State-Feedback Control of Fuzzy Discrete-Event Systems

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Abstract

In a 2002 paper, we combined fuzzy logic with discrete-event systems (DESs) and established an automaton model of fuzzy DESs (FDESs). The model can effectively represent deterministic uncertainties and vagueness, as well as human subjective observation and judgment inherent to many real-world problems, particularly those in biomedicine. We also investigated optimal control of FDESs and applied the results to optimize HIV/AIDS treatments for individual patients. Since then, other researchers have investigated supervisory control problems in FDESs, and several results have been obtained. These results are mostly derived by extending the traditional supervisory control of (crisp) DESs, which are string based. In this paper, we develop state-feedback control of FDESs that is different from the supervisory control extensions. We use state space to describe the system behaviors and use state feedback in control. Both disablement and enforcement are allowed. Furthermore, we study controllability based on the state space and prove that a controller exists if and only if the controlled system behavior is (state-based) controllable. We discuss various properties of the state-based controllability. Aside from novelty, the proposed new framework has the advantages of being able to address a wide range of practical problems that cannot be effectively dealt with by existing approaches. We use the diabetes treatment as an example to illustrate some key aspects of our theoretical results.

Index Terms—

Controllability; discrete-event systems; fuzzy logic; state-feedback control

I. INTRODUCTION

One of the rapidly developing areas in systems and control is that of discrete-event systems (DESs). In a DES, states are logical or symbolic rather than numerical, and events describe significant changes in the system. The supervisory control theory of DES addresses the issues of modeling, control, and optimization of DES [4], [9], [13]. It has been applied to practical engineering systems. In traditional supervisory control theory, both states and events are crisp. Although states and events are crisp in many engineering applications, more likely than not, this is not the case in biomedical systems in which the system's state (e.g., a patient's health status) is always somewhat uncertain and vague, even in a deterministic sense. Subjective observation, judgment, and interpretation by a health-care provider or a patient invariably play a significant role in describing the status of the patient and are rarely crisp. For instance, it is vague when a patient's health is said to be "good." Furthermore, the transition from one state to another is also vague. It is hard to describe how exactly a patient's condition has changed from "good" to "bad."

Vagueness, imprecision, uncertainty, and subjectivity are the norm for most of the complex biomedical systems in existence (e.g., the way that a doctor describes the diagnosis, treatment, and prognosis of a disease). Fuzzy sets and fuzzy logic are tools to handle them in practice [7], [14], [17] because they can quantitatively and effectively represent and compute deterministic uncertainty, as well as subjective and qualitative concepts. A fuzzy set is an extension of a classical set, which allows a partial membership. Representation and calculation in fuzzy set theory are mathematically precise. They have been widely used in various applications.

The first attempt in recent years to introduce fuzzy logic into DESs appeared in our paper [10]. In that paper, we generalized the conventional (crisp) finite-automaton model to fuzzy finite-automaton model. We consider two mechanisms for control of fuzzy DESs (FDESs): disablement and enforcement. The optimal control objective for FDESs is to maximize their effectiveness and minimize their cost at the same time. An online scheme is used in [10] to design a controller that achieves the optimal control objective. Observability of FDESs is also investigated in [10]. Recently, we applied optimal control of FDESs to HIV/AIDS treatments and obtained some promising results [15], [16].

Since our work was published, other researchers have investigated control problems in FDESs, and several results have been obtained. In [12], the author deals with supervisory control of FDESs. Fuzzy languages are used for modeling FDESs. The paper [2] investigates supervisory control of FDESs under partial observation. Observability, normality, and co-observability of crisp languages are extended to fuzzy languages. The work [3] investigates the state-based control of FDESs.

The approach used in this paper is different from those in [1], [2], and [12] because we use state spaces rather than fuzzy languages to describe the system behaviors. State spaces provide a better alternative because, unlike crisp DESs whose state spaces are usually finite, the state spaces of FDESs are usually infinite. Therefore, it will be more efficient to directly deal with it using state spaces rather than indirectly using fuzzy languages. This also allows us to use state feedback in supervisory control. We introduce state-based controllability, which is not based on languages but on state spaces. We show that such a definition is more intuitive, more natural, and, hence, more suitable for solving a range of practical problems. We prove that a controller exists if and only if the desired system behavior is state-based controllable. We also discuss various properties of this state-based controllability, including the existence of the supremal controllable subset of a given state space. Although state spaces are also used in [3] to control FDESs, our control mechanism is different because we allow both disablement and enforcement, while [3] uses only disablement. Furthermore, our definition of state-based controllability is completely different from that of [3]. The controllability defined in [3] is not suitable for state-feedback control.

The rest of this paper is organized as follows. In Section II, we will briefly review the background for FDESs modeled by fuzzy automata. We will illustrate the underlying idea by relating it to diabetes treatment. In Section III, we will introduce control mechanisms and state feedback, specify control objectives, and derive an existence condition for control. This condition is related to state-based controllability. When state-based controllability is not satisfied, we show how to modify the control objective to achieve the supremal controllable subset. The diabetes treatment problem will be further studied in Section IV as an example to illustrate the main theoretical results.

