Extension of the matrix Bartlett's formula to the third and fourth order and to noisy linear models with application to parameter estimation

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Abstract

This paper focuses on the extension of the asymptotic covariance of the sample covariance (denoted *Bartlett's formula*) of linear processes to third- and fourth-order sample cumulant and to noisy linear processes. Closed-form expressions of the asymptotic covariance and cross-covariance of the sample second-, third- and fourth-order cumulants are derived in a relatively straightforward manner thanks to a matrix polyspectral representation and a symbolic calculus akin to a high level language. As an application of these extended formulae, we underscore the sensitivity of the asymptotic performance of estimated ARMA parameters by an arbitrary third- or fourth order-based algorithm with respect to the signal to noise ratio, the spectra of the linear process, and the colored additive noise.

Index terms: Statistical performance analysis, Bartlett's formula, third-order cumulant, fourth-order cumulant, noisy linear process.

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1 Introduction

The problem of estimating the parameters of linear time-invariant nonminimum phase systems when only output data are available from higher-order statistics has been intensively studied. The use of the cumulants in time series analysis has a long-back history, starting with the classical paper of Brillinger and Rosenblatt [1] (see also the Brillinger's book [2]). Giannakis [3] was the first to show that the parameters of a qth-order MA system can be calculated from only the system's output cumulant with his third- and fourth-order formulae. From this pioneering work, many contributions have dealt with higher-order statistics based algorithms to estimate the MA, AR, ARMA parameters of linear systems driven by an independent and identically distributed non-Gaussian sequence corrupted (or not) by additive Gaussian noise that may be colored (see e.g., [4]-[8] and the reference therein).

The statistical performance of the proposed algorithms has been analyzed only by Monte-Carlo simulations except to our knowledge in the work by Porat and Friedlander [5] and by Dandawaté and Giannakis [9][10]. This former work gives closed-form expressions for the asymptotic variances and covariances of the sample third-order moments of ARMA processes, thanks to a state-space representation focused on the noise-free case only. The latter work is dedicated to estimates of the asymptotic variances and covariances of sample kth-order cumulants of arbitrary mixtures of deterministic, stationary and non-stationary processes satisfying a mixing condition, based on smoothed cross periodograms.

The purpose of this paper is to give closed-form expressions of the asymptotic variances and covariances of the sample third- and fourth-order cumulants of linear processes corrupted by an additive white or colored Gaussian or non-Gaussian noise. In addition, naturally, this work provides tools for performance evaluation and comparison of identification algorithms based on sample third- or fourth-order cumulants in these conditions. The computation of each asymptotic variance/covariance in the noisy case turns to be a very tedious task. For example, for zero-mean real-valued processes, the number of terms is 222 [resp. 6022] to express variance/covariance of the sample third-order [resp. fourth-order] moments in the noisy case to 41 [resp. 715] terms in the noise-free case. To overcome this computational difficulty, we propose in this paper to start from another point of view, and to derive these variance/covariance via a matrix polyspectral approach. As a result, the complexity of the derivation of these different terms will not increase from the noise-free to the noisy case. Furthermore, to avoid overly laborious calculations, we use a symbolic calculus akin to a high level language.

This paper is organized as follows. After the data model and some notations are given in Section 2, the second-order Bartlett's formula is recalled in Section 3 and expressed in a matrix polyspectral closed-form in the noisy case for real-valued processes. This approach is extended in Section 4 and 5 to the third- and fourth-order respectively. Because the derivation developped for the second-order would be very tedious, a symbolic algorithm based on a few well defined rules is used. Matrix closed-form expressions of the asymptotic covariance of the third-order sample moment and the asymptotic cross-covariance between the second and third-order sample moments are given in the noisy case for zero-mean real-valued processes in Section 4. For the fourth-order, we focus on zero-mean complex processes circular up to the fourth-order as examples in Section 5, where we get closed-form expressions of the asymptotic covariance of the second and fourth-order sample cumulants and the asymptotic cross-covariances between the second and fourth-order sample cumulants in the noisy case. Finally, the sensitivity of the asymptotic performance of the estimated ARMA parameters by an arbitrary third or fourth order-based algorithm to the signal to noise ratio (SNR), the spectra of the linear process, and the colored additive noise is addressed in Section 6. As an example, the asymptotic lower bound for the variances of third- or fourth-order algorithms are compared to the asymptotic variances given by the so-called $C(k,q)$ algorithms for non-Gaussian first or second-order moving average processes [abbreviated as MA(1) and MA(2) in the sequel] corrupted by a Gaussian first-order autoregressive process (abbreviated as AR(1) in the sequel).

The following notations are used throughout the paper. The range of all summations is understood to be from $-\infty$ to ∞ except the specified summations and the range of all integrations is $\triangle = [-1/2, 1/2]$. $\mathrm{Cov}(\mathbf{x},\mathbf{y})$ is used for real and complex-valued random vectors and means $\mathrm{E}(\mathbf{x}\mathbf{y}^T) - \mathrm{E}(\mathbf{x})\mathrm{E}(\mathbf{y}^T)$.

2 Data model

Consider the following linear process:
$$x_t = \sum_n h_n u_{t-n}$$
where h_n is real-valued in Sections 3 and 4 [resp. complex-valued in Section 5] with $\sum_n |h_n| < \infty$ and the observation of x_t is noisy:
$$y_t = x_t + \epsilon_t.$$
The input sequence u_t is zero-mean, independent and identically distributed, and non-Gaussian, real-valued with $\kappa_{3_u} \stackrel{\text{def}}{=} \mathrm{E}(u_t^3) \neq 0$ and $\mathrm{E}(u_t^6) < \infty$ in Sections 3 and 4 [resp. complex-valued, circular up to the fourth-order with

$\kappa_{4_u} \stackrel{\text{def}}{=} \mathrm{E}|u_t^4| - 2(\mathrm{E}|u_t^2|)^2 \neq 0$ and $\mathrm{E}|u_t|^8 < \infty$ in Section 5]. The measurement noise sequence ϵ_t is assumed to be zero-mean, colored stationary with unknown power spectrum and is independent of u_t. In Sections 3 and 4, ϵ_t is real-valued Gaussian or non-Gaussian, with $\mathrm{E}(\epsilon_t^2) \stackrel{\text{def}}{=} \sigma_\epsilon^2$ and with sixth-order cumulants $c^\epsilon_{t_2-t_1,\ldots,t_6-t_1} \stackrel{\text{def}}{=} \mathrm{Cum}(\epsilon_{t_1},\ldots,\epsilon_{t_6})$ satisfying $\sum_{t_1}\ldots\sum_{t_5}|c^\epsilon_{t_1,t_2,\ldots,t_5}| < \infty$. So, the polyspectra of y_t, x_t and ϵ_t are defined up to the fifth-order

$$S_\epsilon(f_1,f_2,\ldots,f_p) \stackrel{\text{def}}{=} \sum_{t_1}\ldots\sum_{t_p} c^\epsilon_{t_1,\ldots,t_p} e^{-i2\pi(t_1f_1+\ldots+t_pf_p)}.$$

And in Sections 5, ϵ_t is circular complex-valued with $\mathrm{E}|\epsilon_t^2| \stackrel{\text{def}}{=} \sigma_\epsilon^2$ and with eighth-order cumulants $c^\epsilon_{t_2-t_1,\ldots,t_8-t_1} \stackrel{\text{def}}{=} \mathrm{Cum}(\epsilon_{t_1},\epsilon^*_{t_2},\ldots,\epsilon_{t_{2k-1}},\epsilon^*_{t_{2k}}\ldots,\epsilon^*_{t_8})$ satisfying $\sum_{t_1}\ldots\sum_{t_7}|c^\epsilon_{t_1,t_2,\ldots,t_7}| < \infty$. So, the polyspectra of y_t, x_t and ϵ_t are defined up to the seventh-order

$$S_\epsilon(f_1,f_2,\ldots,f_{2p-1}) \stackrel{\text{def}}{=} \sum_{t_1}\ldots\sum_{t_{2p-1}} c^\epsilon_{t_1,\ldots,t_{2p-1}} e^{-i2\pi(t_1f_1+\ldots+t_{2p-1}f_{2p-1})}.$$

In Sections 3 and 4, the second-order moment $c^x_k \stackrel{\text{def}}{=} \mathrm{E}(x_tx_{t+k})$ and third-order moment $c^x_{k,l} \stackrel{\text{def}}{=} \mathrm{E}(x_tx_{t+k}x_{t+l})$ are estimated from T consecutive mesurements by the associated sample moments: $c^x_k(T) \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^T x_tx_{t+k}$ and $c^x_{k,l}(T) \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^T x_tx_{t+k}x_{t+l}$ [1]. These moments are stacked in the vectors $\mathbf{c}^x_2 \stackrel{\text{def}}{=} (c^x_0,\ldots,c^x_{L_2-1})^T$, $\mathbf{c}^x_3 \stackrel{\text{def}}{=} (c^x_{0,0},c^x_{0,1},\ldots,c^x_{0,L_3-1},c^x_{1,0},\ldots,c^x_{L_3-1,L_3-1})^T$, $\mathbf{c}^x_2(T)$ and $\mathbf{c}^x_3(T)$ are defined in the same way. In Section 5, the second-order moment $c^x_k \stackrel{\text{def}}{=} \mathrm{E}(x_tx^*_{t+k})$ and fourth-order cumulant $c^x_{k,l,m} \stackrel{\text{def}}{=} \mathrm{E}(x_tx^*_{t+k}x_{t+l}x^*_{t+m}) - \mathrm{E}(x_tx^*_{t+k})\mathrm{E}(x_{t+l}x^*_{t+m}) - \mathrm{E}(x_tx^*_{t+m})\mathrm{E}(x_{t+l}x^*_{t+k})$ are estimated from T consecutive mesurements by the associated sample cumulants: $c^x_k(T)$ and $c^x_{k,l,m}(T)$. These cumulants are stacked in increasing order in the vectors $\mathbf{c}^x_4 \stackrel{\text{def}}{=} (c^x_{0,0,0},c^x_{0,0,1},\ldots,c^x_{L_4-1,L_4-1,L_4-1})^T$ and $\mathbf{c}^x_4(T)$ is defined in the same way.

3 Second-order Bartlett's formula

3.1 Noise-free case

Under the above assumptions, $\mathbf{c}^x_2(T)$ is asymptotically Gaussian (see e.g., [11, Th. 3.3]):

$$\sqrt{T}\ \left(\mathbf{c}^x_2(T) - \mathbf{c}^x_2\right) \xrightarrow{\mathcal{L}} \mathcal{N}(\mathbf{0};\mathbf{C}^x_{2,2})$$

where $\xrightarrow{\mathcal{L}}$ stands for convergence in distribution and $\mathbf{C}^x_{2,2} = \lim_{T\to\infty} T\mathrm{Cov}\left(\mathbf{c}^x_2(T),\mathbf{c}^x_2(T)\right)$ is given by Bartlett's formula (see e.g. [12, rel. 6, p. 255]):

$$\lim_{T\to\infty} T\mathrm{Cov}\left(c^x_k(T),c^x_l(T)\right) = \sum_t \left(c^x_tc^x_{t+k-l} + c^x_{t+k}c^x_{t-l} + c^x_{t+k,t,l}\right).$$

Using Parseval's theorem and the Fourier relationship between the covariance and the spectral density $S_x(f)$ of x_t and between the fourth-order cumulant and the trispectrum $S_x(f_1,f_2,f_3)$ of x_t, we get the following alternative matrix polyspectral Bartlett's formula:

$$\mathbf{C}^x_{2,2} = \int_\triangle S^2_x(f)[\mathbf{e}(f)\mathbf{e}^H(f) + \mathbf{e}(f)\mathbf{e}^T(f)]df + \int_{\triangle^2} S_x(f_1,-f_1,f_2)\mathbf{e}(f_1)\mathbf{e}^T(f_2)df_1df_2 \tag{3.1}$$

with $\mathbf{e}(f) \stackrel{\text{def}}{=} (1,e^{i2\pi f},\ldots,e^{i2\pi(L_2-1)f})^T$ where T and H stand for transpose and conjugate transpose respectively. We note that an elementwise counterpart of this relation was derived in [2, rel. 5.10.15] by another approach.

3.2 Noisy case

Under the assumptions of Section 2, the asymptotic normality and (3.1) apply in the noisy case, by replacing x_t by $y_t = x_t + \epsilon_t$. Furthermore, from the independence assumption, $S_y(f) = S_x(f) + S_\epsilon(f)$ and $S_y(f_1,f_2,f_3) =$

[1] We note that both $c^x_k(T)$ and $c^x_{k,l}(T)$ can be defined in several other ways, differing in the manner in which the end data are treated. But all these definitions are asymptotically equivalent.

$S_x(f_1, f_2, f_3) + S_\epsilon(f_1, f_2, f_3)$, and thus there holds:

$$\begin{aligned}\mathbf{C}^y_{2,2} &= \mathbf{C}^x_{2,2} + 2\int_\triangle S_\epsilon(f)S_x(f)[\mathbf{e}(f)\mathbf{e}^H(f) + \mathbf{e}(f)\mathbf{e}^T(f)]df + \int_\triangle S^2_\epsilon(f)[\mathbf{e}(f)\mathbf{e}^H(f) + \mathbf{e}(f)\mathbf{e}^T(f)]df \\ &+ \int_{\triangle^2} S_\epsilon(f_1, -f_1, f_2)\mathbf{e}(f_1)\mathbf{e}^T(f_2)df_1 df_2. \end{aligned} \quad (3.2)$$

We note that the last term of (3.2) vanishes if the additive noise is Gaussian.

