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Abstract—
This paper examines asymptotic performance of MUSIC-like algorithms for estimating directions of arrival (DOA) of narrowband complex non-circular sources. Using closed-form expressions of the covariance of the asymptotic distribution of different projection matrices, it provides a unifying framework for investigating the asymptotic performance of arbitrary subspace-based algorithms valid for Gaussian or non Gaussian and complex circular or non-circular sources. We also derive different robustness properties from the asymptotic covariance of the estimated DOA given by such algorithms. These results are successively applied to four algorithms: to two attractive MUSIC-like algorithms previously introduced in the literature, to an extension of these algorithms, and to an optimally weighted MUSIC algorithm proposed in this paper. Numerical examples illustrate the performance of the studied algorithms compared to the asymptotically minimum variance (AMV) algorithms introduced as benchmarks.

Index terms: MUSIC algorithm, subspace-based algorithms, DOA estimation, complex non-circular sources, asymptotically minimum variance.

I. Introduction

THERE is considerable literature about second-order statistics-based algorithms for estimating the DOA of narrowband sources impinging on an array of sensors. Among these algorithms, subspace-based algorithms, i.e., algorithms obtained by exploiting the orthogonality between a sample subspace and a DOA parameter-dependent subspace, have been proved very interesting. However, up to now these algorithms have been mainly designed under the complex circular Gaussian assumption only (see e.g., [1],[2]).

In mobile communications, after frequency down-shifting the sensor signals to baseband, the paired in-phase and quadrature components may be complex non-circular (for example, binary phase shift keying (BPSK) and offset quadrature phase shift keying (OQPSK) modulated signals). Because the second-order statistical characteristics are also contained in the unconjugated spatial covariance matrix for non-circular signals, second-order AMV algorithms [3] and Gaussian maximum likelihood algorithms [4] must be based on the two covariance matrices. In [3], the potential benefits due to the non-circular property have been evaluated using a closed-form expression of the lower bound on the asymptotic covariance of estimators given by arbitrary second-order algorithms. However the generalized covariance matching algorithm that attains this bound requires a multidimensional nonlinear optimization which is computationally demanding. Consequently, we need suboptimal monodimensional optimization algorithms that could benefit from the non-circular property. Such algorithms have been introduced in the context of uncorrelated sources of maximum non-circularity rate impinging on a uniform linear array in [5], [6], [7], [8] where their performance was observed by simulation only. The aim of this paper is to extend these algorithms, to provide generic asymptotic results for subspace-based estimates of the DOA for non-circular sources based on closed-form expressions of the covariance of the asymptotic distribution of extended projection matrices and to apply these results to specific MUSIC-like algorithms.

The paper is organized as follows. The array signal model and the statement of the problem are given in Section 2. The potential benefit due to the non-circularity property is underscored by the help of subspace-based algorithms built from the unconjugated spatial covariance matrix only in Section 3. The four subspace-based algorithms that we shall study are described in Section 4. Their performance is analyzed in Section 5 using a general functional methodology. Finally, numerical illustrations and Monte Carlo simulations of the performance of the algorithms are given in Section 6.

The following notations are used throughout the paper. Matrices and vectors are represented by bold upper case and bold lower case characters, respectively. Vectors are by default in column orientation, while T, H, $*$ and $\#$ stand for transpose, conjugate transpose, conjugate and Moore Penrose inverse, respectively. E(.), Tr(.), Det(.) $\|.\|_{\mathrm{Fro}}$, $\Re(.)$ and $\Im(.)$ are the expectation, trace, determinant, Frobenius norm, real and imaginary part operators respectively. $\mathbf{I}_K$ is the identity matrix. $\mathrm{vec}(\cdot)$ is the "vectorization" operator that turns a matrix into a vector by stacking the columns of the matrix one below another which is used in conjunction with the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ as the block matrix whose (i,j) block element is $a_{i,j}\mathbf{B}$ and with the vec-permutation matrix $\mathbf{K}_M$ which transforms $\mathrm{vec}(\mathbf{C})$ to $\mathrm{vec}(\mathbf{C}^T)$ for any $M \times M$ matrix $\mathbf{C}$.

II. Statement of the problem

Let an array of M sensors receive the signals emitted by K narrowband sources. The observation vectors are modeled as

$$\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \mathbf{n}_t, \qquad t = 1, \ldots, T,$$

where $(\mathbf{y}_t)_{t=1,\ldots,T}$ are independent and identically distributed. $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_K]$ is the steering matrix, where each vector $\mathbf{a}_k = \mathbf{a}(\theta_k)$ is parameterized by the real scalar parameter θ_k. $\mathbf{x}_t = (x_{t,1}, \ldots, x_{t,K})^T$ and $\mathbf{n}_t$ model signals transmitted by sources and additive measurement noise respectively. $\mathbf{x}_t$ and $\mathbf{n}_t$ are multivariate independent, zero-mean, $\mathbf{n}_t$ is assumed to be Gaussian complex circular, spatially uncorrelated with $\mathrm{E}(\mathbf{n}_t\mathbf{n}_t^H) = \sigma_n^2\mathbf{I}_M$, while $\mathbf{x}_t$ is complex non-circular, not necessarily Gaussian and possibly spatially correlated with nonsingular covariance matrices $\mathbf{R}_x \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{x}_t\mathbf{x}_t^H)$ and $\mathbf{R}'_x \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{x}_t\mathbf{x}_t^T)$. Consequently, this leads to two covariance matrices of $\mathbf{y}_t$ that contain information about : $\Theta \stackrel{\text{def}}{=} (\theta_1, \ldots, \theta_K)^T$

$$\mathbf{R}_y = \mathbf{A}\mathbf{R}_x\mathbf{A}^H + \sigma_n^2\mathbf{I}_M \ \text{ and } \ \mathbf{R}'_y = \mathbf{A}\mathbf{R}'_x\mathbf{A}^T \neq \mathbf{O}. \quad \text{(II.1)}$$

These covariance matrices are traditionally estimated by $\mathbf{R}_{y,T} = \frac{1}{T}\sum_{t=1}^T \mathbf{y}_t\mathbf{y}_t^H$ and $\mathbf{R}'_{y,T} = \frac{1}{T}\sum_{t=1}^T \mathbf{y}_t\mathbf{y}_t^T$, respectively. The parameter vector Θ is assumed identifiable from $(\mathbf{R}_y, \mathbf{R}'_y)$.

For a performance analysis, we suppose that the signal waveforms have finite fourth-order moments. The fourth-order cumulants of the sources $(x_{t,i}, x_{t,j}, x_{t,k}, x_{t,l})_{i,j,k,l=1,\ldots,K}$ are gathered into the $K^2 \times K^2$ quadrivariance matrix $\mathbf{Q}_x$ defined by $(\mathbf{Q}_x)_{i+(j-1)K,k+(l-1)K} = \mathrm{Cum}\left((\mathbf{x}_t)_i, (\mathbf{x}_t)^*_j, (\mathbf{x}_t)^*_k, (\mathbf{x}_t)_l\right)$. The non-circularity rate ρ_k of the kth source is defined by $\mathrm{E}(x_{t,k}^2) = \rho_k e^{i\phi_k}\mathrm{E}|x_{t,k}^2| = \rho_k e^{i\phi_k}\sigma_k^2$ where ϕ_k is its noncircularity phase and satisfies $0 \le \rho_k \le 1$ (from the Cauchy Schwartz inequality).

The problem addressed in this paper is to estimate the DOA Θ from the two sample covariance matrices $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$ by using subspace-based algorithms. The number K of sources is assumed to be known.

III. Subspace-based algorithms based on $\mathbf{R}'_{y,T}$ only

We prove in this section the potential benefit due to the non-circularity property by proposing a MUSIC-like algorithm based on the unconjugated spatial covariance matrix only. Because $\mathbf{R}'_y$ and $\mathbf{R}_y$ have a common noise subspace (see (II.1)) with associated orthogonal projection matrices $\mathbf{\Pi}' = \mathbf{\Pi}$, the first idea for estimating Θ from $\mathbf{R}'_{y,T}$ alone, is to apply the following steps: Estimate the projection matrix $\mathbf{\Pi}'_T$ associated with the noise subspace of $\mathbf{R}'_{y,T}$ using the singular value decomposition (SVD) of the symmetric complex-valued matrix $\mathbf{R}'_{y,T}$ and then, use the standard MUSIC algorithm based on $\mathbf{\Pi}'_T$ where the DOA $(\theta_{k,T})_{k=1,\ldots,K}$ are estimated as the locations of the K smallest minima of the function:

$$\theta_{k,T}^{\mathrm{Algo}} = \arg\min_\theta g_{0,T}(\theta) \ \text{ with } \ g_{0,T}(\theta) \stackrel{\text{def}}{=} \mathbf{a}^H(\theta)\mathbf{\Pi}'_T\mathbf{a}(\theta). \quad \text{(III.1)}$$

Compared to the standard MUSIC algorithm based on $\mathbf{\Pi}_T$ associated with the noise subspace of $\mathbf{R}_{y,T}$, whose performance is given e.g., in [1, rel. (3.11a)], we prove in Appendix A the following

Theorem 1: The sequences $\sqrt{T}(\Theta_T - \Theta)$, where Θ_T is the DOA estimate given by these two MUSIC algorithms, converge in distribution to the zero-mean Gaussian distribution of covariance matrix given by

$$(\mathbf{C}_\Theta)_{k,l} = \frac{2}{\alpha_k\alpha_l}\Re\left(\left(\mathbf{a}_l^H\mathbf{U}\mathbf{a}_k\right)\left(\mathbf{a}'^H_k\mathbf{\Pi}\mathbf{a}'_l\right)\right) \quad \text{(III.2)}$$

with $\mathbf{a}'_k \stackrel{\text{def}}{=} \frac{d\mathbf{a}_k}{d\theta_k}$ and $\alpha_k \stackrel{\text{def}}{=} 2\mathbf{a}'^H_k\mathbf{\Pi}\mathbf{a}'_k$, where $\mathbf{U} \stackrel{\text{def}}{=} \sigma_n^2\mathbf{S}^\#\mathbf{R}_y\mathbf{S}^\#$ with $\mathbf{S} \stackrel{\text{def}}{=} \mathbf{A}\mathbf{R}_x\mathbf{A}^H$ and $\mathbf{U} \stackrel{\text{def}}{=} \sigma_n^2\mathbf{S}'^{*\#}\mathbf{R}_y^T\mathbf{S}'^\#$ with $\mathbf{S}' \stackrel{\text{def}}{=} \mathbf{A}\mathbf{R}'_x\mathbf{A}^T$ for the MUSIC algorithms built on $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$, respectively.

As a result of the similar structure of $\mathbf{C}_\Theta$, given by these two MUSIC algorithms, the asymptotic performance of their estimates can be very similar. In particular for only one source, it is proved in Appendix A that these asymptotic variances are respectively given by:

$$C_{\theta_1} = \frac{1}{\alpha_1}\left[\frac{\sigma_n^2}{\sigma_1^2} + \frac{1}{\|\mathbf{a}_1\|^2}\frac{\sigma_n^4}{\sigma_1^4}\right]$$

and

$$C_{\theta_1} = \frac{1}{\alpha_1\rho_1^2}\left[\frac{\sigma_n^2}{\sigma_1^2} + \frac{1}{\|\mathbf{a}_1\|^2}\frac{\sigma_n^4}{\sigma_1^4}\right].$$

We note that for $\rho_1 = 1$ (e.g., for an unfiltered BPSK modulated source), these two variances are equal. Naturally when ρ_1 approaches zero, C_{θ_1} is unbounded and the unconjugated spatial covariance matrix $\mathbf{R}'_y$ conveys no information about θ_1. In consequence of Theorem 1, the following query is raised: how does one combine the statistics $\mathbf{\Pi}_T$ and $\mathbf{\Pi}'_T$ to improve the estimate of Θ? A possible solution is proposed in the following section.

