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Authors
Sun, Qinqin
Wang, Xiuye
Yang, Guolai
et al.

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Optimal Constraint Following for Fuzzy Mechanical Systems Based on a Time-Varying β-Measure and Cooperative Game Theory

Qinqin Sun, Xiuye Wang, Guolai Yang, Ye-Hwa Chen, and Fai Ma

***Abstract*—This article addresses a cooperative game-oriented optimal constraint-following problem for fuzzy mechanical systems. The state of the concerned system is affected by possibly (fast) time-varying uncertainty. The fuzzy set theory is adopted to describe such uncertainty. The task is to drive the system to obey a set of prescribed constraints optimally. Since the control objective may be changing along with the system uncertainty, a time-varying β-measure is defined to gauge the constraint-following error; based on which, an adaptive robust control scheme with two tunable parameters is then proposed to render it to be uniform boundedness and uniform ultimate boundedness. For the seeking of the optimal design parameters, two cost functions, each of which is dominated by one tunable parameter, are developed with the fuzzy information, and thereout a two-player cooperative game is formulated. Finally, the optimal design problem is successfully solved: with the *existence*, *uniqueness*, and *analytical expression* of the Pareto optimality.**

***Index Terms*—Constraint following, cooperative game theory, fuzzy uncertainty, mechanical systems, optimal design.**

I. Introduction

UNCERTAINTY, including unknown or imprecise system parameters, unmodeled dynamics characteristics, changing equilibrium position, sensors noise, external disturbance, etc., exists widely in practical problems and brings to serious affection on system performance. Uncertainty management in the control design has attracted many scholars' attention in past years. There mainly are two categories approaches for the control design of uncertain dynamical systems: 1) the deterministic control approach (such as the H_2/H_∞ approach [1], Lyapunov-based control [2]–[5], sliding mode control [6], [7], etc.) and 2) the stochastic control approach (such as linear-quadratic-Gaussian (LQG) control [8]). In the control design of uncertain dynamical systems, all the past scholars reached to the same problem of how to describe the system uncertainty accurately to assist the control design better. It is frankly to say that uncertainty description is really one important issue in the control of uncertain systems.

In the past, uncertainty in dynamical systems is usually described in two manners: one probability and one fuzzy. This article falls into the fuzzy-based manner. There are two branches of fuzzy theory: one fuzzy *set* theory and one fuzzy *logic* theory. The fuzzy *set* theory was developed by Zadeh [9]. Fuzzy *logic* theory, on the other hand, was developed by Zadeh [10], and the core of it is fuzzy if-then inference rules. The comparisons between the fuzzy set theory and fuzzy if-then inference rules are as follows. First, the fuzzy set theory serves as a valid tool in representing uncertainty. This may be viewed as an alternative uncertainty theory, besides the probability theory. Second, the if-then inference rules, based on fuzzy logic theory, were introduced to relate objects with unsharp (fuzzy) boundaries using if-then connections, that is, making a connection between an object (the cause) and another object (the consequence). The fuzzy if-then inference rules are best used to mimic human or subjective reasoning. Third, most of other research in control applies fuzzy if-then rules to represent system models (such as Takagi–Sugeno modeling [11]–[15]) or the control scheme (such as Mamdani-type control architecture [16]). These are excellent research when it comes to the representations related to reasoning. Our research, on the other hand, does not seek to represent reasoning. Rather, we endeavor to explore an alternative tool for uncertainty representations, other than probability, in the system theory and control. Therefore, the fuzzy set theory is selected to describe the system uncertainty in this article.

As a novel alternative in the *fuzzy*-based manner, Chen *et al.* [17]–[23] have made some pioneering contributions on the fuzzy control design. In their studies, fuzzy but bounded uncertainty is considered. The uncertainty is addressed by the *fuzzy set* but not the usual *if-then* rules. They put the emphasis on two layers: 1) advanced formulation of controllers and 2) optimal design of control parameters. For the

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(Corresponding author: Xiuye Wang.)

Qinqin Sun is with the School of Energy and Power Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, Jiangsu, China (e-mail: sun421529@126.com).

Xiuye Wang and Guolai Yang are with the School of Mechanical Engineering, Nanjing University of Science and Technology, Nanjing 210094, Jiangsu, China (e-mail: xiuyewang@126.com; yyanggl@njust.edu.cn).

Ye-Hwa Chen is with the George W. Woodruff School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA 30332 USA, and also with the Key Laboratory of Road Construction Technology and Equipment of MOE, Chang'an University, Xi'an 710064, Shanxi, China (e-mail: yehwa.chen@me.gatech.edu).

Fai Ma is with the Department of Mechanical Engineering, University of California at Berkeley, Berkeley, CA 94720 USA (e-mail: fma@berkeley.edu).
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control design, they addressed the system uncertainty with the fuzzy set theory, but ignored that the control objective may be also affected by system uncertainty. However, in practical engineering problems, it is more promising to address the uncertainty influence on both system dynamics and control objective simultaneously. For the parameter optimal design, they mainly focus on *single* parameter optimization. By this, more explorations are excepted, especially for uncertainty management and multiparameter optimal design.

The above inadequacy in fuzzy control motivates us to do the following improvements in this article. First, for fuzzy uncertainty handling, the fuzzy set theory is adopted to describe the system uncertainty, by which both the control and the system can be represented by analytic expressions rather than if-then rule based, and the system performance can be analyzed and be optimized in a deterministic way. Second, for motion control of the fuzzy mechanical system, an adaptive robust control scheme based on constraint following is proposed, by which a bottom line of performance can be guaranteed. Third, for parameter optimization, a multiple objectives/parameters optimal design problem is formulated and solved by applying the cooperative game theory, by which the performance is enhanced by finding the optima.

It is worth emphasizing that motion control of mechanical systems is guided by constraint following in this article. The main focus of constraint following is to design appropriate servo control for the concerned mechanical system to drive it to follow a desired constraint closely, thereby rendering the desired performance. The concept of constraint following [24] was first proposed by Chen in 2008. Wang *et al.* [17] introduced it into the motion control of fuzzy mechanical systems. Sun *et al.* [25] applied it on an avoidance-arrival problem of uncertain mechanical systems. In recently, Sun *et al.* [26] extended it to underactuated mechanical systems. However, the previous works related to constraint following [24]–[26] only focus on system uncertainty but neglect the constraint uncertainty (i.e., control objective uncertainty), while this study considers both of them. Moreover, the previous works [17] addressed the optimal design problem with one objective and one design parameter, while this study extends it to two objective and two design parameters. This shows the two main differences between this study and our previous works.

The main contributions of this article are as follows.

1) A constraint following task is performed for the motion control of fuzzy mechanical systems to drive the concerned system to follow the prescribed constraints approximatively.
2) Both system uncertainty and constraint uncertainty are described by the fuzzy set and specially handled in the process of control design.
3) An adaptive robust control is proposed to render the β-measure to be uniformly bounded and uniformly ultimately bounded.
4) A two-player cooperative game is formulated for the optimal parameter design by taking the control design parameters as the players.
5) It is proved that the solution of the Pareto-optimality problem always exists and is unique, and the analytical (i.e., closed form) expression of the solution (meaning not just the *numerical* value) is obtained.
6) A novel control scheme with optimization characteristics can be constructed by using the optimal design parameters in the aforementioned adaptive robust control.

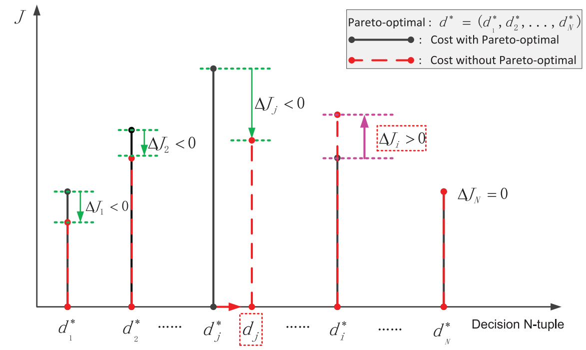

Fig. 1. Pareto optimality.

II. Cooperative Game Theory

The rules of an N-player game impose the following mappings:

$$J_i(\cdot) : \prod_{i=1}^{N} D_i \to \mathbf{R} \quad i = 1, 2, \ldots, N \tag{1}$$

where D_i and $J_i(\cdot)$ are, respectively, the decision set and the cost function for player i. The players in the game may be cooperative with each other toward the same goal or noncooperative [27]. Hence, a game may be a cooperative game or noncooperative game. This article falls to cooperative game.

Pareto optimality (a measure of efficiency) shown as Fig. 1 is an important concept in cooperative game theory.

Definition 1: A decision N-tuple $d^* \in \prod_{i=1}^{N} D_i$ is Pareto optimal (or Pareto optimality) if and only if for every $d \in \prod_{i=1}^{N} D_i$ either

$$J_i(d) = J_i\big(d^*\big) \quad \forall i \in \{1, 2, \ldots, N\} \tag{2}$$

or there is at least one $i \in \{1, 2, \ldots, N\}$ such that

$$J_i(d) > J_i\big(d^*\big). \tag{3}$$

The set of the cost outcomes $(J_1, J_2, \ldots, J_N)$ with different Pareto-optimal decisions is called the *Pareto frontier*.

Theorem 1 [28, Lemma 1.2., p. 9]: Decision N-tuple $d^* \in \prod_{i=1}^{N} D_i$ is Pareto optimal if there exists an $\zeta \in \mathbf{R}^N$ with $\zeta_i > 0$, $i = 1, 2, \ldots, N$, and $\sum_{i=1}^{N} \zeta_i = 1$, such that

$$J\big(d^*\big) \leq J(d) \quad \forall d \in \prod_{i=1}^{N} D_i \tag{4}$$

where $J(d) = \sum_{i=1}^{N} \zeta_i J_i(d)$.

Remark 1: As a summary for the above retrospect, cooperative game theory can be chosen as an effective guide for multiple objectives/parameters optimal design by taking the decisions d_i as the design parameters and the cost functions J_i as the objective functions.

III. Constraint on Fuzzy Mechanical Systems

Consider a mechanical system described as [29, p. 256], [30, p. 286]

$$
\begin{aligned}
& M(q(t), \omega(t), t)\ddot{q}(t) \\
&\quad + C(q(t), \dot{q}(t), \omega(t), t)\dot{q}(t) \\
&\quad + g(q(t), \omega(t), t) + F(q(t), \dot{q}(t), \omega(t), t) = \tau(t) \quad (5)
\end{aligned}
$$

where $q,\ \dot{q},\ \ddot{q} \in \mathbf{R}^n$ are, respectively, the coordinate, the velocity, and the acceleration, $t \in \mathbf{R}$ is the independent variable, and $\tau \in \mathbf{R}^n$ is the control input. $\omega \in \Omega \subset \mathbf{R}^p$ presents the (possibly fast) time-varying uncertainty with the compact bound $\Omega \subset \mathbf{R}^p$, and $M > 0$, $C\dot{q}$, g, and F are, respectively, the inertia matrix, the Coriolis/centrifugal force, the gravitational force, and the friction force or/and other external disturbances.

The system needs to follow the *first-order* form constraints which can be expressed in matrix form as follows:

$$A(q, \omega, t)\dot{q} = c(q, \omega, \dot{\omega}, t) \quad (6)$$

where $A = [A_{li}]_{m\times n}$, $c = [c_1\ c_2\ \cdots\ c_m]^T$. By differentiation, the constraints can be expressed in *second-order* form as follows:

$$A(q, \omega, t)\ddot{q} = b(\dot{q}, q, \omega, \dot{\omega}, \ddot{\omega}, t) \quad (7)$$

where $b = [b_1\ b_2\ \cdots\ b_m]^T$.