4 Third-order Bartlett's formula

4.1 Noise-free case

Under the assumptions of Section 2, $[\mathbf{c}^x_2(T), \mathbf{c}^x_3(T)]$ is asymptotically Gaussian (see e.g., [1]):

$$\sqrt{T}\begin{pmatrix} \mathbf{c}^x_2(T) - \mathbf{c}^x_2 \\ \mathbf{c}^x_3(T) - \mathbf{c}^x_3 \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}\left(\begin{pmatrix} \mathbf{0} \\ \mathbf{0}\end{pmatrix}; \begin{pmatrix} \mathbf{C}^x_{2,2} & \mathbf{C}^x_{2,3} \\ {\mathbf{C}^x_{2,3}}^T & \mathbf{C}^x_{3,3}\end{pmatrix}\right)$$

where $\mathbf{C}^x_{2,3} = \lim_{T\to\infty} T\mathrm{Cov}\left(\mathbf{c}^x_2(T), \mathbf{c}^x_3(T)\right)$ and $\mathbf{C}^x_{3,3} = \lim_{T\to\infty} T\mathrm{Cov}\left(\mathbf{c}^x_3(T), \mathbf{c}^x_3(T)\right)$, and it is straightforward to get (see e.g., [13, rel.10.5.2]):

$$\lim_{T\to\infty} T\mathrm{Cov}\left(c^x_k(T), c^x_{l,m}(T)\right) = \sum_t \left[\mathrm{E}(x_0 x_k x_t x_{t+l} x_{t+m}) - c^x_k c^x_{l,m}\right] \quad (4.1)$$

$$\lim_{T\to\infty} T\mathrm{Cov}\left(c^x_{k,l}(T), c^x_{m,n}(T)\right) = \sum_t \left[\mathrm{E}(x_0 x_k x_l x_t x_{t+m} x_{t+n}) - c^x_{k,l} c^x_{m,n}\right]. \quad (4.2)$$

To proceed, deducing relations similar to eq. (3.1) along the same lines is possible in principle but such a derivation would be extremely tedious. A much more interesting approach, which consists of devising a symbolic calculus akin to a high level language, is used. Based on a few well defined rules, this algorithm allows us to automatically perform the following steps:

- Generate all partitions given by the cumulants-to-moments formula (Leonov Shiryaev formula) [13, Th.10.1] expressing the fifth- ($\mathrm{E}(x_0 x_k x_t x_{t+l} x_{t+m})$ of (4.1)) and sixth-order moments ($\mathrm{E}(x_0 x_k x_l x_t x_{t+m} x_{t+n})$ of (4.2)) as functions of sums of products of cumulants;
- Eliminate the zero terms using the zero-mean property;
- Construct sets of similar expressions w.r.t. the number of terms x_{t+a} in each product of cumulants. For each such set of similar expressions (3 sets for $\mathrm{E}(x_0 x_k x_t x_{t+l} x_{t+m})$ and 6 sets for $\mathrm{E}(x_0 x_k x_l x_t x_{t+m} x_{t+n})$), a representative term is chosen to be analytically expressed by a polyspectral formula, as proved in Appendix A.

Consequently, using the (.) notation introduced in [14] to avoid listing explicitly all the partitions, we obtain:

$$\begin{aligned}\mathrm{E}(x_0 x_k x_t x_{t+l} x_{t+m}) &= \mathrm{Cum}(x_0, x_k, x_t, x_{t+l}, x_{t+m}) \\ &+ \mathrm{Cum}(x_0, x_k, x_t)\mathrm{Cum}(x_{t+l}, x_{t+m})(10) \\ \mathrm{E}(x_0 x_k x_l x_t x_{t+m} x_{t+n}) &= \mathrm{Cum}(x_0, x_k, x_l, x_t, x_{t+m}, x_{t+n}) \\ &+ \mathrm{Cum}(x_0, x_k, x_l, x_t)\mathrm{Cum}(x_{t+m}, x_{t+n})(15) \\ &+ \mathrm{Cum}(x_0, x_k, x_l)\mathrm{Cum}(x_t, x_{t+m}, x_{t+n})(10) \\ &+ \mathrm{Cum}(x_0, x_k)\mathrm{Cum}(x_l, x_t)\mathrm{Cum}(x_{t+m}, x_{t+n})(15).\end{aligned} \quad (4.3)$$

And an example, the expression $\mathrm{Cum}(x_0, x_k, x_t)\mathrm{Cum}(x_{t+l}, x_{t+m})(10)$ can be broken down into three sets:

$$\begin{aligned}&\mathrm{Cum}(x_0, x_k, x_t)\mathrm{Cum}(x_{t+l}, x_{t+m})(10) \\ &= c^x_{t+m-k}c^x_{l,-t} + c^x_{t+l-k}c^x_{m,-t} + c^x_t c^x_{m-l,-t+k-l} + c^x_{t-k}c^x_{m-l,-t-l} + c^x_{t+l}c^x_{m,-t-k} + c^x_{t+m}c^x_{l,-t+k} \\ &+c^x_{m-l}c^x_{k,t} + c^x_m c^x_{k,t+l} + c^x_l c^x_{k,t+m} \\ &+c^x_k c^x_{l,m}\end{aligned} \quad (4.4)$$

and the last term cancels with $-c_k^x c_{l,m}^x$ in summation (4.1). So, this expression reduces to two sets of similar expressions. Using expressions (4.1), (4.3) and (4.4) the following matrix polyspectral extensions of Bartlett's formula is proved in Appendix A, where each polyspectral integral is associated with a set of the previous similar expressions:

$$\begin{aligned}\lim_{T\to\infty} T\mathrm{Cov}\left(\mathbf{c}_2^x(T),\mathbf{c}_3^x(T)\right) &= \int_{\triangle^3} S_x(f_1,-f_1,f_2,f_3)\mathbf{E}_{1,2,3}df_1df_2df_3 \\ &+ \int_{\triangle^2} S_x(f_1,f_2)S_x(f_1)\mathbf{E}_{1,2}^{(1)}df_1df_2 + \int_{\triangle^2} S_x(f_1,0)S_x(f_2)\mathbf{E}_{1,2}^{(2)}df_1df_2, \end{aligned} \tag{4.5}$$

where matrices $\mathbf{E}_{1,2,3}$, $\mathbf{E}_{1,2}^{(1)}$ and $\mathbf{E}_{1,2}^{(2)}$ are defined in Appendix B.

With the same procedure, the following matrix polyspectral formula is derived from (4.2):

$$\begin{aligned}\lim_{T\to\infty} T\mathrm{Cov}\left(\mathbf{c}_3^x(T),\mathbf{c}_3^x(T)\right) &= \int_{\triangle^4} S_x(f_1,f_2,f_3,f_4,-f_3-f_4)\mathbf{E}_{1,2,3,4}df_1df_2df_3df_4 \\ &+ \int_{\triangle^3} S_x(f_1,f_3)S_x(f_2,f_3)\mathbf{E}_{1,2,3}^{(1)}df_1df_2df_3 + \int_{\triangle^3} S_x(f_1,f_2,0)S_x(f_3)\mathbf{E}_{1,2,3}^{(2)}df_1df_2df_3 \\ &+ \int_{\triangle^3} S_x(f_1,f_2,f_3)S_x(f_2+f_3)\mathbf{E}_{1,2,3}^{(3)}df_1df_2df_3 + \int_{\triangle^2} S_x(0)S_x(f_1)S_x(f_2)\mathbf{E}_{1,2}^{(3)}df_1df_2 \\ &+ \int_{\triangle^2} S_x(f_1)S_x(f_2)S_x(f_1+f_2)\mathbf{E}_{1,2}^{(4)}df_1df_2, \end{aligned} \tag{4.6}$$

where the different matrices $\mathbf{E}_j^{(i)}$ are defined in Appendix B.

4.2 Noisy case

In the noisy case, the third-order relations are derived in the same way as for the second-order, because the independence of x_t and ϵ_t ensures the additivity of their polyspectra. Therefore, for example, for additive Gaussian noise, we get from (4.5) and (4.6):

$$\begin{aligned}\mathbf{C}_{2,3}^y &= \mathbf{C}_{2,3}^x \\ &+ \int_{\triangle^2} S_\epsilon(f_2)S_x(f_1,0)\mathbf{E}_{1,2}^{(1)}df_1df_2 + \int_{\triangle^2} S_\epsilon(f_1)S_x(f_1,f_2)\mathbf{E}_{1,2}^{(2)}df_1df_2 \end{aligned} \tag{4.7}$$

$$\begin{aligned}\mathbf{C}_{3,3}^y &= \mathbf{C}_{3,3}^x \\ &+ \int_{\triangle^3} S_\epsilon(f_2+f_3)S_x(f_1,f_2,f_3)\mathbf{E}_{1,2,3}^{(3)}df_1df_2df_3 + \int_{\triangle^3} S_\epsilon(f_3)S_x(f_1,f_2,0)\mathbf{E}_{1,2,3}^{(3)}df_1df_2df_3 \\ &+ \int_{\triangle^2} [S_\epsilon(f_2)S_x(0)S_x(f_1) + S_\epsilon(f_1)S_x(0)S_x(f_2 + S_\epsilon(0)S_x(f_1)S_x(f_2)]\mathbf{E}_{1,2}^{(3)}df_1df_2 \\ &+ \int_{\triangle^2} [S_\epsilon(f_1)S_\epsilon(f_2)S_x(0) + S_\epsilon(0)S_\epsilon(f_2)S_x(f_1) + S_\epsilon(0)S_\epsilon(f_1)S_x(f_2)]\mathbf{E}_{1,2}^{(3)}df_1df_2 \\ &+ \int_{\triangle^2} [S_\epsilon(f_1+f_2)S_x(f_1)S_x(f_2) + S_\epsilon(f_1)S_x(f_2)S_x(f_1+f_2) + S_\epsilon(f_2)S_x(f_1)S_x(f_1+f_2)]\mathbf{E}_{1,2}^{(4)}df_1df_2 \\ &+ \int_{\triangle^2} [S_\epsilon(f_1)S_\epsilon(f_1+f_2)S_x(f_2) + S_\epsilon(f_2)S_\epsilon(f_1+f_2)S_x(f_1) + S_\epsilon(f_1)S_\epsilon(f_2)S_x(f_1+f_2)]\mathbf{E}_{1,2}^{(4)}df_1df_2 \\ &+ \int_{\triangle^2} S_\epsilon(0)S_\epsilon(f_1)S_\epsilon(f_2)\mathbf{E}_{1,2}^{(3)}df_1df_2 + \int_{\triangle^2} S_\epsilon(f_1)S_\epsilon(f_2)S_\epsilon(f_1+f_2)\mathbf{E}_{1,2}^{(4)}df_1df_2. \end{aligned} \tag{4.8}$$

The influence of the additive colored noise on these asymptotic covariances is difficult to analyse from these expressions. However from the SNR point of view, we note that for a specific distribution of u_t, $\mathbf{C}_{2,3}^x$ and $\mathbf{C}_{3,3}^x$ are proportional to σ_x^5 and σ_x^6 respectively, whereas the noise additive terms are proportional to $\sigma_\epsilon^2\sigma_x^3$ and $\alpha_1\sigma_\epsilon^6 + \alpha_2\sigma_\epsilon^4\sigma_x^2 + \alpha_3\sigma_\epsilon^2\sigma_x^4$ respectively, where the terms $(\alpha_i)_{i=1,2,3}$ depend on the ARMA model and spectral shape of the additive noise. In the case where the noise spectrum has sharp resonances, the dominant term of the previous expression is given by the last term of (4.8). For example, for an AR(1) noise ($\epsilon_t = e_t + b\epsilon_{t-1}$), because it is proved in Appendix C that

$$\int_{\triangle^2} S_\epsilon(f_1)S_\epsilon(f_2)S_\epsilon(f_1+f_2)df_1df_2 = \sigma_\epsilon^6\left(\frac{1+b^3}{1-b^3}\right), \tag{4.9}$$

this dominant term grows unbounded as b approaches $+1$ and, therefore contributes to the degradation of the performance when the SNR is decreasing, as will be seen in Section 6.3.

5 Fourth-order Bartlett's formula

In this section, we focus on zero-mean complex processes circular up to the fourth-order [2] as example.

5.1 Noise-free case

Under the assumptions of Section 2, $[\mathbf{c}_2^x(T), \mathbf{c}_4^x(T)]$ is asymptotically Gaussian [3] (see e.g., [1]): [4]

$$\sqrt{T}\begin{pmatrix}\mathbf{c}_2^x(T)-\mathbf{c}_2^x\\ \mathbf{c}_4^x(T)-\mathbf{c}_4^x\end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}_c\left(\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}; \begin{pmatrix}\mathbf{C}_{2,2}^x & \mathbf{C}_{2,4}^x\\ \mathbf{C}_{2,4}^{x\ H} & \mathbf{C}_{4,4}^x\end{pmatrix}, \begin{pmatrix}\mathbf{C'}_{2,2}^x & \mathbf{C'}_{2,4}^x\\ \mathbf{C'}_{2,4}^{x\ T} & \mathbf{C'}_{4,4}^x\end{pmatrix}\right)$$

where $\mathbf{C}_{2,2}^x = \lim_{T\to\infty} T\mathrm{Cov}\,(\mathbf{c}_2^x(T), \mathbf{c}^{*x}_2(T))$, $\mathbf{C'}_{2,2}^x = \lim_{T\to\infty} T\mathrm{Cov}\,(\mathbf{c}_2^x(T), \mathbf{c}_2^x(T))$, $\mathbf{C}_{2,4}^x = \lim_{T\to\infty} T\mathrm{Cov}\,(\mathbf{c}_2^x(T), \mathbf{c}^{*x}_4(T))$, $\mathbf{C'}_{2,4}^x = \lim_{T\to\infty} T\mathrm{Cov}\,(\mathbf{c}_2^x(T), \mathbf{c}_4^x(T))$, $\mathbf{C}_{4,4}^x = \lim_{T\to\infty} T\mathrm{Cov}\,(\mathbf{c}_4^x(T), \mathbf{c}^{*x}_4(T))$ and $\mathbf{C'}_{4,4}^x = \lim_{T\to\infty} T\mathrm{Cov}\,(\mathbf{c}_4^x(T), \mathbf{c}_4^x(T))$. With the approach used to prove the real-valued Bartlett's formula in [13, sec. 4.1, 4.2], it is straightforward to get:

$$\begin{aligned}\lim_{T\to\infty} T\mathrm{Cov}\,(c_k^x(T), c^{*x}_l(T)) &= \sum_t\left(c_t^x c_{-t+k-l}^x + c_{k,t+l,t}^x\right)\\ \lim_{T\to\infty} T\mathrm{Cov}\,(c_k^x(T), c_l^x(T)) &= \sum_t\left(c_t^x c_{-t+k+l}^x + c_{k,t,t+l}^x\right).\end{aligned}$$

Similarly to the real-valued Bartlett's formula (3.1), we get the following alternative matrix polyspectral Bartlett's formula:

$$\mathbf{C}_{2,2}^x = \int_\triangle S_x^2(f)\mathbf{e}(f)\mathbf{e}^H(f)df + \int_{\triangle^2} S_x(f_1,f_2,-f_2)\mathbf{e}(f_1)\mathbf{e}^T(f_2)df_1df_2 \tag{5.1}$$

$$\mathbf{C'}_{2,2}^x = \int_\triangle S_x^2(f)\mathbf{e}(f)\mathbf{e}^T(f)df + \int_{\triangle^2} S_x(f_1,-f_2,f_2)\mathbf{e}(f_1)\mathbf{e}^T(f_2)df_1df_2. \tag{5.2}$$

Then to express $\mathbf{C}_{2,4}^x, \mathbf{C'}_{2,4}^x, \mathbf{C}_{4,4}^x, \mathbf{C'}_{4,4}^x$ in function of the polyspectra of x_t, we first note that because

$$c_{l,m,n}^x(T) \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^T x_t x_{t+l}^* x_{t+m} x_{t+n}^* - \left(\frac{1}{T}\sum_{t=1}^T x_t x_{t+l}^*\right)\left(\frac{1}{T}\sum_{t=1}^T x_{t+m}x_{t+n}^*\right) - \left(\frac{1}{T}\sum_{t=1}^T x_t x_{t+n}^*\right)\left(\frac{1}{T}\sum_{t=1}^T x_{t+m}x_{t+l}^*\right),$$

we get the following first order expansion:

$$c_{l,m,n}^x(T) - c_{l,m,n}^x = \begin{pmatrix}1 & -c_{n-m}^x & -c_l^x & -c_{l-m}^x & -c_n^x\end{pmatrix}\underbrace{\begin{pmatrix}\frac{1}{T}\sum_t x_t x_{t+l}^* x_{t+m} x_{t+n}^* - \mu_{l,m,n}^x\\ \frac{1}{T}\sum_t x_t x_{t+l}^* - c_l^x\\ \frac{1}{T}\sum_t x_{t+m}x_{t+n}^* - c_{n-m}^x\\ \frac{1}{T}\sum_t x_t x_{t+n}^* - c_n^x\\ \frac{1}{T}\sum_t x_{t+m}x_{t+l}^* - c_{l-m}^x\end{pmatrix}}_{\boldsymbol{\alpha}_T} + \mathrm{o}(\boldsymbol{\alpha}_T),$$

[2] A zero-mean complex processes x_t is circular up to the r-order iff $\mathrm{E}(\prod_{\sum a_k = p} x_{t_k}^{a_k}\prod_{\sum b_l = q} x_{t_l}^{b_l\,*}) = 0$ for all positive integers a_k, b_l, p, q such that $p+q\le r$ and $p\ne q$.

