Design of a Switching Controller for Nonlinear Systems With Unknown Parameters Based on a Fuzzy Logic Approach

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***Abstract*—This paper deals with nonlinear plants subject to unknown parameters. A fuzzy model is first used to represent the plant. An equivalent switching plant model is then derived, which supports the design of a switching controller. It will be shown that the closed-loop system formed by the plant and the switching controller is a linear system. Hence, the system performance of the closed-loop system can be designed. An application example on controlling a two-inverted pendulum system on a cart will be given to illustrate the design procedure of the proposed switching controller.**

***Index Terms*—Nonlinear systems, switching controller, switching plant model, Takagi–Sugeno–Kang (TSK) fuzzy plant model.**

I. Introduction

CONTROL of nonlinear systems is difficult because we do not have systematic mathematical tools to help finding a necessary and sufficient condition to guarantee the stability and performance. The problem will become more complex if some of the parameters of the plant are unknown. By using a Takagi–Sugeno–Kang (TSK) fuzzy plant model [1], [2], [5], [10], a nonlinear system can be expressed as a weighted sum of some simple subsystems. This model gives a fixed structure to some of the nonlinear systems, and facilitates the analysis of the systems. There are two ways to obtain the fuzzy plant model: 1) by using identification methods based on the input-output data of the plant [1], [2], [5], [10] and 2) by direct derivation from the mathematical model of the nonlinear plant [3].

Stability of fuzzy systems formed by a fuzzy plant model and a fuzzy controller has been actively studied. Different stability conditions were obtained [3], [4], [6]–[9]. Linear controllers [11], [17], switching controllers [16] and nonlinear controllers [15], [18] were also proposed to control plants represented by fuzzy models. Most of the fuzzy controllers proposed depend on the grades of membership inside the fuzzy plant model. Hence, the membership functions [1], [2] of the fuzzy plant model must be known. It means that the parameters of the nonlinear plant must be known, or be constant when the identification method is used to derive the fuzzy plant model. Practically, the parameters of many nonlinear plants will change during the operation, e.g., the load of a power system, the number of passengers on board a train. In these cases, the robustness of the fuzzy controller becomes an important concern. Robustness issues of fuzzy control systems have been investigated in [3], [8], [12]–[14].

In this paper, a systematic analysis and design method is proposed to guarantee the system stability, and design the system performance for a class of nonlinear systems subject to unknown parameters with known bounds. The proposed control method makes the closed-loop system become a linear system. This is achieved by a proposed switching plant model, which is used as a design tool. The switching plant model is derived based on the fuzzy plant model of the nonlinear system. It will be proved that the switching plant model is equivalent to the fuzzy plant model. Moreover, the switching plant model has a promising feature that the characteristic of the nonlinear system subject to unknown parameters can be modeled by a limited number of known linear systems switching among them. Only one of the linear systems will be activated at any moment. Based on the switching plant model, a switching controller is proposed to control the nonlinear plant subject to unknown parameters within known bounds. The switching controller, which consists of a number of linear controllers, is designed from the linear systems of the switching plant model. One of the linear controllers will be active at a time according to some switching laws derived from the Lyapunov stability theory. Thanks to the linear closed-loop system formed, the system stability and the performance of the closed-loop system can be determined.

This paper is organized as follows. Section II presents the TSK fuzzy plant model. In Section III, the switching plant model will be derived. In Section IV, the switching controller will be presented. An application example on stabilizing a two-inverted pendulum system will be presented in Section V. A conclusion will be drawn in Section VI.

II. TSK Fuzzy Plant Model

Consider a nonlinear plant of the following form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t), w(t))\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1}$$

where $\mathbf{A}(\mathbf{x}(t), w(t)) \in \Re^{n \times n}$ and $\mathbf{B} \in \Re^{n \times m}$ are the system matrix and input matrix respectively. The system matrix $\mathbf{A}(\mathbf{x}(t), w(t))$ has a known structure but subject to unknown parameters; $\mathbf{x}(t) \in \Re^{n \times 1}$ is the system state vector and $w\ (t)$ is a nonlinear function related to the unknown parameters. The input matrix $\mathbf{B}$ is a known constant matrix. $\mathbf{u}(t) \in \Re^{m \times 1}$ is

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the input vector. The system of (1) can be represented by a fuzzy plant model, which expresses the nonlinear system as a weighted sum of linear systems. Let p be the number of fuzzy rules describing the multivariable nonlinear plant of (1), the ith rule is of the following format:

$$\begin{aligned}&\text{Rule } i: \text{IF } f_1(\mathbf{x}(t)) \text{ is } \mathrm{M}_1^i \text{ and } \ldots \text{ and } f_\Psi(\mathbf{x}(t)) \text{ is } \mathrm{M}_\Psi^i\\ &\qquad \text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\end{aligned} \tag{2}$$

where M_α^i is a fuzzy term of rule i corresponding to the function $f_\alpha(\mathbf{x}(t))$ that contains the unknown parameters, $\alpha = 1, 2, \ldots, \Psi; i = 1, 2, \ldots, p; \Psi$ is a positive integer, p is the number of rules; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B} \in \Re^{n\times m}$ are known system and input matrices, respectively, of the ith rule subsystem. The inferred system is described by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)) \tag{3}$$