IV. Subspace-based algorithms under study

To devise subspace-based algorithms built from both $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$, we consider the extended covariance matrix $\mathbf{R}_{\tilde{y}} \stackrel{\text{def}}{=} \mathrm{E}(\tilde{\mathbf{y}}_t\tilde{\mathbf{y}}_t^H)$ where $\tilde{\mathbf{y}}_t \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{y}_t \\ \mathbf{y}_t{}^* \end{pmatrix}$ for which:

$$\mathbf{R}_{\tilde{y}} = \tilde{\mathbf{A}}\mathbf{R}_{\tilde{x}}\tilde{\mathbf{A}}^H + \sigma_n^2\mathbf{I}_{2M} \quad \text{(IV.1)}$$

with

$$\tilde{\mathbf{A}} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^* \end{pmatrix} \ \text{ and } \ \mathbf{R}_{\tilde{x}} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{R}_x & \mathbf{R}'_x \\ \mathbf{R}'^*_x & \mathbf{R}^*_x \end{pmatrix}. \quad \text{(IV.2)}$$

From the assumptions of Section 2, $K \le \mathrm{rank}(\mathbf{R}_{\tilde{x}}) \le 2K$ and depending on this rank, many situations may be considered. We concentrate first on a particular case (case (1)) for which the sources are uncorrelated and with noncircularity rate ρ_k equal to 1 because very attractive algorithms have been devised for this case [5],[6]. This case corresponds, for example, to unfiltered BPSK or OQPSK uncorrelated modulated signals. In this case, $\mathbf{R}_x = \mathbf{\Delta}_\sigma$ and $\mathbf{R}'_x = \mathbf{\Delta}_\sigma\mathbf{\Delta}_\phi$ with $\mathbf{\Delta}_\sigma \stackrel{\text{def}}{=} \mathrm{Diag}(\sigma_1^2, \ldots, \sigma_K^2)$ and $\mathbf{\Delta}_\phi \stackrel{\text{def}}{=} \mathrm{Diag}(e^{i\phi_1}, \ldots, e^{i\phi_K})$. Consequently

$$\mathbf{R}_{\tilde{x}} = \begin{pmatrix} \mathbf{\Delta}_\sigma & \mathbf{\Delta}_\sigma\mathbf{\Delta}_\phi \\ \mathbf{\Delta}_\sigma\mathbf{\Delta}^*_\phi & \mathbf{\Delta}_\sigma \end{pmatrix} = \begin{pmatrix} \mathbf{I}_K \\ \mathbf{\Delta}^*_\phi \end{pmatrix}\mathbf{\Delta}_\sigma\begin{pmatrix} \mathbf{I}_K & \mathbf{\Delta}_\phi \end{pmatrix}$$

and $\mathrm{rank}(\mathbf{R}_{\tilde{x}}) = K$. Then subsequently, we consider the general case for which $\mathrm{rank}(\mathbf{R}_{\tilde{x}}) = 2K$ (case (2)). This case corresponds for example to filtered BPSK or OQPSK modulated signals. In these two cases, using the structured matrices $\tilde{\mathbf{A}}$ and $\mathbf{R}_{\tilde{x}}$ (IV.2), we prove the following lemma

Lemma 1: In cases (1) and (2), the orthogonal projector matrix $\tilde{\mathbf{\Pi}}$ onto the noise subspace is structured as

$$\tilde{\mathbf{\Pi}} = \begin{pmatrix} \mathbf{\Pi}_1 & \mathbf{\Pi}_2 \\ \mathbf{\Pi}_2^* & \mathbf{\Pi}_1^* \end{pmatrix}$$

where $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ are Hermitian and complex symmetric, respectively, and where $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ are not projection matrices in case (1) and $\mathbf{\Pi}_1$ is the orthogonal projector onto the column space of $\mathbf{A}$ and $\mathbf{\Pi}_2 = \mathbf{O}$ in case (2). Furthermore, the orthogonal projector onto the noise subspace $\tilde{\mathbf{\Pi}}_T$ associated with the sample estimate $\mathbf{R}_{\tilde{y},T}$ of $\mathbf{R}_{\tilde{y}}$ has the same structure

$$\tilde{\mathbf{\Pi}}_T = \begin{pmatrix} \mathbf{\Pi}_{1,T} & \mathbf{\Pi}_{2,T} \\ \mathbf{\Pi}_{2,T}^* & \mathbf{\Pi}_{1,T}^* \end{pmatrix} \quad \text{(IV.3)}$$

where $\mathbf{\Pi}_{1,T}$ and $\mathbf{\Pi}_{2,T}$ are Hermitian and complex symmetric respectively.

Proof: Noting that $\mathbf{R}_{\tilde{y}}$ or $\mathbf{R}_{\tilde{y},T}$ satisfy the relation $\mathbf{R}_{\tilde{y}} = \mathbf{J}_M \mathbf{R}_{\tilde{y}}^* \mathbf{J}_M$ with $\mathbf{J}_M \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{O}_M & \mathbf{I}_M \\ \mathbf{I}_M & \mathbf{O}_M \end{pmatrix}$, if $\begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{pmatrix}$ denotes the partitioned eigenvectors matrix associated with the signal subspace of $\mathbf{R}_{\tilde{y}}$ or $\mathbf{R}_{\tilde{y},T}$, the corresponding signal part of the eigenvalue decomposition of $\mathbf{R}_{\tilde{y}}$ or $\mathbf{R}_{\tilde{y},T}$ is

$$\begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{pmatrix} \mathbf{\Sigma} \begin{pmatrix} \mathbf{U}_1^H & \mathbf{U}_2^H \end{pmatrix} = \mathbf{J}_M \begin{pmatrix} \mathbf{U}_1^* \\ \mathbf{U}_2^* \end{pmatrix} \mathbf{\Sigma} \begin{pmatrix} \mathbf{U}_1^T & \mathbf{U}_2^T \end{pmatrix} \mathbf{J}_M$$

Thanks to the uniqueness of these normalized eigenvectors up to a unit modulus complex constant, we have $\mathbf{U}_2 = \mathbf{U}_1^* \mathbf{\Delta}$ where $\mathbf{\Delta}$ is a diagonal matrix whose diagonal is composed of unit modulus complex terms. Consequently, these orthogonal projector matrices onto the noise subspace are structured as

$$\begin{aligned} \tilde{\mathbf{\Pi}} &= \mathbf{I}_{2M} - \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{pmatrix} \begin{pmatrix} \mathbf{U}_1^H & \mathbf{U}_2^H \end{pmatrix} \\ &= \mathbf{I}_{2M} - \begin{pmatrix} \mathbf{U}_1\mathbf{U}_1^H & \mathbf{U}_1\mathbf{\Delta}^*\mathbf{U}_1^T \\ \mathbf{U}_1^*\mathbf{\Delta}\mathbf{U}_1^H & \mathbf{U}_1^*\mathbf{U}_1^T \end{pmatrix}. \end{aligned}$$

In case (2) specifically, $\tilde{\mathbf{\Pi}} = \mathbf{I}_{2M} - \tilde{\mathbf{A}}\left(\tilde{\mathbf{A}}^H\tilde{\mathbf{A}}\right)^{-1}\tilde{\mathbf{A}}^H = \begin{pmatrix} \mathbf{\Pi}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{\Pi}_1^* \end{pmatrix}$ with $\mathbf{\Pi}_1 \stackrel{\text{def}}{=} \mathbf{I}_M - \mathbf{A}\left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H$. ■

A. Case (1): uncorrelated sources with $\rho_k = 1$

Consider now three subspace-based algorithms for case (1). An algorithm (denoted Alg_1) devised in [5], has been derived from the standard MUSIC algorithm because in this case (IV.1) becomes

$$\mathbf{R}_{\tilde{y}} = \begin{pmatrix} \mathbf{A} \\ \mathbf{A}^*\mathbf{\Delta}_\phi^* \end{pmatrix} \mathbf{\Delta}_\sigma \begin{pmatrix} \mathbf{A}^H & \mathbf{\Delta}_\phi \mathbf{A}^T \end{pmatrix}^H + \sigma_n^2 \mathbf{I}_{2M}. \quad \text{(IV.4)}$$

Specifically, the estimated DOA $(\theta_{k,T})_{k=1,\ldots,K}$ are obtained as the locations of the K smallest minima of the following function:

$$\theta_{k,T}^{\mathrm{Alg}_1} = \arg\min_\theta g_{1,T}(\theta)$$

with

$$\begin{aligned} g_{1,T}(\theta) &\stackrel{\text{def}}{=} \min_\phi \tilde{\mathbf{a}}^H(\theta,\phi)\tilde{\mathbf{\Pi}}_T\tilde{\mathbf{a}}(\theta,\phi) = \mathbf{a}^H(\theta)\mathbf{\Pi}_{1,T}\mathbf{a}(\theta) \\ &- |\mathbf{a}^T(\theta)\mathbf{\Pi}_{2,T}^*\mathbf{a}(\theta)|, \end{aligned} \quad \text{(IV.5)}$$

with the extended steering vector $\tilde{\mathbf{a}}(\theta,\phi) \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{a}(\theta) \\ \mathbf{a}^*(\theta)e^{-i\phi} \end{pmatrix}$.

Noting that $\tilde{\mathbf{a}}(\theta,\phi)^H\tilde{\mathbf{\Pi}}\tilde{\mathbf{a}}(\theta,\phi) = \begin{pmatrix} 1 & e^{i\phi} \end{pmatrix} \mathbf{M} \begin{pmatrix} 1 \\ e^{-i\phi} \end{pmatrix} = 0$ with $\mathbf{M} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{a}^H(\theta) & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{a}^T(\theta) \end{pmatrix} \tilde{\mathbf{\Pi}} \begin{pmatrix} \mathbf{a}(\theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{a}^*(\theta) \end{pmatrix}$, the matrix $\mathbf{M}_T \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{a}^H(\theta) & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{a}^T(\theta) \end{pmatrix} \tilde{\mathbf{\Pi}}_T \begin{pmatrix} \mathbf{a}(\theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{a}^*(\theta) \end{pmatrix}$ is positive definite and a consistent estimate of the rank deficient 2×2 matrix $\mathbf{M}$. Consequently we can propose a new subspace-based algorithms (denoted Alg_2) defined by

$$\theta_{k,T}^{\mathrm{Alg}_2} = \arg\min_\theta g_{2,T}(\theta)$$

with

$$\begin{aligned} g_{2,T}(\theta) &\stackrel{\text{def}}{=} \mathrm{Det}(\mathbf{M}_T) = \left(\mathbf{a}^H(\theta)\mathbf{\Pi}_{1,T}\mathbf{a}(\theta)\right)^2 \\ &- \left(\mathbf{a}^T(\theta)\mathbf{\Pi}_{2,T}^*\mathbf{a}(\theta)\right)\left(\mathbf{a}^H(\theta)\mathbf{\Pi}_{2,T}\mathbf{a}^*(\theta)\right). \end{aligned} \quad \text{(IV.6)}$$

In the particular case of a uniform linear array, replacing the generic steering vector $\mathbf{a}(\theta) = (1, e^{i\theta}, \ldots, e^{i(M-1)\theta})^T$ by $\mathbf{a}(z) \stackrel{\text{def}}{=} (1, z, \ldots, z^{M-1})^T$ in (IV.6), [6] proposed a root-MUSIC-like algorithm (denoted Alg_3) defined by

$$\begin{aligned} \theta_{k,T}^{\mathrm{Alg}_3} &= \arg(z_k) \ \text{ with } z_k \ K \text{ roots}_{|z|<1} \text{ of} \\ & \quad g_{3,T}(z) \ \text{ closest to the unit circle} \end{aligned} \quad \text{(IV.7)}$$

where $g_{3,T}(z)$ is the following polynomial [1] of degree $4(M-1)$ whose roots appear in reciprocal conjugate pairs z_k and $(z_k^*)^{-1}$:

$$\begin{aligned} g_{3,T}(z) &\stackrel{\text{def}}{=} \left(\mathbf{a}^T(z^{-1})\mathbf{\Pi}_{1,T}\mathbf{a}(z)\right)^2 \\ &- \left(\mathbf{a}^T(z)\mathbf{\Pi}_{2,T}^*\mathbf{a}(z)\right)\left(\mathbf{a}^T(z^{-1})\mathbf{\Pi}_{2,T}\mathbf{a}(z^{-1})\right). \end{aligned}$$

B. Case (2): arbitrary full rank spatial extended covariance matrix

Based on $\tilde{\mathbf{\Pi}}\tilde{\mathbf{A}} = \begin{pmatrix} \mathbf{\Pi}_1 & \mathbf{\Pi}_2 \\ \mathbf{\Pi}_2^* & \mathbf{\Pi}_1^* \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^* \end{pmatrix} = \mathbf{O}$, different MUSIC-like algorithms can be proposed. Since $\mathbf{\Pi}_2 = \mathbf{O}$, a natural idea consists in proposing the following algorithm (denoted Alg_4) [2]

$$\theta_{k,T}^{\mathrm{Alg}_4} = \arg\min_\theta \mathbf{a}^H(\theta)\mathbf{\Pi}_{1,T}\mathbf{a}(\theta). \quad \text{(IV.8)}$$

[1] We note that this procedure allows one to estimate up to $2(M-1)$ possible DOA, whereas the upper bound is $2M-1$ [9].

[2] We note that unlike $\mathbf{\Pi}_1$, the positive semidefinite matrix $\mathbf{\Pi}_{1,T}$ is not a projection matrix.