II. Brief Introduction to FDESs

Let us introduce FDESs by considering the following example of diabetes treatment. Diabetes is a disease characterized by a high level of blood glucose resulting from defects in

insulin production, insulin action, or both. There are three types of diabetes. We will focus on type-2 diabetes in this paper because it accounts for about 90%–95% of all diagnosed cases. The following four states can be defined to classify a person.

1) Normal (NM): The person does not have diabetes. His/her fasting blood sugar level is less than 100 mg/dL after an overnight fasting.

2) Prediabetes (P): The person is at an increased risk of developing diabetes. The person's fasting blood sugar level is elevated to 100–125 mg/dL.

3) Diabetes without complications (NC): The clinic definition is that the person's fasting blood sugar level is greater than 125 mg/dL after an overnight fasting.

4) Diabetes with complications (C): The person has not only diabetes but also complications associated with the disease, such as blindness, kidney damage, and lower limb amputations.

Clearly, it is best to describe a person's condition by fuzzy sets that are capable of effectively and accurately characterizing the extent of the condition. For example, if a person's blood sugar level varies between 90 and 105 mg/dL after an overnight fasting, is the person NM or P? Using our FDES model, the person can be in more than one of the four states partially and simultaneously. More specifically, the person's state can be described by a fuzzy state vector, such as $q = [0.6\ 5.0\ 0\ 0]$, which means that the person is in "NM" state with membership 0.6 and in "P" state with membership 0.5 (the membership for the remaining two states is 0). Typical membership functions for fuzzy sets (e.g., triangular or Gaussian) can be used to derive these memberships from the blood sugar level. Specific shapes of membership functions, however, are not needed, and thus not assumed, for the theoretical development in this paper.

An event is represented by a matrix. For example, the event of diabetes progression could be captured by the following matrix:

$$\alpha = \begin{array}{cccc} \text{NM} & \text{P} & \text{NC} & \text{C} \\ \begin{bmatrix} 0.9 & 0.3 & 0.1 & 0 \\ 0 & 0.8 & 0.5 & 0.2 \\ 0 & 0 & 0.7 & 0.3 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} \text{NM} \\ \text{P} \\ \text{NC} \\ \text{C} \end{array} \end{array}.$$

The interpretation is that, for instance, the possibility of transferring from state P to state NM is 0.3, while that from NC to NM is 0.1. Generally speaking, the numerical elements of an event matrix could be obtained from clinical knowledge (see the derivations of similar matrices in the HIV/AIDS treatment problem discussed in [15] and [16] as an example).

In summary, an FDES can be represented by a fuzzy automaton [6], [10]

$$G = (Q, \Sigma, \delta, q_o). \quad (1)$$

The elements of G are as follows.

1) $Q = [0, 1]^n$ is the state space, where n defines the dimension of the state space. A state is represented by an n-dimensional row vector.

2) $q_o \in Q$ is the initial state. At the beginning, G is in its initial states.

3) Σ is a set of events. An event is represented by an $n \times n$ matrix.

4) $\delta : Q \times \Sigma \to Q$ is the transition function from one state to another after the occurrence of an event, i.e.,

$$\delta\left(q,\sigma\right) = \begin{cases} q \circ \sigma, & \text{if } q \in Q_\sigma \\ \text{undefined}, & \text{otherwise} \end{cases} \quad (2)$$

where $\circ$ denotes a fuzzy reasoning operator and $Q_\sigma \subseteq Q$ is the set of states where event σ is defined. For example, we can let $\circ$ be the max-product or max-min operator and $Q_\sigma = Q$ (i.e., σ is defined in all the states). The theoretical results in this paper, including all the theorems, are true for any fuzzy reasoning operator $\circ$ (including the max-min and max-product mentioned earlier but not limited to them), and any $Q_\sigma \subseteq Q$.

Let us denote the set of all sequences of events over Σ as Σ^*. We say that $t \in \Sigma^*$ is a prefix of $s \in \Sigma^*$, denoted by $t \quad s$, if $(\exists u \in \Sigma^*)tu = s$. We can extend the transition function to $\delta : Q \times \Sigma^* \to Q$ recursively as follows:

$$\delta\left(q_o, s\sigma\right) = \delta\left(\delta\left(q_o, s\right), \sigma\right). \quad (3)$$

Given an FDES G, we denote $Q_{\text{ac}}(q_o, \delta)$, the set of states accessible from q_o under δ, as

$$Q_{\text{ac}}\left(q_o, \delta\right) = \left\{\delta\left(q_o, s\right) : s \in \Sigma^*\right\}. \quad (4)$$

With the aforementioned model of the FDES, we can now study its state-feedback control.

III. State-Feedback Control of Fdes

In this section, we study state-feedback control of FDES. We use the following two control mechanisms [8].

1) Disablement: The events in $\Sigma_c \subseteq \Sigma$ can be disabled by a controller. These events, $\sigma \in \Sigma_c$, are called controllable events.

2) Enforcement: The events in $\Sigma_f \subseteq \Sigma$ can be enforced by a controller. These events, $\sigma \in \Sigma_f$, are called enforceable events.

Note that no restriction is imposed on Σ_c and Σ_f. Hence, an event may be controllable, enforceable, both, or neither. We assume that the control is "crisp" and deterministic, i.e., if an event is enforced (e.g., a treatment is applied to a disease), then it will definitely occur. On the other hand, if it is disabled, then it will definitely not occur.