with the properties of (4) and (5) shown at the bottom of the page, $\mu_{\mathrm{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))), \alpha = 1, 2, \ldots, \Psi$, are membership functions of unknown values. Thus, the nonlinear plant with unknown parameters is modeled by (3) such that the unknown parameters are reflected in $w_i(\mathbf{x}(t))$. Reshuffling the terms, (3) can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}_p\mathbf{x}(t) + \sum_{i=1}^{p-1} w_i(\mathbf{x}(t))\tilde{\mathbf{A}}_i\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{6}$$

where

$$\tilde{\mathbf{A}}_i = \mathbf{A}_i - \mathbf{A}_p \quad \text{for } i = 1, 2, \ldots, p-1. \tag{7}$$

III. Switching Plant Model

In this section, we shall present a switching plant model, which will be proved to be equivalent to the TSK fuzzy plant model of (6) as time $t \to \infty$. The following equation describes the switching plant model:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}_p\mathbf{x}(t) + \sum_{i=1}^{p-1} \hat{w}_i(t)\tilde{\mathbf{A}}_i\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + K\mathbf{e}(t) \tag{8}$$

where $\hat{\mathbf{x}}(t) \in \Re^{n\times 1}$ is a state vector, $\hat{w}_i(t) = 1$ or 0 according to some switching laws to be defined, $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$. In the following, the switching laws that govern the values (either 0 or 1) of $\hat{w}_i(\mathbf{x}(t)), i = 1, 2, \ldots, p-1$, for the switching plant model of (8) will be derived. For simplicity, let $w_i(\mathbf{x}(t))$ and $\hat{w}_i(\mathbf{x}(t))$ be written as w_i and $\hat{w}_i$ respectively. From (6) and (8), we have

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) = \sum_{i=1}^{p-1} (w_i - \hat{w}_i)\tilde{\mathbf{A}}_i\mathbf{x}(t) - K\mathbf{e}(t) \tag{9}$$

where K is a nonzero positive scalar. To investigate the convergence of $\mathbf{e}(t)$, we consider the following Lyapunov's candidate function:

$$V_p(\mathbf{e}(t)) = \frac{1}{2}\mathbf{e}^{\mathrm{T}}(t)\mathbf{e}(t) \tag{10}$$

$$\Rightarrow \dot{V}_p(\mathbf{e}(t)) = \mathbf{e}^{\mathrm{T}}(t)\dot{\mathbf{e}}(t). \tag{11}$$

If $\dot{V}_p(\mathbf{e}(t))$ is negative definite, it can be concluded that $\mathbf{e}(t) \to 0$ as $t \to \infty$. In the following, $\dot{V}_p(\mathbf{e}(t))$ will be shown to be negative definite under some defined switching laws.

Proof: From (9) and (11), we have

$$\begin{aligned}\dot{V}_p(\mathbf{e}(t)) &= \mathbf{e}^{\mathrm{T}}(t)\left(\sum_{i=1}^{p-1}(w_i - \hat{w}_i)\tilde{\mathbf{A}}_i\mathbf{x}(t) - K\mathbf{e}(t)\right)\\ &= \sum_{i=1}^{p-1}(w_i - \hat{w}_i)\mathbf{e}^{\mathrm{T}}(t)\tilde{\mathbf{A}}_i\mathbf{x}(t) - K\mathbf{e}^{\mathrm{T}}(t)\mathbf{e}(t)\\ &= \sum_{i=1}^{p-1}(w_i - \hat{w}_i)\mathbf{e}^{\mathrm{T}}(t)\tilde{\mathbf{A}}_i\mathbf{x}(t) - K\|\mathbf{e}(t)\|^2\end{aligned} \tag{12}$$

where $\|\cdot\|$ denotes the l_2 vector norm. Let the switching laws be

$$\hat{w}_i = \frac{1}{2}[1 + \mathrm{sgn}(\mathbf{e}^{\mathrm{T}}(t)\tilde{\mathbf{A}}_i\mathbf{x}(t))] \quad \text{for } i = 1, 2, \ldots, p-1 \tag{13}$$

where $\mathrm{sgn}(\cdot)$ is a function defined by

$$\mathrm{sgn}(z) = \begin{cases}1, & \text{when } z > 0\\ -1, & \text{when } z \le 0.\end{cases} \tag{14}$$