But it is shown in Section V, that this algorithm is always outperformed by the standard MUSIC algorithm based on $\mathbf{R}_{y,T}$ only. Using the ideas of the weighted MUSIC algorithm introduced for DOA estimation [2], then applied for frequency estimation [10],[11], we propose the following column weighting[3] MUSIC (denoted Alg_5):

$$\theta_{k,T}^{\text{Alg}_5} = \arg\min_{\theta} g_{5,T}(\theta),$$

with

$$g_{5,T}(\theta) \stackrel{\text{def}}{=} \text{Tr}\left(\mathbf{W}\bar{\mathbf{A}}^H(\theta)\tilde{\mathbf{\Pi}}_T\bar{\mathbf{A}}(\theta)\right),$$

where $\mathbf{W}$ is a 2×2 non-negative definite weighting matrix whose optimal value will be specified in Theorem 7, and $\bar{\mathbf{A}}(\theta)$ is the steering matrix $\begin{pmatrix} \mathbf{a}(\theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{a}^*(\theta) \end{pmatrix}$. To derive the optimal weighting matrix $\mathbf{W} = \begin{pmatrix} w_{1,1} & w_{1,2} \\ w_{1,2}^* & w_{2,2} \end{pmatrix}$ in the next section, the weighted MUSIC cost function can be written as

$$g_{5,T}(\theta) = (w_{1,1}+w_{2,2})\left(\mathbf{a}^H(\theta)\mathbf{\Pi}_{1,T}\mathbf{a}(\theta) + \Re(z\mathbf{a}^T(\theta)\mathbf{\Pi}_{2,T}\mathbf{a}(\theta))\right), \tag{IV.9}$$

with $z \stackrel{\text{def}}{=} \frac{2w_{1,2}^*}{w_{1,1}+w_{2,2}}$. Consequently the performance of this algorithm depends only on z. By choosing $\mathbf{W}$ diagonal, we have $z=0$ and this algorithm reduces to Alg_4.

V. Performance analysis

A. Second-order algorithms based on $\mathbf{R}_{y,T}$ only

Considering first the influence of the non-circularity on the performance of an arbitrary second-order algorithm based on $\mathbf{R}_{y,T}$ only, we prove the following theorem.

Theorem 2: All DOA consistent estimates given by an arbitrary second-order algorithms based on $\mathbf{R}_{y,T}$ only, that do not explicitly suppose the sources to be spatially uncorrelated, are robust to the distribution and to the non-circularity of the sources; i.e., the asymptotic performances are those of the standard complex circular Gaussian case.

Proof: Based on these assumptions, the Jacobian matrix $\mathbf{D}_{\Theta}^{\text{Alg}}$ of the mapping $(\mathbf{R}_{y,T} \longmapsto \Theta_T = \text{Alg}(\mathbf{R}_{y,T}))$ that associates the estimate Θ_T to $\mathbf{R}_{y,T}$ satisfies the constraint (see [12])

$$\mathbf{D}_{\Theta}^{\text{Alg}}(\mathbf{A}^*\otimes\mathbf{A}) = \mathbf{O},$$

and because the covariance matrix $\mathbf{C}_{r_y}$ of the asymptotic distribution of $\text{vec}(\mathbf{R}_{y,T})$ is given by [3]

$$\begin{aligned}\mathbf{C}_{r_y} &= (\mathbf{A}^*\otimes\mathbf{A})\mathbf{C}_{r_x}(\mathbf{A}^T\otimes\mathbf{A}^H) + \sigma_n^4\mathbf{I}_{M^2} \\ &+ \sigma_n^2\mathbf{I}_M\otimes\mathbf{A}\mathbf{R}_x\mathbf{A}^H + \mathbf{A}^*\mathbf{R}_x^*\mathbf{A}^T\otimes\sigma_n^2\mathbf{I}_M\end{aligned}$$

with $\mathbf{C}_{r_x} = \mathbf{R}_x^*\otimes\mathbf{R}_x + \mathbf{K}_K(\mathbf{R}'_x\otimes\mathbf{R}_x^{'*}) + \mathbf{Q}_x$ where $\mathbf{Q}_x$ is the quadrivariance matrix defined in Section II, the first term of $\mathbf{C}_{r_y}$ (which contains $\mathbf{R}'_x$ and $\mathbf{Q}_x$) disappears in the expression of the covariance $\mathbf{C}_{\Theta}^{\text{Alg}} = \mathbf{D}_{\Theta}^{\text{Alg}}\mathbf{C}_{r_y}\left(\mathbf{D}_{\Theta}^{\text{Alg}}\right)^H$ of the asymptotic distribution of the estimated DOA Θ_T given by the algorithm Alg(.). ∎

[3]Because $\tilde{\mathbf{\Pi}}_T$ is an orthogonal projector, the cost function $g_{5,T}(\theta)$ reduces to $\|\tilde{\mathbf{\Pi}}_T\bar{\mathbf{A}}(\theta)\mathbf{W}^{1/2}\|_{\text{Fro}}^2$.

B. Subspace-based algorithms built from $\mathbf{R}_{\tilde{y},T}$

To consider the asymptotic performance of an arbitrary subspace-based algorithms built from $\mathbf{R}_{\tilde{y},T}$, we adopt a functional analysis which consists of recognizing that the whole process of constructing an estimate Θ_T of Θ is equivalent to defining a functional relation linking this estimate Θ_T to the statistics $\tilde{\mathbf{\Pi}}_T$ from which it is inferred. This functional dependence is denoted $\Theta_T = \text{Alg}(\tilde{\mathbf{\Pi}}_T)$. By assumption, $\Theta = \text{Alg}(\tilde{\mathbf{\Pi}})$, so arbitrary sufficiently "regular" subspace-based algorithms built from $\mathbf{R}_{\tilde{y},T}$ constitute distinct extensions of the mapping $\tilde{\mathbf{\Pi}} \longmapsto \Theta$. For the different algorithms Alg(.) defined in Section IV, we note that this mapping is differentiable with respect to $(\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{\Pi}_2^*)$. With this approach, the asymptotic distributions of the estimates given by these algorithms are directly related to the asymptotic distributions of $\tilde{\mathbf{\Pi}}_T$ or $(\mathbf{\Pi}_{1,T}, \mathbf{\Pi}_{2,T}, \mathbf{\Pi}_{2,T}^*)$ for which we prove the following theorem in Appendix B

Theorem 3: The sequence of statistics

$$\sqrt{T}\text{vec}\left(\tilde{\mathbf{\Pi}}_T - \tilde{\mathbf{\Pi}}\right) \quad \text{and} \quad \sqrt{T}\begin{pmatrix} \text{vec}(\mathbf{\Pi}_{1,T}-\mathbf{\Pi}_1) \\ \text{vec}(\mathbf{\Pi}_{2,T}-\mathbf{\Pi}_2) \\ \text{vec}(\mathbf{\Pi}_{2,T}^*-\mathbf{\Pi}_2^*) \end{pmatrix}$$

converge in distribution to the zero-mean Gaussian distributions of first covariance matrices

$$\mathbf{C}_{\tilde{\Pi}} = (\mathbf{I}_{4M^2} + \mathbf{K}_{2M}(\mathbf{J}_M\otimes\mathbf{J}_M))\left((\tilde{\mathbf{\Pi}}^*\otimes\tilde{\mathbf{U}}) + (\tilde{\mathbf{U}}^*\otimes\tilde{\mathbf{\Pi}})\right) \tag{V.1}$$

and

$$\mathbf{C}_{\Pi_1,\Pi_2,\Pi_2^*} = \begin{pmatrix} \mathbf{C}_{\Pi_1} & \mathbf{C}_{\Pi_2,\Pi_1}^H & \mathbf{C}_{\Pi_2^*,\Pi_1}^H \\ \mathbf{C}_{\Pi_2,\Pi_1} & \mathbf{C}_{\Pi_2} & \mathbf{C}_{\Pi_2^*,\Pi_2}^H \\ \mathbf{C}_{\Pi_2^*,\Pi_1} & \mathbf{C}_{\Pi_2^*,\Pi_2} & \mathbf{C}_{\Pi_2^*} \end{pmatrix} \tag{V.2}$$

with

$$\begin{aligned}\mathbf{C}_{\Pi_1} &= (\mathbf{\Pi}_1^*\otimes\mathbf{U}_1) + (\mathbf{U}_1^*\otimes\mathbf{\Pi}_1) \\ &+ \mathbf{K}_M\left((\mathbf{\Pi}_2\otimes\mathbf{U}_2^*) + (\mathbf{U}_2\otimes\mathbf{\Pi}_2^*)\right), \quad\quad \text{(V.3)}\\ \mathbf{C}_{\Pi_2} &= (\mathbf{I}_{M^2}+\mathbf{K}_M)\left((\mathbf{\Pi}_1\otimes\mathbf{U}_1) + (\mathbf{U}_1\otimes\mathbf{\Pi}_1)\right), \\ \mathbf{C}_{\Pi_2,\Pi_1} &= (\mathbf{I}_{M^2}+\mathbf{K}_M)\left((\mathbf{\Pi}_2\otimes\mathbf{U}_1) + (\mathbf{U}_2\otimes\mathbf{\Pi}_1)\right), \\ \mathbf{C}_{\Pi_2^*,\Pi_1} &= (\mathbf{I}_{M^2}+\mathbf{K}_M)\left((\mathbf{\Pi}_1^*\otimes\mathbf{U}_2^*) + (\mathbf{U}_1^*\otimes\mathbf{\Pi}_2^*)\right), \\ \mathbf{C}_{\Pi_2^*,\Pi_2} &= (\mathbf{I}_{M^2}+\mathbf{K}_M)\left((\mathbf{\Pi}_2^*\otimes\mathbf{U}_2^*) + (\mathbf{U}_2^*\otimes\mathbf{\Pi}_2^*)\right),\end{aligned}$$

where $\tilde{\mathbf{U}} \stackrel{\text{def}}{=} \sigma_n^2\tilde{\mathbf{S}}^{\#}\mathbf{R}_{\tilde{y}}\tilde{\mathbf{S}}^{\#} = \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 \\ \mathbf{U}_2^* & \mathbf{U}_1^* \end{pmatrix}$ with $\tilde{\mathbf{S}} \stackrel{\text{def}}{=} \tilde{\mathbf{A}}\mathbf{R}_{\tilde{x}}\tilde{\mathbf{A}}^H$.

We note that Theorem 2 does not extend to arbitrary second-order algorithms based on $\mathbf{R}_{\tilde{y},T}$ because here $\mathbf{D}_{\Theta}^{\text{Alg}}\left(\tilde{\mathbf{A}}^*\otimes\tilde{\mathbf{A}}\right) \neq \mathbf{O}$ due to the constraints on $\mathbf{R}_{\tilde{x}}$ (see the proof in [12]). However, since expression (V.1) of $\mathbf{C}_{\tilde{\Pi}}$ does not depend on the fourth-order moments of the sources, we have proved the following

Theorem 4: The asymptotic performance given by an arbitrary subspace-based algorithm built from $\mathbf{R}_{\tilde{y},T}$ depends on the distribution of the sources through their second-order moments only.

More specifically, regarding the algorithms described in Section IV, we prove the following

Theorem 5: The sequences $\sqrt{T}(\Theta_T - \Theta)$, where Θ_T are the DOA estimates given by the first three subspace-based algorithms [resp., algorithms 1 and 2] described in Section IV for a uniform linear array [resp., arbitrary array], converge in distribution to the same zero-mean Gaussian distribution [4] with covariance matrix

$$(\mathbf{C}_\Theta)_{k,l} = \frac{1}{\gamma_k \gamma_l} \begin{pmatrix} \alpha^{(k)}_{\phi,\phi} & -\alpha^{(k)}_{\theta,\phi} \end{pmatrix} \mathbf{B}^{(k,l)} \begin{pmatrix} \alpha^{(l)}_{\phi,\phi} \\ -\alpha^{(l)}_{\theta,\phi} \end{pmatrix} \quad \text{(V.4)}$$

with $\left(\mathbf{B}^{(k,l)}\right)_{i,j} \stackrel{\text{def}}{=} 4\Re\left((\tilde{\mathbf{a}}_k^T \tilde{\mathbf{U}}^* \tilde{\mathbf{a}}_l^*)(\tilde{\mathbf{a}'}^H_{i,k} \tilde{\mathbf{\Pi}} \tilde{\mathbf{a}}'_{j,l})\right)$, $i,j = \theta,\phi$ where $\tilde{\mathbf{a}}_k \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{a}_k \\ \mathbf{a}_k^* e^{-i\phi_k} \end{pmatrix}$, $\tilde{\mathbf{a}}'_{\theta,k} \stackrel{\text{def}}{=} \frac{d\tilde{\mathbf{a}}_k}{d\theta_k}$, $\tilde{\mathbf{a}}'_{\phi,k} \stackrel{\text{def}}{=} \frac{d\tilde{\mathbf{a}}_k}{d\phi_k}$ and with $\left(\alpha^{(k)}_{i,j}\right)_{i,j=\theta,\phi}$ and γ_k are the purely geometric factors: $\alpha^{(k)}_{i,j} \stackrel{\text{def}}{=} \Re(\tilde{\mathbf{a}'}^H_{i,k} \tilde{\mathbf{\Pi}} \tilde{\mathbf{a}}'_{j,k})$ and $\gamma_k \stackrel{\text{def}}{=} \alpha^{(k)}_{\theta,\theta}\alpha^{(k)}_{\phi,\phi} - (\alpha^{(k)}_{\theta,\phi})^2$. In particular:

$$(\mathbf{C}_\Theta)_{k,k} = \frac{2\alpha^{(k)}_{\phi,\phi}}{\gamma_k}(\tilde{\mathbf{a}}_k^H \tilde{\mathbf{U}} \tilde{\mathbf{a}}_k), \ k = 1, \ldots, K \quad \text{(V.5)}$$

which gives in the case of a single source:

$$C_{\theta_1} = \frac{1}{\alpha_1}\left[\frac{\sigma_n^2}{\sigma_1^2} + \frac{1}{2\|\mathbf{a}_1\|^2}\frac{\sigma_n^4}{\sigma_1^4}\right], \quad \text{(V.6)}$$

where α_1 is the purely geometric factor $2{\mathbf{a}'_1}^H \mathbf{\Pi} \mathbf{a}'_1$ with $\mathbf{a}'_1 \stackrel{\text{def}}{=} \frac{d\mathbf{a}_1}{d\theta_1}$.