To formally define state-feedback control, we first define a set of feasible control patterns as follows [8], [11]:

$$\Gamma = \left\{\gamma \subseteq \Sigma : \left(\Sigma - \Sigma_c \subseteq \gamma\right) \vee \left(\gamma \subseteq \Sigma_f\right)\right\}. \quad (5)$$

This set of control patterns allows two types of control: 1) disabling some controllable events (i.e., those in $\Sigma - \gamma$) if the first disjunction is satisfied and 2) enforcing some enforceable events (i.e., those in γ) if the second disjunction is met. The semantics of the controlled patterns should be interpreted as follows; If $\gamma \subseteq \Sigma_f$, then one of the events in γ is enforced; otherwise, only the events in γ are enabled. Note that $\gamma \not\subseteq \Sigma_f$ implies that $\Sigma - \Sigma_c \subseteq \gamma$. Now, a state-feedback control is defined as a mapping

$$\varphi : Q \to \Gamma \quad (6)$$

where $\phi(q)$ denotes the events to be enforced or enabled. Under this control, the controlled system (or the corresponding closed-loop system) is more restricted than the uncontrolled system G because the controller will enable or enforce only some of the events and/or disable some other events. The controlled system is denoted by

$$\varphi/G = (Q, \Sigma, \varphi/\delta, q_o) \quad (7)$$

where state space Q, event set Σ, and initial state q_o are the same as in G, and the transition function is modified as

$$(\varphi/\delta)(q, \sigma) = \begin{cases} \delta(q, \sigma), & \text{if } \sigma \in \varphi(q) \\ \text{undefined}, & \text{otherwise.} \end{cases} \quad (8)$$

Let us denote $Q_{\text{ac}}(q_o, \phi/\delta)$, the set of states accessible from q_o under control ϕ, as

$$Q_{\text{ac}}(q_o, \varphi/\delta) = \{(\varphi/\delta)(q_o, s) : s \in \Sigma^*\} \quad (9)$$

where $(\phi/\delta)(q_o, s)$ is defined similarly to $\delta(q_o, s)$ before.

The control objective of a controller may vary. The objective that we consider is safety. We assume that there is a set of states that we do not wish the system to visit. We call these states "illegal states" and denote them by $Q_b \subset Q$. We use disablement and enforcement to ensure that the controlled system never enters these illegal states. We want to design a state-feedback controller ϕ such that

$$Q_{\text{ac}}(q_o, \varphi/\delta) \subseteq Q - Q_b. \quad (10)$$

We call such a controller the "legal controller." A legal controller ϕ is said to be "least restrictive" if, for any other controller ϕ'

$$\begin{aligned} & Q_{\text{ac}}\left(q_o, \varphi'/\delta\right) \subseteq Q - Q_b \\ \Rightarrow \ & Q_{\text{ac}}\left(q_o, \varphi'/\delta\right) \subseteq Q_{\text{ac}}(q_o, \varphi/\delta). \end{aligned} \quad (11)$$

In other words, the least restrictive legal controller allows for more behavior than any other legal controller. Obviously, the best solution is to find a controller ϕ such that $Q_{\text{ac}}(q_o, \phi/\delta) = Q - Q_b$. But is this possible? More generally, we want to answer the following existence question: Given a subset of state space $X \subseteq Q_{\text{ac}}(q_o, \delta)$, can one find a controller ϕ such that

$$Q_{\text{ac}}(q_o, \varphi/\delta) = X? \quad (12)$$

To answer this question, we first need to define the closed and controllable subspaces.

1) *Definition of Prefix Closure:* A subset of state space $X \subseteq Q_{\text{ac}}(q_o, \delta)$ is prefix closed (or closed for short) with respect to system G if

$$(\forall x \in X)(\exists s \in \Sigma^*) \, x \in \delta(q_o, s) \wedge (\forall t \leq s)\, \delta(q_o, t) \in X. \quad (13)$$

2) *Definition of State-Based Controllability:* A closed subset of state space $X \subseteq Q_{\text{ac}}(q_o, \delta)$ is state-based controllable (or controllable for short) with respect to system G and control pattern Γ if

$$\left(\forall x \in X\right)\left(\exists \gamma \in \Gamma\right)\left(\underset{\sigma \in \Sigma}{\cup} \delta\left(x, \sigma\right)\right) \cap X = \underset{\sigma \in \gamma}{\cup} \delta\left(x, \sigma\right). \quad (14)$$

The following theorem states that controllability is a necessary and sufficient condition for the existence of a state-feedback control.

Theorem 1: Given a subset of state space $X \subseteq Q_{\mathrm{ac}}(q_o, \delta)$, there exists a state-feedback controller ϕ such that

$$Q_{\mathrm{ac}}\left(q_o, \varphi/\delta\right) = X \quad (15)$$

if and only if X is closed and controllable with respect to G and Γ.