From (12) and (13), we have

$$\begin{aligned}\dot{V}_p(\mathbf{e}(t)) &= \sum_{i=1}^{p-1}\left(w_i - \frac{1}{2}[1 + \mathrm{sgn}(\mathbf{e}^{\mathrm{T}}(t)\tilde{\mathbf{A}}_i\mathbf{x}(t))]\right)\\ &\quad \times \mathbf{e}^{\mathrm{T}}(t)\tilde{\mathbf{A}}_i\mathbf{x}(t) - K\|\mathbf{e}(t)\|^2\\ &= \sum_{i=1}^{p-1}\left(w_i - \frac{1}{2}\right)\mathbf{e}^{\mathrm{T}}(t)\tilde{\mathbf{A}}_i\mathbf{x}(t)\\ &\quad - \sum_{i=1}^{p-1}\frac{1}{2}\mathrm{sgn}(\mathbf{e}^{\mathrm{T}}(t)\tilde{\mathbf{A}}_i\mathbf{x}(t))\mathbf{e}^{\mathrm{T}}(t)\tilde{\mathbf{A}}_i\mathbf{x}(t)\\ &\quad - K\|\mathbf{e}(t)\|^2\\ &\le \sum_{i=1}^{p-1}\left|w_i - \frac{1}{2}\right||\mathbf{e}^{\mathrm{T}}(t)\tilde{\mathbf{A}}_i\mathbf{x}(t)|\\ &\quad - \sum_{i=1}^{p-1}\frac{1}{2}|\mathbf{e}^{\mathrm{T}}(t)\tilde{\mathbf{A}}_i\mathbf{x}(t)| - K\|\mathbf{e}(t)\|^2\\ &= \sum_{i=1}^{p-1}\left(\left|w_i - \frac{1}{2}\right| - \frac{1}{2}\right)|\mathbf{e}^{\mathrm{T}}(t)\tilde{\mathbf{A}}_i\mathbf{x}(t)|\\ &\quad - K\|\mathbf{e}(t)\|^2 \le 0.\end{aligned} \tag{15}$$

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1, \quad 0 \le w_i(\mathbf{x}(t)) \le 1 \quad \text{for all } i \tag{4}$$

$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{M}_\Psi^i}(f_\Psi(\mathbf{x}(t)))}{\sum_{k=1}^{p}\left(\mu_{\mathrm{M}_1^k}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^k}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{M}_\Psi^k}(f_\Psi(\mathbf{x}(t)))\right)} \tag{5}$$

From (15), as $|w_i - (1/2)| - (1/2) \leq 0$ for all $i = 1, 2, \ldots, p-1$ (which is a property of the TSK fuzzy plant model as revealed by (4)), we have

$$\dot{V}_p(\mathbf{e}(t)) \leq -K\|\mathbf{e}(t)\|^2 \leq 0. \tag{16}$$

Equality holds only when $\mathbf{e}(t) = \mathbf{0}$. **QED**

As a result, we can conclude that $\mathbf{e}(t) \to \mathbf{0}$ as $t \to \infty$, and $\mathbf{e}(t) = 0$ is an equilibrium point, which is achieved by $\hat{w}_i$ based on (13). Thus, when the switching laws of (13) are used, the switching plant model (8) becomes equivalent to the TSK fuzzy plant model (6) when $t \to \infty$. From (8)

$$\dot{\hat{\mathbf{x}}}(t) = \left(\mathbf{A}_p + \sum_{i=1}^{p-1} \hat{w}_i \tilde{\mathbf{A}}_i\right) \hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{m}_\mathbf{e}(t) \tag{17}$$

where

$$\mathbf{m}_\mathbf{e}(t) = \left(\mathbf{A}_p(t) + \sum_{i=1}^{p-1} \hat{w}_i(t) \tilde{\mathbf{A}}_i\right) \mathbf{e}(t) + K\mathbf{e}(t) \tag{18}$$

is a modeling error vector, which has the property that $\lim_{t\to\infty} \mathbf{m}_\mathbf{e}(t) = \mathbf{0}$. This is because $\mathbf{e}(t) \to \mathbf{0}(\hat{\mathbf{x}}(t) \to \mathbf{x}(t))$ when $t \to \infty$. To ensure that $\mathbf{m}_\mathbf{e}(t)$ is bounded, the initial condition of $\mathbf{e}(t)$ has to be bounded, which is often valid in practice.

IV. Switching Controller

In this section, a switching controller is developed based on the switching plant model of (17). From (17), since $\hat{w}_i$ can take a value of 0 or 1, a total of 2^{p-1} linear subsystems can be formed. Therefore

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \left(\mathbf{A}_p + \sum_{i=1}^{p-1} \hat{w}_i \tilde{\mathbf{A}}_i\right) \hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{m}_\mathbf{e}(t)\\ &= \sum_{k=1}^{2^{p-1}} v_k(\hat{w}_1, \hat{w}_2, \ldots, \hat{w}_{p-1})(\hat{\mathbf{A}}_k \hat{\mathbf{x}}(t)\\ &\quad + \mathbf{B}\mathbf{u}(t)) + \mathbf{m}_\mathbf{e}(t)\end{aligned} \tag{19}$$

where

$$\hat{\mathbf{A}}_k = \mathbf{A}_p + \sum_{i=1}^{p-1} \hat{w}_i \tilde{\mathbf{A}}_i \quad \text{for } k = N(\hat{\mathbf{W}}(t)) \tag{20}$$

$$\hat{\mathbf{W}}(t) = [\hat{w}_{p-1} \quad \cdots \quad \hat{w}_2 \quad \hat{w}_1] \tag{21}$$

$$N(\hat{\mathbf{W}}(t)) = 1 + \sum_{\lambda=1}^{p-1} 2^{\lambda-1} \hat{w}_\lambda \tag{22}$$

which is an integer between 1 and 2^{p-1} inclusively

$$v_k(\hat{w}_1, \hat{w}_2, \ldots, \hat{w}_{p-1}) = \begin{cases} 1, & \text{if } k = N(\hat{\mathbf{W}}(t)) \\ 0, & \text{if } k \neq N(\hat{\mathbf{W}}(t)) \end{cases} \quad \text{for } k = 1, 2, \ldots, 2^{p-1}. \tag{23}$$