Remark: If the case of a single non-circular complex Gaussian distributed source of maximum noncircularity rate ($\rho_1 = 1$), asymptotic variance (V.6) attains the non-circular Gaussian Cramer-Rao bound given in [4]. Consequently, the first three subspace-based algorithms described in Section IV are efficient for a single source.

Proof: First, we note that the cost functions $g_{1,T}(\alpha)$ and $g_{2,T}(\alpha)$ given in (IV.5) and (IV.6) respectively, satisfy the relation $g_{2,T}(\alpha) = g_{1,T}(\alpha) r_T(\alpha)$ with $r_T(\alpha) \stackrel{\text{def}}{=} (\mathbf{a}^H(\alpha)\mathbf{\Pi}_{1,T}\mathbf{a}(\alpha)) + |\mathbf{a}^T(\alpha)\mathbf{\Pi}^H_{2,T}\mathbf{a}(\alpha)|$, where in exact statistics $r(\theta_k) \neq 0$ (because if $r(\theta_k)$ were to vanish, we would have $\mathbf{a}^H(\theta_k)\mathbf{\Pi}_1\mathbf{a}(\theta_k) = 0$ and $|\mathbf{a}^H(\theta_k)\mathbf{\Pi}_2\mathbf{a}^*(\theta_k)| = 0$, and consequently $\begin{pmatrix} \mathbf{a}_k \\ \mathbf{a}_k^* e^{i\beta} \end{pmatrix}$ would belong to the signal space of $\mathbf{R}_{\tilde{y}}$ for all values of β, which leads to a contradiction with (IV.4). Then, applying the proof [1, Theorem 3.2], the estimates minimizing $g_{1,T}$ and $g_{3,T}$ have the same asymptotic distribution and consequently algorithms 1 and 2 have the same asymptotic performances.

Then, to prove that algorithms 2 and 3 have the same asymptotic performances, we consider the first order perturbation expansions of $\delta\theta_{k,T} \stackrel{\text{def}}{=} \theta_{k,T} - \theta_k$ as a function of $\delta\mathbf{\Pi}_{1,T} \stackrel{\text{def}}{=} \mathbf{\Pi}_{1,T} - \mathbf{\Pi}_1$ and $\delta\mathbf{\Pi}_{2,T} \stackrel{\text{def}}{=} \mathbf{\Pi}_{2,T} - \mathbf{\Pi}_2$ given by these two algorithms. Following the lines of the derivation given in [13] where the standard MUSIC and root-MUSIC algorithms are replaced by algorithms 1 and 3 respectively, we prove in Appendix C that these algorithms satisfy the same perturbation expansion:

$$\begin{aligned} \theta_{k,T} &= \theta_k + \mathbf{A}_{1,k}\mathrm{vec}(\delta\mathbf{\Pi}_{1,T}) + \mathbf{A}_{2,k}\mathrm{vec}(\delta\mathbf{\Pi}_{2,T}) \\ &+ \mathbf{A}^*_{2,k}\mathrm{vec}(\delta\mathbf{\Pi}^*_{2,T}) + o(\delta\mathbf{\Pi}_{1,T}) + o(\mathbf{\Pi}_{2,T}). \quad \text{(V.7)} \end{aligned}$$

The proof is completed in Appendix C where the DOA estimate given by algorithm 1 is proved to converge in distribution to a Gaussian distribution whose covariance matrice is given with (V.4),(V.5) and (V.6). ■

[4]These three algorithms have different behavior outside the asymptotic regime, as will be stressed in Section VI.

In case (2), it is straightforward to prove the following Theorem

Theorem 6: The sequence $\sqrt{T}(\Theta_T - \Theta)$, where Θ_T is the DOA estimate given by the MUSIC-like algorithm (IV.8) described in Section IV, converges in distribution to the zero-mean Gaussian distribution with covariance matrix

$$\begin{aligned} (\mathbf{C}_\Theta)_{k,l} &= \left(\mathbf{D}_\Theta^{\mathrm{Alg}_4} \mathbf{C}_{\Pi_1} \mathbf{D}_\Theta^{\mathrm{Alg}_4}\right)_{k,l} \\ &= \frac{2}{\alpha_k \alpha_l}\Re\left(\left(\mathbf{a}_l^H \mathbf{U}_1 \mathbf{a}_k\right)\left({\mathbf{a}'}_k^H \mathbf{\Pi} \mathbf{a}'_l\right)\right) \quad \text{(V.8)} \end{aligned}$$

where $\mathbf{D}_\Theta^{\mathrm{Alg}_4} = \mathbf{D}_\Theta^{\mathrm{Alg}_0}$ is given in (A.3). Because $\begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 \\ \mathbf{U}_2^* & \mathbf{U}_1^* \end{pmatrix} = \sigma_n^2 \tilde{\mathbf{S}}^\# \mathbf{R}_{\tilde{y}} \tilde{\mathbf{S}}^\#$ with $\tilde{\mathbf{S}} = \tilde{\mathbf{A}} \mathbf{R}_{\tilde{x}} \tilde{\mathbf{A}}^H$, we note that the performance of this algorithm is critical when $\mathbf{R}_{\tilde{x}}$ which interacts in $\tilde{\mathbf{S}}$ approaches singularity. This is particularly the case when the sources are uncorrelated with at least a non-circularity rate that tends to one (because in this case, $\det(\mathbf{R}_{\tilde{x}}) = \prod_{k=1}^K \left(\sigma_k^4(1-\rho_k^2)\right)$).

For a single source, (V.8) gives

$$C_{\theta_1}^{\mathrm{Alg}_4} = \frac{1}{\alpha_1(1-\rho_1^2)}\left[\frac{\sigma_n^2}{\sigma_1^2} + \frac{1}{\|\mathbf{a}_1\|^2}\frac{(1+\rho_1^2)}{(1-\rho_1^2)}\frac{\sigma_n^4}{\sigma_1^4}\right]. \quad \text{(V.9)}$$

and consequently

$$\begin{aligned} C_{\theta_1}^{\mathrm{Alg}_4} \geq C_{\theta_1}^{\mathrm{MUSIC}} &= \frac{1}{\alpha_1}\left[\frac{\sigma_n^2}{\sigma_1^2} + \frac{1}{\|\mathbf{a}_1\|^2}\frac{\sigma_n^4}{\sigma_1^4}\right] \\ \lim_{\rho_1 \to 1} C_{\theta_1}^{\mathrm{Alg}_4} &= \infty. \quad \text{(V.10)} \end{aligned}$$

Thus this algorithm is always outperformed by the standard MUSIC algorithm. This critical property will be studied for two sources, through numerical examples in Section VI.

Then considering the second algorithm proposed in case (2), we prove in Appendix D the following

Theorem 7: The sequence $\sqrt{T}(\Theta_T - \Theta)$ where Θ_T is the DOA estimate given by the weighted MUSIC algorithm introduced in Section IV converges in distribution to the zero-mean Gaussian distribution with covariance matrix

$$\begin{aligned} (\mathbf{C}_\Theta)_{k,l} &= \frac{1}{2\alpha_k\alpha_l}(1 \ z^* \ z \ 1)\left((\bar{\mathbf{A}}_k^T \tilde{\mathbf{U}}^* \bar{\mathbf{A}}_l^*) \otimes (\bar{\mathbf{A}'}^H_k \tilde{\mathbf{\Pi}} \bar{\mathbf{A}}'_l)\right. \\ &\left. + (\bar{\mathbf{A}'}^T_k \tilde{\mathbf{\Pi}}^* \bar{\mathbf{A}'}^*_l) \otimes (\bar{\mathbf{A}}_k^H \tilde{\mathbf{U}} \bar{\mathbf{A}}_l)\right)(1 \ z^* \ z \ 1)^H, \quad \text{(V.11)} \end{aligned}$$

with $z \stackrel{\text{def}}{=} \frac{2w^*_{1,2}}{w_{1,1}+w_{2,2}}$, $\bar{\mathbf{A}}_k \stackrel{\text{def}}{=} \bar{\mathbf{A}}(\theta_k)$ and $\bar{\mathbf{A}}'_k \stackrel{\text{def}}{=} \frac{d\bar{\mathbf{A}}_k}{d\theta_k}$. Furthermore, the value z_k^{opt} that minimizes $(\mathbf{C}_\Theta)_{k,k}$ is given by

$$z_k^{\mathrm{opt}} = -\frac{\mathbf{a}_k^T \mathbf{U}_2^* \mathbf{a}_k}{\mathbf{a}_k^H \mathbf{U}_1 \mathbf{a}_k}, \tag{V.12}$$

for which the minimum value of $(\mathbf{C}_\Theta)_{k,k}$ is

$$\min_z (\mathbf{C}_\Theta)_{k,k} = \frac{\mathrm{Det}(\bar{\mathbf{A}}_k^H \tilde{\mathbf{U}} \bar{\mathbf{A}}_k)}{2(\mathbf{a}_k^H \mathbf{U}_1 \mathbf{a}_k)(\mathbf{a}'^H_k \mathbf{\Pi}_1 \mathbf{a}'_k)}. \tag{V.13}$$

For a single source, we prove in Appendix E

Corollary 1: The asymptotic variance of the DOA estimate given by the optimal weighting MUSIC algorithm attains the non-circular Gaussian Cramer Rao bound for all values of the non-circularity rate in the single source case.

Remark 1: The optimal value of the weight previously derived depends on the specific DOA whose variance is to be minimized, which means that the optimal weight is not the same for all DOAs. This, however, might have been expected as MUSIC estimates the DOAs one by one. In addition, it should be noted that z_k^{opt} is sample dependent. Consequently, this value ought to be replaced by a consistent estimate in the implementation of the optimal weighting MUSIC algorithm. This point will be described in the next section. We note that this replacement of z_k^{opt} by a consistent estimate has no effect on the asymptotic variance of the weighting MUSIC algorithm as it is proved in Appendix E.

Remark 2: For circular sources, $\mathbf{R}_{\tilde{y}}$ is block diagonal. This successively implies that $\tilde{\mathbf{S}}$, $\tilde{\mathbf{S}}^{\#}$ and $\tilde{\mathbf{U}}$ are block diagonal. Consequently, $\mathbf{U}_2 = \mathbf{O}$, $z_k^{\mathrm{opt}} = 0$, $\mathbf{W}_{\mathrm{opt}}$ is diagonal and the optimal weighting MUSIC algorithm reduces to the standard MUSIC algorithm. Then (V.13) becomes $\min_z (\mathbf{C}_\Theta)_{k,k} = \frac{\mathbf{a}_k^H \mathbf{U} \mathbf{a}_k}{2\mathbf{a}'^H_k \mathbf{\Pi}_1 \mathbf{a}'_k}$, which is the asymptotic variance given by (III.2).

To implement this optimal weighted MUSIC algorithm, we propose to use the following multistep procedure described in [11, Section 7].

1. Determine standard MUSIC estimates of $(\theta_k)_{k=1,...,K}$ from $\mathbf{R}_{y,T}$.
2. For $k = 1, ..., K$, perform the following: Let $\theta^0_{k,T}$ denote the estimates obtained in step 1. Use $(\theta^0_{k,T})_{k=1,...,K}$ and the estimate $\mathbf{U}_{1,T}$ and $\mathbf{U}_{2,T}$ of $\mathbf{U}_1$ and $\mathbf{U}_2$ derived from $\tilde{\mathbf{R}}_{y,T}$ to obtain consistent estimates $z_{k,T}$ of z_k^{opt}. Then determine improved estimates $\theta^1_{k,T}$ by locally minimizing the weighted MUSIC cost function (IV.9) associated with $z_{k,T}$ around $\theta^0_{k,T}$.

VI. Illustrative examples

In this section, we provide numerical illustrations and Monte Carlo simulations of the performance of the different algorithms presented in Section IV and numerical comparisons of the variances of these DOA estimates to the asymptotic variance of AMV estimators based on $\mathbf{R}_{\tilde{y},T}$ (i.e., $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$) and on $\mathbf{R}_{y,T}$ alone [3].

We consider throughout this section two uncorrelated [5] equipowered (SNR $\stackrel{\mathrm{def}}{=} \frac{\sigma_1^2}{\sigma_n^2}$) filtered or unfiltered BPSK modulated signals with identical non-circularity rate ($\rho \stackrel{\mathrm{def}}{=} \rho_1 = \rho_2$) with phases of noncircularity ϕ_1 and ϕ_2. These signals impinge on a uniform linear array with $M = 6$ sensors separated by a half-wavelength for which $\mathbf{a}_k = (1, e^{i\theta_k}, \ldots, e^{i(M-1)\theta_k})^T$ where $\theta_k = \pi \sin(\alpha_k)$, with α_k the DOAs relative to the normal of array broadside. 1000 independent simulation runs have been performed to obtain the estimated variances and the number of snapshots is $T = 500$ [resp. $T = 1000$] in case (1) [resp. in case (2)].