Remark 2: Equation (7) is derived from (6) by taking differentiation with respect to t: $(d/dt) : A\dot{q} = c \Longrightarrow A\ddot{q} = -\dot{A}\dot{q} + \dot{c} =: b$. The A matrix is from the prescribed constraint. For example, suppose we impose the desired performance as $q = \bar{q}(t)$, where $\bar{q}(t)$ is the desired trajectory. Then, we can construct the constraint as $\dot{e} + le = 0$ with $e = q - \bar{q}$, where $l > 0$ is a constant. The constraint can be cast into the form (6) and (7) with $A = 1$ ($c = \dot{\bar{q}} - l(q - \bar{q})$ and $b = \ddot{\bar{q}} - l(\dot{q} - \dot{\bar{q}})$).

Remark 3: The constraint as (6) or (7) just shows a general form of the constraint, but does not implies that the constraint should be *only* about velocity or acceleration. Actually, no limitation on the practical implication of the constraint exists here, and it could be about the coordinate, the velocity, the acceleration, or even a combination of them.

As ω in the system (5) is uncertain, $\dot{\omega}$ and $\ddot{\omega}$ in the constraints (6) and (7) are also uncertain; thereout, uncertainty in both the system and the constraints should be considered thereafter. To express these two types of uncertainty uniformly, we introduce a new (augmented) uncertain parameter as $\sigma := [\omega, \dot{\omega}, \ddot{\omega}]^T$, with which the constraints as (6) and (7) can be reexpressed as follows:

$$A(q, \sigma, t)\dot{q} = c(q, \sigma, t) \quad (8)$$

$$A(q, \sigma, t)\ddot{q} = b(q, \dot{q}, \sigma, t). \quad (9)$$

Meanwhile, replacing the uncertain parameter ω in (5) with the new uncertain parameter σ, the system (5) can be reexpressed as follows:

$$
\begin{aligned}
& M(q, \sigma, t)\ddot{q} + C(q, \dot{q}, \sigma, t)\dot{q} + g(q, \sigma, t) \\
&\quad + F(q, \dot{q}, \sigma, t) = \tau. \quad (10)
\end{aligned}
$$

Remark 4: As the control objective may be affected by the system uncertainty in practical problem, this article considers uncertain constraint as (9) which is possibly (fast) time varying but bounded. Such work is very different from the past studies on constraint following, which only consider determinate constraint (such as [24] and [25] and their bibliographies).

We now consider the description of the uncertainty, including the initial conditions and the uncertainty parameter, as follows.

1) Suppose the initial state is $x_0 = [q^T(t_0)\dot{q}^T(t_0)]^T, x_0 \in \mathbf{R}^{2n}$. For each entry of x_0, namely, x_{0i}, $i = 1, 2, \ldots, 2n$, there exists a fuzzy set X_{0i} in a universe of discourse $\Xi_i \subset \mathbf{R}$ characterized by a membership function $\mu_{x_{0i}} : \Xi_i \to [0, 1]$. That is
$$X_{0i} = \left\{\left(x_{0i}, \mu_{x_{0i}}(x_{0i})\right)|x_{0i} \in \Xi_i\right\} \quad (11)$$
where Ξ_i is known and compact.
2) For each entry of the vector σ, namely, σ_i, $i = 1, 2, \ldots, p$, the function $\sigma_i(\cdot)$ is Lebesgue measurable, and there exists a fuzzy set S_i in a universe of discourse $\Sigma_i \subset \mathbf{R}$ characterized by a membership function $\mu_{\sigma_i} : \Sigma_i \to [0, 1]$. That is
$$S_i = \left\{\left(\sigma_i, \mu_{\sigma_i}(\sigma_i)\right)|\sigma_i \in \Sigma_i\right\} \quad (12)$$
where Σ_i is known and compact.

Remark 5: This article handles time-varying uncertainty by utilizing the fuzzy set theory. As a completely different way, many past works address time-varying uncertainty by introducing the neural network theory. For example, a varying-parameter convergent neural network (VP-CDNN) was verified useful and solid in some applications by [31]. The VP-CDNN is then proved to be global convergent and the super-exponential convergence rate is obtained in [32]. A time-varying parameter is designed with the neural dynamic method in [33]. A novel reinforcement learning-based optimal tracking control scheme is established for an unmanned surface vehicle with a neural network approximator in [34].

IV. Adaptive Robust Control Design: Time-Varying β−Measure-Based Approach

By previous analysis, an adaptive robust control is explored to drive the *whole* system (10) to follow a anticipated set of *uncertain* constraint approximately. First, the uncertain system (10) and the uncertain constraint (8), (9) can be decomposed into *nominal* and *uncertain* portions as follows:

$$M(q, \sigma, t) = \bar{M}(q, t) + \Delta M(q, \sigma, t) \quad (13)$$

$$C(q, \dot{q}, \sigma, t) = \bar{C}(q, \dot{q}, t) + \Delta C(q, \dot{q}, \sigma, t) \quad (14)$$

$$g(q, \sigma, t) = \bar{g}(q, t) + \Delta g(q, \sigma, t) \quad (15)$$

$$F(q, \dot{q}, \sigma, t) = \bar{F}(q, \dot{q}, t) + \Delta F(q, \dot{q}, \sigma, t) \quad (16)$$

$$A(q, \sigma, t) = \bar{A}(q, t) + \Delta A(q, \sigma, t) \quad (17)$$

$$c(q, \sigma, t) = \bar{c}(q, t) + \Delta c(q, \sigma, t) \quad (18)$$

$$b(q, \dot{q}, \sigma, t) = \bar{b}(q, \dot{q}, t) + \Delta b(q, \dot{q}, \sigma, t). \quad (19)$$

Here, $\bar{M} > 0$, and the functions $\bar{(\cdot)}$ and $\Delta(\cdot)$ are continuous. Let $D := \bar{M}^{-1}$, $\Delta D := M^{-1} - \bar{M}^{-1}$, and $E := \bar{M}M^{-1} - I$; hence, $\Delta D = DE$.

With the constraint (8), define a measure (i.e., the *tracking error*)

$$\beta(q, \dot{q}, \sigma, t) := A(q, \sigma, t)\dot{q} - c(q, \sigma, t) \tag{20}$$

where $\beta = [\beta_1, \beta_2, \ldots, \beta_m]^T$ and β_i is the ith component of the vector β. With (17) and (18), decompose the uncertain β-measure into $\beta = \bar{\beta} + \Delta\beta$ with

$$\bar{\beta}(q, \dot{q}, t) := \bar{A}(q, t)\dot{q} - \bar{c}(q, t) \tag{21}$$

$$\Delta\beta(q, \dot{q}, \sigma, t) := \Delta A(q, \sigma, t)\dot{q} - \Delta c(q, \sigma, t). \tag{22}$$

Note that, $\bar{A}$ and $\bar{c}$ can be determined according to the control objective and the constraint decomposition. First, construct the desired constraints as (8) to describe the control objective, by which the matrix/vector A and c can be determined. Second, decompose the matrix/vector A and c into *nominal* and *uncertain* portions as (17) and (18), by which the nominal portions $\bar{A}$ and $\bar{c}$ can be determined. The $\bar{\beta}$-dynamics is then given by

$$\begin{aligned}\dot{\bar{\beta}} &= \bar{A}\ddot{q} - \bar{b}\\ &= \bar{A}\big[D\big(-\bar{C}\dot{q} - \bar{g} - \bar{F}\big) + D\tau\\ &\quad + D(-\Delta C\dot{q} - \Delta g - \Delta F)\\ &\quad + \Delta D(-C\dot{q} - g - F + \tau)\big] - \bar{b}.\end{aligned} \tag{23}$$

Assumption 1: There exist a constant $\kappa \in (0, \infty)$ and a function $f(\kappa, \cdot) : \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}^m$ such that: 1) the function $f(\kappa, \cdot)$ is in the range space of A and 2) there are a C^1 function $V(\cdot) : \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}_+$, and strictly increasing functions $\gamma_i(\cdot) : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying

$$\begin{aligned}&\gamma_i(0) = 0\\ &\lim_{r\to\infty} \gamma_i(r) = \infty, \quad i = 1, 2, 3\end{aligned} \tag{24}$$

such that for all $(\kappa, \beta, q, \dot{q}, t) \in (0, \infty) \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$

$$\gamma_1(\|\beta\|) \le V(\beta, t) \le \gamma_2(\|\beta\|) \tag{25}$$

$$\frac{\partial V(\beta, t)}{\partial t} + \frac{\partial^T V(\beta, t)}{\partial \beta} f\big(\kappa, \bar{\beta}, t\big) \le -\kappa\gamma_3(\|\beta\|). \tag{26}$$

Remark 6: Without loss of generality, to subject to Assumption 1, we can always choose quadratic function $V(\beta) = \beta^T P\beta$, with $P > 0, P \in \mathbf{R}^{m\times m}, \bar{f}(\kappa, \bar{\beta}) = -\kappa\bar{\beta}$ and quadratic functions $\gamma_1(\|\beta\|) = \lambda_m(P)\|\beta\|^2, \gamma_2(\|\beta\|) = \lambda_M(P)\|\beta\|^2$, and $\gamma_3(\|\beta\|) = \kappa\lambda_M(P)\|\beta\|^2$; hence, Assumption 1 is reasonable.

A control input equal to constraint force Q_c as in [35] can drive the *nominal* system to meet a certain constraint, while driving the *whole* uncertain system (10) to meet uncertain constraint as (9) requires a more realistic control design with the consideration of uncertainty influence, which can be applied in practical problems. By this, we first choose

$$\begin{aligned}p_1\big(\kappa, \bar{\beta}, q, \dot{q}, t\big) &= \bar{M}^{1/2}(q, t)\Big(\bar{A}(q, t)\bar{M}^{-1/2}(q, t)\Big)^+\\ &\quad \times \Big[f\big(\kappa, \bar{\beta}, t\big) + \bar{b}(\dot{q}, q, t) + \bar{A}(q, t)\bar{M}^{-1}(q, t)\\ &\quad \times \big(\bar{C}(q, \dot{q}, t)\dot{q} + \bar{g}(q, t) + \bar{F}(q, t)\big)\Big].\end{aligned} \tag{27}$$

Theorem 2: Subject to Assumption 1, the control $\tau = p_1(\kappa, \bar{\beta}, q, \dot{q}, t)$ renders [36]

$$\bar{A}\big[D\big(-\bar{C}\dot{q} - \bar{g} - \bar{F}\big) + D\tau\big] - \bar{b} = f. \tag{28}$$

Assumption 2:

1) There exist two matrices $G(q, \sigma, t)$ and $Q(q, \sigma, t)$ such that

$$\Delta A(q, \sigma, t) = G(q, \sigma, t)\bar{A}(q, t) \tag{29}$$

$$\Delta\beta(q, \sigma, t) = Q(q, \sigma, t)\bar{\beta}(q, t). \tag{30}$$

2) Subject to Assumption 2(1), let

$$\Psi(q, \sigma, t) := (I + Q(q, \sigma, t))^T \tag{31}$$

$$\begin{aligned}W(q, \sigma, t) &:= \bar{A}(q, t)D(q, t)E(q, \sigma, t)\\ &\quad \times D^{-1}(q, t)\bar{A}^{-1}(q, t)\Psi(q, \sigma, t)\end{aligned} \tag{32}$$

$$\Lambda(q, \sigma, t) := G(q, \sigma, t)\Psi(q, \sigma, t) \tag{33}$$

$$\begin{aligned}\Gamma(q, \sigma, t) &:= G(q, \sigma, t)\bar{A}(q, t)D(q, t)\\ &\quad \times E(q, \sigma, t)D^{-1}(q, t)\bar{A}^{-1}(q, t)\Psi(q, \sigma, t).\end{aligned} \tag{34}$$