The value of the integer $N(\hat{\mathbf{W}}(t))$ is determined by the switching laws of (13). $\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}_k \hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t), k = 1, 2, \ldots, 2^{p-1}$, is the kth switched linear system. The system matrix $\hat{\mathbf{A}}_k$ and input matrix $\mathbf{B}$ of each switched linear system are known. For each switched system, a switched controller is designed for it. Thus, writing $v_k(\hat{w}_1, \hat{w}_2, \ldots, \hat{w}_{p-1})$ as v_k, the switching controller is given by

$$\begin{aligned}\mathbf{u}(t) &= \sum_{k=1}^{2^{p-1}} v_k \mathbf{u}_k(t)\\ &= \sum_{k=1}^{2^{p-1}} v_k \mathbf{G}_k \mathbf{x}(t) + \mathbf{r}(t)\end{aligned} \tag{24}$$

where $\mathbf{u}_k(t) = \mathbf{G}_k \mathbf{x}(t) + \mathbf{r}(t)$ is the kth switched linear state feedback controller, $\mathbf{G}_k \in \Re^{m\times n}$ is the state feedback gain to be designed, and $\mathbf{r}(t) \in \Re^{m\times 1}$ is the reference input. It should be noted that at any time, among all $v_k, k = 1, 2, \ldots, 2^{p-1}$, only one of them is equal to 1 which has $k = N(\hat{\mathbf{W}}(t))$. All other $v_k, k \neq N(\hat{\mathbf{W}}(t))$, are equal to zero. Clearly, the switching controller applies the $N(\hat{\mathbf{W}}(t))$th switched controller to tackle the $N(\hat{\mathbf{W}}(t))$th switched linear system when it is active. In the following, we shall analyze the stability of the switching control system, which is formed by the switching plant model of (17) and the switching controller of (24) connected in closed-loop. From (19) and (24), the switching control system is given by

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \sum_{k=1}^{2^{p-1}} v_k(\hat{\mathbf{A}}_k + \mathbf{B}\mathbf{G}_k)\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{r}(t) + \mathbf{m}_\mathbf{e}(t)\\ &\quad + \sum_{k=1}^{2^{p-1}} v_k \mathbf{B}\mathbf{G}_k \mathbf{e}(t)\\ &= \sum_{k=1}^{2^{p-1}} v_k \hat{\mathbf{H}}_k \hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{r}(t) + \mathbf{m}_\mathbf{e}(t)\\ &\quad + \sum_{k=1}^{2^{p-1}} v_k \mathbf{B}\mathbf{G}_k \mathbf{e}(t)\end{aligned} \tag{25}$$

where

$$\hat{\mathbf{H}}_k = \hat{\mathbf{A}}_k + \mathbf{B}\mathbf{G}_k \tag{26}$$

It can be seen that if $\hat{\mathbf{H}} = \hat{\mathbf{H}}_k$ for all $k = 1, 2, \ldots, 2^{p-1}$, we have

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{H}}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{r}(t) + \mathbf{m}_\mathbf{e}(t) + \sum_{k=1}^{2^{p-1}} v_k \mathbf{B}\mathbf{G}_k \mathbf{e}(t). \tag{27}$$

In (27), if $\hat{\mathbf{H}}$ is designed to be a stable matrix and $\mathbf{m}_\mathbf{e}(t) + \sum_{k=1}^{2^{p-1}} v_k \mathbf{B}\mathbf{G}_k \mathbf{e}(t)$ is treated as an external input, the system is guaranteed to be stable by linear control theory. (27) is equivalent to a linear system of $\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{H}}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{r}(t)$ when $\mathbf{m}_\mathbf{e}(t) + \sum_{k=1}^{2^{p-1}} v_k \mathbf{B}\mathbf{G}_k \mathbf{e}(t)$ approaches zero. As $\mathbf{e}(t) \to \mathbf{0}$ as $t \to \infty$, it can be concluded that the nonlinear system of (1) can also be stabilized by the proposed switching controller. The above results and the design of the switching controller can be summarized by the following Lemma.

Lemma 1: The nonlinear system subject to unknown parameters with known bounds of (3) can be represented by the following switching plant model:

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}_p \mathbf{x}(t) + \sum_{i=1}^{p-1} \hat{w}_i(t) \tilde{\mathbf{A}}_i \mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + K\mathbf{e}(t),\\ \mathbf{e}(t) &= \mathbf{x}(t) - \hat{\mathbf{x}}(t).\end{aligned}$$