[3] The distribution of a zero-mean Gaussian complex multivariate random variable $\mathbf{x}$ is characterized by the two covariance matrices $\boldsymbol{\Sigma}_1 \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{x}\mathbf{x}^H)$ and $\boldsymbol{\Sigma}_2 \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{x}\mathbf{x}^T)$. This distribution is denoted $\mathcal{N}(\mathbf{0};\boldsymbol{\Sigma}_1,\boldsymbol{\Sigma}_2)$.

[4] We note that despite the cumulants are rich of symmetries, the non-circular complex Gaussian asymptotic distribution of $\mathbf{c}_2^x(T)$ [resp. $\mathbf{c}_4^x(T)$] is not characterized by $\mathbf{C}_{2,2}^x$ [resp. $\mathbf{C}_{4,4}^x$] only.

where $\mu^x_{l,m,n} \stackrel{\text{def}}{=} \mathrm{E}(x_t x^*_{t+l} x_{t+m} x^*_{t+n})$. Using $c^x_k(T) - c^x_k = \frac{1}{T}\sum_t x_t x^*_{t+k} - c^x_k$ and the previous expansion, we get thanks to the asymptotic covariances of the sample moments derived from [13, rel.10.5.2] extended to the complex case, the asymptotic cross-covariance of the second- and fourth-order sample cumulants.

$$\lim_{T\to\infty} T\mathrm{Cov}\left(c^x_k(T), c^x_{l,m,n}(T)\right) = \begin{pmatrix} 1 & -c^x_{n-m} & -c^x_l & -c^x_{l-m} & -c^x_n \end{pmatrix} \begin{pmatrix} \sum_t \left[\mathrm{E}(x_0 x^*_k x_t x^*_{t+l} x_{t+m} x^*_{t+n}) - c^x_k \mu^x_{l,m,n}\right] \\ \sum_t \left[\mathrm{E}(x_0 x^*_k x_t x^*_{t+l}) - c^x_k c^x_l\right] \\ \sum_t \left[\mathrm{E}(x_0 x^*_k x_{t+m} x^*_{t+n}) - c^x_k c^x_{n-m}\right] \\ \sum_t \left[\mathrm{E}(x_0 x^*_k x_t x^*_{t+n}) - c^x_k c^x_n\right] \\ \sum_t \left[\mathrm{E}(x_0 x^*_k x_{t+m} x^*_{t+l}) - c^x_k c^x_{l-m}\right] \end{pmatrix}. \tag{5.3}$$

To get $\lim_{T\to\infty} T\mathrm{Cov}\left(c^x_{k,l,m}(T), c^x_{n,p,q}(T)\right)$, we use the same approach for which we have:

$$\lim_{T\to\infty} T\mathrm{Cov}\left(c^x_{k,l,m}(T), c^x_{n,p,q}(T)\right) = \left(1, -c^x_{m-l}, -c^x_k, -c^x_{k-l}, -c^x_m\right) \begin{pmatrix} c_1 & \mathbf{c}_3^T \\ \mathbf{c}_2 & \mathbf{C} \end{pmatrix} \left(1, -c^x_{q-p}, -c^x_n, -c^x_{n-p}, -c^x_q\right)^T$$

with

$$c_1 = \sum_t \left[\mathrm{E}(x_0 x^*_k x_l x^*_m x_t x^*_{t+n} x_{t+p} x^*_{t+q}) - \mu^x_{k,l,m}\mu^x_{n,p,q}\right],$$

$$\mathbf{c}_2 = \begin{bmatrix} \sum_t \left[\mathrm{E}(x_0 x^*_k x_t x^*_{t+n} x_{t+p} x^*_{t+q}) - c^x_k \mu^x_{n,p,q}\right] \\ \sum_t \left[\mathrm{E}(x_l x^*_m x_t x^*_{t+n} x_{t+p} x^*_{t+q}) - c^x_{m-l} \mu^x_{n,p,q}\right] \\ \sum_t \left[\mathrm{E}(x_0 x^*_m x_t x^*_{t+n} x_{t+p} x^*_{t+q}) - c^x_m \mu^x_{n,p,q}\right] \\ \sum_t \left[\mathrm{E}(x_l x^*_k x_t x^*_{t+n} x_{t+p} x^*_{t+q}) - c^x_{k-l} \mu^x_{n,p,q}\right] \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} \sum_t \left[\mathrm{E}(x_0 x^*_n x_t x^*_{t+k} x_{t+l} x^*_{t+m}) - c^x_n \mu^x_{k,l,m}\right] \\ \sum_t \left[\mathrm{E}(x_p x^*_q x_t x^*_{t+k} x_{t+l} x^*_{t+m}) - c^x_{q-p} \mu^x_{k,l,m}\right] \\ \sum_t \left[\mathrm{E}(x_0 x^*_q x_t x^*_{t+k} x_{t+l} x^*_{t+m}) - c^x_q \mu^x_{k,l,m}\right] \\ \sum_t \left[\mathrm{E}(x_p x^*_n x_t x^*_{t+k} x_{t+l} x^*_{t+m}) - c^x_{n-p} \mu^x_{k,l,m}\right] \end{bmatrix}$$

$$\begin{aligned}
(\mathbf{C})_{1,1} &= \begin{bmatrix} \sum_t \left[\mathrm{E}(x_0 x^*_k x_t x^*_{t+n}) - c^x_k c^x_n\right] & \sum_t \left[\mathrm{E}(x_0 x^*_k x_{t+p} x^*_{t+q}) - c^x_k c^x_{q-p}\right] \\ \sum_t \left[\mathrm{E}(x_l x^*_m x_t x^*_{t+n}) - c^x_{m-l} c^x_n\right] & \sum_t \left[\mathrm{E}(x_l x^*_m x_{t+p} x^*_{t+q}) - c^x_{m-l} c^x_{q-p}\right] \end{bmatrix} \\
(\mathbf{C})_{2,1} &= \begin{bmatrix} \sum_t \left[\mathrm{E}(x_0 x^*_m x_t x^*_{t+n}) - c^x_m c^x_n\right] & \sum_t \left[\mathrm{E}(x_0 x^*_m x_{t+p} x^*_{t+q}) - c^x_m c^x_{q-p}\right] \\ \sum_t \left[\mathrm{E}(x_l x^*_k x_t x^*_{t+n}) - c^x_{k-l} c^x_n\right] & \sum_t \left[\mathrm{E}(x_l x^*_k x_{t+p} x^*_{t+q}) - c^x_{k-l} c^x_{q-p}\right] \end{bmatrix} \\
(\mathbf{C})_{1,2} &= \begin{bmatrix} \sum_t \left[\mathrm{E}(x_0 x^*_k x_t x^*_{t+q}) - c^x_k c^x_q\right] & \sum_t \left[\mathrm{E}(x_0 x^*_k x_{t+p} x^*_{t+n}) - c^x_k c^x_{n-p}\right] \\ \sum_t \left[\mathrm{E}(x_l x^*_m x_t x^*_{t+q}) - c^x_{m-l} c^x_q\right] & \sum_t \left[\mathrm{E}(x_l x^*_m x_{t+p} x^*_{t+n}) - c^x_{m-l} c^x_{n-p}\right] \end{bmatrix} \\
(\mathbf{C})_{2,2} &= \begin{bmatrix} \sum_t \left[\mathrm{E}(x_0 x^*_m x_t x^*_{t+q}) - c^x_m c^x_q\right] & \sum_t \left[\mathrm{E}(x_0 x^*_m x_{t+p} x^*_{t+n}) - c^x_m c^x_{n-p}\right] \\ \sum_t \left[\mathrm{E}(x_l x^*_k x_t x^*_{t+q}) - c^x_{k-l} c^x_q\right] & \sum_t \left[\mathrm{E}(x_l x^*_k x_{t+p} x^*_{t+n}) - c^x_{k-l} c^x_{n-p}\right] \end{bmatrix}.
\end{aligned}$$

To proceed, the moments in the four last summations of (5.3) and in $\mathbf{C}$ are expressed by cumulants. For example:

$$\begin{aligned}
\sum_t \left[\mathrm{E}(x_0 x^*_k x_t x^*_{t+l}) - c^x_k c^x_l\right] &= \sum_t \left(c^x_t c^x_{-t+k+l} + c^x_{k,t,t+l}\right) \\
\sum_t \left[\mathrm{E}(x_0 x^*_k x_{t+m} x^*_{t+n}) - c^x_k c^x_{n-m}\right] &= \sum_t \left(c^x_{t+n} c^x_{-t+k-m} + c^x_{k,t+m,t+n}\right) \\
\sum_t \left[\mathrm{E}(x_0 x^*_k x_t x^*_{t+n}) - c^x_k c^x_n\right] &= \sum_t \left(c^x_t c^x_{-t+k+n} + c^x_{k,t,t+n}\right) \\
\sum_t \left[\mathrm{E}(x_0 x^*_k x_{t+m} x^*_{t+l}) - c^x_k c^x_{l-m}\right] &= \sum_t \left(c^x_{t+l} c^x_{-t+k-m} + c^x_{k,t+m,t+l}\right),
\end{aligned}$$

then these sommations are evaluated similarly to the real-valued polyspectral Bartlett's formula (3.1).

The summations of sixth-order moments in $\mathbf{c}_2$, $\mathbf{c}_3$ and in the first term of (5.3), and summations of eighth-order moments in c_1 are expressed as functions of polyspectra of x_t from our symbolic calculus akin to a high level language based on a few well defined rules used in Section 4. Here, the zero terms are eliminated according to the circularity property up to the fourth-order, and the sets of similar expressions are constructed w.r.t. the number of terms x_{t+a} and x^*_{t+b} in each product of cumulants. For each such set of similar expressions (6 sets for $\mathrm{E}(x_0 x^*_k x_t x^*_{t+l} x_{t+m} x^*_{t+n})$ and 21 sets for $\mathrm{E}(x_0 x^*_k x_l x^*_m x_t x^*_{t+n} x_{t+p} x^*_{t+q})$), a representative term is chosen to be analytically expressed by a polyspectral formula derived in the same way as for the third-order case proved in Appendix A. We get:

$$\begin{aligned} \mathrm{E}(x_0 x_k^* x_t x_{t+l}^* x_{t+m} x_{t+n}^*) &= \mathrm{Cum}(x_0, x_k^*, x_t, x_{t+l}^*, x_{t+m}, x_{t+n}^*) \\ &+ \mathrm{Cum}(x_0, x_k^*, x_t, x_{t+l}^*)\mathrm{Cum}(x_{t+m}, x_{t+n}^*)(9) \\ &+ \mathrm{Cum}(x_0, x_k^*)\mathrm{Cum}(x_t, x_{t+l}^*)\mathrm{Cum}(x_{t+m}, x_{t+n}^*)(6). \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} \mathrm{E}(x_0 x_k^* x_l x_m^* x_t x_{t+n}^* x_{t+p} x_{t+q}^*) &= \mathrm{Cum}(x_0, x_k^*, x_l, x_m^*, x_t, x_{t+n}^*, x_{t+p}, x_{t+q}^*) \\ &+ \mathrm{Cum}(x_0, x_k^*, x_l, x_m^*, x_t, x_{t+n}^*)\mathrm{Cum}(x_{t+p}, x_{t+q}^*)(16) \\ &+ \mathrm{Cum}(x_0, x_k^*, x_l, x_m^*)\mathrm{Cum}(x_t, x_{t+n}^*, x_{t+p}, x_{t+q}^*)(18) \\ &+ \mathrm{Cum}(x_0, x_k^*, x_l, x_m^*)\mathrm{Cum}(x_t, x_{t+n}^*)\mathrm{Cum}(x_{t+p}, x_{t+q}^*)(72) \\ &+ \mathrm{Cum}(x_0, x_k^*)\mathrm{Cum}(x_l, x_m^*)\mathrm{Cum}(x_t, x_{t+n}^*)\mathrm{Cum}(x_{t+p}, x_{t+q}^*)(24), \end{aligned} \tag{5.5}$$

where for example, the second line of (5.4) gives:

$$\begin{aligned} \mathrm{Cum}(x_0, x_k^*, x_t, x_{t+l}^*)\mathrm{Cum}(x_{t+m}, x_{t+n}^*)(9) &= c^x_{k,t,t+l} c^x_{n-m} + c^x_{k,t,t+n} c^x_{l-m} + c^x_{k,t+m,t+l} c^x_n + c^x_{k,t+m,t+n} c^x_l \\ &+ c^x_{l,-t,n} c^x_{t+m-k} + c^x_{l-m,-t-m,n-m} c^x_{t-k} \\ &+ c^x_{n,m,-t+k} c^x_{t+l} + c^x_{l,m,-t+k} c^x_{t+n} \\ &+ c^x_k c^x_{l,m,n}, \end{aligned}$$

and because the last term cancels with $-c^x_k \mu^x_{l,m,n}$ in the summation $\sum_t \left[\mathrm{E}(x_0 x_k^* x_t x_{t+l}^* x_{t+m} x_{t+n}^*) - c^x_k \mu^x_{l,m,n}\right]$, this line gives three sets of similar expressions.