Proof: We first prove the "if" part. If X is closed and controllable with respect to G and Γ, then, for any $q \in X$, we can find at least one $\gamma \in \Gamma$ such that $(\cup_{\sigma \in \Sigma}\delta(q, \sigma)) \cap X = \cup_{\sigma \in \gamma} \delta(q, \sigma)$. Let such a γ be the feedback control at q, i.e., let $\phi(q) = \gamma$. Also, for any $q \in Q - X$, let $\phi(q) = \Sigma - \Sigma_c$ (i.e., enabling all uncontrollable events). We want to show that, for such a state-feedback controller ϕ, for all $x \in Q$

$$x \in Q_{\mathrm{ac}}\left(q_o, \varphi/\delta\right) \iff x \in X.$$

To this end, let us define

$$Q_{\mathrm{ac}}^n\left(q_o, \varphi/\delta\right) = \left\{\left(\varphi/\delta\right)\left(q_o, s\right) : s \in \Sigma^n\right\}.$$

Clearly

$$Q_{\mathrm{ac}}^0\left(q_o, \varphi/\delta\right) = \{q_o\}$$

$$Q_{\mathrm{ac}}^{n+1}\left(q_o, \varphi/\delta\right) = \underset{\sigma \in \Sigma}{\cup}\left(\varphi/\delta\right)\left(Q_{\mathrm{ac}}^n\left(q_o, \delta\right), \sigma\right)$$

$$Q_{\mathrm{ac}}\left(q_o, \varphi/\delta\right) = \overset{\infty}{\underset{n=0}{\cup}} Q_{\mathrm{ac}}^n\left(q_o, \varphi/\delta\right).$$

Similarly, we can define $Q_{\mathrm{ac}}^n\left(q_o, \delta\right)$. We first prove by induction on n that, for all $x \in Q$

$$x \in Q_{\mathrm{ac}}^n\left(q_o, \varphi/\delta\right) \iff x \in X \cap Q_{\mathrm{ac}}^n\left(q_o, \delta\right).$$

    Base:

$$\text{Since } Q_{\mathrm{ac}}^0\left(q_o, \varphi/\delta\right) = \{q_o\} \text{ and } q_o \in Q_{\mathrm{ac}}^0\left(q_o, \delta\right) \subseteq X$$

$$x \in Q^0_{\mathrm{ac}}(q_o, \varphi/\delta) \iff x{=}q_o \iff x \in Q^0_{\mathrm{ac}}(q_o, \delta) \cap X.$$

Induction hypothesis:

Assume that, for all $x \in Q$

$$x \in Q^n_{\mathrm{ac}}(q_o, \varphi/\delta) \iff x \in X \cap Q^n_{\mathrm{ac}}(q_o, \delta).$$

Induction step:

For all $y \in Q$

$$y \in Q^{n+1}_{\mathrm{ac}}(q_o, \varphi/\delta)$$

$$\iff y \in \underset{\sigma \in \Sigma}{\cup}(\varphi/\delta)\left(Q^n_{\mathrm{ac}}(q_o, \varphi/\delta), \sigma\right)$$

$$\iff \left(\exists x \in Q^n_{\mathrm{ac}}(q_o, \varphi/\delta)\right)\left(\exists \sigma \in \Sigma\right) y{=}(\varphi/\delta)(x, \sigma)$$

$$\iff \left(\exists x \in X \cap Q^n_{\mathrm{ac}}(q_o, \delta)\right)\left(\exists \sigma \in \Sigma\right) y{=}(\varphi/\delta)(x, \sigma)$$

$$\iff \left(\exists x \in X \cap Q^n_{\mathrm{ac}}(q_o, \delta)\right)\left(\exists \sigma \in \Sigma\right) y{=}\delta(x, \sigma) \wedge \sigma \in \varphi(x)$$

$$\iff \left(\exists x \in X \cap Q^n_{\mathrm{ac}}(q_o, \delta)\right)\left(\exists \sigma \in \Sigma\right) y{=}\delta(x, \sigma) \wedge \sigma \in \gamma \wedge \left(\underset{\sigma \in \Sigma}{\cup}\delta(x, \sigma)\right) \cap X{=}\underset{\sigma \in \gamma}{\cup}\delta(x, \sigma)$$

$$\iff \left(\exists x \in X \cap Q^n_{\mathrm{ac}}(q_o, \delta)\right)\left(\exists \sigma \in \gamma\right) y{=}\delta(x, \sigma) \wedge \left(\underset{\sigma \in \Sigma}{\cup}\delta(x, \sigma)\right) \cap X{=}\underset{\sigma \in \gamma}{\cup}\delta(x, \sigma)$$

$$\iff \left(\exists x \in X \cap Q^n_{\mathrm{ac}}(q_o, \delta)\ y \in \underset{\sigma \in \gamma}{\cup}\delta(x, \sigma) \wedge \left(\underset{\sigma \in \Sigma}{\cup}\delta(x, \sigma)\right) \cap X{=}\underset{\sigma \in \gamma}{\cup}\delta(x, \sigma)\right)$$

$$\iff \left(\exists x \in X \cap Q^n_{\mathrm{ac}}(q_o, \delta)\right) y \in \left(\underset{\sigma \in \Sigma}{\cup}\delta(x, \sigma)\right) \cap X \wedge \left(\underset{\sigma \in \Sigma}{\cup}\delta(x, \sigma)\right) \cap X{=}\underset{\sigma \in \gamma}{\cup}\delta(x, \sigma)$$

$$\Longleftrightarrow \; (\exists x \in X \cap Q_{\mathrm{ac}}^{n}(q_o, \delta)) \, y \in \left(\underset{\sigma \in \Sigma}{\cup} \delta(x, \sigma) \right) \cap X$$

$$\Longleftrightarrow \; y \in X \cap Q_{\mathrm{ac}}^{n+1}(q_o, \delta).$$

Therefore, we have shown that

$$Q_{\mathrm{ac}}^{n}(q_o, \varphi/\delta) = X \cap Q_{\mathrm{ac}}^{n}(q_o, \delta).$$