It can be stabilized by the following switching controlle:

$$\mathbf{u}(t) = \sum_{k=1}^{2^{p-1}} v_k \mathbf{G}_k \mathbf{x}(t) + \mathbf{r}(t)$$

if $\mathbf{G}_k$ is designed such that $\hat{\mathbf{H}} = \hat{\mathbf{H}}_k = \hat{\mathbf{A}}_k + \mathbf{B}\mathbf{G}_k$

$$\hat{\mathbf{A}}_k = \mathbf{A}_p + \sum_{i=1}^{p-1} \hat{w}_i \tilde{\mathbf{A}}_i \quad \text{for } k = N(\hat{\mathbf{W}}(t))$$

$$= 1 + \sum_{\lambda=1}^{p-1} 2^{\lambda-1} \hat{w}_\lambda$$

$$v_k(\hat{w}_1, \hat{w}_2, \ldots, \hat{w}_{p-1}) = \begin{cases} 1, & \text{if } k = N(\hat{\mathbf{W}}(t)) \\ 0, & \text{if } k \neq N(\hat{\mathbf{W}}(t)) \end{cases}$$

$\hat{w}_i = (1/2)[1 + \mathrm{sgn}(\mathbf{e}^{\mathrm{T}}(t)\tilde{\mathbf{A}}_i \mathbf{x}(t))]$ for $i = 1, 2, \ldots, p-1$. The closed-loop system then becomes a linear system.

The design procedure can be summarized by the following steps.

Step I) Obtain the fuzzy plant model of a nonlinear plant by means of the fuzzy modeling methods in [1], [2], [5], [10] or other methods.

Step II) Design the switching controller by choosing the switching laws of (13) and $\mathbf{G}_k$ for each switched controller such that $\hat{\mathbf{H}} = \hat{\mathbf{H}}_k$.

Referring to (26), a sufficient condition for $\hat{\mathbf{H}} = \hat{\mathbf{H}}_k$ for all k is that $\mathbf{B}$ is an invertible square matrix. Then $\mathbf{G}_k = \mathbf{B}^{-1}(\hat{\mathbf{H}}_k - \hat{\mathbf{A}}_k)$. When $\mathbf{B}$ is not a square matrix, one way of finding the feedback gains $\mathbf{G}_k$ is given as follows:

$$\begin{aligned} &\mathbf{B}\mathbf{G}_k = \hat{\mathbf{H}}_k - \hat{\mathbf{A}}_k \\ &\Rightarrow \mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{G}_k = \mathbf{B}^{\mathrm{T}}(\hat{\mathbf{H}}_k - \hat{\mathbf{A}}_k) \\ &\Rightarrow \mathbf{G}_k = (\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}(\hat{\mathbf{H}}_k - \hat{\mathbf{A}}_k) \\ &\qquad k = 1, 2, \ldots, 2^{p-11}. \end{aligned} \tag{28}$$

However, we have to verify if $\mathbf{G}_k$ of (28) gives $\mathbf{B}\mathbf{G}_k = \hat{\mathbf{H}}_k - \hat{\mathbf{A}}_k$. If this is the case for all $k, \hat{\mathbf{H}} = \hat{\mathbf{H}}_k$ can be achieved. It should be noted that for $\mathbf{G}_k$ to be designed under a given $\hat{\mathbf{H}} = \hat{\mathbf{H}}_k$ for all k, a necessary condition is that the linear system of $[\hat{\mathbf{A}}_k, \mathbf{B}]$ is controllable.

V. Application Example

A two-inverted pendulum system [19] is shown in Fig. 1. It consists of two cart-pole inverted pendulums. The inverted pendulums are linked by a spring moving along the pendulums governed by a function $a(t)$. The carts will move to and fro according to some functions. The control objective is to balance the inverted pendulums vertically despite the moving of the spring and carts by applying torques at the tips of the pendulums properly. Referring to Fig. 1, $M = 10$ kg and $m = 10$ kg are the masses of the carts and the pendulums respectively. $L = 0.5$ m is the length of the pendulums. $y_1(t)$ and $y_2(t)$ are the displacements of the moving carts. $u_1(t)$ and $u_2(t)$ are the control torques applied to the pendulums. $\theta_1(t)$ and $\theta_2(t)$ are the angular displacements of the pendulums measured from the vertical. The dynamics of the two-inverted pendulum system is governed by

$$\dot{\mathbf{x}}(t) = \bar{\mathbf{A}}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \bar{\mathbf{E}} \tag{29}$$

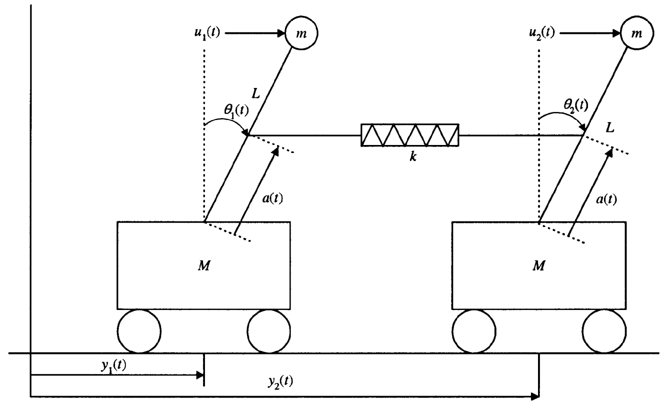


Fig. 1. Two-inverted pendulum system.