The limits $\lim_{T\to\infty} T\mathrm{Cov}\left(c^x_k(T), c^{*x}_{l,m,n}(T)\right)$ and $\lim_{T\to\infty} T\mathrm{Cov}\left(c^x_{k,l,m}(T), c^{*x}_{n,p,q}(T)\right)$ are evaluated similarly. Finally, our symbolic calculus delivers:

- LaTeX polyspectral expressions of $\mathbf{C}^x_{2,4}$, $\mathbf{C'}^x_{2,4}$, $\mathbf{C}^x_{4,4}$ and $\mathbf{C'}^x_{4,4}$ similar to (4.5)(4.6), but not reproduced here due to lack of space. They are available from the authors upon request.
- Matlab function files allowing one to compute the numerical values of these expressions (see subsection 6.2).

We have chosen to consider zero-mean processes which are complex circular up to the fourth-order. Naturally our methodology can be applied to the cases of zero-mean real-valued processes or zero-mean complex processes circular up to the second-order. The only difference is due to distinct rules of elimination of the zero terms.

5.2 Noisy case

In the noisy case, the fourth-order relations are derived in the same way as for the second and third-order, thanks to the additivity of the polyspectra of x_t and ϵ_t.

6 Application to estimation of ARMA parameters

It is beyond the scope of this paper to analyze the statistical performance of the identification algorithms based on sample third- or fourth-order cumulants proposed in the literature. Instead, we unveil the influence of colored additive noise on the potential asymptotic performance of such an arbitrary algorithm. In that purpose, asymptotic lower bound for the covariance of third- or fourth-order estimators and asymptotic covariance of an arbitrary third- or fourth order-based algorithm are considered where a special attention is given on the statistics involved.

6.1 Asymptotic lower bound on the covariance

To apply the notion of asymptotic minimum variance (AMV) estimators [5] (also called asymptotically best consistent estimators in [15]), the involved sample cumulants $\mathbf{c}^y(T)$ must satisfy three conditions:

- If Θ denotes the real-valued parameters (real and imaginary parts in the case of complex processes) of noisy ARMA model, Θ must be identifiable from $\mathbf{c}^y(\Theta)$ in the following sense: $\mathbf{c}^y(\Theta) = \mathbf{c}^y(\Theta') \Rightarrow \Theta = \Theta'$ [5].

[5] We note that the definition of Θ depends on the choice of the cumulants c^y and the a priori knowledge on the distribution of the measurement noise ϵ_t.

- The involved third- or fourth-order algorithms considered as mappings which associate to $\mathbf{c}^y(T)$, the estimate $\Theta(T)$: $\mathbf{c}^y(T) \longmapsto \Theta(T) = \text{alg}(\mathbf{c}^y(T))$ must be real [resp. complex] differentiable w.r.t. $\mathbf{c}^y(T)$ at the point $\mathbf{c}^y(\Theta)$ for real-[resp. complex] valued processes.
- The covariance $\mathbf{\Sigma}$ of the asymptotic distribution of the sample cumulants $\mathbf{c}^y(T)$ must be nonsingular.

These two latter conditions do not raise any problem for real-valued processes. However, for complex-valued processes, $\mathbf{c}^y(T)$ must collect real-valued cumulants (e.g., $c^y_0(T)$ and $c^y_{0,0,0}(T)$) and complex valued cumulants and their conjugate (e.g., $c^y_k(T)$ and $c^{y*}_k(T)$ for $k \neq 0$) to satisfy the second condition (see [16]). In addition, to satisfy the third condition, redundant cumulant samples must be withdrawn. In these conditions, the asymptotic covariance $\mathbf{C}_\Theta$ of an estimator of Θ given by an arbitrary third- or fourth-order algorithm is bounded below by the real symmetric positive definite matrix $[\mathbf{F}^H(\Theta)\mathbf{\Sigma}^{-1}(\Theta)\mathbf{F}(\Theta)]^{-1}$:

$$\mathbf{C}_\Theta \geq [\mathbf{F}^H(\Theta)\mathbf{\Sigma}^{-1}(\Theta)\mathbf{F}(\Theta)]^{-1} \tag{6.1}$$

where $\mathbf{F}(\Theta) \stackrel{\text{def}}{=} \frac{d\mathbf{c}^y(\Theta)}{d\Theta}$. Furthermore, there exists a nonlinear least square algorithm (dubbed the AMV algorithm [5]) whose covariance of the asymptotic distribution of the estimate of Θ satisfies (6.1) with equality. In practice, ϵ_t is Gaussian distributed and if third- or fourth-order cumulants are considered, the parametrization Θ can be partitioned as $\Theta = [\Theta_1^T, \Theta_2^T]^T$ where Θ_1 collects the parameters of the ARMA filter of interest and where Θ_2 collects the parameters σ_u^2, κ_{3_u},...,σ_e^2 of the sequences u_t and e_t. Consequently, the covariance of the asymptotic distribution of the minimum variance third- or fourth-order ARMA estimator is given by the top left “ARMA corner” of $[\mathbf{F}^H(\Theta)\mathbf{\Sigma}^{-1}(\Theta)\mathbf{F}(\Theta)]^{-1}$. Then, because $\mathbf{c}^y(\Theta)$ is linear with respect to Θ_2, i.e., $\mathbf{c}^y(\Theta) = \mathbf{\Psi}(\Theta_1)\Theta_2$ implies $\mathbf{F} = [\mathbf{F}_1, \mathbf{\Psi}]$ with $\mathbf{F}_1 \stackrel{\text{def}}{=} \frac{\partial\mathbf{c}^y(\Theta)}{\partial\Theta_1}$, the matrix inversion lemma gives

$$\begin{aligned}
\mathbf{C}_{\Theta_1} &= \left(\mathbf{F}_1^H\mathbf{\Sigma}^{-1}\mathbf{F}_1 - \mathbf{F}_1^H\mathbf{\Sigma}^{-1}\mathbf{\Psi}\left[\mathbf{\Psi}^H\mathbf{\Sigma}^{-1}\mathbf{\Psi}\right]^{-1}\mathbf{\Psi}^H\mathbf{\Sigma}^{-1}\mathbf{F}_1\right)^{-1} \\
&= \left(\mathbf{F}_1^T\mathbf{\Sigma}^{-1/2}\mathbf{P}^\perp_{\mathbf{\Sigma}^{-1/2}\mathbf{\Psi}}\mathbf{\Sigma}^{-1/2}\mathbf{F}_1\right)^{-1},
\end{aligned} \tag{6.2}$$

where $\mathbf{P}^\perp_{\mathbf{\Sigma}^{-1/2}\mathbf{\Psi}}$ denotes the projector onto the orthogonal complement of the columns of $\mathbf{\Sigma}^{-1/2}\mathbf{\Psi}$.

6.2 Asymptotic covariance of an arbitrary third- or fourth-order estimator

The asymptotic performance of an arbitrary third or fourth order-based algorithm that estimates the ARMA parameters Θ_1 of a noisy ARMA model can be derived (see e.g., [13, Th.3.16]) from the asymptotic normality of $[\mathbf{c}^y_2(T), \mathbf{c}^y_3(T)]$ or $[\mathbf{c}^y_2(T), \mathbf{c}^y_4(T)]$:

$$\sqrt{T}\ (\Theta_1(T) - \Theta_1) \xrightarrow{\mathcal{L}} \mathcal{N}(\mathbf{0}; \mathbf{G}(\Theta)\mathbf{\Sigma}(\Theta)\mathbf{G}^H(\Theta))$$

where $\mathbf{G}(\Theta)$ is the differential of the algorithm considered as a mapping, evaluated at point Θ, and $\mathbf{\Sigma}(\Theta)$ is the asymptotic covariance matrix of the sample cumulants involved in the algorithm deduced from $\mathbf{C}^y_{2,2}$, $\mathbf{C}^y_{2,3}$, $\mathbf{C}^y_{3,3}$ for the third-order real case or $\mathbf{C}^y_{2,2}$, $\mathbf{C'}^y_{2,2}$, $\mathbf{C}^y_{2,4}$, $\mathbf{C'}^y_{2,4}$, $\mathbf{C}^y_{4,4}$, $\mathbf{C'}^y_{4,4}$ for the fourth-order complex case.

Our symbolic calculus translates the polyspectral expressions of these different asymptotic covariance matrices into rational fraction expressions w.r.t. the ARMA process x_t and AR(1) process ϵ_t coefficients, under the form of matlab function files. These files allow one to compute the numerical values of these matrices for particular values of the parameters and are available from the authors upon request. They allow the interested practitioner to evaluate the performance of third- or fourth-order algorithms by simple computation of the differential $\mathbf{G}(\Theta)$ of the algorithm and selection of the $\mathbf{\Sigma}(\Theta)$ involved. The programs giving the numerical values of $(\mathbf{C}^y_{i,j})_{i,j=2,3}$ and $(\mathbf{C}^y_{i,j}, \mathbf{C'}^y_{i,j})_{i,j=2,4}$ are built along the following steps. First, each polyspectral integral expression obtained in Section 4 and 5 is symbolically expressed as functions of $H(f) \stackrel{\text{def}}{=} \sum_n h_n e^{-i2\pi nf}$ and of the transfer function $G(f)$ of the measurement noise generator driven by the independent Gaussian sequence e_t of power σ_e^2, thanks to the relations (see e.g., [4, rel. (C-24)]) extended to the complex case:

$$\begin{aligned}
S_y(f) &= \sigma_u^2 H^*(-f)H(-f) + \sigma_e^2 G^*(-f)G(-f), \\
S_y(f_1,\ldots,f_p) &= \kappa_{p+1_u} H(f_1)H(f_2)\ldots H(f_p)H(-f_1 - f_2 \ldots - f_{p-1}), \quad p > 1 \text{ (real case)} \\
S_y(f_1,\ldots,f_{2p-1}) &= \kappa_{2p_u} H^*(-f_1)H(f_2)\ldots H^*(-f_{2p-1})H(-f_1 - f_2 \ldots - f_{2p-1}), \quad p > 1 \text{ (complex case).}
\end{aligned}$$

Second, the transfer functions $H(f)$ and $G(f)$ are expressed in terms of the parameters of the MA process x_t and AR process ϵ_t (e.g., a and b in Section 6.3) and, finally, the polyspectral integrals are symbolically computed as functions of the MA and AR parameters for $(L_i)_{i=1,\dots,4}$ fixed, thanks to the relation

$$\int_{\triangle^k} \sum_{\alpha_1} \dots \sum_{\alpha_k} c_{\alpha_1 \dots \alpha_k} e^{-i2\pi(\alpha_1 f_1 + \dots + \alpha_k f_k)} df_1 df_2 \dots df_k = c_{0 \dots 0},$$

where $\sum_{\alpha_1} \dots \sum_{\alpha_k} c_{\alpha_1 \dots \alpha_k} e^{-i2\pi(\alpha_1 f_1 + \dots + \alpha_k f_k)}$ are deduced from the polyspectral expressions evaluated as functions of $H(f)$ and $G(f)$.

6.3 Illustrative numerical examples

As examples, three experiments are proposed for which noisy MA(1) or MA(2) process are considered. The SNR is defined as $SNR(dB) = 10log_{10}(\mathrm{E}(x_t^2)/\mathrm{E}(\epsilon_t^2))$. In the two first experiments, the processes are real-valued, the input u_t is exponentially distributed with mean adjusted to zero, power σ_u^2 and $\kappa_{k_u} = (k-1)!\sigma_u^k$ and the measurement noise ϵ_t is either Gaussian i.i.d. or Gaussian AR(1), ($\epsilon_t = e_t + b\epsilon_{t-1}$ where e_t is Gaussian i.i.d.).

In the first experiment an MA(1) is considered where $\Theta_1 = a$ and $\Theta_2 = \kappa_{3_u}$. Fig.1 shows the normalized asymptotic lower bound [6] for the asymptotic variance of estimates of a based on the third-order diagonal [7] cumulants $\{c_{k,k}^y(T); k = 0, \dots, L-1\}$ and $\{c_{0,k}^y(T); k = 1, \dots, L-1\}$ [8] as a function of the SNR for different values of L for white noise. For $k \geq 2$, the sample cumulants $c_{k,k}^y(T)$ and $c_{0,k}^y(T)$ are consistent estimates of zero. But nevertheless, Fig.1 shows that they contribute to improve the performance. This extends to noisy processes an observation shown in the noise-free case in [5].

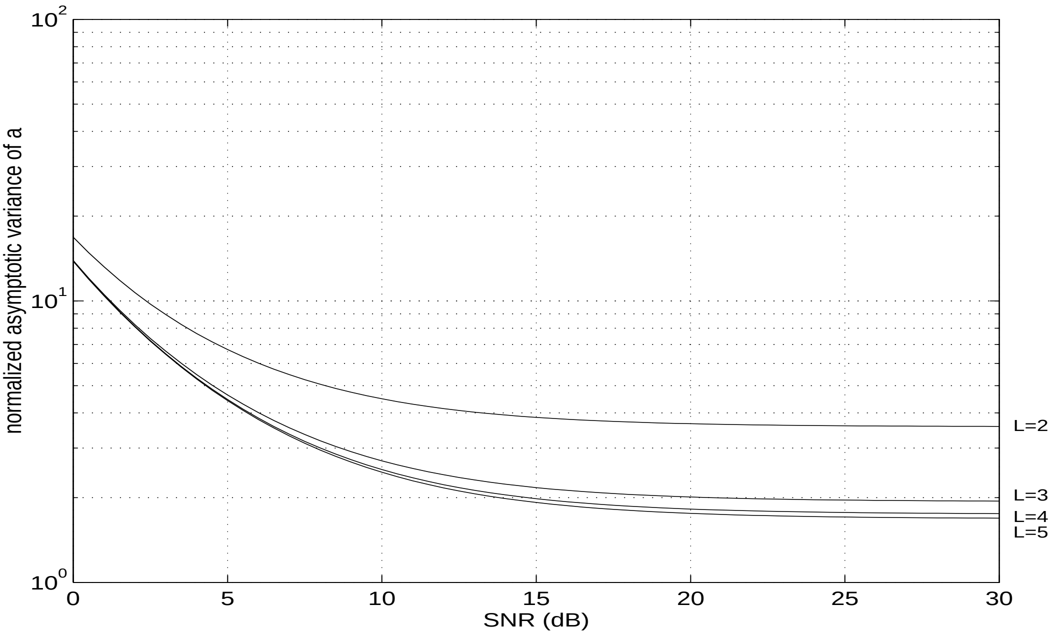

Fig.1 Third-order normalized lower bound for the asymptotic variance of estimates of a as a function of the SNR for different values of L.