There exist $\hat{\rho}_Q, \hat{\rho}_E, \hat{\rho}_G, \hat{\rho}_{GE} \in \mathbf{R}$ such that for all $(q, t) \in \mathbf{R}^n \times \mathbf{R}$

$$\frac{1}{2}\min_{\sigma\in\Sigma} \lambda_m\big(\Psi(q, \sigma, t) + \Psi^T(q, \sigma, t)\big) \ge \hat{\rho}_Q \tag{35}$$

$$\frac{1}{2}\min_{\sigma\in\Sigma} \lambda_m\big(W(q, \sigma, t) + W^T(q, \sigma, t)\big) \ge \hat{\rho}_E \tag{36}$$

$$\frac{1}{2}\min_{\sigma\in\Sigma} \lambda_m\big(\Lambda(q, \sigma, t) + \Lambda^T(q, \sigma, t)\big) \ge \hat{\rho}_G \tag{37}$$

$$\frac{1}{2}\min_{\sigma\in\Sigma} \lambda_m\big(\Gamma(q, \sigma, t) + \Gamma^T(q, \sigma, t)\big) \ge \hat{\rho}_{GE} \tag{38}$$

and $\hat{\rho}_G + \hat{\rho}_E + \hat{\rho}_G + \hat{\rho}_{GE} > 0$.

Note that, when no uncertainty exists in M and A, we can select $\hat{\rho}_Q = 1, \hat{\rho}_E = 0, \hat{\rho}_G = 0$, and $\hat{\rho}_{GE} = 0$.

Assumption 3:

1) There exist a constant vector $\alpha \in \mathbf{R}_+^k$ whose exact value may be unknown, and a known function $\Pi(\cdot) : \mathbf{R}_+^k \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}_+$ such that for all $(q, \dot{q}, t) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}, \sigma \in \Sigma$

$$\begin{aligned}&\big\|\bar{A}(q, t)\Delta D(q, \sigma, t)(-C(q, \dot{q}, \sigma, t)\dot{q}\\ &\quad - g(q, \sigma, t) - F(q, \dot{q}, \sigma, t) + p_1(\kappa, \bar{\beta}, q, \dot{q}, t)\big)\\ &\quad - \bar{A}(q, t)D(q, t)(\Delta C(q, \dot{q}, \sigma, t)\dot{q} + \Delta g(q, \sigma, t)\\ &\quad + \Delta F(q, \dot{q}, \sigma, t)) + \Delta A(q, \sigma, t)(\Delta D(q, \sigma, t)\\ &\quad + D(q, t))(-C(q, \dot{q}, \sigma, t)\dot{q} - g(q, \sigma, t)\\ &\quad - F(q, \dot{q}, \sigma, t) + p_1(\kappa, \bar{\beta}, q, \dot{q}, t)\big) - \Delta b(q, \dot{q}, \sigma, t)\big\|\\ &\le \Pi(\alpha, q, \dot{q}, t).\end{aligned} \tag{39}$$

2) For each $(q, \dot{q}, t) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$, the function $\Pi(\cdot, q, \dot{q}, t) : \mathbf{R}_+^k \to \mathbf{R}_+$ is: a) C^1; b) concave; that is, for any $\alpha_1 \in \mathbf{R}_+^k$ and $\alpha_2 \in \mathbf{R}_+^k$

$$\Pi(\alpha_1, q, \dot{q}, t) - \Pi(\alpha_2, q, \dot{q}, t) \le \frac{\partial \Pi}{\partial \alpha}(\alpha_2, q, \dot{q}, t)(\alpha_1 - \alpha_2) \tag{40}$$

and c) nondecreasing with respect to each component of its argument α. Note that, α_1 and α_2 denote any two different values of the variable α in function $\Pi(\cdot)$ and the formula (40) shows the property of $\Pi(\cdot)$ when it is a concave function.

Remark 7: Assumptions 2 imposes the effect of uncertainty on the possible deviations of M from $\bar{M}$ as well as A from $\bar{A}$ to be within certain thresholds which are unidirectional. Assumptions 3 imposes a comprehensive bound $\Pi(\cdot)$ for the comprehensive effect of uncertainty. As all the functions related are continuous, if the uncertainty is bounded, the comprehensive effect of uncertainty is always bounded; hence, we can always find such a comprehensive bound. By this, the rationality of both Assumptions 2 and 3 depends on the boundedness of uncertainty. Assuming the uncertainty is bounded and then carrying out the control design with its bound is the primary idea of robust control. It is actually a very mature manner to handle uncertainty in control of an uncertain system. What is more, in practice, any parameter that has real physical meaning is bounded. Therefore, Assumptions 2 and 3 are quite reasonable in practice.

The parameter α depends on the bound of the uncertainty σ, for which we propose the following leakage-type adaptive law:

$$\dot{\hat{\alpha}}(t) = k_1 \frac{\partial^T \Pi}{\partial \alpha}(\hat{\alpha}(t), q(t), \dot{q}(t), t)\|\bar{\beta}(t)\| - k_2\gamma^{-1}\hat{\alpha}(t) \quad (41)$$

where $\hat{\alpha}_i(t_0) > 0$, $i = 1, \ldots, k$, $k_{1,2} \in \mathbf{R}$, $k_{1,2},\ \gamma > 0$.

Remark 8: We choose positive initial value $\hat{\alpha}_i(t_0) > 0$ because it will guarantee $\hat{\alpha}_i(t) > 0$ for all $t \geq t_0$; that is, lead to positive adaptive parameter $\hat{\alpha}$. This is important in the development. The adaptive parameter stays positive since the adaptive law as (41) is of leakage type. We try to keep the adaptive parameter $\hat{\alpha}$ to be positive that is because it is an estimation of α that denotes a combined effect of the uncertainty bound. As the uncertainty bound is positive, α is positive; hence, its estimation $\hat{\alpha}$ should be positive.

We now propose an adaptive robust control as follows:

$$\begin{aligned}\tau(t) = {}& p_1\big(\kappa, \bar{\beta}(t), q(t), \dot{q}(t), t\big)\\ &+ p_2\big(\gamma, \kappa, \hat{\alpha}(t), \bar{\beta}(t), q(t), \dot{q}(t), t\big)\end{aligned} \quad (42)$$

with p_1 as (27) and p_2 as follows:

$$\begin{aligned}&p_2\big(\gamma, \kappa, \hat{\alpha}, \bar{\beta}, q, \dot{q}, t\big)\\ &\quad= -\gamma D^{-1}(q,t)\bar{A}^{-1}(q,t)\frac{\partial V(\bar{\beta},t)}{\partial \bar{\beta}}\Pi^2\big(\hat{\alpha}, q, \dot{q}, t\big).\end{aligned} \quad (43)$$

Remark 9: In summary, there are two types of design parameters in the proposed control scheme: one type is used in the adaptive law (41) (i.e., k_1 and k_1) and the other type is used in the control (42) (i.e., γ and κ), in which k_1 and k_1 can be selected as any positive scalars, and γ and κ also are positive scalars and can be optimally selected by solving the later proposed two-player cooperative game in Section V, that is, they can be selected as the resulting Pareto optimality.

Theorem 3: Consider the system (10) and constraint (9). Let $\varsigma(t) := [\beta^T(q(t), \dot{q}(t), t), (\hat{\alpha}(t) - \alpha)^T]^T \in \mathbf{R}^{m+k}$. Subject to Assumptions 1–3, the control (42) renders the performance of uniform boundedness and uniform ultimate boundedness for $\varsigma(t)$.

Proof: Subject to Assumption 1, we consider a Lyapunov function candidate as follows:

$$\mathcal{L}(\beta, \hat{\alpha} - \alpha, t) = V(\beta, t) + \frac{1}{2}k_1^{-1}(\hat{\alpha} - \alpha)^T(\hat{\alpha} - \alpha). \quad (44)$$

Its derivative is evaluated as follows:

$$\dot{\mathcal{L}} = \frac{\partial V}{\partial t} + \frac{\partial^T V}{\partial \beta}\dot{\beta} + k_1^{-1}\big(\hat{\alpha} - \alpha\big)^T\dot{\hat{\alpha}}. \quad (45)$$

Based on Assumption 1 and by (28), we have

$$\begin{aligned}&\frac{\partial V}{\partial t} + \frac{\partial^T V}{\partial \beta}\big\{\bar{A}\big[D\big(-\bar{C}\dot{q} - \bar{g} - \bar{F}\big) + Dp_1\big] - \bar{b}\big\}\\ &\leq -\kappa\gamma_3(\|\beta\|).\end{aligned} \quad (46)$$

Next, by (39) and Assumption 3(2), we have

$$\begin{aligned}&\frac{\partial^T V}{\partial \beta}\big[\bar{A}\Delta D(-C\dot{q} - g - F + p_1) - \bar{A}D\\ &\quad\times(\Delta C\dot{q} + \Delta g + \Delta F) + \Delta A(\Delta D + D)\\ &\quad\times(-C\dot{q} - g - F + p_1) - \Delta b\big] \leq \left\|\frac{\partial^T V}{\partial \beta}\right\|\Pi(\alpha, q, \dot{q}, t)\\ &\leq \left\|\frac{\partial^T V}{\partial \beta}\right\|\Pi\big(\hat{\alpha}, q, \dot{q}, t\big) + \left\|\frac{\partial^T V}{\partial \beta}\right\|\frac{\partial \Pi}{\partial \alpha}\big(\hat{\alpha}, q, \dot{q}, t\big)(\alpha - \hat{\alpha}).\end{aligned} \quad (47)$$

By (43) and with $\Delta A = G\bar{A}$

$$\begin{aligned}\frac{\partial^T V}{\partial \beta}A(D + \Delta D)p_2 = {}& \frac{\partial^T V}{\partial \beta}\bar{A}Dp_2 + \frac{\partial^T V}{\partial \beta}\bar{A}\Delta Dp_2\\ &+ \frac{\partial^T V}{\partial \beta}\Delta ADp_2 + \frac{\partial^T V}{\partial \beta}\Delta A\Delta Dp_2.\end{aligned} \quad (48)$$

Recalling $\beta = (I + Q)\bar{\beta}$, $\Psi = (I + Q)^T$, and p_2 as (43), yields

$$p_2 = -\gamma D^{-1}\bar{A}^{-1}\Psi\frac{\partial V}{\partial \beta}\Pi^2\big(\hat{\alpha}, q, \dot{q}, t\big). \quad (49)$$