The first experiment illustrates Theorem 5 for which $\rho = 1$. Figs.1, 2 and 3 exhibit the dependence of $\mathrm{var}(\theta_{1,T})$ given by algorithms 1, 2 and 3, and by the AMV algorithm based on $\mathbf{R}_{\tilde{y},T}$ (i.e., on $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$), with the SNR, the DOA separation $\Delta\theta = \theta_2 - \theta_1$ and the noncircularity phase separation $\Delta\phi = \phi_2 - \phi_1$ [6].

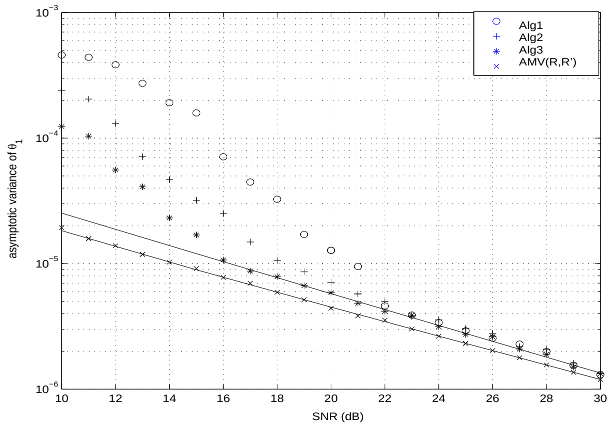

Fig.1 Theoretical and empirical asymptotic variances given by algorithms 1, 2, 3 and AMV algorithm based on $(\mathbf{R}_{y,T}, \mathbf{R}'_{y,T})$ as a function of the SNR for $\Delta\theta = 0.05$ radians, $\Delta\phi = \pi/6$ radians.

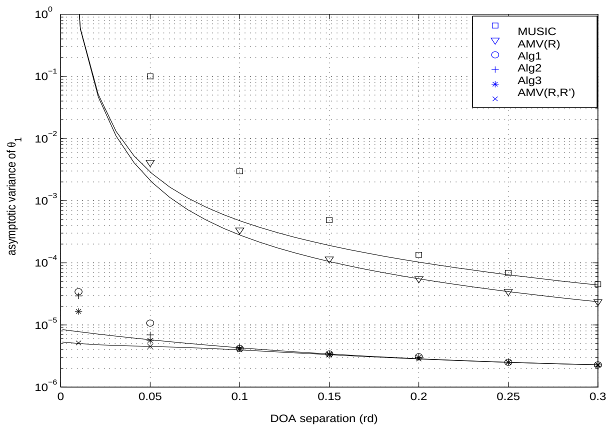

Fig.2 Theoretical and empirical asymptotic variances given by algorithms 1, 2, 3, standard MUSIC and AMV algorithms based on $\mathbf{R}_{y,T}$ only and on $(\mathbf{R}_{y,T}, \mathbf{R}'_{y,T})$ as a function of the DOA separation for SNR=$20dB$, $\Delta\phi = \pi/6$ radians.

Figs.1 and 2 show that the domain of validity of our asymptotic analysis depends on the algorithm. Below a SNR threshold that is algorithm-dependent, al-

[5] We concentrate on uncorrelated sources because it was shown in [3] that expected benefits due to the non-circular property happens mainly for uncorrelated sources.

[6] By virtue of numerical examples, the different theoretical variances depend on θ_1, θ_2, ϕ_1, ϕ_2 by only $\Delta\theta = \theta_2 - \theta_1$ and $\Delta\phi = \phi_2 - \phi_1$ in case (1) [only $\Delta\theta = \theta_2 - \theta_1$ in case (2)] for two equipowered sources with identical non-circularity rates.

gorithm 3 (root-MUSIC-like algorithm) outperforms algorithm 2 which outperforms algorithm 1, and naturally all three algorithms clearly outperform the standard MUSIC and the AMV algorithm based on $\mathbf{R}_{y,T}$ alone.

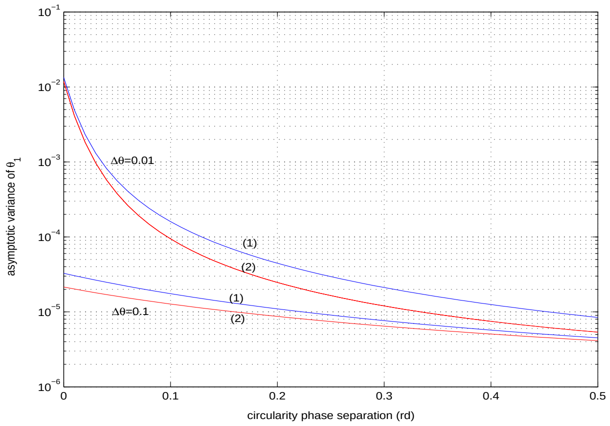

Fig.3 Theoretical asymptotic variances given by algorithms 1, 2, 3 (1) and by the AMV algorithm based on $(\mathbf{R}_{y,T}, \mathbf{R}'_{y,T})$ (2) as a function of the noncircularity phase separation for two DOA separations and SNR=20dB.

In Fig.2, we note that the asymptotic variances given by algorithms 1, 2 and 3 and the AMV algorithm tend to a finite limit when the DOA separation decreases to zero. For algorithms 1, 2 and 3, this strange behavior is explained by the two non-zero eigenvalues $(\lambda_k)_{k=1,2}$ of $\tilde{\mathbf{S}}$ which interact in $\tilde{\mathbf{U}} \stackrel{\text{def}}{=} \sigma_n^2 \tilde{\mathbf{S}}^{\#} \mathbf{R}_{\tilde{y}} \tilde{\mathbf{S}}^{\#}$ that appears in (V.5) of Theorem 5. With $\lambda_k = 2M\sigma_1^2 \left(1 + (-1)^k \cos((M-1)\frac{\Delta\theta}{2} - \Delta\phi) \frac{\sin(M\frac{\Delta\theta}{2})}{M \sin(\frac{\Delta\theta}{2})}\right)$ $k = 1, 2$, we see that one of these eigenvalues approaches zero and consequently the asymptotic variances increases without limit *only if both* $\Delta\theta$ and $\Delta\phi$ tend to zero. For the AMV algorithm, $\mathbf{C}_\Theta = \left[(\mathcal{S}^H \mathbf{C}_s^{-1} \mathcal{S})^{-1}\right]_{(1:K,1:K)}$ (see the notations of [3]) and $\mathcal{S}$ is column rank deficient *only if both* $\Delta\theta$ and $\Delta\phi$ tend to zero as well. Fig.3 illustrates the sensitivity of the performances to the noncircularity phase separation $\Delta\phi$, which is particularly prominent for low DOA separations. Figs.1 and 2 show the good efficiency of these three algorithms compared to the AMV estimator based on $\mathbf{R}_{\tilde{y},T}$, particularly for large DOA separations.

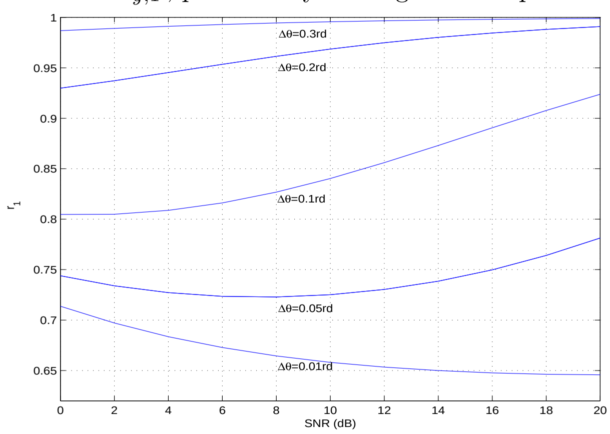

Fig.4 Ratio $r_1 \stackrel{\text{def}}{=} \text{Var}_{\theta_1}^{\text{AMV(R,R}')} / \text{Var}_{\theta_1}^{\text{Alg}_{1,2,3}}$ as a function of the SNR for different DOA separations, $\Delta\phi = \pi/6$ radians.

To specify this point, Fig.4 exhibits the ratio $r_1 \stackrel{\text{def}}{=} \text{Var}_{\theta_1}^{\text{AMV(R,R}')} / \text{Var}_{\theta_1}^{\text{Alg}_{1,2,3}}$ as a function of the SNR for different DOA separations. It shows that algorithms 1, 2 and 3 are very efficient, except for low DOA separations and low SNRs.

The second experiment considers arbitrary non-circularity rates ρ (case (2)). Fig.5 exhibits the ratio $r_2 \stackrel{\text{def}}{=} \text{Var}_{\theta_1}^{\text{MUSIC(R)}} / \text{Var}_{\theta_1}^{\text{Alg}_4}$ as a function of the non-circularity rate for different DOA separations. It shows that algorithm 4 is worse than the standard MUSIC algorithm based on $\mathbf{R}_{y,T}$ alone, for all scenarios. This extends that a property proved by (V.10) in the single source case.

In the following, we concentrate on the optimal weighted MUSIC algorithm (alg5) introduced in Subsection IV-B. Compared to the standard MUSIC algorithm based on $\mathbf{R}_{y,T}$, Figs.6 and 7 show that algorithm 5 outperforms the standard MUSIC algorithm, particularly for low SNRs and DOA separations when the noncircularity rate ρ increases. The efficiency of this optimal weighted MUSIC algorithm is exhibited in Fig.8 through the ratio $r_4 \stackrel{\text{def}}{=} \text{Var}_{\theta_1}^{\text{AMV(R,R}')} / \text{Var}_{\theta_1}^{\text{Alg}_5}$. We show that, despite the fact that algorithm 5 improves the performance of the standard MUSIC algorithm based on $\mathbf{R}_{y,T}$ for low SNRs and DOA separations when the noncircularity rate ρ increases, its efficiency decreases in these circumstances.

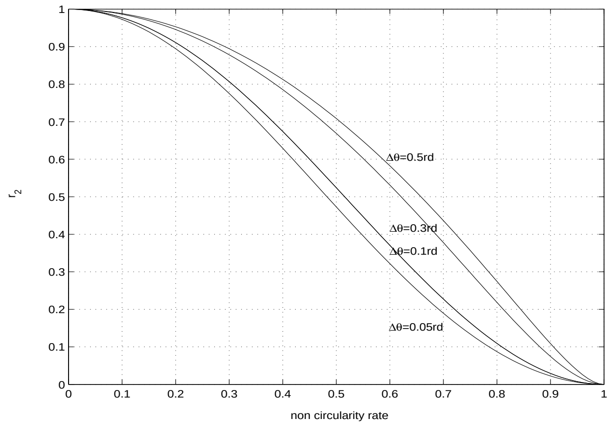

Fig.5 Ratio $r_2 \stackrel{\text{def}}{=} \text{Var}_{\theta_1}^{\text{MUSIC(R)}} / \text{Var}_{\theta_1}^{\text{Alg}_4}$ as a function of the non-circularity rate for different DOA separations for SNR=5dB, $\Delta\phi = \pi/6$ radians.

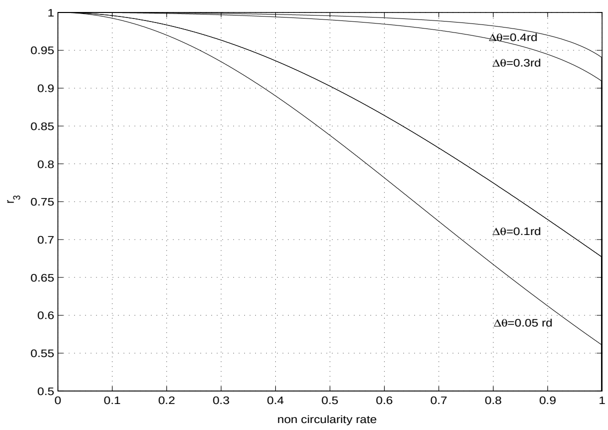

Fig.6 Ratio $r_3 \stackrel{\text{def}}{=} \text{Var}_{\theta_1}^{\text{Alg}_5} / \text{Var}_{\theta_1}^{\text{MUSIC(R)}}$ as a function of the non-circularity rate for different DOA separations for SNR=5dB.

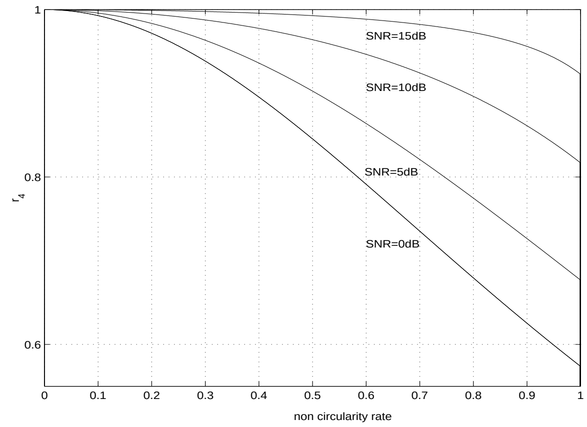

Fig.7 Ratio $r_4 \stackrel{\text{def}}{=} \mathrm{Var}_{\theta_1}^{\mathrm{Alg}_5}/\mathrm{Var}_{\theta_1}^{\mathrm{MUSIC(R)}}$ as a function of the non-circularity rate for different SNRs for $\Delta\theta = 0.1$ radians.

