\right]=0 \tag{61}
\end{equation*}
$$

Similarly to Sec. III, assuming that all agents are confronted with the same prospect lattice, we introduce the notion of a typical agent, whose decisions are described by the average prospect probabilities

$$
\begin{equation*}
p\left(\pi_{j}, \mu\right)=\frac{1}{N} \sum_{\alpha=1}^{N} p\left(\pi_{\alpha j}, \mu\right) \tag{62}
\end{equation*}
$$

The property of quasi-isolation (61) makes it possible to show that the prospect probabilities 62 acquire the form

$$
\begin{equation*}
p\left(\pi_{j}, \mu\right)=f\left(\pi_{j}\right)+q\left(\pi_{j}, \mu\right) \tag{63}
\end{equation*}
$$

similar to Eq. (22). Here, the first term is the utility factor that is the same as in Eqs. (23) and (26). The second term is the attraction factor that can be represented as

$$
\begin{equation*}
q\left(\pi_{j}, \mu\right)=q\left(\pi_{j}\right) D(\mu) \tag{64}
\end{equation*}
$$

where

$$
\begin{equation*}
q\left(\pi_{j}\right)=q\left(\pi_{j}, 0\right) \tag{65}
\end{equation*}
$$

is the attraction factor at the initial state, when no additional information has yet been digested, and $D(\mu)$ is a decoherence factor. This name comes from the fact that, technically, the attraction factor appears under the interference of composite prospects [58]. Decoherence implies that the interference effects fade away, so that the prospect probabilities tend to their classical values defined by the utility factors. In other words, this means that

$$
\begin{equation*}
\lim _{\mu \rightarrow \infty} p\left(\pi_{j}, \mu\right)=f\left(\pi_{j}\right) \tag{66}
\end{equation*}
$$

Treating the agent interactions in analogy with a scattering process over random scatterers, with the width $\mu_{c}$ in the Lorentzian distribution of scatterer defects [93], we have

$$
\begin{equation*}
D(\mu)=\exp \left(-\frac{\mu}{\mu_{c}}\right) \tag{67}
\end{equation*}
$$

The meaning of $\mu_{c}$ is the amount of information required for the reduction of the attraction factor by a ratio of $e=2.718 \ldots$.

The dependence of the attraction factors on the available information suggests that it is admissible to vary these factors by regulating the amount of information. Respectively, by varying the attraction factors, it is possible to influence decisions. For instance, suppose that, at $\mu=0$, the prospect $\pi_{1}$ is preferred to $\pi_{2}$. By providing additional information, one can reduce the attraction factors according to Eq. (67). As a result, the preference can be reversed, with the prospect $\pi_{2}$ becoming preferable to $\pi_{1}$.

In a series of experimental studies, it has been found that decisions essentially change when the agents are allowed to consult with each other, increasing in this way their mutual information [81]-[87], or when they can get additional information by learning from their own experience [88]. In these experiments [81]-[88], it has been proved that additional information does decrease the errors in decision making, which in
our notation correspond to the diminishing decoherence factor $D(\mu)$. However, in those experiments, one considers twostep procedures. The knowledge of only two points does not allow for defining the whole function. Therefore more detailed experiments analyzing multi-step procedures are needed for comparing empirical results with the form of the theoretical decoherence factor.

Note that it is possible to provide correct information as well as incorrect one, the latter corresponding to the process of confusing decision makers and forcing them to accept some desired decision. The effect, similar to providing negative information, can be achieved if decision makers are asked to deliberate concentrating of the uncertainty contained in the considered prospects [94].

It is worth stressing that, while the attraction factor and hence the decision makers choice can be influenced by the provided additional information, decision makers can never become completely rational. This is because the amount of information cannot be infinite. Therefore, the attraction factor is never exactly zero.

## VII. Conclusion

We have studied how the choice of decision makers can be influenced under the presence of risk and uncertainty. Our analysis is based on the Quantum Decision Theory that has been previously developed by the authors for individual decision makers. The term "quantum" does not imply that decision makers are assumed to be quantum objects, but it reflects the use of mathematical techniques that are common for quantum theory, in particular, for the definition of event probabilities. In quantum theory, these mathematical techniques make it possible to take into account unknown hidden variables, at the same time, avoiding their explicit consideration. Similarly, in decision theory, these techniques allow for taking into account such hidden variables as subconscious feelings, emotions, and behavioral biases.

We have suggested a generalization of the theory to the case of decision makers that are members of a society. The social decision makers interact with each other by exchanging information. The notion of a typical decision maker, representing the average society behavior, has been introduced and characterized.

Under the given utility of prospects, the typical behavior of agents can be influenced. Changing the results of decision making can be realized by influencing the attraction factor of decision makers. This can be done in two ways. One method is to arrange the payoff weights so as to induce the required changes of the attraction factors. The variation of the payoff weights can invert the attraction-factor values and reverse the decision preferences. The second method of influencing is by providing information to decision makers or by allowing consultations between the agents of the society. The attraction factors can be either decreased, when decision makers obtain correct information, or increased if the delivered information is wrong. The variation of the attraction factors, induced by positive or negative information, can lead to the reversal of preferences. Since the amount of information is never infinite,
the attraction factors cannot be reduced exactly to zero. This means that decision makers cannot become absolutely rational and will always exhibit some behavioral biases.

The possibility of influencing decision makers is, of course, not a novelty. What is principally new in the present paper is the mathematical description of the process allowing for quantitative predictions. By treating several concrete decision problems, we have illustrated that our theory yields theoretical predictions that, within experimental accuracy, coincide with empirical results.

In the present paper, we have considered the application of the Quantum Decision Theory to human decision making. This, however, is only one of the admissible applications. Having in hands a well developed mathematical theory, it is possible to apply it to the problem of creating artificial quantum intelligence [55] and to use it for quantum information processing [95]. Understanding the logic of functioning of the human brain can give us hints on the optimal ways of arranging the functioning of artificial machine devices. In that sense, the Quantum Decision Theory plays a special role. From one side, it makes it possible to give unambiguous interpretation of human decision making. And from another side, this theory can be used for organizing artificial processes imitating the logic of humans. Some ideas on the feasibility of creating artificial quantum intelligence have been advanced in Ref. [55] and a model of quantum information processing has been analyzed in Ref. [95]. The detailed consideration of such artificial processes is a separate problem that needs additional investigations.

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