By (35) of Assumption 2(2) and adopting the Rayleigh's principle [37], we have

$$\begin{aligned}\frac{\partial^T V}{\partial \beta}\bar{A}Dp_2 &= -\frac{1}{2}\gamma\frac{\partial^T V}{\partial \beta}\big(\Psi + \Psi^T\big)\frac{\partial V}{\partial \beta}\Pi^2\big(\hat{\alpha}, q, \dot{q}, t\big)\\ &\leq -\frac{1}{2}\gamma\frac{\partial^T V}{\partial \beta}\lambda_m\big(\Psi + \Psi^T\big)\frac{\partial V}{\partial \beta}\Pi^2\big(\hat{\alpha}, q, \dot{q}, t\big)\\ &= -\gamma\hat{\rho}_Q\left\|\frac{\partial^T V}{\partial \beta}\right\|^2\Pi^2\big(\hat{\alpha}, q, \dot{q}, t\big).\end{aligned} \quad (50)$$

Similarly, we have

$$\frac{\partial^T V}{\partial \beta}\bar{A}\Delta Dp_2 \leq -\gamma\hat{\rho}_E\left\|\frac{\partial^T V}{\partial \beta}\right\|^2\Pi^2\big(\hat{\alpha}, q, \dot{q}, t\big) \quad (51)$$

$$\frac{\partial^T V}{\partial \beta}\Delta ADp_2 \leq -\gamma\hat{\rho}_G\|\frac{\partial^T V}{\partial \beta}\Pi^2\big(\hat{\alpha}, q, \dot{q}, t\big) \quad (52)$$

$$\frac{\partial^T V}{\partial \beta}\Delta A\Delta Dp_2 \leq -\gamma\hat{\rho}_{GE}\left\|\frac{\partial^T V}{\partial \beta}\right\|^2\Pi^2\big(\hat{\alpha}, q, \dot{q}, t\big). \quad (53)$$

Combining (50)–(53) yields

$$\frac{\partial^T V}{\partial \beta} A(D+\Delta D)p_2 \le -\gamma\big(\hat{\rho}_Q+\hat{\rho}_E+\hat{\rho}_G+\hat{\rho}_{GE}\big) \times \left\|\frac{\partial^T V}{\partial \beta}\right\|^2 \Pi^2\big(\hat{\alpha},q,\dot{q},t\big). \tag{54}$$

Recalling $\dot{\hat{\alpha}}$ as (41), we have

$$\begin{aligned} &k_1^{-1}\big(\hat{\alpha}-\alpha\big)^T\dot{\hat{\alpha}} \\ &\le \big(\hat{\alpha}-\alpha\big)^T\frac{\partial \Pi^T}{\partial \alpha}\big(\hat{\alpha},q,\dot{q},t\big)\|\bar{\beta}\| \\ &\quad -\frac{1}{2}k_1^{-1}k_2\gamma^{-1}\|\hat{\alpha}-\alpha\|^2+k_1^{-1}k_2\gamma^{-1}\|\alpha\|^2. \end{aligned} \tag{55}$$

With (45)–(55), we have

$$\begin{aligned} \dot{\mathcal{L}} \le &-\kappa\gamma_3(\|\beta\|)-\frac{1}{2}k_1^{-1}k_2\gamma^{-1}\|\hat{\alpha}-\alpha\|^2 \\ &+\ k_1^{-1}k_2\gamma^{-1}\|\alpha\|^2+\frac{1}{4\big(\hat{\rho}_Q+\hat{\rho}_E+\hat{\rho}_G+\hat{\rho}_{GE}\big)\gamma}. \end{aligned} \tag{56}$$

Recalling $\varsigma(t)=[\beta^T,(\hat{\alpha}(t)-\alpha)^T]^T$, there exits a function

$$\hat{\gamma}_3(\|\varsigma\|)\le\gamma_3(\|\beta\|)+\frac{1}{2}\kappa^{-1}k_1^{-1}k_2\gamma^{-1}\|\hat{\alpha}-\alpha\|^2 \tag{57}$$

with which we have

$$\begin{aligned} \dot{\mathcal{L}} &\le -\kappa\hat{\gamma}_3(\|\varsigma\|)+k_1^{-1}k_2\gamma^{-1}\|\alpha\|^2 \\ &\quad+\frac{1}{4\big(\hat{\rho}_Q+\hat{\rho}_E+\hat{\rho}_G+\hat{\rho}_{GE}\big)\gamma} \\ &=: -\kappa\hat{\gamma}_3(\|\varsigma\|)+\frac{\delta}{\gamma} \end{aligned} \tag{58}$$

where

$$\delta=k_1^{-1}k_2\|\alpha\|^2+\frac{1}{4\big(\hat{\rho}_Q+\hat{\rho}_E+\hat{\rho}_G+\hat{\rho}_{GE}\big)}. \tag{59}$$

This shows us that $\dot{\mathcal{L}}$ is negative definite for all ς such that

$$-\kappa\hat{\gamma}_3(\|\varsigma\|)+\frac{\delta}{\gamma}<0. \tag{60}$$

Therefore, according to [38], the performance of uniform boundedness and uniform ultimate boundedness could be achieved as follows:

$$d(r)=\begin{cases}\big(\gamma_1^{-1}\circ\gamma_2\big)(r), & \text{if } r>R\\ \big(\gamma_1^{-1}\circ\gamma_2\big)(R), & \text{if } r\le R\end{cases} \tag{61}$$

$$T(\bar{d},r)=\begin{cases}0, & \text{if } r\le\overline{R}\\ \frac{\gamma_2(r)-\gamma_1(\bar{R})}{\hat{\gamma}_3(\bar{R})-\frac{\delta}{\kappa\gamma}}, & \text{otherwise}\end{cases} \tag{62}$$

where $R=\hat{\gamma}_3^{-1}(\delta/\kappa\gamma)$, $\bar{d}>(\gamma_1^{-1}\circ\gamma_2)(R)$, and $\bar{R}=(\gamma_2^{-1}\circ\gamma_1)(\bar{d})$. ■

V. Cooperative Game-Oriented Optimal Design

A. Cost Functions

For the optimal parameter design, we construct a cooperative game, in which κ and γ are the players, $D_1=(0,\infty)$ and $D_2=(0,\infty)$ are the decision sets, and the cost functions are derived as follows. First, inspired by the performance of uniform ultimate boundedness, we define a measure of *performance cost* as follows:

$$\eta_\infty(\delta,\kappa,\gamma):=\Big(\gamma_1^{-1}\circ\gamma_2\circ\hat{\gamma}_3^{-1}\Big)\Big(\frac{\delta}{\kappa\gamma}\Big). \tag{63}$$

Second, inspired by the finite entering time T as (62), we define a measure of *time cost* as follows:

$$\eta_T(\delta,\kappa,\gamma):=\Big(\gamma_1\circ\hat{\gamma}_3^{-1}\Big)\Big(\frac{\delta}{\kappa\gamma}\Big). \tag{64}$$

Third, as κ and γ affect the control effort directly, we define two measures of *control cost*: 1) κ^2 and 2) γ^2.

For κ and γ, we now propose the fuzzy-theoretic cost functions as follows:

$$\begin{aligned} &J_1(\kappa,\gamma)\\ &:=\mathcal{D}\Big[\eta_\infty^2(\delta,\kappa,\gamma)\Big]-\mathcal{D}\Big[\eta_T^2(\delta,\kappa,\gamma)\Big]+\kappa^2\\ &=:\underbrace{J_{11}(\kappa,\gamma)}_{\text{performance cost}}+\underbrace{J_{12}(\kappa,\gamma)}_{\text{time cost}}+\underbrace{J_{13}(\kappa)}_{\text{control cost}} \end{aligned} \tag{65}$$

and

$$J_2(\gamma):=\underbrace{\gamma^2}_{\text{control cost}} \tag{66}$$

where $\mathcal{D}[\cdot]$ denotes the $\mathcal{D}$-operation as in [17].

Assumption 4: There exists a twice continuously differentiable function $g(\cdot):(0,\ \infty)\to(0,\ \infty)$ such that $\mathcal{D}[\eta_\infty^2(\delta,\kappa,\gamma)]-\mathcal{D}[\eta_T^2(\delta,\kappa,\gamma)]=:g(1/\kappa\gamma)$. Furthermore, $g_1(\cdot):(0,\ \infty)\to(0,\ \infty)$, $g_2(\cdot):(0,\ \infty)\to\mathbf{R}_+$ satisfy that $\lim_{x\to+\infty}g_1(x)>0$ and $\lim_{x\to0^+}g_1(x)$ exist, where $g_1(x):=\partial g(x)/\partial x$ and $g_2(x):=\partial^2 g(x)/\partial x^2$.

Remark 10: Assumptions 4 aims at an appropriate function $g(\cdot)$ that can be determined by $\mathcal{D}[\eta_\infty^2]$ and $\mathcal{D}[\eta_T^2]$, in which η_∞ and η_T are exactly defined as (63) and (64) and can be determined with predetermined functions $\gamma_{1,2,3}$; hence, we can always find such function $g(\cdot)$. Assumptions 4 is quite reasonable in practice.

With Assumption 4, the cost function of player κ as (65) then can be simplified as follows:

$$J_1(\kappa,\gamma)=g\Big(\frac{1}{\kappa\gamma}\Big)+\kappa^2. \tag{67}$$

B. Formulation of the Optimal Design Problem

For the seeking of Pareto optimality (κ^*,γ^*), according to Theorem 1, we propose a *comprehensive index* as follows:

$$J(\kappa,\gamma)=\zeta_1J_1(\kappa,\gamma)+\zeta_2J_2(\gamma) \tag{68}$$

where $\zeta_{1,2}>0$ and $\zeta_1+\zeta_2=1$. With (68), the following constrained optimization problem is formulated as: for any given $\zeta_{1,2}$:

$$\min_{\kappa,\gamma}J(\kappa,\gamma)\quad\text{subject to: }\kappa\in D_1,\gamma\in D_2. \tag{69}$$

Remark 11: The problem formulated here is with *dual* objectives/design parameters, in which the uncertainty influences on both the system motion and the control objective are considered. This is very different from the past studies that merely refer to uncertain system motion as well as single objective/design parameter optimization (such as [17] and [19]–[22]).

C. Solution: Pareto Optimality

Taking a partial differential of $J(\kappa, \gamma)$, we have

$$\frac{\partial J}{\partial \kappa} = \frac{1}{\gamma\kappa^2}\left(2\zeta_1\gamma\kappa^3 - \zeta_1 g_1\left(\frac{1}{\kappa\gamma}\right)\right) \tag{70}$$

$$\frac{\partial J}{\partial \gamma} = \frac{1}{\kappa\gamma^2}\left(2\zeta_2\kappa\gamma^3 - \zeta_1 g_1\left(\frac{1}{\kappa\gamma}\right)\right). \tag{71}$$

The (*necessary*) stationary condition $((\partial J/\partial\kappa) = 0$ and $(\partial J/\partial\gamma) = 0)$ leads to

$$\begin{cases} 2\zeta_1\gamma\kappa^3 - \zeta_1 g_1\left(\frac{1}{\kappa\gamma}\right) = 0 \\ 2\zeta_2\kappa\gamma^3 - \zeta_1 g_1\left(\frac{1}{\kappa\gamma}\right) = 0. \end{cases} \tag{72}$$

Taking a second-order partial differential of $J(\kappa, \gamma)$, and with $\zeta_{1,2}, \kappa, \gamma, g(1/\kappa\gamma), g_1(1/\kappa\gamma) > 0$, $g_2(1/\kappa\gamma) \geq 0$, we have

$$\frac{\partial^2 J}{\partial \kappa^2} > 0,\ \ \frac{\partial^2 J}{\partial \kappa^2}\frac{\partial^2 J}{\partial \gamma^2} - \frac{\partial^2 J}{\partial \kappa\partial\gamma}\frac{\partial^2 J}{\partial \gamma\partial\kappa} > 0; \tag{73}$$

hence $\begin{bmatrix} \frac{\partial^2 J}{\partial\kappa^2} & \frac{\partial^2 J}{\partial\kappa\gamma} \\ \frac{\partial^2 J}{\partial\gamma\kappa} & \frac{\partial^2 J}{\partial\gamma^2} \end{bmatrix} > 0$ (i.e., the *sufficient* condition). Thus, the solution of (72) globally minimizes $J(\kappa, \gamma)$ as in (69).