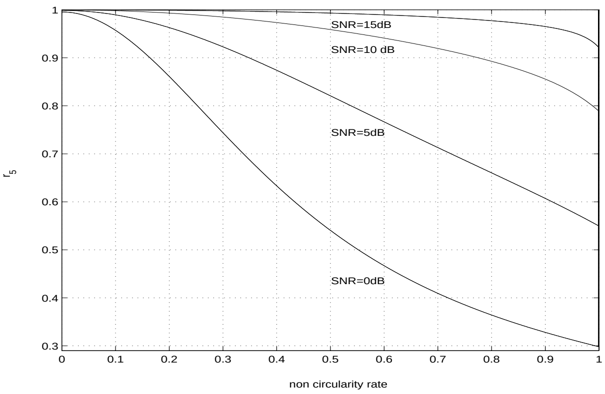

Fig.8 Ratio $r_5 \stackrel{\text{def}}{=} \mathrm{Var}_{\theta_1}^{\mathrm{AMV(R,R')}}/\mathrm{Var}_{\theta_1}^{\mathrm{Alg}_5}$ as a function of the non-circularity rate for different SNRs for $\Delta\theta = 0.1$ radians, $\Delta\phi = \pi/6$ radians.

Tables 1 and 2 compare our theoretical asymptotic variance expressions with empirical mean square errors (MSEs) obtained from Monte Carlo simulations for the standard MUSIC and the optimal weighted MUSIC algorithms for $\rho = 0.9$, $\Delta\theta = 0.2$rd. We see that there is an agreement between the theoretical and empirical results beyond a SNR threshold. Below this threshold, the optimal weighted MUSIC algorithm largely outperforms the standard MUSIC algorithm.

(dB)	MUSIC		Weighted MUSIC	
SNR	est MSE(θ_1)	th Var(θ_1)	est MSE(θ_1)	th Var(θ_1)
6	$4.452.10^{-3}$	$4.589.10^{-4}$	$3.154.10^{-4}$	$4.151.10^{-4}$
8	$1.600.10^{-3}$	$2.604.10^{-4}$	$2.344.10^{-4}$	$2.449.10^{-4}$
10	$2.899.10^{-4}$	$1.527.10^{-4}$	$1.561.10^{-4}$	$1.474.10^{-4}$
20	$1.338.10^{-5}$	$1.348.10^{-5}$	$1.337.10^{-5}$	$1.347.10^{-5}$

TABLE I
$\Delta\theta = 0.2rd$

VII. Conclusion

This paper has provided a unifying framework to investigate the asymptotic performance of arbitrary subspace-based algorithms for estimating DOAs of narrowband complex non-circular sources by giving closed-form expressions of the covariance of the asymptotic distribution of extended projection matrices. Different robustness properties of the asymptotic covariance of the estimated DOA given by such algorithms are proved. These results are applied to different MUSIC-like algorithms. We have proved that such specific algorithms largely outperform the standard MUSIC algorithm in the case of uncorrelated sources with maximum non-circularity rate. In the general case of nonsingular extended spatial covariance of the sources, the optimal weighted MUSIC that we have introduced outperforms the standard MUSIC algorithm as well, but the offered performance gain is noticeable for low SNRs and DOA separations only. Furthermore this optimal weighted MUSIC is computationally more demanding than the standard MUSIC algorithm. Consequently from an application viewpoint this gain in performance may not motivate the extra computational complexity. In this general case of nonsingular extended spatial covariance of the sources, only multidimensional non-linear optimization algorithms such as the subspace-based AMV estimator seems to be able to totally benefit of the non-circular property. A study to deal with this issue is underway.

	MUSIC		Weighted MUSIC	
	est MSE(θ)	th Var(θ)	est MSE(θ)	th Var(θ)
θ_1	$1.600.10^{-3}$	$2.604.10^{-4}$	$2.344.10^{-4}$	$2.449.10^{-4}$
θ_2	$1.800.10^{-3}$	$2.604.10^{-4}$	$2.457.10^{-4}$	$2.449.10^{-4}$

TABLE II
$\Delta\theta = 0.2rd$ AND $SNR = 8dB$

Appendix

I. Appendix: Proof of Theorem 1

Because $\mathbf{\Pi}'_T$ is the orthogonal projector onto the noise subspace of the Hermitian matrix $\mathbf{R}'_{y,T}\mathbf{R}'^H_{y,T}$, the standard perturbation result (B.1) for orthogonal projectors associated with invariant subspaces of Hermitian matrices can be applied:

$$\begin{aligned}\delta(\mathbf{\Pi}') &= -\mathbf{\Pi}'\delta(\mathbf{R}'_y\mathbf{R}'^H_y)(\mathbf{R}'_y\mathbf{R}'^H_y)^{\#} \\ &- (\mathbf{R}'_y\mathbf{R}'^H_y)^{\#}\delta(\mathbf{R}'_y\mathbf{R}'^H_y)\mathbf{\Pi}' + o(\delta(\mathbf{R}'_y\mathbf{R}'^H_y)).\end{aligned}$$

Using $\delta(\mathbf{R}'_y\mathbf{R}'^H_y) = \delta(\mathbf{R}'_y)\mathbf{R}'^H_y + \mathbf{R}'_y\delta(\mathbf{R}'^H_y) + o(\delta(\mathbf{R}'_y))$, $\mathbf{\Pi}'\mathbf{R}'_y = \mathbf{O}$ and $\mathbf{R}'^H_y(\mathbf{R}'_y\mathbf{R}'^H_y)^{\#} = \mathbf{R}'^{\#}_y$, we obtain:

$$\delta(\mathbf{\Pi}') = -\mathbf{\Pi}'\delta(\mathbf{R}'_y)\mathbf{R}'^{\#}_y - \mathbf{R}'^{H\#}_y\delta(\mathbf{R}'^H_y)\mathbf{\Pi}' + o\left(\delta(\mathbf{R}'_y)\right).$$

Then using the standard theorem of continuity (see e.g., [18, p. 122]) on regular functions of asymptotically Gaussian statistics, the asymptotic behaviors of $\mathbf{\Pi}'_T$ and $\mathbf{R}'_{y,T}$ are directly related and the first covariance matrix of the asymptotically Gaussian distribution of $\mathbf{\Pi}'_T$ can be written as

$$\mathbf{C}_{\Pi'} = \begin{pmatrix} \mathbf{R'}_y^{\#} \otimes \mathbf{\Pi'} & \mathbf{\Pi'}^* \otimes \mathbf{R'}_y^{*\#} \end{pmatrix} \begin{pmatrix} \mathbf{C}_{r'_y} & \mathbf{C}'_{r'_y} \\ \mathbf{C}'^*_{r'_y} & \mathbf{C}^*_{r'_y} \end{pmatrix} \begin{pmatrix} \mathbf{R'}_y^{*\#} \otimes \mathbf{\Pi'} \\ \mathbf{\Pi'}^* \otimes \mathbf{R'}_y^{\#} \end{pmatrix}, \quad \text{(A.1)}$$

where the expressions of $\mathbf{C}_{r'_y} = \mathrm{E}\left((\mathbf{y}_t \otimes \mathbf{y}_t - \mathrm{vec}(\mathbf{R}'_y))(\mathbf{y}_t \otimes \mathbf{y}_t - \mathrm{vec}(\mathbf{R}'_y))^H\right)$ and $\mathbf{C}'_{r'_y} = \mathrm{E}\left((\mathbf{y}_t \otimes \mathbf{y}_t - \mathrm{vec}(\mathbf{R}'_y))(\mathbf{y}_t \otimes \mathbf{y}_t - \mathrm{vec}(\mathbf{R}'_y))^T\right)$ are given in [3, Lemma 2]. Inserting these expressions in (A.1) and using $\mathbf{\Pi}'\mathbf{A} = \mathbf{O}$, $\mathbf{R'}_y^{\#}\mathbf{R}_y\mathbf{\Pi}' = \mathbf{O}$, we obtain after tedious but simple algebra manipulations:

$$\mathbf{C}_{\Pi'} = \mathbf{\Pi'}^* \otimes \mathbf{U} + \mathbf{U}^* \otimes \mathbf{\Pi'}, \quad \text{(A.2)}$$

where $\mathbf{U}$ is given here by $\sigma_n^2 \mathbf{S}'^{*\#}\mathbf{R}_y^T\mathbf{S}'^{\#}$. Using again the standard theorem of continuity, the DOAs estimated by the MUSIC algorithm based on $\mathbf{\Pi}'_T$ are asymptotically Gaussian distributed with covariance $\mathbf{C}_\Theta = \mathbf{D}_\Theta^{\mathrm{Algo}}\mathbf{C}_{\Pi'}(\mathbf{D}_\Theta^{\mathrm{Algo}})^H$ where the Jacobian matrix $\mathbf{D}_\Theta^{\mathrm{Algo}}$ of the mapping that associates Θ_T to $\mathbf{\Pi}'_T$ is given by

$$\mathbf{D}_\Theta^{\mathrm{Algo}} = \begin{pmatrix} \mathbf{d}_1^T \\ \vdots \\ \mathbf{d}_K^T \end{pmatrix} \quad \text{with} \quad \mathbf{d}_k^T = \frac{-1}{\alpha_k}\left(\mathbf{a}_k'^{\,T} \otimes \mathbf{a}_k^H + \mathbf{a}_k^T \otimes \mathbf{a}_k'^{\,H}\right) \quad \text{(A.3)}$$

straightforwardly obtained from a first-order expansion of $\frac{\partial g_{0,T}(\theta)}{\partial\theta}_{|\theta=\theta_k+\delta\theta_{k,T}} = 0$. Using expression (A.2) of $\mathbf{C}_{\Pi'}$, expression (III.2) of $\mathbf{C}_\Theta$ is straightforwardly deduced.

In the case of a single source, $\mathbf{U} = \frac{\sigma_n^2(\sigma_1^2\|\mathbf{a}_1\|^2+\sigma_n^2)}{\sigma_1^4\rho_1^2\|\mathbf{a}_1\|^6}\mathbf{a}_1\mathbf{a}_1^H$ and the expression of C_{θ_1} follows.

II. Appendix: Proof of Theorem 3

The proof relies on the standard central limit theorem applied to the independent equidistributed complex non-circular random variables $\tilde{\mathbf{y}}_t^* \otimes \tilde{\mathbf{y}}_t$ with $\tilde{\mathbf{y}}_t = \tilde{\mathbf{A}}\tilde{\mathbf{x}}_t + \tilde{\mathbf{n}}_t$. Thanks to simple algebraic manipulations of $\mathbf{C}_{r_{\tilde{y}}} = \mathrm{E}\left((\tilde{\mathbf{y}}_t^* \otimes \tilde{\mathbf{y}}_t - \mathrm{vec}(\mathbf{R}_{\tilde{y}}))(\tilde{\mathbf{y}}_t^* \otimes \tilde{\mathbf{y}}_t - \mathrm{vec}(\mathbf{R}_{\tilde{y}}))^H\right)$, we straightforwardly obtain

$$\mathbf{C}_{r_{\tilde{y}}} = \mathbf{R}_{\tilde{y}}^* \otimes \mathbf{R}_{\tilde{y}} + \mathbf{K}_{2M}\left(\mathbf{R}'_{\tilde{y}} \otimes \mathbf{R}'^{*}_{\tilde{y}}\right) + (\tilde{\mathbf{A}}^* \otimes \tilde{\mathbf{A}})\mathbf{Q}_{\tilde{x}}(\tilde{\mathbf{A}}^T \otimes \tilde{\mathbf{A}}^H)$$

with $\mathbf{R}'_{\tilde{y}} \stackrel{\mathrm{def}}{=} \mathrm{E}(\tilde{\mathbf{y}}_t\tilde{\mathbf{y}}_t^T)$ and where $(\mathbf{Q}_{\tilde{x}})_{i+(j-1)2K,k+(l-1)2K} = \mathrm{Cum}\left((\tilde{\mathbf{x}}_t)_i, (\tilde{\mathbf{x}}_t)_j^*, (\tilde{\mathbf{x}}_t)_k^*, (\tilde{\mathbf{x}}_t)_l\right)$. Then using the standard perturbation result for orthogonal projectors [17] (see also [13]) applied to $\tilde{\mathbf{\Pi}}$ associated with the noise subspace of $\mathbf{R}_{\tilde{y}}$

$$\delta(\tilde{\mathbf{\Pi}}) = -\tilde{\mathbf{\Pi}}\delta(\mathbf{R}_{\tilde{y}})\tilde{\mathbf{S}}^{\#} - \tilde{\mathbf{S}}^{\#}\delta(\mathbf{R}_{\tilde{y}})\tilde{\mathbf{\Pi}} + o\left(\delta(\mathbf{R}_{\tilde{y}})\right), \quad \text{(B.1)}$$