Figs.2 and 3 show the lower bound for the asymptotic variance of estimates of a ($a = 0.5$) based on the sample third-order cumulants $(c_{0,0}^y(T), c_{0,1}^y(T), c_{1,1}^y(T))$ and the asymptotic theoretical and empirical variance given by the so-called $C(k,q)$ formula of Giannakis [3] ($a(T) = c_{1,1}^y(T)/c_{0,1}^y(T)$) respectively for white and AR(1) noise ($b = 0.99$) as a function of the SNR. Because $\mathbf{G}(\Theta) = \left[-\frac{1}{\kappa_{3_u}}, \frac{1}{a\kappa_{3_u}}\right] = \frac{(1+a^2)^{3/2}}{2\sigma_x^3}\left[-1, \frac{1}{a}\right]$, we note that from the

[6]The normalized asymptotic lower bounds and asymptotic theoretical variances are computed for $T = 1$. That means that the actual asymptotic lower bounds and asymptotic theoretical variances are obtained from the results given here by dividing by T.

[7]We restrict this example to diagonal cumulants because the off-diagonal third-order cumulants carry almost no additional information beyond the information in the diagonal ones in the MA(1) process case, as it was shown in the noise-free case in [5].

[8]The cumulants in the second set are called diagonal because of the relationship $c_{0,k}^y = c_{-k,-k}^y$.

expression of $\mathbf{C}^y_{3,3}$ given in Section 4.2, we get for the $C(k,q)$ estimator:

$$\lim_{T\to\infty} T\mathrm{Var}(a(T)) = v_0 + \left[\beta_1(\frac{\sigma_\epsilon^2}{\sigma_x^2}) + \beta_2(\frac{\sigma_\epsilon^4}{\sigma_x^4}) + \beta_3(\frac{\sigma_\epsilon^6}{\sigma_x^6})\right] \tag{6.3}$$

where v_0 does not depend on the powers of x_t and ϵ_t. We see from these two figures that below a certain threshold (which is typically related to the noise spectrum), the asymptotic variance of the estimate of a grows rapidly with the noise level. Beyond this threshold (for SNR$>$ $10dB$ [resp. SNR$>$ $20dB$] for white noise [resp. AR(1) noise]), the asymptotic variance is approximately constant. This proves that the AMV3 and the $C(k,q)$ algorithms are insensitive to noise in a large domain. Furthermore, we note that contrary to the asymptotic lower bound, the asymptotic variance given by the $C(k,q)$ algorithm strongly degrades for sharp resonant AR(1) noise compared to white noise of the same power.

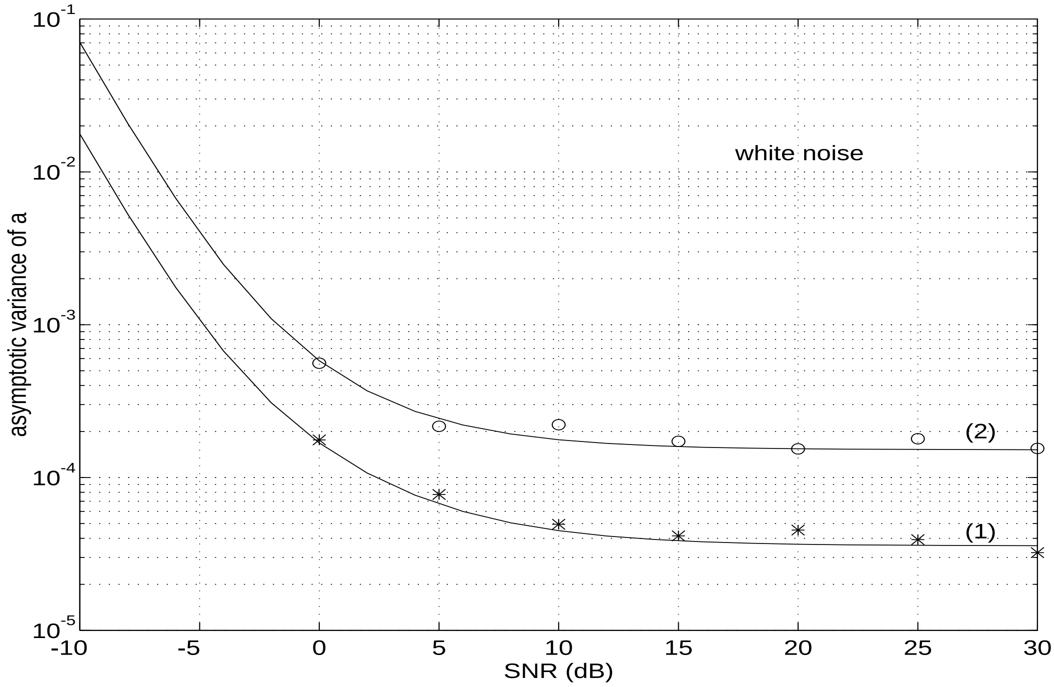

Fig.2 Third-order asymptotic lower bound (1) and asymptotic theoretical and empirical (averaged on 100 independent Monte Carlo runs) variance given by the $C(k,q)$ formula (2) as a function of the SNR for $T = 10^5$.

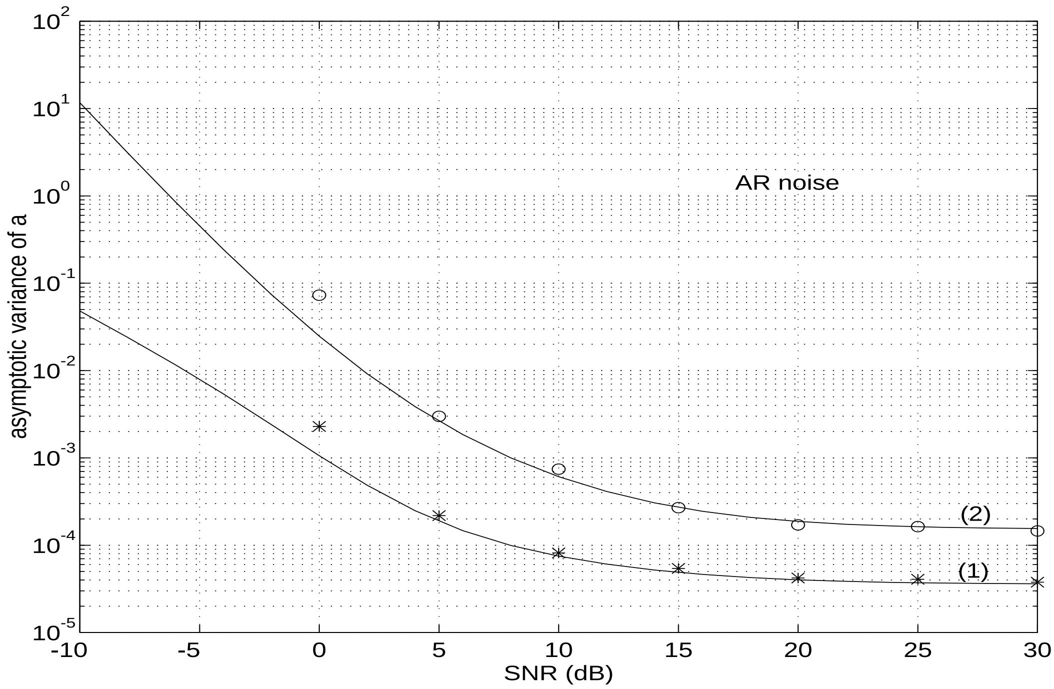

Fig.3 Third-order asymptotic lower bound (1) and asymptotic theoretical and empirical (averaged on 100 independent Monte Carlo runs) variance given by the $C(k,q)$ formula (2) as a function of the SNR for $T = 10^5$.

Figs.4 and 5 show the asymptotic lower bound and the asymptotic theoretical and empirical variance given by the $C(k,q)$ algorithm as a function of the parameter b of the AR(1) noise ϵ_t and of the parameter a of the MA(1) process x_t, respectively. We see that the performance is very sensitive to the spectrum of the MA(1) process x_t, but relatively insensitive to the spectrum of the additive noise ϵ_t except when b approaches 1 [see (4.9)] where the performance of any third order-based algorithm dramatically degrades. We note that the asymptotic variance given by the $C(k,q)$ algorithm attains the asymptotic lower bound for $a = 1$ [9] and is inadequate to estimate a parameter a close to zero.

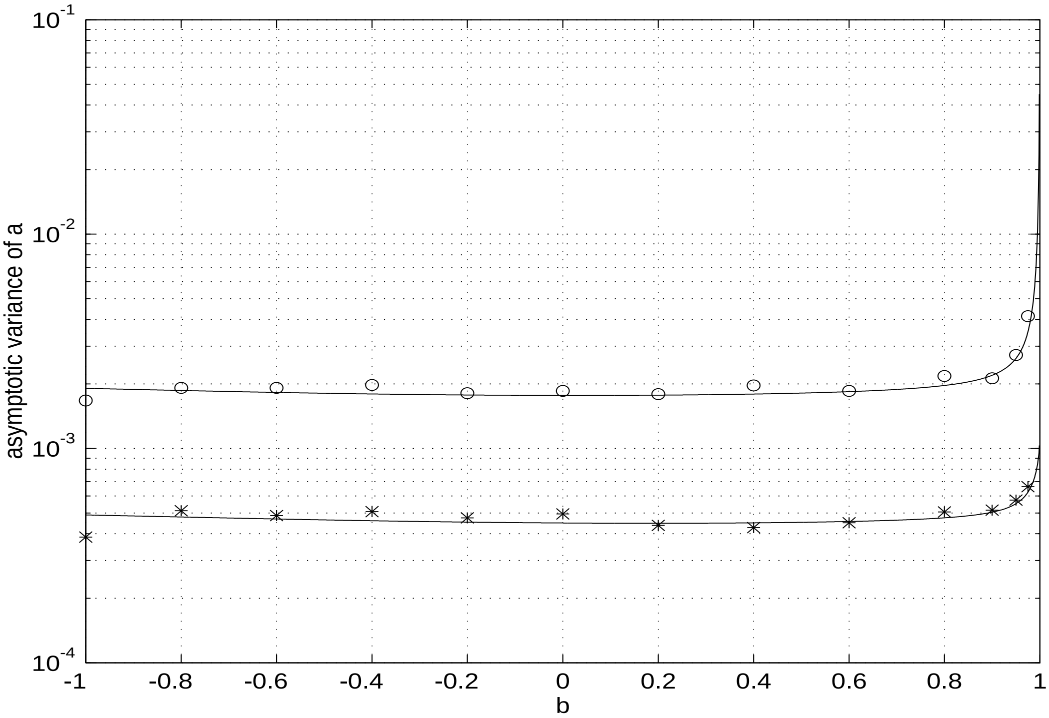

Fig.4 Third-order asymptotic lower bound (1) and asymptotic theoretical and empirical (averaged on 1000 independent Monte Carlo runs) variance given by the $C(k,q)$ formula (2) as a function of the noise parameter b for $a = 0.5$, $SNR = 10dB$ and $T = 10^4$.

[9]This property has been confirmed for all values of b and SNR, but we have not succeded in proving it analytically.

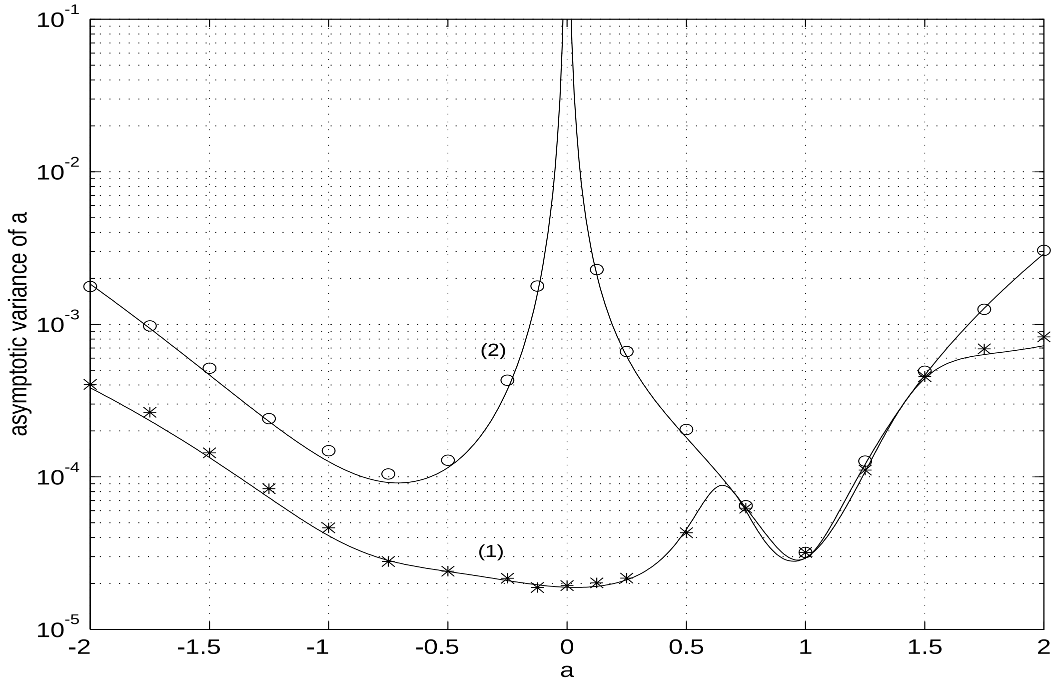

Fig.5 Third-order asymptotic lower bound (1) and asymptotic theoretical and empirical (averaged on 1000 independent Monte Carlo runs) variance given by the $C(k,q)$ formula (2) as a function of the MA parameter a for $b = 0.5$, $SNR = 10dB$ and $T = 10^4$.

Figs.6a and 6.b show the theoretical asymptotic lower bound and the empirical asymptotic variance given by the AMV3 and the $C(k,q)$ algorithms as a function of the number T of samples for two values of SNR. We see that the domain of validity of our asymptotic analysis roughly do not depend on the algorithm, but is sensitive to the SNR. Naturally this domain of validity increases with increasing SNR ($T > 4000$ for $SNR = 0dB$ and $T > 1000$ for $SNR = 20dB$).