Theorem 4: The solution $(\kappa, \gamma) > 0$ to (72) (i.e., the Pareto optimality of the proposed cooperative game) always exists and is unique under Assumption 4.

Proof: With (72), we have $\kappa = \sqrt{\zeta_2/\zeta_1}\gamma$, and $\gamma = \sqrt{\zeta_1/\zeta_2}\kappa$. Taking it into (72), and (72) can be rewritten as follows:

$$2\zeta_1\sqrt{\frac{\zeta_1}{\zeta_2}}(\kappa)^4 - \zeta_1 g_1\left(\sqrt{\frac{\zeta_2}{\zeta_1}}\frac{1}{(\kappa)^2}\right) = 0 \tag{74a}$$

$$2\zeta_2\sqrt{\frac{\zeta_2}{\zeta_1}}(\gamma)^4 - \zeta_1 g_1\left(\sqrt{\frac{\zeta_1}{\zeta_2}}\frac{1}{(\gamma)^2}\right) = 0. \tag{74b}$$

For (74a), let

$$\theta_1(\kappa) := 2\zeta_1\sqrt{\frac{\zeta_1}{\zeta_2}}(\kappa)^4 - \zeta_1 g_1\left(\sqrt{\frac{\zeta_2}{\zeta_1}}\frac{1}{(\kappa)^2}\right). \tag{75}$$

Since $\zeta_{1,2} > 0$, $g_2(x) \geq 0$, taking its first-order derivative, yields

$$\begin{aligned}\frac{\partial\theta_1(\kappa)}{\partial\kappa} &= 8\zeta_1\sqrt{\frac{\zeta_1}{\zeta_2}}(\kappa)^3 + 2\zeta_1\sqrt{\frac{\zeta_2}{\zeta_1}}g_2\left(\sqrt{\frac{\zeta_2}{\zeta_1}}\frac{1}{(\kappa)^2}\right)\frac{1}{(\kappa)^3} \\ &> 0\end{aligned} \tag{76}$$

such that $\theta_1(\cdot)$ is strictly increasing in κ. By Assumption 4, we have $\lim_{\kappa\to 0^+}\theta_1(\kappa) < 0$, $\lim_{\kappa\to+\infty}\theta_1(\kappa) = +\infty$. By this, the solution $\kappa > 0$ to (74a) always exists and is unique. Similarly, we reach the same conclusion for (74b).

As $\begin{bmatrix} \frac{\partial^2 J}{\partial\kappa^2} & \frac{\partial^2 J}{\partial\kappa\gamma} \\ \frac{\partial^2 J}{\partial\gamma\kappa} & \frac{\partial^2 J}{\partial\gamma^2} \end{bmatrix} > 0$, the solution $(\kappa, \gamma) > 0$ to (72) is the Pareto optimality of the proposed cooperative game. ■

D. Special Case

Consider $\gamma_1(\|\beta\|) = a_1\|\beta\|^2$, $\gamma_2(\|\beta\|) = a_2\|\beta\|^2$, and $\gamma_3(\|\beta\|) = a_3\|\beta\|^2$ for the mechanical system (10). Here, $a_{1,2,3} > 0$ are constants subjected to $a_2 > a_1^2$ and $a_3 \leq (1/2)\kappa^{-1}k_1^{-1}k_2\gamma^{-1}$. We have

$$\begin{aligned}&\gamma_3(\|\beta\|) + \frac{1}{2}\kappa^{-1}k_1^{-1}k_2\gamma^{-1}\left\|\hat{\varsigma} - \varsigma\right\|^2 \\ &= a_3\|\beta\|^2 + \frac{1}{2}\kappa^{-1}k_1^{-1}k_2\gamma^{-1}\left\|\hat{\varsigma} - \varsigma\right\|^2 \\ &\geq a_3\|\varsigma\|^2.\end{aligned} \tag{77}$$

By this, (57) is met by choosing $\hat{\gamma}_3(\|\varsigma\|) = a_3\|\varsigma\|^2$, with which we have

$$\eta_\infty(\delta, \kappa, \gamma) = \sqrt{\frac{a_2\delta}{a_1 a_3\kappa\gamma}} \tag{78}$$

$$\eta_T(\delta, \kappa, \gamma) = \sqrt{\frac{a_1\delta}{a_3\kappa\gamma}}. \tag{79}$$

By $\mathcal{D}$-operation, we have

$$g\left(\frac{1}{\kappa\gamma}\right) = \frac{(a_2 - a_1^2)\mathcal{D}[\delta]}{a_1 a_3}\frac{1}{\kappa\gamma} \tag{80}$$

and $g_1(x) = (a_2 - a_1^2)\mathcal{D}[\delta]/(a_1 a_3)$, $g_2(x) = 0$. We then obtain the cost functions for κ and γ as follows:

$$\begin{aligned}J_1(\kappa, \gamma) &= \frac{(a_2 - a_1^2)\mathcal{D}[\delta]}{a_1 a_3}\frac{1}{\kappa\gamma} + \kappa^2 \\ J_2(\gamma) &= \gamma^2\end{aligned} \tag{81}$$

and the comprehensive index as follows:

$$J(\kappa, \gamma) = \zeta_1\left(\frac{(a_2 - a_1^2)\mathcal{D}[\delta]}{a_1 a_3}\frac{1}{\kappa\gamma} + \kappa^2\right) + \zeta_2\gamma^2. \tag{82}$$

Introducing $g(1/\kappa\gamma)$ and $g_1(1/\kappa\gamma)$ into (72) and solving it, we obtain

$$\begin{cases} \kappa = \sqrt[8]{\frac{\zeta_2}{\zeta_1}}\sqrt[4]{\frac{(a_2 - a_1^2)\mathcal{D}[\delta]}{2a_1 a_3}} \\ \gamma = \sqrt[8]{\left(\frac{\zeta_1}{\zeta_2}\right)^3}\sqrt[4]{\frac{(a_2 - a_1^2)\mathcal{D}[\delta]}{2a_1 a_3}}. \end{cases} \tag{83}$$

Note that, since $a_2 > a_1^2$, $g(x), g_1(x) > 0$, Assumption 4 is satisfied. The solution as (83) is the Pareto optimality (κ^*, γ^*).

Remark 12: The two most prominent control methods/approaches about the quadratic programming framework are linear-quadratic regulator (LQR) [39], [40] and LQG [41]. LQR was originally designed for linear systems without uncertainty subject to quadratic cost functional minimization. It went to l_2, H_∞, etc., extension to address its robustness with respect to uncertainty. In that framework, the uncertainty is constant or time varying and bounded. No further prescription of the uncertainty characteristics is used nor needed. LQG was originally designed for linear systems under time-varying

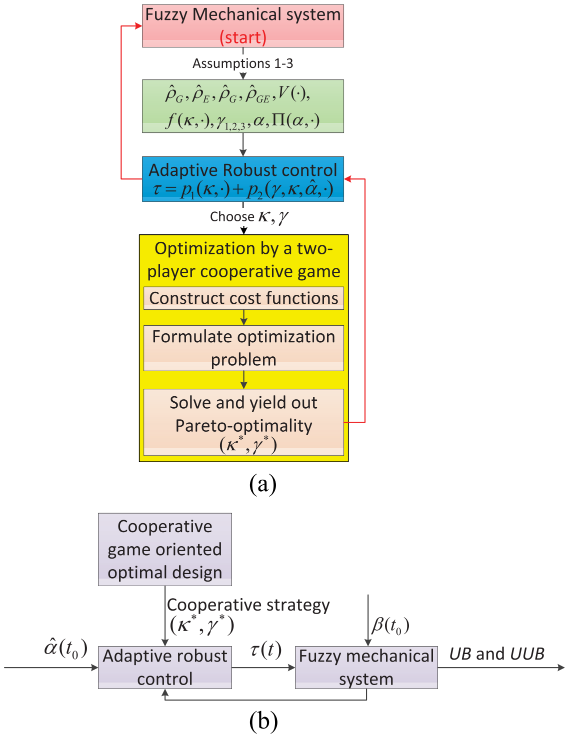

Fig. 2. (a) Design procedure and (b) control structure.

external disturbance and measurement noise. The design is also subject to quadratic cost functional minimization. But the uncertainty is under probability prescriptions, mostly Gaussian and white. With Gaussian, the uncertainty is not bounded. With white, there is no correlation of the signals in the temporal dimension. In our uncertainty treatment, the uncertainty is prescribed by to what *degree* it belongs to a set (i.e., the fuzzy set theory), as opposed to what *frequency* it happens (i.e., the probability theory). This is a major difference with LQG. Our method is also different from LQR in that some prescription of the nature of the uncertainty is utilized; whereas in LQR, there is none. This uncertainty treatment, compounded with other features of our method (that it applies to nonlinear mechanical systems with constraints; while for LQR/LQG, one needs to (Jacobian) linearize the system first, which results in a loss of information), renders a rather unique framework in control.

E. Design Procedure

As shown in Fig. 2, the design procedure is summarized as follows.

Step 1: Choose $V(\cdot)$, $f(\kappa, \cdot)$, and $\gamma_i(\cdot)$, $i = 1, 2, 3$ according to Assumption 1 and $\hat{\rho}_Q$, $\hat{\rho}_E$, $\hat{\rho}_G$, and $\hat{\rho}_{GE}$ according to Assumption 2 and determine $\bar{\beta}$, $\bar{A}$, and $\bar{b}$ according the control objective.

Step 2: Choose $\Pi(\hat{\alpha}, \cdot)$, α, and $(\partial\Pi/\partial\alpha)(\hat{\alpha}, \ \cdot)$ according to Assumption 3 and determine $\hat{\alpha}(\cdot)$ as (41).

Step 3: Take $f(\kappa, \cdot)$, $\bar{\beta}$, $\bar{A}$, and $\bar{b}$ into (27) to yield p_1 and take $\bar{\beta}$, $\bar{A}$, $V(\cdot)$, $\Pi(\hat{\alpha}$, and $\cdot)$ into (43) to yield p_2. The control law (42) can be determined by combing p_1 and p_2.

Step 4: Calculate δ with the previously determined $\hat{\rho}_Q$, $\hat{\rho}_E$, $\hat{\rho}_G$, $\hat{\rho}_{GE}$, and α. Calculate η_∞ (63) and η_T (64) and construct cost functions with (65) and (66).

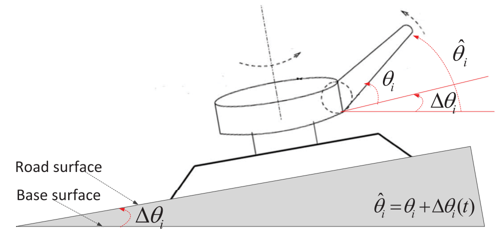

Fig. 3. Gun turret-barrel system under the influence of road excitation.