Taking the union on both sides, we have

$$\begin{aligned}
\underset{n=0}{\overset{\infty}{\cup}} Q_{\mathrm{ac}}^{n}(q_o, \varphi/\delta) &= \underset{n=0}{\overset{\infty}{\cup}} (X \cap Q_{\mathrm{ac}}^{n}(q_o, \delta)) \\
&= X \cap \left(\underset{n=0}{\overset{\infty}{\cup}} Q_{\mathrm{ac}}^{n}(q_o, \delta) \right)
\end{aligned}$$

$$\Longleftrightarrow \; Q_{\mathrm{ac}}(q_o, \varphi/\delta) = X \cap Q_{\mathrm{ac}}(q_o, \delta) = X.$$

This proves the "if" part.

We now prove the "only if" part. Assume that there exists a state-feedback controller ϕ such that $Q_{\mathrm{ac}}(q_o, \phi/\delta) = X$. We need to show the following: 1) X is closed, and 2) X is controllable with respect to G and Γ.

1) X is closed because, by definition, $Q_{\mathrm{ac}}(q_o, \phi/\delta)$ is closed.

2)   To show that X is controllable, for any $x \in X$, let $\gamma = \phi(x)$, and we prove by induction on n that

$$(\forall x \in X \cap Q_{\mathrm{ac}}^{n}(q_o, \delta)) \left(\underset{\sigma \in \Sigma}{\cup} \delta(x, \sigma) \right) \cap X = \underset{\sigma \in \Sigma(x)}{\cup} \delta(x, \sigma).$$

Base:

Since $Q_{\mathrm{ac}}^{0}(q_o, \varphi/\delta) = \{q_o\}$, for $x \in X \cap Q_{\mathrm{ac}}^{0}(q_o, \delta) = \{q_o\}$

$$y \in \left(\underset{\sigma \in \Sigma}{\cup} \delta(x, \sigma) \right) \cap X$$

$$\Longleftrightarrow \; y \in \left(\underset{\sigma \in \Sigma}{\cup} \delta(q_o, \sigma) \right) \cap X$$

$$\Longleftrightarrow \ y \in X \wedge (\exists \sigma \in \Sigma)\, y{=}\delta\,(q_o, \sigma)$$

$$\Longleftrightarrow \ (\exists \sigma \in \Sigma)\, y{=}\delta\,(q_{o,\sigma}) \wedge \sigma \in \varphi\,(q_o)$$

$$\Longleftrightarrow \ (\exists \sigma \in \varphi\,(q_o))\, y{=}\delta\,(q_o, \sigma)$$

$$\Longleftrightarrow \ y \in \underset{\sigma \in \varphi(q_o)}{\cup} \delta\,(q_o, \sigma)$$

$$\Longleftrightarrow \ y \in \underset{\sigma \in \varphi(x)}{\cup} \delta\,(q_o, \sigma)\,.$$

Induction hypothesis:

Assume that

$$(\forall x \in X \cap Q_{\mathrm{ac}}^n\,(q_o, \delta)) \left(\underset{\sigma \in \Sigma}{\cup} \delta\,(x, \sigma) \right) \cap X {=} \underset{\sigma \in \varphi(x)}{\cup} \delta\,(x, \sigma)\,.$$

Induction step:

For all $x \in X \cap Q_{\mathrm{ac}}^n\,(q_o, \delta)$

$$y \in \left(\underset{\sigma \in \Sigma}{\cup} \delta\,(x, \sigma) \right) \cap X$$

$$\Longleftrightarrow \ y \in X \wedge (\exists \sigma \in \Sigma)\, y{=}\delta\,(x, \sigma)$$

$$\Longleftrightarrow \ (\exists \sigma \in \Sigma)\, y{=}\delta\,(x, \sigma) \wedge \sigma \in \varphi\,(x)$$

$$\Longleftrightarrow \ (\exists \sigma \in \varphi\,(x))\, y{=}\delta\,(x, \sigma)$$

$$\Longleftrightarrow \ y \in \underset{\sigma \in \varphi(x)}{\cup} \delta\,(x, \sigma)\,.$$

Because $(\forall x \in X \cap Q_{\mathrm{ac}}^n\,(q_o, \delta))\,(\cup_{\sigma \in \Sigma}\delta\,(x, \sigma)) \cap X {=} \cup_{\sigma \in \varphi(x)}\delta\,(x, \sigma)$ is true for all n

$$(\forall x \in X)\,(\exists \gamma{=}\varphi\,(x) \in \Gamma)\left(\underset{\sigma \in \Sigma}{\cup}\delta\,(x, \sigma)\right) \cap X = \underset{\sigma \in \gamma}{\cup}\delta\,(x, \sigma).$$

That is, X is controllable with respect to G and Γ. Q.E.D.