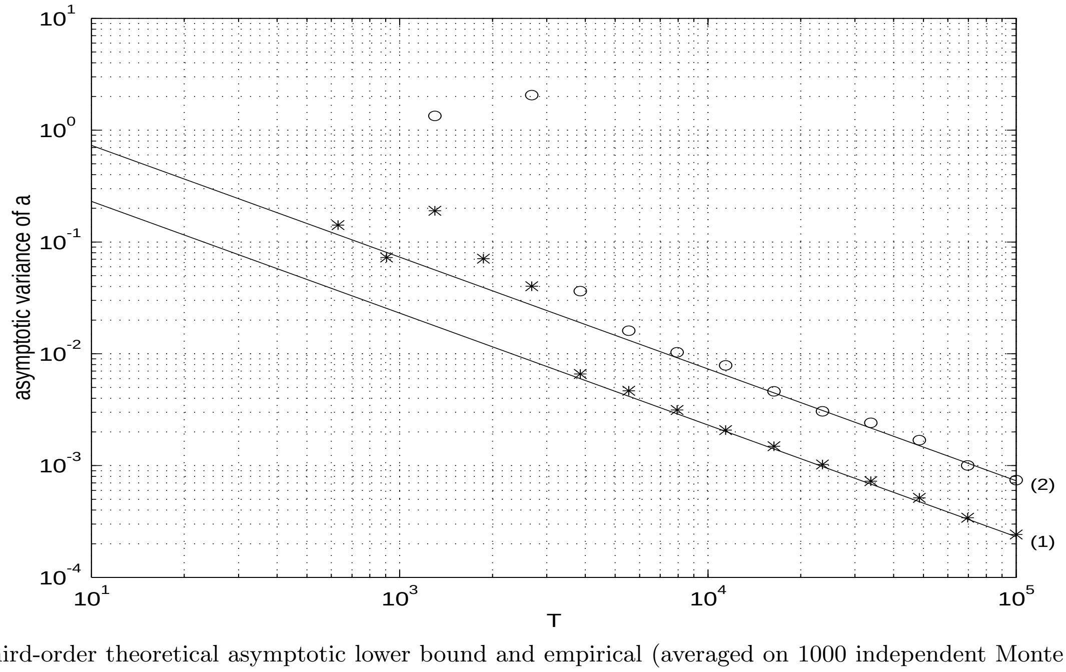

Fig.6a Third-order theoretical asymptotic lower bound and empirical (averaged on 1000 independent Monte Carlo runs) asymptotic variance given by the AMV3 (1) and $C(k,q)$ (2) algorithms as a function of T, for $a = 0.5$, $b = 0.5$ and SNR=0dB.

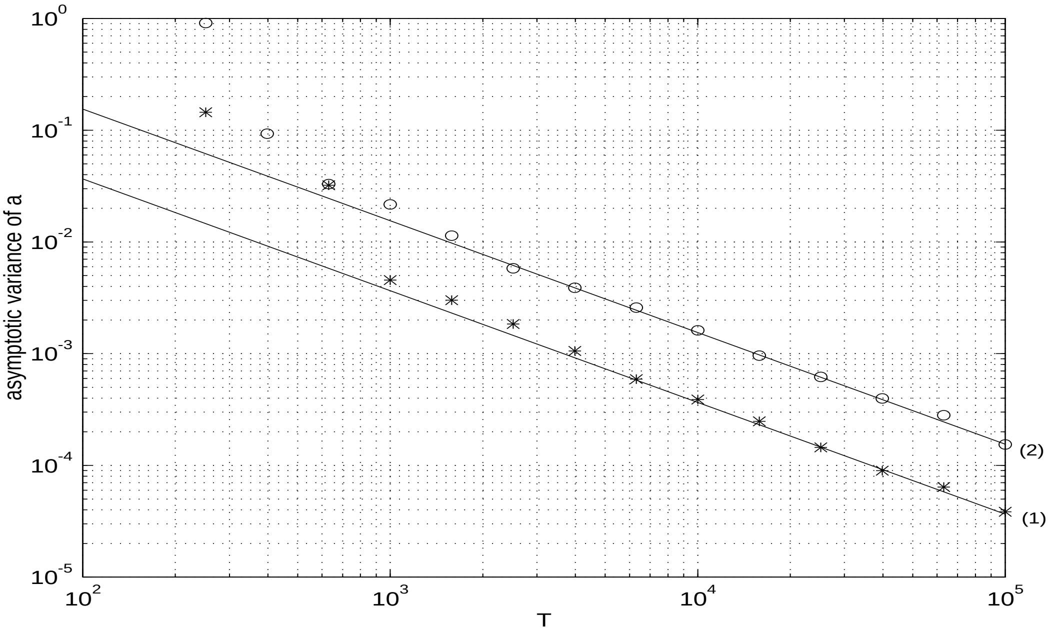

Fig.6b Third-order theoretical asymptotic lower bound and empirical (averaged on 1000 independent Monte Carlo runs) asymptotic variance given by the AMV3 (1) and $C(k,q)$ (2) algorithms as a function of T, for $a = 0.5$,$b = 0.5$ and SNR=20dB.

In the second experiment an MA(2) process is considered where $\Theta_1 = (a_1, a_2)^T$ and $\Theta_2 = \kappa_{3_u}$. Fig.7a [resp. 7b] shows the normalized asymptotic lower bound and theoretical variance given by the $C(k,q)$ algorithm for the estimated parameters a_1 [resp. a_2] as a function of the parameters a_2 [resp. a_1] of the MA(2) process x_t. As for the MA(1) case, we see that the performance is very sensitive to the value of the MA parameters. As for MA(1), we see that the performance is very sensitive to the spectrum of x_t. We note that contrary to the MA(1) case, no algorithm is adequate to estimate a parameter a_2 close to zero. Furthermore, the performance of the $C(k,q)$ algorithm is practically uniformly optimal among the class of third order-based algorithms.

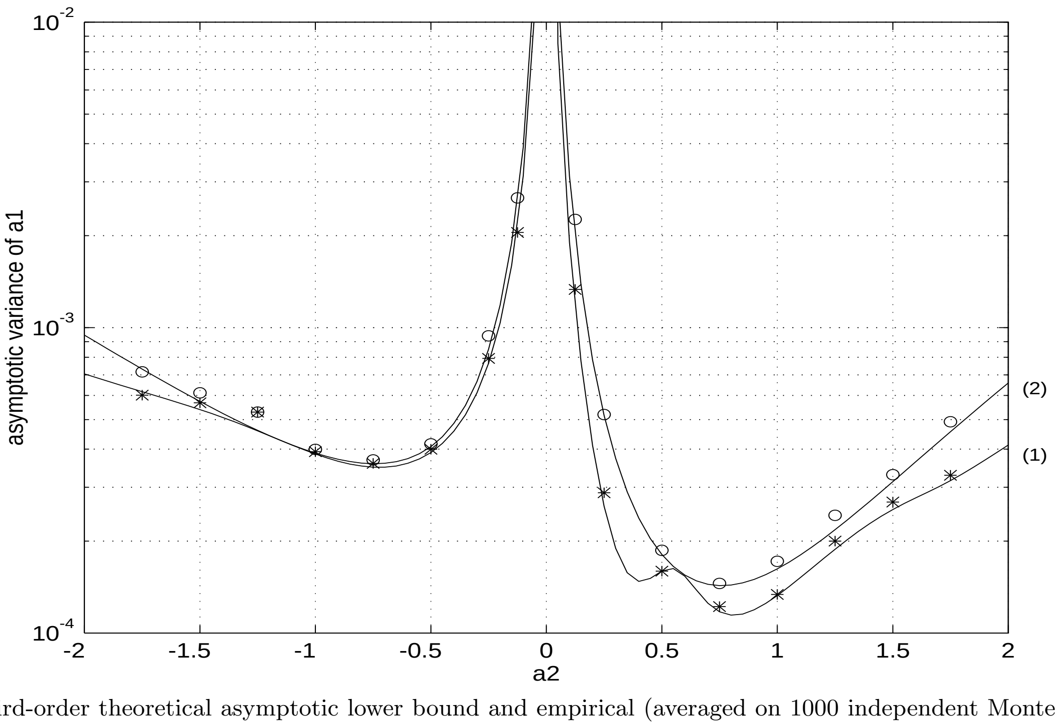

Fig.7a Third-order theoretical asymptotic lower bound and empirical (averaged on 1000 independent Monte Carlo runs) asymptotic variance given by the AMV3 and $C(k,q)$ algorithms for the estimated parameter a_1 as a function of a_2 for

$a_1 = 0.8$, $b = 0.5$, $SNR = 10dB$ and $T = 10^5$.

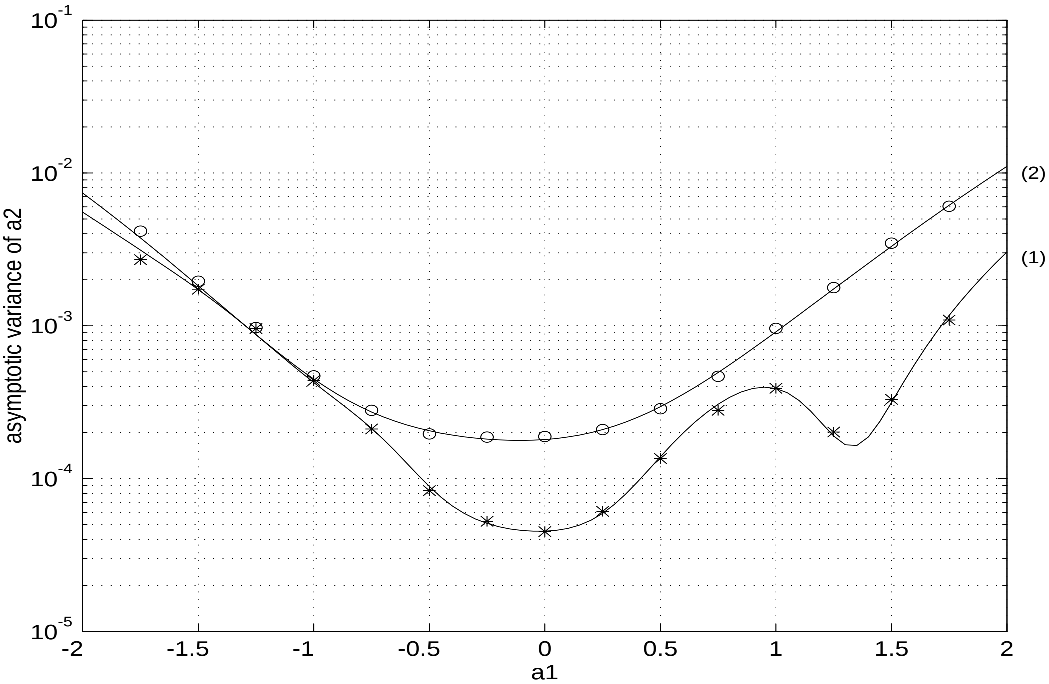

Fig.7b Third-order theoretical asymptotic lower bound and empirical (averaged on 1000 independent Monte Carlo runs) asymptotic variance given by the AMV3 and $C(k,q)$ algorithms for the estimated parameter a_2 as a function of a_1 for $a_2 = 0.5$, $b = 0.5$, $SNR = 10dB$ and $T = 10^5$.

In the third experiment, the processes are complex-valued, the input u_t is a 8PSK modulation with power σ_u^2 and $\kappa_{4_u} = -\sigma_u^4$, $\kappa_{6_u} = 4\sigma_u^6$, $\kappa_{8_u} = -33\sigma_u^8$ [10] and ϵ_t is either complex circular Gaussian i.i.d. or AR(1) ($\epsilon_t = \epsilon_{t-1} + be_t$ where e_t complex circular Gaussian i.i.d.).

Figs.8 and 9 show the normalized lower bound for the asymptotic variance ($\mathrm{Tr}(\mathbf{C}_{\Theta_1})$) of estimates of a ($a = 0.5e^{i\frac{\pi}{4}}$) based on the sample fourth-order cumulants $\mathbf{c}^y(T) \stackrel{\text{def}}{=} \{c^y_{0,0,0}(T), c^y_{0,0,1}(T), c^y_{0,1,0}(T), c^y_{0,1,1}(T), c^y_{1,0,1}(T), c^y_{1,1,1}(T)\}$ [11] and the normalized asymptotic variances given by the so-called $C(k,q)$ formula (1) extended to the complex case and by the following modified $C(k,q)$ formulae (2)(3):

- $a(T) = c^y_{0,1,1}(T)/c^y_{0,0,1}(T))$ (1),
- $a(T) = c^y_{1,0,1}(T)^*/c^y_{0,0,1}(T)^*$ because $c_{1,0,1} = \kappa_{4_u} a^{*2}$ and $c_{0,0,1} = \kappa_{4_u} a^*$ (2),
- $a(T) = c^y_{1,1,1}(T)/c^y_{1,0,1}(T)$ because $c_{1,1,1} = \kappa_{4_u} a^* |a|^2$ and $c_{1,0,1} = \kappa_{4_u} a^{*2}$ (3),

for white and AR(1) noise ($b = 0.999$) as a function of the SNR. We note that here $\Theta_1 = (\Re(a), \Im(a))^T$ and $\Theta_2 = \kappa_{4_u}$. Similarly to the third-order case, we see from these two figures that beyond a certain threshold, the asymptotic lower bound and the asymptotic variance of the estimate of a grows rapidly with the noise level. Beyond this threshold (for about SNR$> 15dB$), the asymptotic variance is approximately constant. Furthermore, we see

[10]Because for up to fourth-order circular processes $\kappa_{6_u} = \mathrm{E}|u_t^6| - 9\ \mathrm{E}|u_t^4|\mathrm{E}|u_t^2| + 2 \times 6\ (\mathrm{E}|u_t^2|)^3$ and $\kappa_{8_u} = \mathrm{E}|u_t^8| - 16\ \mathrm{E}|u_t^6|\mathrm{E}|u_t^2| - 18\ \mathrm{E}|u_t^4|\mathrm{E}|u_t^4| + 2 \times 72\ \mathrm{E}|u_t^4|(\mathrm{E}|u_t^2|)^2 - 6 \times 24\ (\mathrm{E}|u_t^2|)^4$.

[11]Because $c^y_{k_1,k_2,k_3} \stackrel{\text{def}}{=} \mathrm{Cum}(y_0, y^*_{k_1}, y_{k_2}, y^*_{k_3})$, we note that, $c^y_{0,0,0}(T)$, $c^y_{0,1,1}(T) = c^y_{1,1,0}(T)$ are real valued, $c^y_{0,0,1}(T) = c^y_{1,0,0}(T) = {c^y_{0,1,0}}^*(T)$ are complex valued. Consequently the statistic $c^y(T)$ used for the AMV4 estimator is composed of $\{c^y_{0,0,0}(T), c^y_{0,1,1}(T), c^y_{0,0,1}(T), c^y_{1,0,1}(T), c^y_{1,1,1}(T), {c^y_{0,0,1}}^*(T), {c^y_{1,0,1}}^*(T), {c^y_{1,1,1}}^*(T)\}$.

that below this threshold, the asymptotic variances given by all the $C(k,q)$ algorithms grow more rapidly than in the third-order with the noise level as anticipated from the relation (6.3) extented to the fourth-order case

$$\lim_{T\to\infty} T\mathrm{Var}(a(T)) = v_0 + \left[\beta_1(\frac{\sigma_\epsilon^2}{\sigma_x^2}) + \beta_2(\frac{\sigma_\epsilon^4}{\sigma_x^4}) + \beta_3(\frac{\sigma_\epsilon^6}{\sigma_x^6}) + \beta_4(\frac{\sigma_\epsilon^8}{\sigma_x^8})\right].$$

We see that the $C(k,q)$ formula outperforms the modified $C(k,q)$ formulae except for small SNR where their performance is similar. Furthermore, we note that contrary to all the $C(k,q)$ formulae whose performance degrades for AR(1) noise compared to white noise of the same power, the AMV estimate improves for small SNR. Consequently, there must be fourth-order algorithms much more efficient than the $C(k,q)$ formulae for small SNR.