Step 5: According to (65), (66), and (72), calculate Pareto optimality (κ^*, γ^*) and $J_{\min}$. The optimal adaptive robust control is finally determined by taking predetermined $p_{1,2}$ and (κ^*, γ^*) into (42).

Remark 13: It only takes five steps to obtain the proposed controller, which is carried out offline, not online. The design is straightforward and easy to implement. The cooperative game is also solved offline, not online. All it takes is to solve the two algebraic scalar equations (74a) and (74b). For online computations, only algebraic operations (addition/subtraction and multiplication/division) are needed. No complicated operations (such as high-dimensional matrices operations, eigenvalues, eigenvectors, pseudo inverse, etc.) are required. As to sensors, we only need position and velocity sensors. There is no need for other feedback signals (such as force feedback, image feedback, etc.). Therefore, the computational burden of the proposed control system actually is rather low.

Remark 14: There are three major distinct features (advantages) of the proposed control method, comparing with others. First, it represents the fuzzy characteristics of both the control and the system by analytic expressions, rather than the usual fuzzy if-then rule. Second, it proposes a control scheme under the consideration of both the system uncertainty and the control objective uncertainty, while the past-related researches usually focus on only the system uncertainty. Third, it proposes an optimal design problem with the dual objective and dual design parameter, while the past-related researches usually focus on single ones.

VI. Illustrative Example: Bidirectional Stability Control for Gun Turret-Barrel System

We consider a gun turret-barrel system shown as Fig. 3, which can be generalized to a two-link manipulator system. With $q = [\theta_1, \theta_2]^T$ and $\tau = [\tau_1, \tau_2]^T$, the equation of motion [42] can be rewritten in form of (10), in which

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}, \quad C\dot{q} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$
$$g = \begin{bmatrix} 0 \\ \frac{1}{2} m_2 \tilde{g} R_2 \cos\theta_2 \end{bmatrix}, \quad F = [F_1, F_2]^T \tag{84}$$

where $M_1 = (1/2)m_1R_1^2 + m_2R_1^2 + m_2R_1R_2\cos\theta_2 + (1/3)m_2R_2^2\cos^2\theta_2$, $M_2 = (1/3)m_2R_2^2$, $C_1 = -(m_2R_1R_2\sin\theta_2 + (1/3)m_2R_2^2\sin(2\theta_2))\dot{\theta}_1\dot{\theta}_2$, and $C_2 = (1/2)m_2R_1R_2\sin\theta_2\dot{\theta}_1^2 + (1/6)m_2R_2^2\sin(2\theta_2)\dot{\theta}_1^2$. Here, $m_{1,2}$ are the masses, $R_{1,2}$ are the radius, $\tilde{g}$ is the gravitational constant, $\theta_{1,2}$ are the angular positions relative to the joint space, of the turret load and the barrel load, $F_{1,2}$ are the external disturbances, and $\tau_{1,2}$ are the torque inputs.

Note that, we choose the gun turret-barrel system as an illustrative example for three reasons. First, it is an important system in contemporary intelligent transportation. Second, despite its importance, there has been far less attention devoted to its control design, comparing with other more traditional vehicle systems with perhaps higher dimensions (such as excavators, cranes, etc.). Third, the system possesses all important nonlinear and uncertainty characteristics for a significant and representative demonstration of the proposed method.

A. Uncertainty Analysis

We consider three types of uncertainty. For the system modeling error and the external disturbance, we suppose the mass m_1 and the disturbance $F_{1,2}$ are uncertain: $m_1 = \bar{m}_1 + \Delta m_1(t)$ and $F_{1,2} = \bar{F}_{1,2} + \Delta F_{1,2}(t)$, where $\bar{F}_{1,2} = 0$. Here, $\Delta m_1(t)$ and $\Delta F_{1,2}$ are unknown portions with the bounds of $\Delta \underline{m}_1 \le \Delta m_1 \le \Delta \overline{m}_1$ and $\Delta \underline{F}_{1,2} \le \Delta F_{1,2} \le \Delta \overline{F}_{1,2}$. With the system modeling error and the external disturbance, we have

$$\bar{M} = \begin{bmatrix} \bar{M}_1 & 0 \\ 0 & M_2 \end{bmatrix}, \quad D = \bar{M}^{-1} = \begin{bmatrix} \bar{M}_1^{-1} & 0 \\ 0 & M_2^{-1} \end{bmatrix}$$
$$\Delta C\dot{q} = 0, \quad \Delta g = 0, \quad \bar{F} = 0, \quad \Delta F = \begin{bmatrix} \Delta F_1 \\ \Delta F_2 \end{bmatrix} \tag{85}$$

where $\bar{M}_1 = (1/2)\bar{m}_1 R_1^2 + m_2 R_1^2 + m_2 R_1 R_2 \cos\theta_2 + (1/3) m_2 R_2^2 \cos^2\theta_2$. We further have

$$\Delta D = \begin{bmatrix} M_1^{-1} - \bar{M}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{86}$$

Especially, we consider the influence of road excitation for the system state as Fig. 3, in which $\hat{\theta}_i = \theta_i + \Delta\theta_i (i = 1, 2)$, $\hat{\theta}_{1,2}$ are the angular positions of the turret load and the barrel load relative to the base space, and $\Delta\theta_{1,2}$ are the angular position fluctuation resulted by the road excitation. We suppose the angular position errors $\Delta\theta_{1,2}$ as well as their first and second-order derivatives $\Delta\dot{\theta}_{1,2}$ (i.e., the fluctuation velocity) and $\Delta\ddot{\theta}_{1,2}$ (i.e., the fluctuation acceleration) are time varying but bounded.

In the simulations, for the system modeling uncertainty, high frequency is considered by choosing $\Delta m_1 = 100\sin(10t)$. For the external disturbance, sinusoidal fluctuations with attenuation characteristics is considered by choosing $\Delta F_1 = 1500e^{-2t}\sin(6t) + 500e^{-2t}\theta_1$ and $\Delta F_2 = 1000e^{-2t}\sin(6t) + 500e^{-2t}\theta_2$. For the influence of road excitation, we choose the angular position errors $\Delta\theta_{1,2} = 20\sin(10t)$, the fluctuation velocity $\Delta\dot{\theta}_{1,2} = \sin(10t)$, and the fluctuation acceleration $\Delta\ddot{\theta}_{1,2} = 0.1\sin(10t)$.

B. Control Objective: Bidirectional Stability

For precise shot, we desire the gun barrel-barrel system to be stable at fixed horizontal angular position $\bar{\theta}_1$ and fixed elevation angular position $\bar{\theta}_2$ relative to the base space. This is the so-called *bidirectional stability control* [43].

Let $e_1 := \hat{\theta}_1 - \bar{\theta}_1 = (\theta_1 + \Delta\theta_1) - \bar{\theta}_1$, and $e_2 := \hat{\theta}_2 - \bar{\theta}_2 = (\theta_2 + \Delta\theta_2) - \bar{\theta}_2$. We consider the gun barrel-barrel system is desired to be constrained by $\dot{e}_1 + l_1 e_1 = 0$, $\dot{e}_2 + l_2 e_2 = 0$, $l_{1,2} > 0$ are scalars; hence, the performance measure is $\beta = [\beta_1 \ \beta_2]^T$ with

$$\begin{aligned} \beta_1 &= \dot{e}_1 + l_1 e_1 = \dot{\theta}_1 + \Delta\dot{\theta}_1 + l_1\big(\theta_1 - \bar{\theta}_1 + \Delta\theta_1\big) \\ \beta_2 &= \dot{e}_2 + l_2 e_2 = \dot{\theta}_2 + \Delta\dot{\theta}_2 + l_2\big(\theta_2 - \bar{\theta}_2 + \Delta\theta_2\big). \end{aligned} \tag{87}$$

Recalling the decomposition as (21), we have

$$\bar{\beta} = \begin{bmatrix} \dot{\theta}_1 + l_1\big(\theta_1 - \bar{\theta}_1\big) \\ \dot{\theta}_2 + l_2\big(\theta_2 - \bar{\theta}_2\big) \end{bmatrix} \Delta\beta = \begin{bmatrix} \Delta\dot{\theta}_1 + l_1\Delta\theta_1 \\ \Delta\dot{\theta}_2 + l_2\Delta\theta_2 \end{bmatrix}. \tag{88}$$

Taking the derivation of $\beta_{1,2}$, we have

$$\begin{aligned} \dot{\beta}_1 &= \ddot{\theta}_1 + \Delta\ddot{\theta}_1 + l_1\big(\dot{\theta}_1 + \Delta\dot{\theta}_1\big) \\ \dot{\beta}_2 &= \ddot{\theta}_2 + \Delta\ddot{\theta}_2 + l_2\big(\dot{\theta}_2 + \Delta\dot{\theta}_2\big). \end{aligned} \tag{89}$$

Recalling the constraint, we have $A = \mathrm{Diag}\{1, 1\}$ and

$$\begin{aligned} c &= \begin{bmatrix} \Delta\dot{\theta}_1 + l_1\big(\theta_1 - \bar{\theta}_1 + \Delta\theta_1\big) \\ \Delta\dot{\theta}_2 + l_2\big(\theta_2 - \bar{\theta}_2 + \Delta\theta_2\big) \end{bmatrix} \\ b &= \begin{bmatrix} \Delta\ddot{\theta}_1 + l_1\big(\dot{\theta}_1 + \Delta\dot{\theta}_1\big) \\ \Delta\ddot{\theta}_2 + l_2\big(\dot{\theta}_2 + \Delta\dot{\theta}_2\big) \end{bmatrix}. \end{aligned} \tag{90}$$

Recalling the decomposition as (17)–(19), we have $\bar{A} = \mathrm{Diag}\{1, 1\}$, $\Delta A = 0$, and

$$\begin{aligned} \bar{c} &= \begin{bmatrix} l_1\big(\theta_1 - \bar{\theta}_1\big) \\ l_2\big(\theta_2 - \bar{\theta}_2\big) \end{bmatrix}, \Delta c = \begin{bmatrix} \Delta\dot{\theta}_1 + l_1\Delta\theta_1 \\ \Delta\dot{\theta}_2 + l_2\Delta\theta_2 \end{bmatrix} \\ \bar{b} &= \begin{bmatrix} l_1\dot{\theta}_1 \\ l_2\dot{\theta}_2 \end{bmatrix}, \Delta b = \begin{bmatrix} \Delta\ddot{\theta}_1 + l_1\Delta\dot{\theta}_1 \\ \Delta\ddot{\theta}_2 + l_2\Delta\dot{\theta}_2 \end{bmatrix}. \end{aligned} \tag{91}$$

C. Assumptions Verification

For Assumption 1, $V(\beta) = \beta^T P \beta$ is selected as the Lyapunov function, along with the selection of $f(\kappa, \bar{\beta}) = -k\kappa\bar{\beta}$, where $P > 0 \in \mathbf{R}^{m\times m}$, $k > 0$. Furthermore, select the corresponding functions $\gamma_{1,2,3}$ as $\gamma_1(\|\beta\|) = \lambda_m(P)\|\beta\|^2$, $\gamma_2(\|\beta\|) = \tilde{k}\lambda_M(P)\|\beta\|^2$, and $\gamma_3(\|\beta\|) = k\lambda_M(P)\|\beta\|^2$, with $\tilde{k} > 1$ and $k \le (1/2)\kappa^{-1}k_1^{-1}k_2\gamma^{-1}$. Based on this, we further have $\hat{\gamma}_3(\|\varsigma\|) = k\lambda_M(P)\|\varsigma\|^2$.