the asymptotic behaviors of $\tilde{\mathbf{\Pi}}_T$ and $\mathbf{R}_{\tilde{y},T}$ are directly related. The standard theorem (see e.g., [18, p. 122]) on regular functions of asymptotically Gaussian statistics applies and the first covariance matrix of the asymptotically Gaussian distribution of $\tilde{\mathbf{\Pi}}_T$ can be written as

$$\begin{aligned}
\mathbf{C}_{\tilde{\Pi}} &= \left((\tilde{\mathbf{\Pi}}^* \otimes \tilde{\mathbf{S}}^{\#}) + (\tilde{\mathbf{S}}^{\#^*} \otimes \tilde{\mathbf{\Pi}})\right)\mathbf{C}_{r_{\tilde{y}}}\left((\tilde{\mathbf{\Pi}}^* \otimes \tilde{\mathbf{S}}^{\#}) + (\tilde{\mathbf{S}}^{\#^*} \otimes \tilde{\mathbf{\Pi}})\right) \quad \text{(B.2)}\\
&= \left((\tilde{\mathbf{\Pi}}^* \otimes \tilde{\mathbf{S}}^{\#}) + (\tilde{\mathbf{S}}^{\#^*} \otimes \tilde{\mathbf{\Pi}})\right)\left(\mathbf{R}_{\tilde{y}}^* \otimes \mathbf{R}_{\tilde{y}} + \mathbf{K}_{2M}(\mathbf{R}'_{\tilde{y}} \otimes \mathbf{R}'^{*}_{\tilde{y}})\right)\left((\tilde{\mathbf{\Pi}}^* \otimes \tilde{\mathbf{S}}^{\#}) + (\tilde{\mathbf{S}}^{\#^*} \otimes \tilde{\mathbf{\Pi}})\right)\\
&= (\mathbf{I}_{4M^2} + \mathbf{K}_{2M}(\mathbf{J}_M \otimes \mathbf{J}_M))\left((\tilde{\mathbf{\Pi}}^* \otimes \tilde{\mathbf{U}}) + (\tilde{\mathbf{U}}^* \otimes \tilde{\mathbf{\Pi}})\right)
\end{aligned}$$

where $\tilde{\mathbf{\Pi}}\tilde{\mathbf{A}} = \mathbf{O}$, and $\mathbf{R}'^{*}_{\tilde{y}} = \mathbf{J}_M\mathbf{R}_{\tilde{y}}$ and $\tilde{\mathbf{S}}^{\#}\mathbf{R}_{\tilde{y}}\tilde{\mathbf{\Pi}} = \mathbf{O}$ are used in the second and third equalities respectively.

Proving the convergence in distribution of the second statistic follows the same lines where the terms of

$$\mathbf{C}_{\Pi_1,\Pi_2,\Pi_2^*} = \lim_{T\to\infty}\frac{1}{T}\mathrm{E}\left[\begin{pmatrix} \mathrm{vec}(\mathbf{\Pi}_{1,T} - \mathbf{\Pi}_1) \\ \mathrm{vec}(\mathbf{\Pi}_{2,T} - \mathbf{\Pi}_2) \\ \mathrm{vec}(\mathbf{\Pi}^*_{2,T} - \mathbf{\Pi}^*_2) \end{pmatrix}\begin{pmatrix} \mathrm{vec}(\mathbf{\Pi}_{1,T} - \mathbf{\Pi}_1) \\ \mathrm{vec}(\mathbf{\Pi}_{2,T} - \mathbf{\Pi}_2) \\ \mathrm{vec}(\mathbf{\Pi}^*_{2,T} - \mathbf{\Pi}^*_2) \end{pmatrix}^H\right] = \begin{pmatrix} \mathbf{C}_{\Pi_1} & \mathbf{C}^H_{\Pi_2,\Pi_1} & \mathbf{C}^H_{\Pi_2^*,\Pi_1} \\ \mathbf{C}_{\Pi_2,\Pi_1} & \mathbf{C}_{\Pi_2} & \mathbf{C}^H_{\Pi_2^*,\Pi_2} \\ \mathbf{C}_{\Pi_2^*,\Pi_1} & \mathbf{C}_{\Pi_2^*,\Pi_2} & \mathbf{C}_{\Pi_2^*} \end{pmatrix}$$

can be deduced from the expression of

$$\begin{aligned}
\mathbf{C}_{\tilde{\Pi}} &= \lim_{T\to\infty}\frac{1}{T}\mathrm{E}\left[\mathrm{vec}\begin{pmatrix} \mathbf{\Pi}_{1,T} - \mathbf{\Pi}_1 & \mathbf{\Pi}_{2,T} - \mathbf{\Pi}_2 \\ \mathbf{\Pi}^*_{2,T} - \mathbf{\Pi}^*_2 & \mathbf{\Pi}^*_{1,T} - \mathbf{\Pi}^*_1 \end{pmatrix}\mathrm{vec}^H\begin{pmatrix} \mathbf{\Pi}_{1,T} - \mathbf{\Pi}_1 & \mathbf{\Pi}_{2,T} - \mathbf{\Pi}_2 \\ \mathbf{\Pi}^*_{2,T} - \mathbf{\Pi}^*_2 & \mathbf{\Pi}^*_{1,T} - \mathbf{\Pi}^*_1 \end{pmatrix}\right] \quad \text{(B.3)}\\
&= \begin{pmatrix} \mathbf{G} & \mathbf{O} \\ \mathbf{O} & \mathbf{G} \end{pmatrix}\lim_{T\to\infty}\frac{1}{T}\mathrm{E}\left(\begin{bmatrix} \mathrm{vec}(\mathbf{\Pi}_{1,T} - \mathbf{\Pi}_1) \\ \mathrm{vec}(\mathbf{\Pi}^*_{2,T} - \mathbf{\Pi}^*_2) \\ \mathrm{vec}(\mathbf{\Pi}_{2,T} - \mathbf{\Pi}_2) \\ \mathrm{vec}(\mathbf{\Pi}^*_{1,T} - \mathbf{\Pi}^*_1) \end{bmatrix}\begin{bmatrix} \mathrm{vec}(\mathbf{\Pi}_{1,T} - \mathbf{\Pi}_1) \\ \mathrm{vec}(\mathbf{\Pi}^*_{2,T} - \mathbf{\Pi}^*_2) \\ \mathrm{vec}(\mathbf{\Pi}_{2,T} - \mathbf{\Pi}_2) \\ \mathrm{vec}(\mathbf{\Pi}^*_{1,T} - \mathbf{\Pi}^*_1) \end{bmatrix}^H\right)\begin{pmatrix} \mathbf{G}^T & \mathbf{O} \\ \mathbf{O} & \mathbf{G}^T \end{pmatrix} \quad \text{(B.4)}\\
&= \begin{pmatrix} \mathbf{G} & \mathbf{O} \\ \mathbf{O} & \mathbf{G} \end{pmatrix} \quad \text{(B.5)}\\
&\quad \begin{pmatrix} \mathbf{C}_{\Pi_1} & \mathbf{C}^H_{\Pi_2^*,\Pi_1} & \mathbf{C}^H_{\Pi_2,\Pi_1} & \mathbf{C}_{\Pi_1,\Pi_1^*} \\ \mathbf{C}_{\Pi_2^*,\Pi_1} & \mathbf{C}_{\Pi_2^*} & \mathbf{C}_{\Pi_2^*,\Pi_2} & \mathbf{C}^*_{\Pi_2,\Pi_1} \\ \mathbf{C}_{\Pi_2,\Pi_1} & \mathbf{C}^H_{\Pi_2^*,\Pi_2} & \mathbf{C}_{\Pi_2} & \mathbf{C}^*_{\Pi_2^*,\Pi_1} \\ \mathbf{C}^H_{\Pi_1,\Pi_1^*} & \mathbf{C}^T_{\Pi_2,\Pi_1} & \mathbf{C}^T_{\Pi_2^*,\Pi_1} & \mathbf{C}_{\Pi_1^*} \end{pmatrix}\begin{pmatrix} \mathbf{G}^T & \mathbf{O} \\ \mathbf{O} & \mathbf{G}^T \end{pmatrix}
\end{aligned}$$

where $\mathbf{G}$ is the block permutation matrix defined by $\mathrm{vec}\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} = \mathbf{G}\begin{pmatrix} \mathrm{vec}(\mathbf{A}) \\ \mathrm{vec}(\mathbf{B}) \end{pmatrix}$. With expressions (B.5) and

(V.1) of $\tilde{\mathbf{\Pi}}$, we obtain

$$\begin{pmatrix} \mathbf{C}_{\Pi_1} & \mathbf{C}^H_{\Pi_2^*,\Pi_1} & \mathbf{C}^H_{\Pi_2,\Pi_1} & \mathbf{C}_{\Pi_1,\Pi_1^*} \\ \mathbf{C}_{\Pi_2^*,\Pi_1} & \mathbf{C}_{\Pi_2^*} & \mathbf{C}_{\Pi_2^*,\Pi_2} & \mathbf{C}^*_{\Pi_2,\Pi_1} \\ \mathbf{C}_{\Pi_2,\Pi_1} & \mathbf{C}^H_{\Pi_2^*,\Pi_2} & \mathbf{C}_{\Pi_2} & \mathbf{C}^*_{\Pi_2^*,\Pi_1} \\ \mathbf{C}^H_{\Pi_1,\Pi_1^*} & \mathbf{C}^T_{\Pi_2,\Pi_1} & \mathbf{C}^T_{\Pi_2^*,\Pi_1} & \mathbf{C}_{\Pi_1^*} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{G}^T & \mathbf{O} \\ \mathbf{O} & \mathbf{G}^T \end{pmatrix} \left(\begin{bmatrix} \mathbf{\Pi}_1^* & \mathbf{\Pi}_2^* \\ \mathbf{\Pi}_2 & \mathbf{\Pi}_1 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \\ \mathbf{U}_2^* & \mathbf{U}_1^* \end{bmatrix} \right.$$
$$+ \begin{bmatrix} \mathbf{U}_1^* & \mathbf{U}_2^* \\ \mathbf{U}_2 & \mathbf{U}_1 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{\Pi}_1 & \mathbf{\Pi}_2 \\ \mathbf{\Pi}_2^* & \mathbf{\Pi}_1^* \end{bmatrix} \tag{B.6}$$
$$+\mathbf{K}_{2M} \left(\begin{bmatrix} \mathbf{\Pi}_2 & \mathbf{\Pi}_1 \\ \mathbf{\Pi}_1^* & \mathbf{\Pi}_2^* \end{bmatrix} \otimes \begin{bmatrix} \mathbf{U}_2^* & \mathbf{U}_1^* \\ \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix} \right)$$
$$\left. +\mathbf{K}_{2M} \left(\begin{bmatrix} \mathbf{U}_2 & \mathbf{U}_1 \\ \mathbf{U}_1^* & \mathbf{U}_2^* \end{bmatrix} \otimes \begin{bmatrix} \mathbf{\Pi}_2^* & \mathbf{\Pi}_1^* \\ \mathbf{\Pi}_1 & \mathbf{\Pi}_2 \end{bmatrix} \right) \right) \begin{pmatrix} \mathbf{G} & \mathbf{O} \\ \mathbf{O} & \mathbf{G} \end{pmatrix}.$$

Then, using the following two identities deduced from the definition of the permutation matrices $\mathbf{G}$ and $\mathbf{K}_{2M}$ for any $M \times M$ matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$

$$\begin{pmatrix} \mathbf{G}^T & \mathbf{O} \\ \mathbf{O} & \mathbf{G}^T \end{pmatrix} \left(\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{A}^* \end{bmatrix} \otimes \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{D}^* & \mathbf{C}^* \end{bmatrix} \right) \begin{pmatrix} \mathbf{G} & \mathbf{O} \\ \mathbf{O} & \mathbf{G} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{A}\otimes\mathbf{C} & \mathbf{A}\otimes\mathbf{D} & \mathbf{B}\otimes\mathbf{C} & \mathbf{B}\otimes\mathbf{D} \\ \mathbf{A}\otimes\mathbf{D}^* & \mathbf{A}\otimes\mathbf{C}^* & \mathbf{B}\otimes\mathbf{D}^* & \mathbf{B}\otimes\mathbf{C}^* \\ \mathbf{B}^*\otimes\mathbf{C} & \mathbf{B}^*\otimes\mathbf{D} & \mathbf{A}^*\otimes\mathbf{C} & \mathbf{A}^*\otimes\mathbf{D} \\ \mathbf{B}^*\otimes\mathbf{D}^* & \mathbf{B}^*\otimes\mathbf{C}^* & \mathbf{A}^*\otimes\mathbf{D}^* & \mathbf{A}^*\otimes\mathbf{C}^* \end{pmatrix},$$
$$\begin{pmatrix} \mathbf{G}^T & \mathbf{O} \\ \mathbf{O} & \mathbf{G}^T \end{pmatrix} \mathbf{K}_{2M} \begin{pmatrix} \mathbf{G} & \mathbf{O} \\ \mathbf{O} & \mathbf{G} \end{pmatrix} = \begin{pmatrix} \mathbf{K}_M & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{K}_M & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_M & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{K}_M \end{pmatrix},$$

the expressions of $\mathbf{C}_{\Pi_1}$, $\mathbf{C}_{\Pi_2}$, $\mathbf{C}_{\Pi_2,\Pi_1}$, $\mathbf{C}_{\Pi_2^*,\Pi_1}$ and $\mathbf{C}_{\Pi_2^*,\Pi_2}$ of Theorem 3 follow directly from (B.6).