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \theta_1(t) \\ \dot{\theta}_1(t) \\ \theta_2(t) \\ \dot{\theta}_2(t) \end{bmatrix}$$

$$x_{1_{\min}} = -\frac{\pi}{2} \le x_1(t) \le x_{1_{\max}} = \frac{\pi}{2}$$

$$x_{2_{\min}} = -5 \le x_2(t) \le x_{2_{\max}} = 5$$

$$x_{3_{\min}} = -\frac{\pi}{2} \le x_3(t) \le x_{3_{\max}} = \frac{\pi}{2}$$

$$x_{4_{\min}} = -5 \le x_4(t) \le x_{4_{\max}} = 5$$

$$\bar{\mathbf{A}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \bar{f}_1(t) & 0 & \bar{f}_2(t) & 0 \\ 0 & 0 & 0 & 1 \\ \bar{f}_2(t) & 0 & \bar{f}_3(t) & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ \frac{1}{\mathrm{cmL}^2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{\mathrm{cmL}^2} \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\bar{f}_1(t) = \frac{g}{cL} - \frac{ka(t)(a(t) - cL)}{\mathrm{cmL}^2} - \frac{m}{M}\frac{\sin(x_1(t))}{x_1(t)} x_2(t)^2$$

$$\bar{f}_2(t) = \frac{ka(t)(a(t) - cL)}{\mathrm{cmL}^2}$$

$$\bar{f}_3(t) = \frac{g}{cL} - \frac{ka(t)(a(t) - cL)}{\mathrm{cmL}^2} - \frac{m}{M}\frac{\sin(x_3(t))}{x_3(t)} x_4(t)^2$$

$$\bar{\mathbf{E}} = \begin{bmatrix} 0 \\ \frac{k(a(t)-cL)}{\mathrm{cmL}^2}(y_2(t) - y_1(t)) \\ 0 \\ \frac{k(a(t)-cL)}{\mathrm{cmL}^2}(y_1(t) - y_2(t)) \end{bmatrix}, \quad g = 9.8\ \mathrm{kg}^{-2}$$

is the acceleration due to gravity; $k = 1\ \mathrm{Nm}^{-1}$ is the stiffness constant of the spring, $c = (m)/(M + m) < 1$, the state-dependent $y_1(t) = -(\mathrm{cmL}^2)/(k(a(t) - cL))k_1 x_1(t)^2 x_3(t)^2$ and $y_2(t) = (\mathrm{cmL}^2)/(k(a(t) - cL))(k_2 L + k_1 x_1(t)^2 x_3(t)^2) \ge L, k_1 = 1\ \mathrm{s}^{-2}$ and $k_2 = 1\ \mathrm{m}^{-1}\mathrm{s}^{-2}$ are constants, $a(t) \in [0\ L]$ is an unknown function, which is shown as $a(t) = (L - 1.1cL)/(2) - (L - 1.1cL)/(2), \sin(2t) + cL$ based on doing the simulation later. Based on the values of the plant parameters, (29) can be rewritten as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{E} \tag{30}$$

where

$$\mathbf{A}=\begin{bmatrix}0&1&0&0\\f_1(t)&0&f_2(t)&0\\0&0&0&1\\f_2(t)&0&f_3(t)&0\end{bmatrix},\quad \mathbf{E}=\begin{bmatrix}0\\k_2L\\0\\-k_2L\end{bmatrix}$$
$$f_1(t)=\bar{f}_1(t)+2k_1x_1(t)x_3(t)^2,\quad f_2(t)=\bar{f}_2(t)$$
$$f_3(t)=\bar{f}_3(t)-2k_1x_1(t)^2x_3(t).$$

It can be seen that $6.2484\leq f_1(t),f_3(t)\leq 46.9516, 0\leq f_2(t)\leq 0.2$. Let

$$\mathbf{u}(t)=\bar{\mathbf{u}}(t)+\mathbf{r} \tag{31}$$

where

$$\mathbf{r}=\begin{bmatrix}-k_2\mathrm{cmL}^3\\k_2\mathrm{cmL}^3\end{bmatrix}. \tag{32}$$

From (31) and (32), (30) becomes

$$\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\bar{\mathbf{u}}(t). \tag{33}$$

Based on the proposed approach, a switching controller can be designed to give $\bar{u}(t)$.

Step I) The ith rule of the TSK fuzzy plant model of (33) is given by

Rule i: IF $f_1(t)$ is M_1^i and $f_2(t)$ is M_2^i and $f_3(t)$ is M_3^i
THEN $\dot{\mathbf{x}}(t)=\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}\bar{\mathbf{u}}(t)$ (34)

where M_α^i is a fuzzy term of rule $i, i=1,2,\ldots,8, \alpha=1,2,3$. The system dynamics is described by

$$\begin{aligned}\dot{\mathbf{x}}(t)&=\sum_{i=1}^{8}w_i[\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}\bar{\mathbf{u}}(t)]\\&=\mathbf{A}_8\mathbf{x}(t)+\sum_{i=1}^{7}w_i\tilde{\mathbf{A}}_i\mathbf{x}(t)+\mathbf{B}\bar{\mathbf{u}}(t)\end{aligned} \tag{35}$$