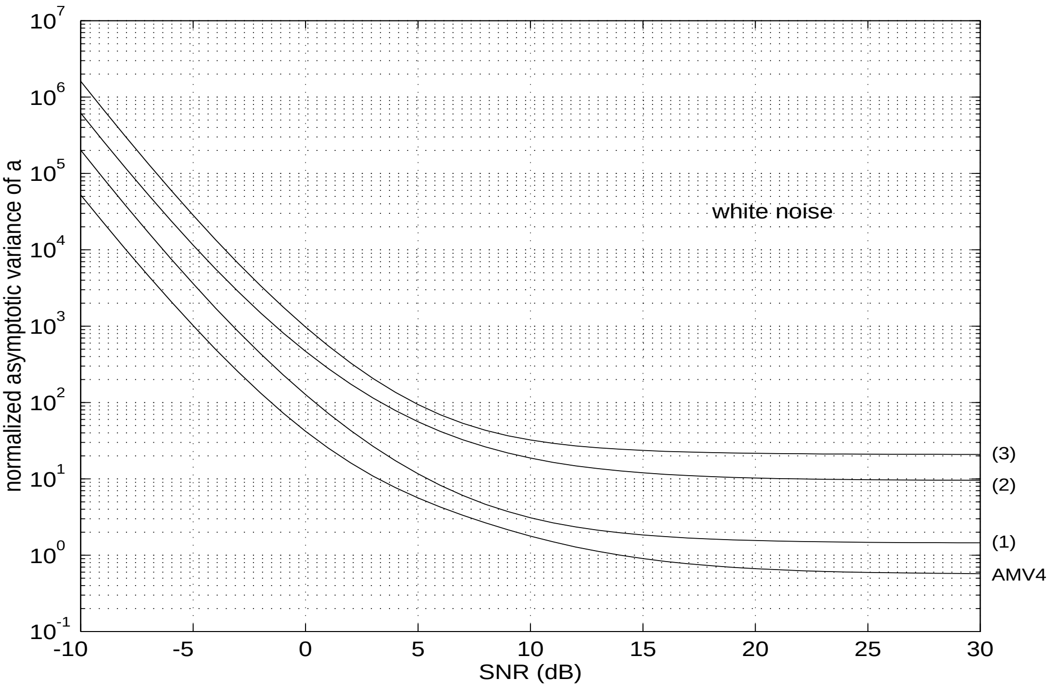

Fig.8 Fourth-order normalized asymptotic lower bound (AMV4) and variances given by the $C(k,q)$ formulae (1)(2)(3) as a function of the SNR.

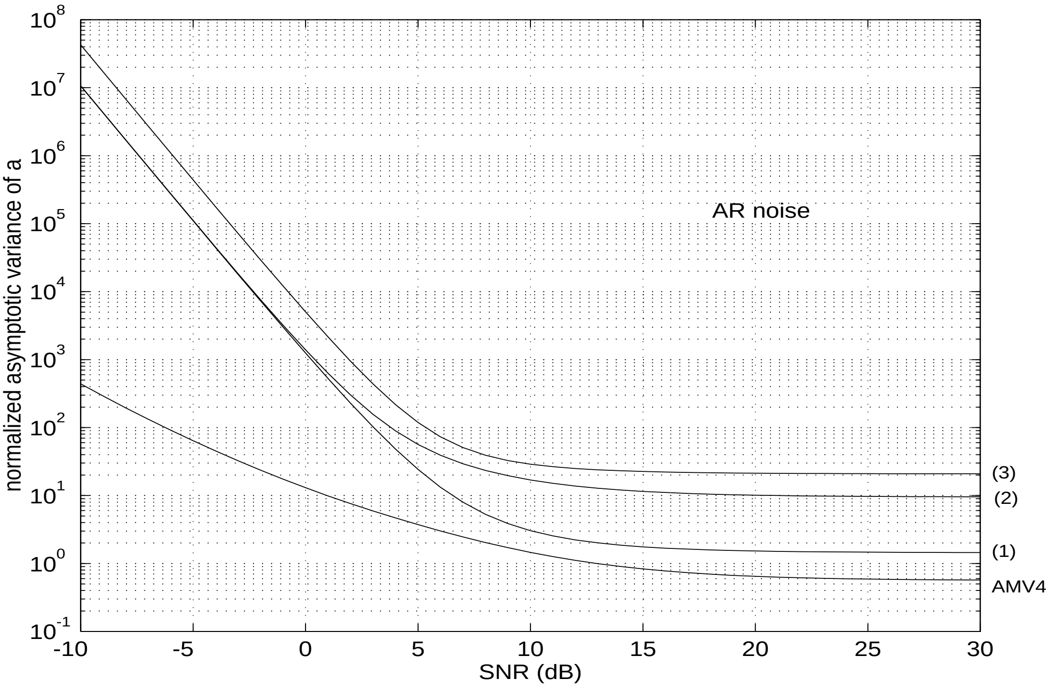

Fig.9 Fourth-order normalized asymptotic lower bound (AMV4) and variances given by the $C(k,q)$ formulae (1)(2)(3) as a function of the SNR.

Fig.10 shows the normalized asymptotic lower bound based on $\mathbf{c}^y(T)$ (AMV4), the normalized asymptotic lower bound based on the statistics used in the $C(k,q)$ formula (1) and the modified $C(k,q)$ formula (2), i.e., based on $\mathbf{c}^{y'}(T) \stackrel{\text{def}}{=} \{c^y_{0,1,1}(T), c^y_{0,0,1}(T), c^y_{1,0,1}(T), {c^y_{0,0,1}}^*(T), {c^y_{1,0,1}}^*(T)\}$ (AMV4') and the normalized asymptotic variance given by the $C(k,q)$ formula (1) and by the modified $C(k,q)$ formulae (2) and (3) as a function of the parameter b of the AR(1) noise ϵ_t. Because these variances are relatively constant and symmetric w.r.t. zero, we focus on the [0.999 1] domain of b where only the performance of the $C(k,q)$ formulae degrades when b approaches ± 1. The AMV4 and the AMV4' are relatively insensitive to the spectrum of the additive noise ϵ_t including in the immediate vicinity of ± 1.

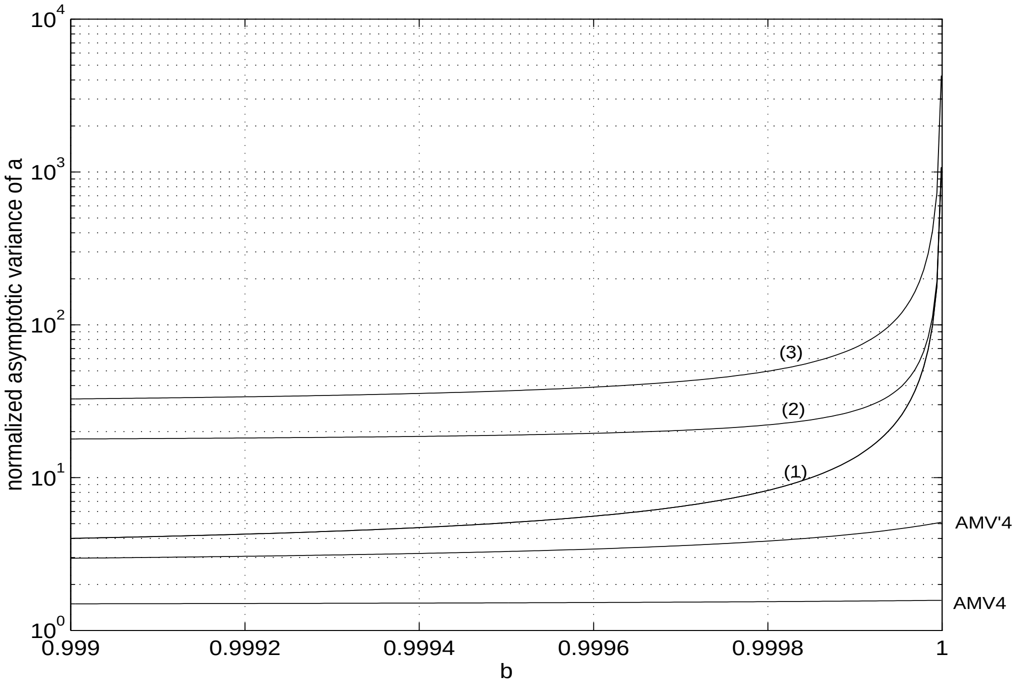

Fig.10 Fourth-order normalized asymptotic lower bounds (AMV4) (AMV4') and normalized asymptotic variance given by the $C(k,q)$ formulae (1)(2)(3) as a function of the noise parameter b for $a = 0.5e^{i\frac{\pi}{4}}$ and $SNR = 10dB$.

Fig.11 shows the asymptotic lower bound based on $c^y(T)$ (AMV4) and the asymptotic theoretical and empirical variance given by the $C(k,q)$ formula (1), by the modified $C(k,q)$ formulae (2) and (3) as a function of the parameter a' ($a = a'e^{i\frac{\pi}{4}}$ with $a' \in (-2,+2)$) of the MA(1) process x_t. We see that the performance is very sensitive to the spectrum of the MA(1) process x_t. Furthermore, we note that there is no uniformly minimum variance estimator among the three $C(k,q)$ formulae which are inadequate to estimate a parameter a close to zero.

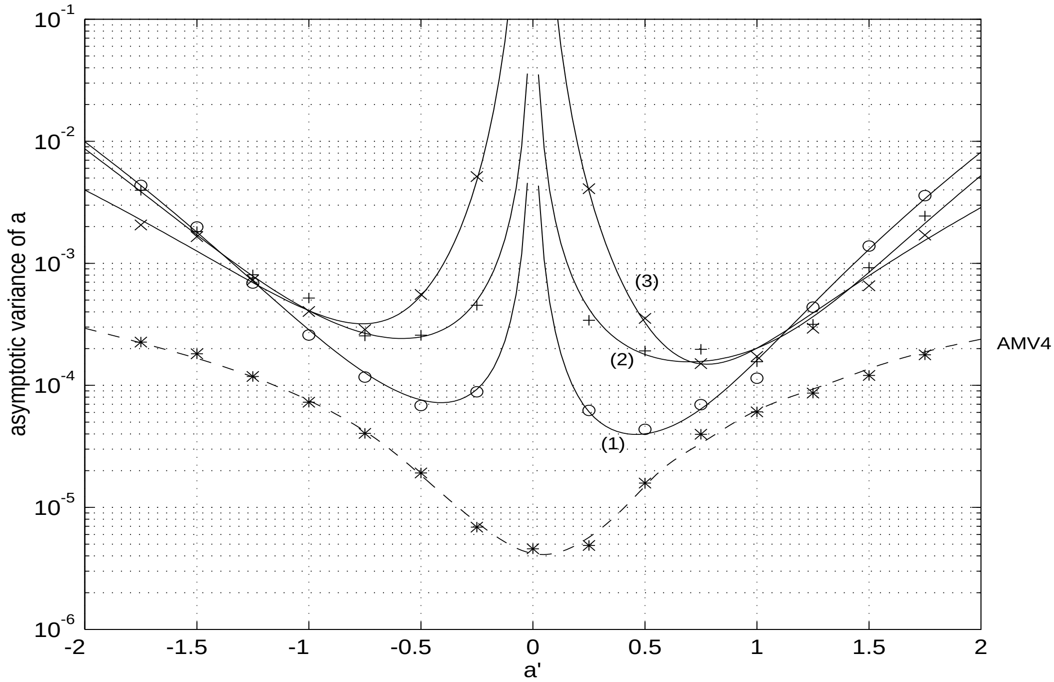

Fig.11 Fourth-order asymptotic lower bound (AMV4) and asymptotic theoretical and empirical (averaged on 100 independent Monte Carlo runs) variance given by the $C(k,q)$ formulae (1),(2) and (3) as a function of the MA parameter a' for $b = 0.999$, $SNR = 10dB$ and $T = 10^5$.

Finally, Fig.12 shows the normalized asymptotic lower bound for the asymptotic variance of estimates of a computed for three different cases: when the statistic $\mathbf{c}^y(T)$ consists of the sample covariances only [12] $\{c_0^y(T), c_1^y(T), c_1^y(T)^*\}$ (AMV2), when the statistic $\mathbf{c}^y(T)$ consists of the sample fourth-order cumulants only $\{c^y_{0,0,0}(T), c^y_{0,1,1}(T), c^y_{0,0,1}(T), c^y_{1,0,1}(T), c^y_{1,1,1}(T), {c^y_{0,0,1}}^*(T),\ {c^y_{1,0,1}}^*(T), {c^y_{1,1,1}}^*(T)\}$ (AMV4) and when the statistic $\mathbf{c}^y(T)$ consists of the preceding sample covariances and sample fourth-order cumulants (AMV24). This figure also exhibits the normalized asymptotic variance given by the $C(k,q)$ formulae (1)(2)(3). In order for the parameter a to be identifiable from the sample covariance only, this figure is drawn in the noise free case. As we see from this figure, there is a considerable amount of information in the fourth-order sample cumulants compared to the information in the sample covariances. Furthermore, we note that contrary to the noisy case, the AMV vanishes for a parameter a close to zero.

[12]As estimation methods based on the sample covariances only cannot distinguish between non-minimum phase and minimum phase processes having the same spectrum, the bound for this case applies only to estimators that are based on prior knowledge of the zero locations within a sufficiently small error.