Since prefix closure and state-based controllability are necessary and sufficient for the existence of feedback control, if X is not closed and controllable, then we need to find the largest subset of X that is closed and controllable. To this end, we present the following result.

Theorem 2: If two subsets of state space, i.e., $X_1 \subseteq Q_{ac}(q_o,\delta)$ and $X_2 \subseteq Q_{ac}(q_o,\delta)$, are closed with respect to G, then their union, i.e., $X_1 \cup X_2 \subseteq Q_{ac}(q_o,\delta)$, is also closed with respect to G.

Proof: Since $X_1 \subseteq Q_{ac}(q_o,\delta)$ and $X_2 \subseteq Q_{ac}(q_o,\delta)$ are closed with respect to system G

$$(\forall x \in X_1)\,(\exists s_1 \in \Sigma^*)\,x \in \delta\,(q_o, s_1) \wedge (\forall t_1 \le s_1)\,\delta\,(q_o, t_1) \in X_1$$

$$(\forall x \in X_2)\,(\exists s_2 \in \Sigma^*)\,x \in \delta\,(q_o, s_2) \wedge (\forall t \le s_2)\,\delta\,(q_o, t) \in X_2.$$

Now, for $x \in X_1 \cup X_2$, either $x \in X_1$ or $x \in X_2$. If $x \in X_1$, then

$$(\exists s{=}s_1 \in \Sigma^*)\,x \in \delta\,(q_o, s) \wedge (\forall t \le s)\,\delta\,(q_o, t) \in X_1$$

$$\Longleftrightarrow (\exists s{=}s_1 \in \Sigma^*)\,x \in \delta\,(q_o, s) \wedge (\forall t \le s)\,\delta\,(q_o, t) \in X_1 \cup X_2.$$

If $x \in X_2$, then

$$(\exists s{=}s_2 \in \Sigma^*)\,x \in \delta\,(q_o, s) \wedge (\forall t \le s)\,\delta\,(q_o, t) \in X_1$$

$$\Longleftrightarrow (\exists s{=}s_2 \in \Sigma^*)\,x \in \delta\,(q_o, s) \wedge (\forall t \le s)\,\delta\,(q_o, t) \in X_1 \cup X_2.$$

Therefore

$$(\forall x \in X_1 \cup X_2)\,(\exists s \in \Sigma^*)\,x \in \delta\,(q_o, s) \wedge (\forall t \le s)\,\delta\,(q_o, t) \in X_1 \cup X_2$$

i.e., $X_1 \cup X_2$ is closed with respect to system G. Q.E.D.

By Theorem 2, for a subset of state space $X \subseteq Q_{ac}(Q_o,\delta)$, the following set of all closed subsets of X is closed under union:

$$P\,(X) = \{Y \subseteq X : Y \text{ is closed w.r.t plant } G\}. \quad (16)$$

Therefore, we can find its supremal element, denoted by $\overline{X}$, as

$$\overline{X} = \underset{Y \in P(X)}{\cup} Y. \quad (17)$$

$\overline{X}$ is called the supremal closed subsets of X.

Theorem 3: If two subsets of state space, namely, $X_1 \subseteq Q_{ac}(q_o,\delta)$ and $X_2 \subseteq Q_{ac}(q_o,\delta)$, are controllable with respect to G Γ, and Γ and is closed under union (i.e., $\gamma_1, \gamma_2 \in \Gamma$ implies that $\gamma_1 \cup \gamma_2 \in \Gamma$), then their union $X_1 \cup X_2 Q_{ac}(q_o, \delta)$ is also controllable controllable with respect to G and Γ.

Proof: Since $X_1 \subseteq Q_{ac}(q_o,\delta)$ and $X_2 \subseteq Q_{ac}(q_o,\delta)$ are controllable with respect to system G and control pattern Γ

$$(\forall x \in X_1)(\exists \gamma_1 \in \Gamma)\left(\underset{\sigma \in \Sigma}{\cup} \delta(x,\sigma)\right) \cap X_1 = \underset{\sigma \in \gamma_1}{\cup} \delta(x,\sigma)$$

$$(\forall x \in X_2)(\exists \gamma_2 \in \Gamma)\left(\underset{\sigma \in \Sigma}{\cup} \delta(x,\sigma)\right) \cap X_2 = \underset{\sigma \in \gamma_2}{\cup} \delta(x,\sigma).$$

Now, for $x \in X_1 \cup X_2$, let $\gamma = \gamma_1 \cup \gamma_2 \in \Gamma$, and then

$$\left(\underset{\sigma \in \Sigma}{\cup} \delta(x,\sigma)\right) \cap X_1 = \underset{\sigma \in \gamma_1}{\cup} \delta(x,\sigma) \wedge \left(\underset{\sigma \in \Sigma}{\cup} \delta(x,\sigma)\right) \cap X_2 = \underset{\sigma \in \gamma_2}{\cup} \delta(x,\sigma)$$

$$\Rightarrow \left(\left(\underset{\sigma \in \Sigma}{\cup} \delta(x,\sigma)\right) \cap X_1\right) \cup \left(\left(\underset{\sigma \in \Sigma}{\cup} \delta(x,\sigma)\right) \cap X_2\right) = \left(\underset{\sigma \in \gamma_1}{\cup} \delta(x,\sigma)\right) \cup \left(\underset{\sigma \in \gamma_2}{\cup} \delta(x,\sigma)\right)$$

$$\Rightarrow \left(\underset{\sigma \in \Sigma}{\cup} \delta(x,\sigma)\right) \cap (X_1 \cup X_2) = \underset{\delta \in \gamma_1 \cup \gamma_2}{\cup} \delta(x,\sigma)$$

$$\Rightarrow \left(\underset{\sigma \in \Sigma}{\cup} \delta(x,\sigma)\right) \cap (X_1 \cup X_2) = \underset{\sigma \in \gamma}{\cup} \delta(x,\sigma).$$