where

$$\mathbf{A}_1=\begin{bmatrix}0&1&0&0\\f_{1_{\min}}&0&f_{2_{\min}}&0\\0&0&0&1\\f_{2_{\min}}&0&f_{3_{\min}}&0\end{bmatrix}$$
$$\mathbf{A}_2=\begin{bmatrix}0&1&0&0\\f_{1_{\min}}&0&f_{2_{\min}}&0\\0&0&0&1\\f_{2_{\min}}&0&f_{3_{\max}}&0\end{bmatrix}$$
$$\mathbf{A}_3=\begin{bmatrix}0&1&0&0\\f_{1_{\min}}&0&f_{2_{\max}}&0\\0&0&0&1\\f_{2_{\max}}&0&f_{3_{\min}}&0\end{bmatrix}$$
$$\mathbf{A}_4=\begin{bmatrix}0&1&0&0\\f_{1_{\min}}&0&f_{2_{\max}}&0\\0&0&0&1\\f_{2_{\max}}&0&f_{3_{\max}}&0\end{bmatrix}$$
$$\mathbf{A}_5=\begin{bmatrix}0&1&0&0\\f_{1_{\max}}&0&f_{2_{\min}}&0\\0&0&0&1\\f_{2_{\min}}&0&f_{3_{\min}}&0\end{bmatrix}$$
$$\mathbf{A}_6=\begin{bmatrix}0&1&0&0\\f_{1_{\max}}&0&f_{2_{\min}}&0\\0&0&0&1\\f_{2_{\min}}&0&f_{3_{\max}}&0\end{bmatrix}$$
$$\mathbf{A}_7=\begin{bmatrix}0&1&0&0\\f_{1_{\max}}&0&f_{2_{\max}}&0\\0&0&0&1\\f_{2_{\max}}&0&f_{3_{\min}}&0\end{bmatrix}$$
$$\mathbf{A}_8=\begin{bmatrix}0&1&0&0\\f_{1_{\max}}&0&f_{2_{\max}}&0\\0&0&0&1\\f_{2_{\max}}&0&f_{3_{\max}}&0\end{bmatrix},\quad \tilde{\mathbf{A}}_i=\mathbf{A}_i-\mathbf{A}_8$$
$$i=1,2,\ldots,7$$

$$w_i=\frac{\mu_{\mathrm{M}_1^i}(f_1)\times\mu_{\mathrm{M}_2^i}(f_2)}{\sum_{i=1}^{8}\left(\mu_{\mathrm{M}_1^i}(f_1)\times\mu_{\mathrm{M}_2^i}(f_2)\right)}$$
$$\mu_{\mathrm{M}_1^\beta}(f_1)=\frac{-f_1+f_{1_{\max}}}{f_{1_{\max}}-f_{1_{\min}}}\quad\text{for }\beta=1,2,3,4$$
$$\mu_{\mathrm{M}_1^\delta}(f_1)=1-\mu_{\mathrm{M}_1^1}(f_1)\text{ for }\delta=5,6,7,8$$
$$\mu_{\mathrm{M}_2^\varepsilon}(f_2)=\frac{-f_2+f_{2_{\max}}}{f_{2_{\max}}-f_{2_{\min}}}\quad\text{for }\varepsilon=1,2,5,6$$
$$\mu_{\mathrm{M}_2^\phi}(f_2)=1-\mu_{\mathrm{M}_2^1}(f_2)\text{ for }\phi=3,4,7,8$$
$$\mu_{\mathrm{M}_3^\eta}(f_3)=\frac{-f_3+f_{3_{\max}}}{f_{3_{\max}}-f_{3_{\min}}}\quad\text{for }\eta=1,3,5,7$$
$$\mu_{\mathrm{M}_3^\rho}(f_3)=1-\mu_{\mathrm{M}_3^1}(f_3)\quad\text{for }\rho=2,4,6,8$$

and the values of these membership functions are unknown.

Step II) A switching controller is designed based on (23) and (24) and is shown as follows:

$$\bar{\mathbf{u}}(t)=\sum_{k=1}^{8}v_k\mathbf{G}_k\mathbf{x}(t) \tag{36}$$

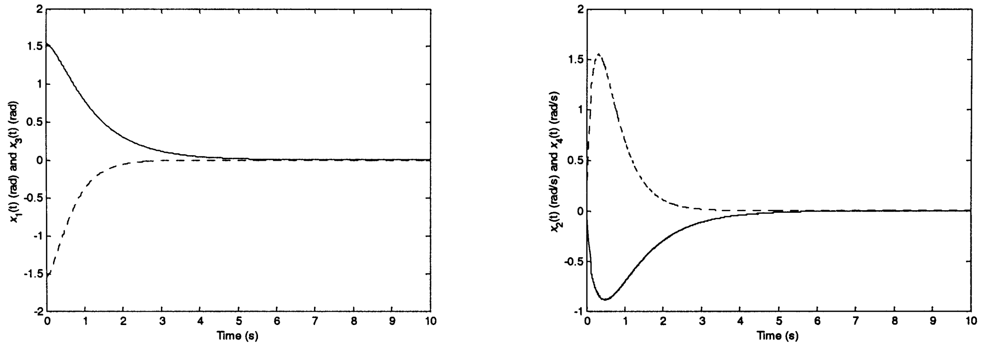
where v_k is determined based on (13), (22), and (23). The feedback gains are designed such that

$$\begin{aligned}\hat{\mathbf{H}}&=\hat{\mathbf{H}}_k\\&=\begin{bmatrix}0&1&0&0\\-3.7984&-4.3183&-1.8160&-0.7268\\0&0&0&1\\-1.8472&-0.7364&-7.2016&-5.6817\end{bmatrix}\end{aligned}$$
for all $k=1,2,\ldots,128$.