For Assumption 2 (1), recalling $\bar{A} = \mathrm{Diag}\{1, 1\} = A$, we have $G = \mathrm{Diag}\{1, 1\}$ to subject (29). Recalling $\bar{\beta}$, $\Delta\beta$ as (88), define

$$Q =: \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}. \tag{92}$$

Choose $Q_{12} = Q_{21} = 0$. To subject $\Delta\beta = Q\bar{\beta}$ in Assumption 2 (1), we have

$$Q =: \begin{bmatrix} \frac{\Delta\dot{\theta}_1 + l_1\Delta\theta_1}{\dot{\theta}_1 + l_1(\theta_1 - \bar{\theta}_1)} & 0 \\ 0 & \frac{\Delta\dot{\theta}_2 + l_2\Delta\theta_2}{\dot{\theta}_2 + l_2(\theta_2 - \bar{\theta}_2)} \end{bmatrix}. \tag{93}$$

For Assumption 2 (2), we choose $\hat{\rho}_Q = 1$, $\hat{\rho}_E = 0$, $\hat{\rho}_G = 0$, and $\hat{\rho}_{GE} = 0$ by the trial-and-error method.

For Assumption 3, with some calculations, we have

$$\begin{aligned} \|h\| &\le \underbrace{\sqrt{f_1^2 + f_2^2 + f_3^2 + f_4^2}}_{=:\alpha} \underbrace{\sqrt{1 + \theta_1^2 + \dot{\theta}_1^2 + \dot{\theta}_1^2\dot{\theta}_2^2}}_{=:\hat{\Pi}} \\ &= \alpha\big(\Delta m_1, \Delta F_1, \Delta F_2, \Delta\dot{\theta}_1, \Delta\dot{\theta}_2, \Delta\ddot{\theta}_1, \Delta\ddot{\theta}_2\big) \\ &\quad \times \hat{\Pi}\big(\theta_1, \dot{\theta}_1, \dot{\theta}_2\big) \end{aligned} \tag{94}$$

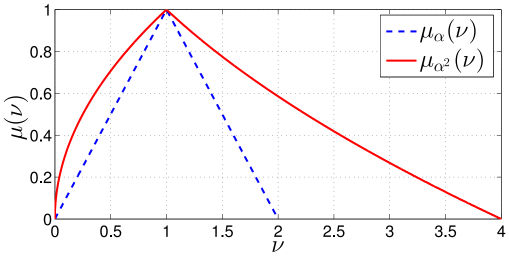

Fig. 4. Membership function: $\mu_{\alpha^2}(\nu)$.

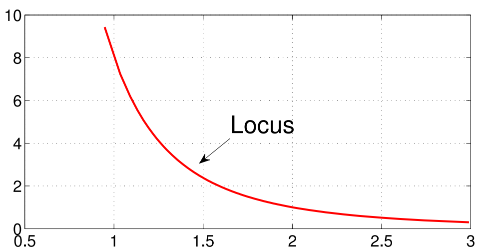

Fig. 5. Locus of the Pareto optimality.

where $f_1(\cdot)$ is a function of Δm_1, ΔF_1, ΔF_2, $\Delta\dot{\theta}_1$, $\Delta\dot{\theta}_2$, $\Delta\ddot{\theta}_1$, $\Delta\ddot{\theta}_2$, and $f_{2,3,4}(\cdot)$ are functions of Δm_1. By this, Assumption 3 can be met by choosing

$$\Pi(\alpha, \theta_1, \dot{\theta}_1, \dot{\theta}_2) = \alpha\sqrt{1 + \theta_1^2 + \dot{\theta}_1^2 + \dot{\theta}_1^2\dot{\theta}_2^2} = \alpha\hat{\Pi}(\theta_1, \dot{\theta}_1, \dot{\theta}_2). \tag{95}$$

D. Optimal Design Parameters: Pareto Optimality

We choose the system parameters $\bar{m}_1 = 5200$ kg, $m_2 = 2088.15$ kg, $R_1 = 1.05$ m, and $R_2 = 5.12$ m, set the crisp initial condition $\theta_{1,2}(0) = 5°$, $\dot{\theta}_{1,2}(0) = 0°/s$, and aim at desired horizontal angular position $\bar{\theta}_1 = 17.2°$ (≈ 0.3 rad) and elevation angular position $\bar{\theta}_2 = 11.46°$ (≈ 0.2 rad). We choose the control parameters $k_1 = 0.1$, $k_2 = 10$, $k = 10$, $\tilde{k} = 2$, $l_1 = 3$, $l_2 = 3$, $\hat{\rho}_Q = 1$, $\hat{\rho}_E = 0$, $\hat{\rho}_G = 0$, $\hat{\rho}_{GE} = 0$, and $\hat{\alpha}_0 = 0$. Recalling α as a measure of combined effect of uncertainty bound, we choose the membership function of it as follows:

$$\mu_\alpha(\nu) = \begin{cases} \nu & 0 \le \nu \le 1 \\ -\nu + 2 & 1 \le \nu \le 2. \end{cases} \tag{96}$$

By using the fuzzy arithmetic, we obtain the membership function of α^2 as $\mu_{\alpha^2}(\nu)$ (shown as in Fig. 4), and then $\mathcal{D}$-operation yields $D[\|\alpha\|^2] = 1.6049$. With (59), we have $\mathcal{D}[\delta] = 160.74$. We obtain the Pareto optimality

$$\begin{cases} \kappa^* = 1.6837\sqrt[8]{\frac{\zeta_2}{\zeta_1}} \\ \gamma^* = 1.6837\sqrt[8]{\left(\frac{\zeta_1}{\zeta_2}\right)^3}. \end{cases} \tag{97}$$

For the minimum comprehensive index, according to (82), we have

$$J_{\min} = \zeta_1\left(\frac{16.074}{\kappa^*\gamma^*} + (\kappa^*)^2\right) + \zeta_2(\gamma^*)^2. \tag{98}$$

Fig. 5 shows the locus of the Pareto optimality. Table I shows the Pareto optimality and the minimum comprehensive index with five different sets of Pareto parameters. Fig. 6 presents the dimensional relationship between $J_{\min}$, κ^*, and γ^* with $\zeta_{1,2} = 0.5$. Theorem 4 is verified that there is unique solution $(\kappa^*, \gamma^*) = (1.6837, 1.6837)$ to render $J_{\min} = 5.6699$. Fig. 7 shows $J_{\min}$ with five sets $\zeta_{1,2}$.

TABLE I
PARETO OPTIMALITY

(ζ_1, ζ_2)	(κ^*, γ^*)	J_{min}
(0.1,0.9)	(2.2159,0.7386)	1.9641
(0.3,0.7)	(1.8718,1.2254)	4.2046
(0.5,0.5)	(1.6837,1.6837)	5.6699
(0.7,0.3)	(1.5145,2.3135)	6.4226
(0.9,0.1)	(1.2794,3.8381)	5.8924

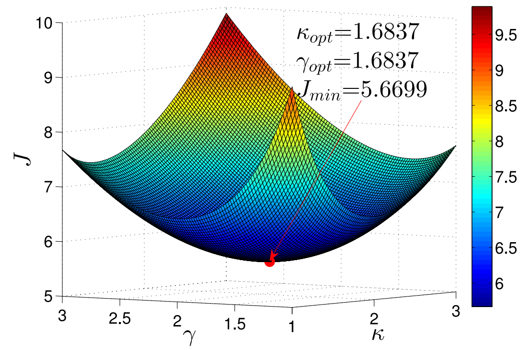

Fig. 6. Three-dimensional relationship of J, κ, and γ.

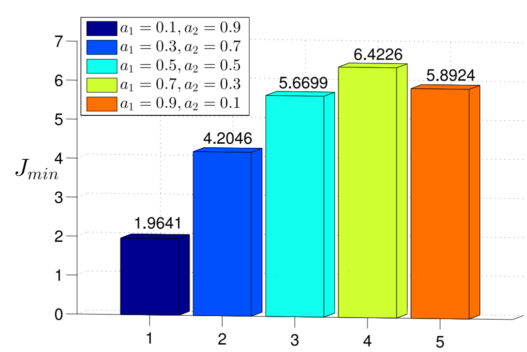

Fig. 7. $J_{\min}$ with five sets $\zeta_{1,2}$.

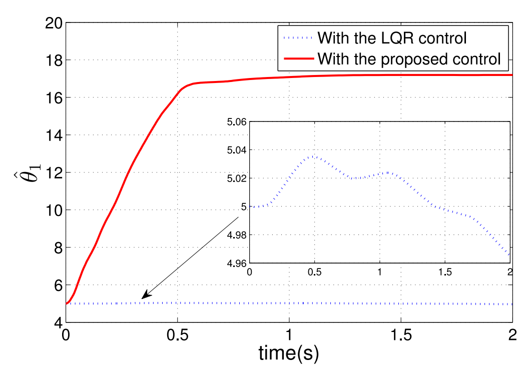

Fig. 8. Angular positions of the turret load $\hat{\theta}_1$ with the proposed control and LQR control.

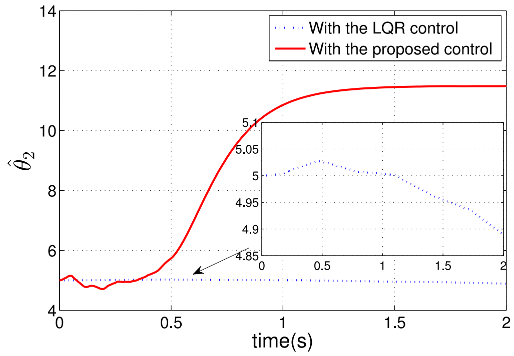

Fig. 9. Angular positions of the barrel load $\hat{\theta}_2$ with the proposed control and LQR control.

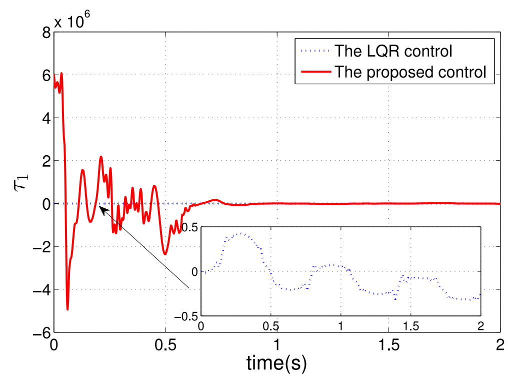

Fig. 10. Proposed control and LQR control τ_1.

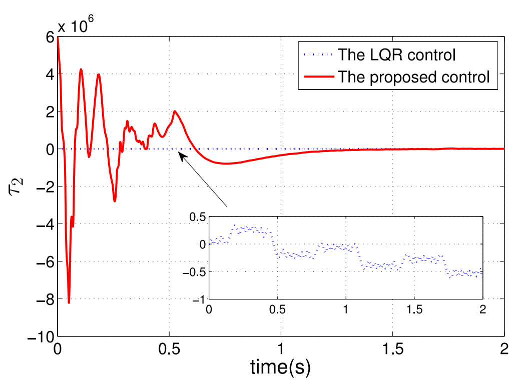

Fig. 11. Proposed control and LQR control τ_2.