Therefore

$$(\forall x \in X_1 \cup X_2)(\exists \gamma \in \Gamma)\left(\underset{\sigma \in \Sigma}{\cup} \delta(x,\sigma)\right) \cap (X_1 \cup X_2) = \underset{\sigma \in \gamma}{\cup} \delta(x,\sigma).$$

That is, $X_1 \cup X_2$ is controllable with respect to system G and control pattern Γ. Q.E.D.

Based on Theorem 3, for a subset of state space $X \subseteq Q_{ac}(Q_o,\delta)$, the following set of all controllable subsets of X is closed under union:

$$C(X) = \{Y \subseteq X : Y \text{ is controllable w.r.t plant } G \text{ and control pattern } \Gamma\}.$$

Consequently, we can find its supremal element, denoted by $X^{\uparrow}$, as

$$X^{\uparrow} = \bigcup_{Y \in C(X)} Y.$$

$X^{\uparrow}$ is called the supremal controllable subsets of *X*.

IV. Example—Diabetes Treatment

We now illustrate how the theoretical results in Section III can be applied to practical problems by continuing the example of diabetes treatment in Section II. As shown, the patient's state can be described in general by a fuzzy state vector

$$q = [\mu_{\mathrm{NM}} \ \mu_P \ \mu_{\mathrm{NC}} \ \mu_C] \quad (18)$$

where μ_{NM}, μ_P, μ_{NC}, and μ_C are the possibilities or membership grades of being at "NM," "P," "NC," and "C" states, respectively. An event is represented by

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{bmatrix} \quad (19)$$

where element σ_{ij} is the possibility of the patient changing from state *i* to state *j* after the occurrence of event σ. To describe the progression and treatment of diabetes, we define the following six events.

1) Event α represents disease progression, i.e., how diabetes progresses from state NM to state C via states P and NC. The numerical elements of the matrix should be determined based on domain expert knowledge. For illustration purposes, supposedly, α is the following matrix:

$$\alpha = \begin{bmatrix} 0.9 & 0.3 & 0.1 & 0 \\ 0 & 0.8 & 0.5 & 0.2 \\ 0 & 0 & 0.7 & 0.3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2) If the disease is relatively mild, it may be treated by diet, exercise, and weight loss. Event β represents treatment by diet. The matrix for β is given by

$$\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0.8 & 0 & 0 \\ 0.1 & 0.4 & 0.7 & 0 \\ 0 & 0.1 & 0.2 & 0.9 \end{bmatrix}.$$

3) Event ρ, representing treatment by exercise, is

$$\rho = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 \\ 0.2 & 0.4 & 0.7 & 0 \\ 0 & 0.1 & 0.3 & 0.8 \end{bmatrix}.$$

4)   Event λ represents treatment by weight loss, which is

$$\lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0.1 & 0.1 & 0.6 & 0 \\ 0 & 0.1 & 0.3 & 0.7 \end{bmatrix}.$$

5)   If diet, exercise, and weight loss cannot effectively control the disease, then taking oral medication may become necessary. The event for taking oral medication is denoted by θ:

$$\theta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.1 & 0.4 & 0.5 \end{bmatrix}.$$

6)   If taking oral medication still cannot effectively govern the blood sugar level, then insulin may have to be used in order to prevent patients from developing complications. This event is represented by η with the following matrix:

$$\eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0.9 & 0.2 \end{bmatrix}.$$

The controllable and enforceable events can be defined as follows. Event α (disease progression) is neither controllable nor enforceable, i.e., $\alpha \notin \Sigma_c$ and $\alpha \notin \Sigma_f$. Diet, exercise, and weight loss are controllable events because they can be disabled, but they are not enforceable because it may be challenging for a patient to control diet, exercise regularly, and lose weight. Therefore, $\beta, \rho, \lambda \in \Sigma_c$ and $\beta, \rho, \lambda \notin \Sigma_f$. Using oral medication and insulin are both controllable and enforceable. Hence, $\theta, \eta \in \Sigma_c$ and $\theta, \eta \in \Sigma_f$. In summary

$$\Sigma_c = \{\beta, \rho, \lambda, \theta, \eta\} \quad \Sigma_f = \{\theta, \eta\}. \quad (20)$$

Supposedly, our control objective is to make sure that the state of the patient will always satisfy the following condition: The membership of being in the state of C is less than 0.5. In other words, we define the illegal state space as

$$Q_b = \left\{ [\mu_{\mathrm{NM}} \ \mu_P \ \mu_{\mathrm{NC}} \ \mu_C] \in [0,1]^4 : \mu_C \geq 0.5 \right\}. \quad (21)$$

To achieve this objective, let us define the subset of the state space to be achieved by the state-feedback control as

$$X = \overline{Q_{ac}(q_o, \delta)} \cap (Q - Q_b) \quad (22)$$

i.e., X is the closed subset of the accessible state space that is not in Q_b. Since X is infinite, we cannot express X explicitly. However, we can still design a state-feedback control to achieve $X^\uparrow$.