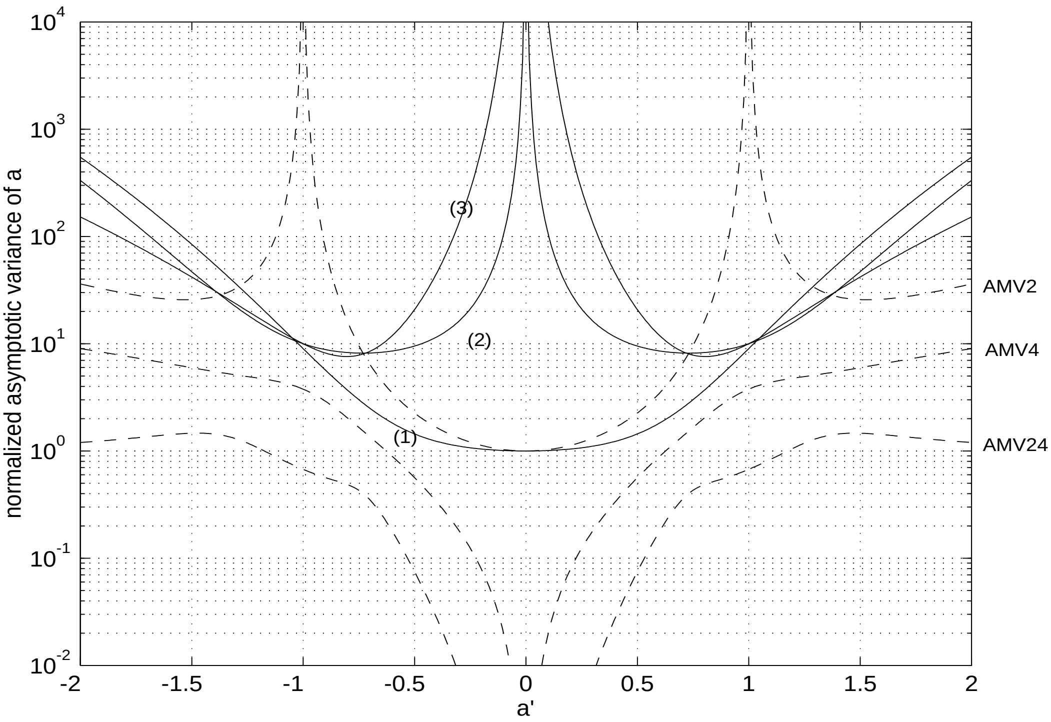

Fig.12 Fourth-order normalized asymptotic lower bound based on sample covariances only (AMV2), on sample fourth-order cumulants only (AMV4) and on sample covariances and fourth-order cumulants (AMV24) and normalized asymptotic variance given by the $C(k,q)$ formulae (1)(2)(3) as a function of the MA parameter a'.

Naturally, these examples are totally inadequate for predicting the asymptotic performance of a specific third- or fourth-order algorithm in the presence of additive colored noise. They simply show the potentially large sensitivity of the asymptotic performance of an arbitrary algorithm to the spectra of the linear process and of the colored noise.

7 Conclusion

This paper has extended Bartlett's formula to the third- and fourth-order and to noisy linear processes thanks to a polyspectral approach and a symbolic calculus akin a high level language. As an application of these closed-form expressions, the sensitivity of the asymptotic performance of the estimated ARMA parameters by an arbitrary third or fourth order-based algorithm to the SNR, the spectra of the linear process, and the colored additive noise is addressed. Such sensitivity analysis has been possible thanks to the numerical expressions derived from our theoretical expressions, whereas Monte-Carlo simulations have accommodated only particular scenarios until now. As an example, the asymptotic lower bound for the variances of third- or fourth-order algorithms are compared to the asymptotic variances given by the so-called $C(k,q)$ algorithms for non-Gaussian first or second-order MA processes corrupted by a Gaussian first-order AR process with respect to the SNR and to the MA and AR parameters. In particular we have shown that the performance presents a threshold effect with respect to the SNR and are very sensitive to the spectrum of the MA process, but relatively insensitive to the AR spectrum except for sharp resonances.

A Appendix: Proof of eq. (4.5)

Because $\int_\triangle e^{i2\pi f\tau}df=1$ if $\tau=0$ and $\int_\triangle e^{i2\pi f\tau}df=0$ if $\tau\neq 0$,

$$\begin{aligned}\sum_t \mathrm{Cum}(x_0,x_k,x_t,x_{t+l},x_{t+m}) &= \int_{\triangle^3}\sum_{t_1}\sum_t\sum_{t_2}\sum_{t_3} c^x_{t_1,t,t_2,t_3}e^{-i2\pi[f_1(t_1-k)+f_2(t_2-t-l)+f_3(t_3-t-m)]}df_1df_2df_3\\ &= \int_{\triangle^3} S_x(f_1,-f_2-f_3,f_2,f_3)e^{i2\pi[f_1k+f_2l+f_3m]}df_1df_2df_3\end{aligned}$$

and $S_x(f_1,-f_2-f_3,f_2,f_3)=S_x(f_1,-f_1,f_2,f_3)$ from [4].
Then looking at each term of $\sum_t \mathrm{Cum}(x_0,x_k,x_t)\mathrm{Cum}(x_{t+l},x_{t+m})$[10], there appears a term $c^x_kc^x_{l,m}$ which vanishes in (4.1), three terms of the form $\sum_t c^x_\alpha c^x_{\beta,t+\gamma}$ and six terms of the form $\sum_t c^x_{t+\alpha}c^x_{-t+\beta,\gamma}$ which become respectively

$$\begin{aligned}\sum_t c^x_\alpha c^x_{\beta,t+\gamma} &= c^x_\alpha\int_\triangle\sum_{t_1}\sum_t c^x_{t_1,t}e^{-i2\pi[f_1(t_1-\beta)+0t]}df_1\\ &= c^x_\alpha\int_\triangle S_x(f_1,0)e^{i2\pi f_1\beta}df_1=\int_{\triangle^2}S_x(f_1,0)S_x(f_2)e^{i2\pi[f_1\beta+f_2\alpha]}df_1df_2,\\ \sum_t c^x_{t+\alpha}c^x_{-t+\beta,\gamma} &= \int_\triangle[S_x(f)\int_\triangle S_x(f,f_1)e^{i2\pi[f\beta+f_1\gamma]}df_1]e^{i2\pi f\alpha}df=\int_{\triangle^2}S_x(f_1)S_x(f_1,f_2)e^{i2\pi[f_1(\alpha+\beta)+f_2\gamma]}df_1df_2\end{aligned}$$

because the second term is a convolution product of c^x_τ and $c^x_{\tau+\beta,\gamma}$ at the point $\tau=\alpha$, since $\int_\triangle S_x(f,f_1)e^{i2\pi[f\beta+f_1\gamma]}df_1$ is the Fourier transform of the sequence $(c^x_{\tau+\beta,\gamma})_{\tau\in Z}$. Combining these groups of terms, the matrix expression (4.5) is obtained.

B Appendix: Expressions of the different matrices $\mathbf{E}^{(i)}_j$ of Section 4

Our symbolic algorithm gives rels. (4.5) and (4.6) where matrices $\mathbf{E}^{(i)}_j$ are composed of finite sums of $e^{i2\pi(f_1a_1+f_2a_2+f_3a_3)}$ defined by the following:

$$\begin{aligned}
\mathbf{E}_{1,2,3} &\stackrel{\text{def}}{=} \mathbf{e}_1(\mathbf{e}_2^T\otimes\mathbf{e}_3^T)\\
\mathbf{E}^{(1)}_{1,2} &\stackrel{\text{def}}{=} \mathbf{e}_1^*(\mathbf{e}_2^T\otimes\mathbf{e}_1^T+\mathbf{e}_1^T\otimes\mathbf{e}_2^T+\mathbf{e}_{1,2}^H\otimes\mathbf{e}_2^T)+\mathbf{e}_1(\mathbf{e}_2^T\otimes\mathbf{e}_1^T+\mathbf{e}_1^T\otimes\mathbf{e}_2^T+\mathbf{e}_{1,2}^H\otimes\mathbf{e}_2^T)\\
\mathbf{E}^{(2)}_{1,2} &\stackrel{\text{def}}{=} \mathbf{e}_1(\mathbf{e}_2^T\otimes\mathbf{e}_2^H+\mathbf{e}_0^T\otimes\mathbf{e}_2^T+\mathbf{e}_2^T\otimes\mathbf{e}_0^T)\\
\mathbf{E}_{1,2,3,4} &\stackrel{\text{def}}{=} \mathbf{e}_1\mathbf{e}_3^T\otimes\mathbf{e}_2\mathbf{e}_4^T\\
\mathbf{E}^{(1)}_{1,2,3} &\stackrel{\text{def}}{=} \mathbf{e}_2\mathbf{e}_1^T\otimes\mathbf{e}_{2,3}^*\mathbf{e}_3^T+\mathbf{e}_2\mathbf{e}_3^T\otimes\mathbf{e}_{2,3}^*\mathbf{e}_1^T+\mathbf{e}_1\mathbf{e}_2^T\otimes\mathbf{e}_3\mathbf{e}_{2,3}^H+\mathbf{e}_3\mathbf{e}_2^T\otimes\mathbf{e}_1\mathbf{e}_{2,3}^H\\
&+ \mathbf{e}_1\mathbf{e}_3^T\otimes\mathbf{e}_3\mathbf{e}_2^T+\mathbf{e}_1\mathbf{e}_2^T\otimes\mathbf{e}_3\mathbf{e}_3^T+\mathbf{e}_3\mathbf{e}_3^T\otimes\mathbf{e}_1\mathbf{e}_2^T+\mathbf{e}_3\mathbf{e}_2^T\otimes\mathbf{e}_1\mathbf{e}_3^T+\mathbf{e}_{2,3}^*\mathbf{e}_{1,3}^H\otimes\mathbf{e}_2\mathbf{e}_1^T\\
\mathbf{E}^{(2)}_{1,2,3} &\stackrel{\text{def}}{=} \mathbf{e}_3\mathbf{e}_1^T\otimes\mathbf{e}_3^*\mathbf{e}_2^T+\mathbf{e}_0\mathbf{e}_1^T\otimes\mathbf{e}_3\mathbf{e}_2^T+\mathbf{e}_3\mathbf{e}_1^T\otimes\mathbf{e}_0\mathbf{e}_2^T+\mathbf{e}_1\mathbf{e}_3^T\otimes\mathbf{e}_2\mathbf{e}_3^H+\mathbf{e}_1\mathbf{e}_3^T\otimes\mathbf{e}_2\mathbf{e}_0^T+\mathbf{e}_1\mathbf{e}_0^T\otimes\mathbf{e}_2\mathbf{e}_3^T\\
\mathbf{E}^{(3)}_{1,2,3} &\stackrel{\text{def}}{=} \mathbf{e}_1\mathbf{e}_{2,3}^H\otimes\mathbf{e}_{2,3}\mathbf{e}_3^T+\mathbf{e}_1\mathbf{e}_3^T\otimes\mathbf{e}_{2,3}\mathbf{e}_{2,3}^H+\mathbf{e}_1\mathbf{e}_2^T\otimes\mathbf{e}_{2,3}\mathbf{e}_3^T+\mathbf{e}_{2,3}\mathbf{e}_{2,3}^H\otimes\mathbf{e}_1\mathbf{e}_3^T\\
&+ \mathbf{e}_{2,3}\mathbf{e}_3^T\otimes\mathbf{e}_1\mathbf{e}_{2,3}^H+\mathbf{e}_{2,3}\mathbf{e}_2^T\otimes\mathbf{e}_1\mathbf{e}_3^T+\mathbf{e}_1\mathbf{e}_2^T\otimes\mathbf{e}_{1,2,3}^*\mathbf{e}_3^T+\mathbf{e}_1\mathbf{e}_{2,3}^H\otimes\mathbf{e}_{1,2,3}^*\mathbf{e}_3^T\\
&+ \mathbf{e}_1\mathbf{e}_3^T\otimes\mathbf{e}_{1,2,3}^*\mathbf{e}_{2,3}^H\\
\mathbf{E}^{(3)}_{1,2} &\stackrel{\text{def}}{=} \mathbf{e}_1\mathbf{e}_2^T\otimes\mathbf{e}_0\mathbf{e}_2^H+\mathbf{e}_1\mathbf{e}_0^T\otimes\mathbf{e}_0\mathbf{e}_2^T+\mathbf{e}_1\mathbf{e}_2^T\otimes\mathbf{e}_0\mathbf{e}_0^T+\mathbf{e}_0\mathbf{e}_2^T\otimes\mathbf{e}_1\mathbf{e}_2^H+\mathbf{e}_0\mathbf{e}_0^T\otimes\mathbf{e}_1\mathbf{e}_2^T+\mathbf{e}_0\mathbf{e}_2^T\otimes\mathbf{e}_1\mathbf{e}_0^T\\
&+ \mathbf{e}_1\mathbf{e}_2^T\otimes\mathbf{e}_1^*\mathbf{e}_2^H+\mathbf{e}_1\mathbf{e}_0^T\otimes\mathbf{e}_1^*\mathbf{e}_2^T+\mathbf{e}_1\mathbf{e}_2^T\otimes\mathbf{e}_1^*\mathbf{e}_0^T\\
\mathbf{E}^{(4)}_{1,2} &\stackrel{\text{def}}{=} \mathbf{e}_2^*\mathbf{e}_2^T\otimes\mathbf{e}_{1,2}\mathbf{e}_{1,2}^H+\mathbf{e}_2^*\mathbf{e}_{1,2}^H\otimes\mathbf{e}_{1,2}\mathbf{e}_2^T\\
&+ \mathbf{e}_2^*\mathbf{e}_1^T\otimes\mathbf{e}_{1,2}\mathbf{e}_{1,2}^H+\mathbf{e}_{1,2}\mathbf{e}_1^T\otimes\mathbf{e}_2^*\mathbf{e}_{1,2}^H+\mathbf{e}_2^*\mathbf{e}_{1,2}^H\otimes\mathbf{e}_{1,2}\mathbf{e}_1^T+\mathbf{e}_{1,2}\mathbf{e}_{1,2}^H\otimes\mathbf{e}_2^*\mathbf{e}_1^T
\end{aligned}$$

where $\mathbf{e}_0\stackrel{\text{def}}{=}(1,\ldots,1)^T$, $\mathbf{e}_k\stackrel{\text{def}}{=}(1,e^{i2\pi f_k},..,e^{i2\pi(L-1)f_k})^T$ and $\mathbf{e}_{k,l}\stackrel{\text{def}}{=}(1,e^{i2\pi(f_k+f_l)},..,e^{i2\pi(L-1)(f_k+f_l)})^T$, $\mathbf{e}_{k,l,m}\stackrel{\text{def}}{=}(1,e^{i2\pi(f_k+f_l+f_m)},..,e^{i2\pi(L-1)(f_k+f_l+f_m)})^T$, $k,l,m=1,2,3$, ($L=L_2$ in the first three eqs. and $L=L_3$ elsewhere).

C Appendix: Proof of eq. (4.9)

Because $S_\epsilon(f_1) = \sum_k \sigma_\epsilon^2 b^{|k|} e^{-i2\pi k f_1}$,

$$\begin{aligned}\int_{\triangle^2} S_\epsilon(f_1)S_\epsilon(f_2)S_\epsilon(f_1+f_2)df_1 df_2 &= \int_{\triangle^2} \sum_k \sum_l \sum_m \sigma_\epsilon^6 b^{|l|} b^{|k|} b^{|m|} e^{-i2\pi[(k+m)f_1+(l+m)f_2]} df_1 df_2 \\ &= \sum_m \sigma_\epsilon^6 b^{|3m|} = \sigma_\epsilon^6 \left(1 + 2\sum_{m=1}^{\infty} b^{3k}\right) = \sigma_\epsilon^6 \left(\frac{1+b^3}{1-b^3}\right).\end{aligned}$$

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