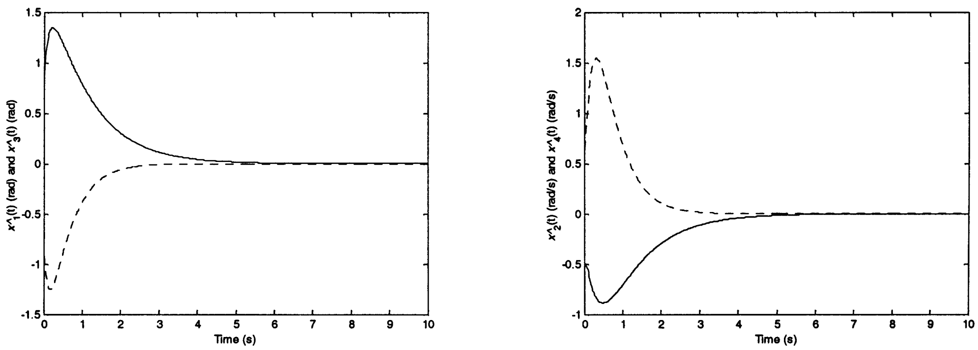
It is a stable matrix with eigenvalues at $-1,-2,-3$, and -4. As $\hat{\mathbf{H}}_k$ is the same for all k and is a stable matrix, it can be concluded that the switching control system is exponentially stable. Fig. 2 shows the responses of system states $(\mathbf{x})$ and the states of the switching plant model $(\hat{\mathbf{x}})$ the under the control of the switching controller with the initial conditions of

$$\mathbf{x}(0)=\begin{bmatrix}\frac{22\pi}{45}&0&-\frac{22\pi}{45}&0\end{bmatrix}^{\mathrm{T}}$$
$$\hat{\mathbf{x}}(0)=\begin{bmatrix}\frac{22\pi}{90}&-0.5&-\frac{22\pi}{90}&0.5\end{bmatrix}^{\mathrm{T}}.$$

It can be seen that the states $\hat{\mathbf{x}}$ of the switching plant model, with K chosen to be 10 arbitrarily, approaches the system state $\mathbf{x}$ in steady state. The control signals are shown in Fig. 3. It can be seen in this example that the system stability is not affected by the unknown system parameter, as the resultant

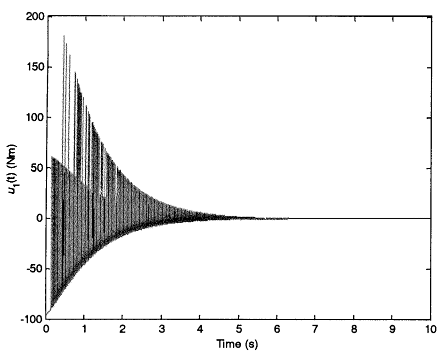


(a)

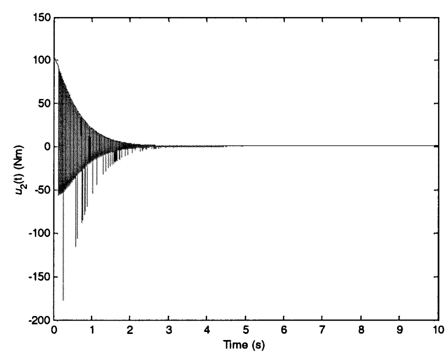


(b)

Fig. 2. Responses of the system states and the states of the switching plant model. (a) $x_1(t)$ (solid line) and $x_3(t)$ (dotted line). (b) $x_2(t)$ (solid line) and $x_4(t)$ (dotted line). (c) $\hat{x}_1(t)$ (solid line) and $\hat{x}_3(t)$ (dotted line). (d) $\hat{x}_2(t)$ (solid line) and $\hat{x}_4(t)$ (dotted line).



(a)



(b)

Fig. 3. Control signals of the system. (a) $u_1(t)$. (b) $u_2(t)$.

switching control system is equivalent to a linear system of $\dot{\mathbf{x}}(t) = \hat{\mathbf{H}}\mathbf{x}(t)$ when $\mathbf{m_e}(t)$ approaches zero as time $t \rightarrow \infty$.

VI. Conclusion

A switching plant model for nonlinear systems has been proposed in this paper. The switching plant model is equivalent to the TSK fuzzy plant model as time $t \rightarrow \infty$. Based on this switching plant model, a switching controller has been proposed to control the nonlinear system. The resultant closed-loop system becomes a linear control system when the modeling error becomes zero. The design procedure of the switching controller has been given. An application example on stabilizing a two-inverted pendulum system using the proposed switching controller has been presented.

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