To achieve $X^\uparrow$, we check, for $x = [\mu_{NM} \ \mu_P \ \mu_{NC} \ \mu_C] \in X$, whether or not $\delta(x, \sigma) \in X$. Using the max-product reasoning, event α

$$
\begin{aligned}
\delta(x, \alpha) \quad &= [\mu_{NM} \ \mu_P \ \mu_{NC} \ \mu_C] \circ
\begin{bmatrix}
0.9 & 0.3 & 0.1 & 0 \\
0 & 0.8 & 0.5 & 0.2 \\
0 & 0 & 0.7 & 0.3 \\
0 & 0 & 0 & 1
\end{bmatrix} \\
&= [\max(0.9\mu_{NM}, 0, 0, 0) \quad \max(0.8\mu_{NM}, 0.3\mu_P, 0, 0) \\
&\qquad \max(0.1\mu_{NM}, 0.9\mu_P, 0.2\mu_{NC}, 0) \\
&\qquad \max(0, 0.2\mu_P, 0.9\mu_{NC}, \mu_C)].
\end{aligned}
$$

One can see that $x \in X$ does not imply $\delta(x, \sigma) \in X$. For example, by letting $x = [0 \ 0.4 \ 0.6 \ 0.1]$, $\delta(x, \alpha) = [0 \ 0.12 \ 0.36 \ 0.54]$. Clearly, $x \in X$ but $\delta(x, \sigma) \notin X$. Since α is not controllable, if there is no event to preempt α, then X is not controllable. In other words, if no treatment is given to the patient, diabetes will get worse and worse; eventually, the system will enter the illegal state space Q_b.

Now, let us consider the different treatments and their consequences. We calculate $\delta(x, \sigma)$ for all the aforementioned five treatment events—$\sigma = \beta, \rho, \lambda, \theta$, or η. It can easily be shown that $x \in X$ implies $\delta(x, \sigma) \in X$. Since events θ (taking oral medication) and η (using insulin) are both enforceable, we can always use them to preempt α if α drives the system out of X. Therefore, the supremal controllable subset of X is

$$X^\uparrow = \{x \in X : \delta(x, \alpha) \in X\} \cup \{\delta(x, \sigma) \in X : \sigma = \theta, \eta \wedge x \in X \wedge \delta(x, \alpha) \notin X\}.$$

In other words, $X^\uparrow$ is divided into two sets: 1) $\{x \in X : \delta(x, \alpha) \in X\}$ when the patient's condition is such that the progression of diabetes will not lead to the illegal state space Q_b, and 2) $\{\delta(x, \sigma) \in X : \sigma = \theta, \eta \wedge x \in X \wedge \delta(x, \alpha) \notin X\}$ when the condition in 1) does not hold oral medication or insulin is necessary. Accordingly, the state-feedback control that achieves $X^\uparrow$ is given by

$$
\varphi(x) =
\begin{cases}
\Sigma, & \text{if } \{x \in X : \delta(x, \alpha) \in X\} \\
\{\theta, \eta\}, & \text{if } \{x \in X : \delta(x, \alpha) \notin X\} \\
\Sigma - \Sigma_c, & \text{otherwise.}
\end{cases}
$$

It is not difficult to see that, under this feedback control, $Q_{ac}(q_o, \phi/\delta) = X^\uparrow$.

We remark that, although the set control pattern Γ is not closed under union in this particular example, the supremal controllable subset of X still exists. By Theorem 3, the condition that the set of control patterns is closed under union is only a sufficient condition for the existence of the supremal controllable subset. This example shows that, indeed, this condition is not a necessary one.

We also remark that the state-feedback control defined earlier for the diabetes treatment is only for ensuring that the patient never enters the illegal states (safety control). It may not be optimal. Optimal control shall be designed separately in addition to safety control.

V. CONCLUSION

We have developed a state-feedback control for FDESs. To find an existence condition for the state-feedback control, we have defined closure and controllability of subset of state space. We have shown that closure and controllability are the necessary and sufficient conditions for the existence of a state-feedback control. We have proved that both closure and controllability are closed under union, and hence, the supremal closed subset and the supremal controllable subset exist. All these results have been illustrated by an example of diabetes treatment.

This paper is the first in state-feedback control for FDESs. More works can be done using state feedback, including the following. 1) Observability: If some events are unobservable to the controller, then observability must be discussed. Observability of a DES is an important concept that was first investigated in [9]. 2) Online control: To reduce the complexity of controller design, particularly under partial observation, online and limited/variable look-ahead control can be used. Online control of a DES, first proposed in [5], can reduce computational complexity significantly. 3) Other control objectives: Control objectives, such as optimal control, can be achieved using state feedback. Optimal control of FDESs is discussed in [11], [15], and [16], but the approach there does not use state feedback.

Acknowledgments

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