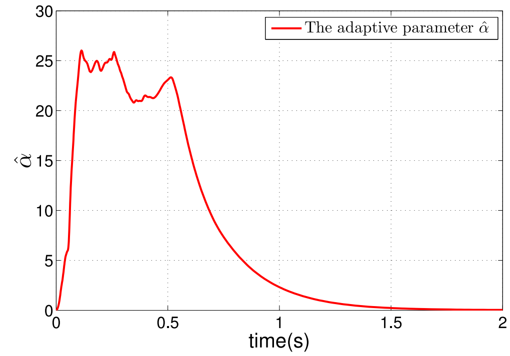

Fig. 12. History of the adaptive parameter $\hat{\alpha}$.

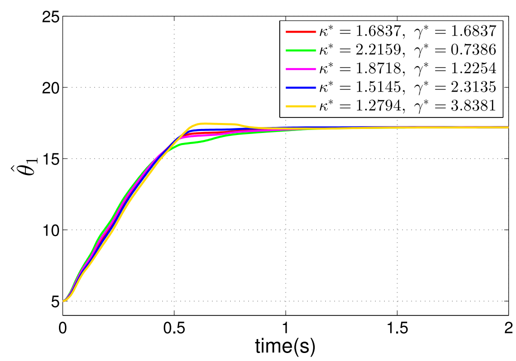

Fig. 13. Angular positions of the turret load $\hat{\theta}_1$ with different (κ^*, γ^*).

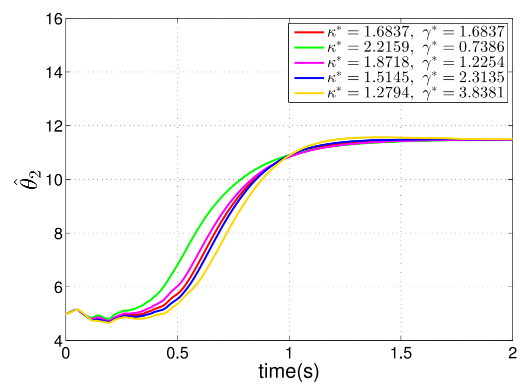

Fig. 14. Angular positions of the barrel load $\hat{\theta}_2$ with different (κ^*, γ^*).

E. Simulation Results

Simulations results are shown in Figs. 8–19. Figs. 8–12 present the comparison of the proposed control with $(\kappa^*, \gamma^*) = (1.6837, 1.6837)$ and a standard LQR control, in which Figs. 8 and 9 present the comparison of the history of angular $\hat{\theta}_{1,2}$, and Figs. 10 and 11 present the comparison of control input. With the proposed (42), the angular $\hat{\theta}_1$ approaches to the desirable angular $\bar{\theta}_1 = 17.2$ before $t_1 = 0.5$ and the angular $\hat{\theta}_2$ approaches to the desirable angular $\bar{\theta}_2 = 11.46$ before $t_1 = 1$, but the LQR control does not lead to desired performance. Fig. 12 shows that the adaptive parameter $\hat{\alpha}$ becomes close to 0 after a self-regulation.

Remark 15: LQR has been critically and thoroughly examined for decades. As many other works have been compared with LQR, our comparison to LQR can be extended to with many other great works; hence, the comparison is representative, unbiased, and objective.

Figs. 13–19 show the comparison of the proposed control with five different sets of (κ^*, γ^*). Figs. 13 and 14 present the comparison of the history of angular $\hat{\theta}_{1,2}$, Figs. 15 and 16 present the corresponding control input, and Fig. 17 presents the corresponding history of adaptive parameter $\hat{\alpha}$. It can be seen that the smaller parameters (i.e., the smaller $\kappa^* + \gamma^*$) renders the smaller ultimate boundedness region with the smaller adaptive parameter and the larger control input. Figs. 18 and 19 present the comparison of the corresponding accumulative performance error (i.e., the area between the angular $\hat{\theta}_{1,2}$ and the desirable angular $\bar{\theta}_{1,2}$). The set of $(\kappa^* = 1.5145, \gamma^* =$

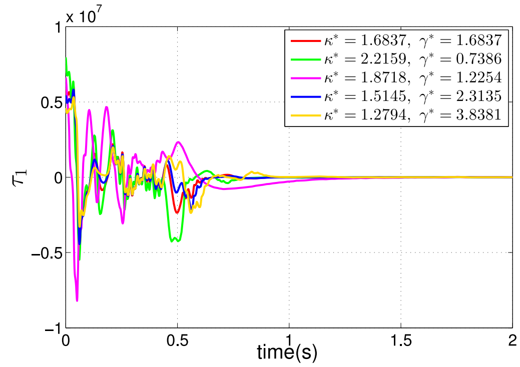

Fig. 15. Control input τ_1 with different $(\kappa^*,\ \gamma^*)$.

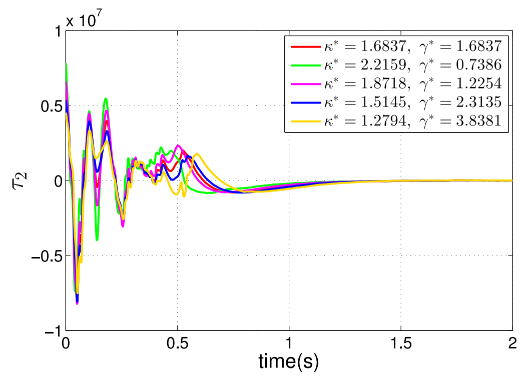

Fig. 16. Control input τ_2 with different $(\kappa^*,\ \gamma^*)$.

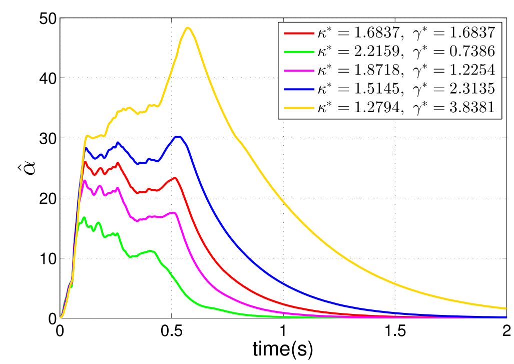

Fig. 17. History of adaptive parameter $\hat{\alpha}$ with different $(\kappa^*,\ \gamma^*)$.

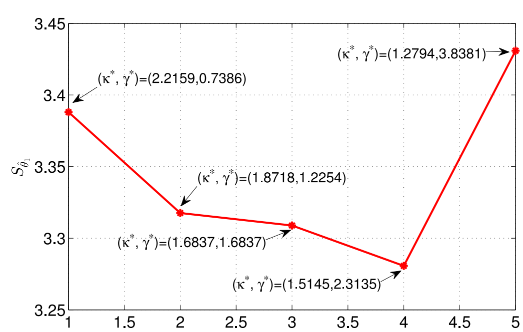

Fig. 18. Accumulative performance error $S_{\hat{\theta}_1}$ with different $(\kappa^*,\ \gamma^*)$.

2.3135) renders the smaller accumulative performance error $S_{\hat{\theta}_1}$, and the set of $(\kappa^* = 2.2159, \gamma^* = 0.7386)$ renders the smaller accumulative performance error $S_{\hat{\theta}_2}$.

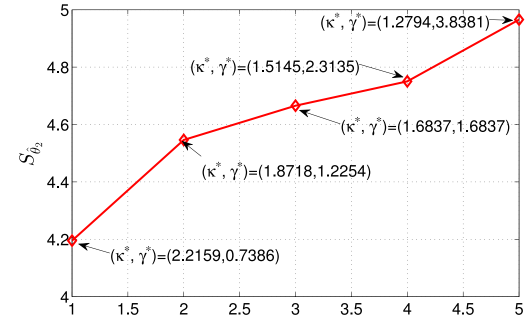

Fig. 19. Accumulative performance error $S_{\hat{\theta}_2}$ with different $(\kappa^*,\ \gamma^*)$.

VII. Conclusion

This article addresses the optimal control problem for mechanical systems through a synthesis of constraint following, fuzzy set theory, and game theory. First, the fuzzy set theory is adopted to describe the uncertainty, and then motion control of the concerned system is converted as a problem of a constraint following with a β-measure as the tracking error, for which an adaptive robust control is explored for stability (uniform boundedness and uniform ultimate boundedness). Second, for the optimal parameter design, an optimization framework is established by the cooperative game theory. With the fuzzy information, two cost functions are constructed to form a comprehensive performance index. By minimizing such an index, the analytical Pareto optimality is achieved. Finally, an optimal control scheme for fuzzy mechanical systems is constructed by using the optimal design parameters in the foregoing adaptive robust control. As an introspection here, this article addresses an optimal parameter design problem with (only) two parameters, however, sometimes there may be more than two parameters that need to be concerned. Inspired by this, we expect to do further explorations on the optimal design problem with multiparameter (more than two parameters) in the future.

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Qinqin Sun received the Ph.D. degree in mechanical engineering from the Nanjing University of Science and Technology, Nanjing, China, in 2021.

She worked with the Department of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA, USA, as a Research Assistant from 2014 to 2016 and visited the Department of Mechanical Engineering, University of California at Berkeley, Berkeley, CA, USA, as a visiting Ph.D. student from 2020 to 2021. She is currently an Associate Research Fellow of Aerospace Propulsion Theory and Engineering with the School of Energy and Power Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing, China. Her research interests include complex system dynamics and control, fuzzy optimal design, and game theory.

Xiuye Wang received the Ph.D. degree in mechanical design and theory from the Hefei University of Technology, Hefei, China, in 2016.

He visited the Department of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA, USA, as a visiting Ph.D. student from 2014 to 2016. He is currently an Assistant Professor of Mechanical Engineering with the School of Mechanical Engineering, Nanjing University of Science and Technology, Nanjing, China. His research interests include multibody system dynamics, robust control of mechanical system, fuzzy dynamical systems, and optimal design.

Guolai Yang received the Ph.D. degree in automatic weapons and ammunition engineering from the Nanjing University of Science and Technology, Nanjing, China, in 1999.

He visited the Department of Mechanical Engineering, Texas Tech University, Lubbock, TX, USA, as a Visiting Professor in 2010. He is currently a Professor of Mechanical Engineering with the School of Mechanical Engineering and serves as the Dean of the School of International Education, Nanjing University of Science and Technology. His research interests include theory and experimental method of time-varying mechanics, mechanical system dynamics, virtual design, and simulation technique.

Ye-Hwa Chen received the Ph.D. degree in control science and engineering from the University of California at Berkeley, Berkeley, CA, USA, in 1985.

He is currently a Professor of Mechanical Engineering with the George W. Woodruff School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA, USA, and also with the Key Laboratory of Road Construction Technology and Equipment of MOE, Chang'an University, Xi'an, Shaanxi, China. His research interests include complex system dynamics, fuzzy dynamical systems, fuzzy reasoning, and modeling and control of mechanical systems.

Prof. Chen received the IEEE TRANSACTIONS ON FUZZY SYSTEMS Outstanding Paper Award from the IEEE Neural Networks Council in 2001. He has been serving as a regional editor and/or an associate editor for six journals.

Fai Ma received the Ph.D. degree in applied mathematics from the California Institute of Technology, Pasadena, CA, USA, in 1981.

He was formerly a Research Engineer with the IBM Thomas J. Watson Research Center, Ossining, NY, USA, and the Standard Oil Company, New York, NY, USA. He is currently a Professor of Applied Mechanics with the Department of Mechanical Engineering, University of California at Berkeley, Berkeley, CA, USA. He has authored or coauthored more than 190 technical publications. His research interests include vibration, nonlinear damping, and system uncertainties.

Prof. Ma is a Fellow of the American Society of Mechanical Engineers.