III. Appendix: Proof of Theorem 5

We first prove that algorithms 2 and 3 satisfy the same first-order perturbation expansion (V.7). For algorithm 2, we note that

$$\begin{aligned} \frac{\partial g_{2,T}(\theta)}{\partial \theta} &= 2\mathrm{Tr}\left(\mathbf{\Pi}_{1,T}\mathbf{M}'(\theta)\mathbf{\Pi}_{1,T}\mathbf{M}(\theta) \right) \\ &- 2\Re\left[\mathrm{Tr}\left(\mathbf{\Pi}^*_{2,T}\mathbf{M}'(\theta)\mathbf{\Pi}_{2,T}\mathbf{M}^*(\theta) \right) \right], \end{aligned}$$

with $\mathbf{M}(\theta) \stackrel{\text{def}}{=} \mathbf{a}(\theta)\mathbf{a}^H(\theta)$ and $\mathbf{M}'(\theta) \stackrel{\text{def}}{=} \frac{d\mathbf{M}(\theta)}{d\theta}$. Because $\theta_{k,T}$ satisfies $\frac{\partial g_{2,T}(\theta)}{\partial\theta}_{|\theta=\theta_{k,T}=\theta_k+\delta\theta_{k,T}} = 0$, we straightforwardly obtain the following first-order perturbation expansion thanks to $\mathbf{\Pi}_{1,T} = \mathbf{\Pi}_1 + \delta\mathbf{\Pi}_{1,T}$, $\mathbf{\Pi}_{2,T} = \mathbf{\Pi}_2 + \delta\mathbf{\Pi}_{2,T}$, $\mathbf{M}(\theta_{k,T}) = \mathbf{M}_k + \mathbf{M}'_k\delta\theta_{k,T} + o(\delta\theta_{k,T})$ and $\mathbf{M}'(\theta_{k,T}) = \mathbf{M}'_k + \mathbf{M}''_k\delta\theta_{k,T} + o(\delta\theta_{k,T})$:

$$\begin{aligned} &\Big[\mathrm{Tr}\left(\mathbf{\Pi}_1\mathbf{M}''_k\mathbf{\Pi}_1\mathbf{M}_k \right) - \Re\left(\mathrm{Tr}\left(\mathbf{\Pi}_2^*\mathbf{M}''_k\mathbf{\Pi}_2\mathbf{M}^*_k \right) \right) \\ &+ \mathrm{Tr}\left(\mathbf{\Pi}_1\mathbf{M}'_k\mathbf{\Pi}_1\mathbf{M}'_k \right) - \Re\left(\mathrm{Tr}\left(\mathbf{\Pi}_2^*\mathbf{M}'_k\mathbf{\Pi}_2\mathbf{M}'^*_k \right) \right) \Big] \delta\theta_{k,T} \\ &= \Re\left(\mathrm{Tr}\left(\delta\mathbf{\Pi}^*_{2,T}\mathbf{M}'_k\mathbf{\Pi}_2\mathbf{M}^*_k \right) \right) + \Re\left(\mathrm{Tr}\left(\mathbf{\Pi}^*_2\mathbf{M}'_k\delta\mathbf{\Pi}_{2,T}\mathbf{M}^*_k \right) \right) \\ &-\mathrm{Tr}\left(\delta\mathbf{\Pi}_{1,T}\mathbf{M}'_k\mathbf{\Pi}_1\mathbf{M}_k \right) - \mathrm{Tr}\left(\mathbf{\Pi}_1\mathbf{M}'_k\delta\mathbf{\Pi}_{1,T}\mathbf{M}_k \right) \qquad \text{(C.1)} \\ &+o(\delta\mathbf{\Pi}_{1,T}) + o(\mathbf{\Pi}_{2,T}) \end{aligned}$$

with $\mathbf{M}_k \stackrel{\text{def}}{=} \mathbf{M}(\theta_k)$, $\mathbf{M}'_k \stackrel{\text{def}}{=} \mathbf{M}'(\theta_k)$ and $\mathbf{M}''_k \stackrel{\text{def}}{=} \frac{d^2\mathbf{M}(\alpha)}{d\alpha^2}_{|\alpha=\theta_k}$. Furthermore, we note that the sum of the first two terms of the left hand side of (C.1) vanishes thanks to the identity $\tilde{\mathbf{\Pi}}\tilde{\mathbf{a}}_k = \mathbf{0}$, which is equivalent to $\mathbf{\Pi}_1\mathbf{a}_k + e^{-i\phi_k}\mathbf{\Pi}_2\mathbf{a}^*_k = \mathbf{0}$, and which implies

$$\mathbf{\Pi}_1\mathbf{M}_k\mathbf{\Pi}_1 = \mathbf{\Pi}_2\mathbf{M}^*_k\mathbf{\Pi}^*_2. \tag{C.2}$$

Consequently

$$\begin{aligned} &\mathrm{Tr}\left(\mathbf{\Pi}_1\mathbf{M}''_k\mathbf{\Pi}_1\mathbf{M}_k \right) - \Re\left(\mathrm{Tr}\left(\mathbf{\Pi}_2^*\mathbf{M}''_k\mathbf{\Pi}_2\mathbf{M}^*_k \right) \right) \\ &= \Re\left(\mathrm{Tr}\left((\mathbf{\Pi}_1\mathbf{M}_k\mathbf{\Pi}_1 - \mathbf{\Pi}_2\mathbf{M}^*_k\mathbf{\Pi}^*_2)\mathbf{M}''_k \right) \right) = 0. \end{aligned}$$

Thus (C.1) becomes

$$\begin{aligned} \delta\theta_{k,T} &= \frac{\mathcal{D}}{\mathrm{Tr}\left(\mathbf{\Pi}_1\mathbf{M}'_k\mathbf{\Pi}_1\mathbf{M}'_k \right) - \Re\left(\mathrm{Tr}\left(\mathbf{\Pi}^*_2\mathbf{M}'_k\mathbf{\Pi}_2\mathbf{M}'^*_k \right) \right)} \\ &+ o(\delta\mathbf{\Pi}_{1,T}) + o(\delta\mathbf{\Pi}_{2,T}) \end{aligned} \tag{C.3}$$

with $\mathcal{D} \stackrel{\text{def}}{=} \Re\left(\mathrm{Tr}\left(\delta\mathbf{\Pi}^*_{2,T}\mathbf{M}'_k\mathbf{\Pi}_2\mathbf{M}^*_k + \mathbf{\Pi}^*_2\mathbf{M}'_k\delta\mathbf{\Pi}_{2,T}\mathbf{M}^*_k \right) \right) - \mathrm{Tr}\left(\delta\mathbf{\Pi}_{1,T}\mathbf{M}'_k\mathbf{\Pi}_1\mathbf{M}_k + \mathbf{\Pi}_1\mathbf{M}'_k\delta\mathbf{\Pi}_{1,T}\mathbf{M}_k \right)$. For algorithm 3, we note that

$$\begin{aligned} g_{3,T}(z) &= \mathrm{Tr}\left(\mathbf{\Pi}_{1,T}\mathbf{M}(z)\mathbf{\Pi}_{1,T}\mathbf{M}(z) \right) \\ &- \mathrm{Tr}\left(\mathbf{\Pi}^*_{2,T}\mathbf{M}(z)\mathbf{\Pi}_{2,T}\mathbf{M}(z^{-1}) \right) \end{aligned} \tag{C.4}$$

with $\mathbf{M}(z) \stackrel{\text{def}}{=} \mathbf{a}(z)\mathbf{a}^T(z^{-1})$. By definition of algorithm 3 (see (IV.7)), $z_{k,T} = \theta_k + \delta\theta_{k,T}$ is solution of

$$g_{3,T}(z_{k,T}) = 0 \text{ with } z_{k,T} = r_{k,T}e^{i\theta_{k,T}} = z_k + \delta z_{k,T} = e^{i\theta_k} + \delta z_{k,T}. \tag{C.5}$$

To relate $\delta\theta_{k,T}$ to $\delta\mathbf{\Pi}_{1,T}$, $\delta\mathbf{\Pi}_{2,T}$ and $\delta\mathbf{\Pi}^*_{2,T}$, a second-order expansion of $\mathbf{a}(z)$, $\mathbf{\Pi}_{1,T}$ and $\mathbf{\Pi}_{2,T}$ is required since the first-order terms in $\delta\theta_{k,T}$ and $\delta r_{k,T}$ vanish, as noted in [13] for the standard root-MUSIC algorithm.

$$\begin{aligned} \mathbf{a}(z_{k,T}) &= \mathbf{a}_k + \mathbf{a}'_k\delta\theta_{k,T} - i\mathbf{a}'_k\delta r_{k,T} + \frac{1}{2}\mathbf{a}''_k(\delta\theta_{k,T})^2 \\ &- i\mathbf{a}''_k\delta\theta_{k,T}\delta r_{k,T} - \frac{1}{2}\mathbf{a}''_k(\delta r_{k,T})^2 + o_2(\delta\theta_{k,T}, \delta r_{k,T}) \\ \mathbf{a}(z^{-1}_{k,T}) &= \mathbf{a}^*_k + \mathbf{a}'^*_k\delta\theta_{k,T} - i\mathbf{a}'^*_k\delta r_{k,T} + \frac{1}{2}\mathbf{a}''^*_k(\delta\theta_{k,T})^2 \\ &- i\mathbf{a}''^*_k\delta\theta_{k,T}\delta r_{k,T} - \frac{1}{2}\mathbf{a}''^*_k(\delta r_{k,T})^2 + o_2(\delta\theta_{k,T}, \delta r_{k,T}) \\ \mathbf{\Pi}_{1,T} &= \mathbf{\Pi}_1 + \delta\mathbf{\Pi}_{1,T} + \delta^2\mathbf{\Pi}_{1,T} + o(\delta^2\mathbf{\Pi}_{1,T}) \\ \mathbf{\Pi}_{2,T} &= \mathbf{\Pi}_2 + \delta\mathbf{\Pi}_{2,T} + \delta^2\mathbf{\Pi}_{2,T} + o(\delta^2\mathbf{\Pi}_{2,T}) \end{aligned}$$

with $\mathbf{a}_k^{'} \stackrel{\text{def}}{=} \frac{d\mathbf{a}_k}{d\theta_k}$ and $\mathbf{a}_k^{''} \stackrel{\text{def}}{=} \frac{d^2\mathbf{a}_k}{d\theta_k^2}$, where $(\delta\mathbf{\Pi}_{i,T})_{i=1,2}$ and $(\delta^2\mathbf{\Pi}_{i,T})_{i=1,2}$ denote the first and second-order terms of the expansion of $(\mathbf{\Pi}_{i,T})_{i=1,2}$ w.r.t. $\delta\mathbf{R}_{\tilde{y},T}$, and where $o_2(\delta\theta_{k,T}, \delta r_{k,T})$ is a third-order term in $(\delta\theta_{k,T}, \delta r_{k,T})$. Consequently

$$
\begin{aligned}
\mathbf{M}(z_{k,T}) &= \mathbf{M}_k + \mathbf{M}_k^{'}\delta\theta_{k,T} - i\mathbf{M}_k^{'}\delta r_{k,T} + \frac{1}{2}\mathbf{M}_k^{''}(\delta\theta_{k,T})^2 \\
&\quad -i\mathbf{M}_k^{''}\delta\theta_{k,T}\delta r_{k,T} - \frac{1}{2}\mathbf{M}_k^{''}(\delta r_{k,T})^2 + o_2(\delta\theta_{k,T}, \delta r_{k,T})) \\
\mathbf{M}(z_{k,T}^{-1}) &= \mathbf{M}_k^* + \mathbf{M}_k^{'*}\delta\theta_{k,T} - i\mathbf{M}_k^{'*}\delta r_{k,T} \\
&+ \frac{1}{2}\mathbf{M}_k^{''*}(\delta\theta_{k,T})^2 - i\mathbf{M}_k^{''*}\delta\theta_{k,T}\delta r_{k,T} \\
&- \frac{1}{2}\mathbf{M}_k^{''*}(\delta r_{k,T})^2 + o_2(\delta\theta_{k,T}, \delta r_{k,T})),
\end{aligned}
$$

Inserting these second-order expansions of $\mathbf{\Pi}_{1,T}$, $\mathbf{\Pi}_{2,T}$, $\mathbf{M}(z_{k,T})$ and $\mathbf{M}(z_{k,T}^{-1})$ in the expression (C.4) of $g_{2,T}(z_{k,T})$ and using identity (C.2) and identity

$$\mathbf{a}_k^H\delta\mathbf{\Pi}_{1,T}\mathbf{a}_k + \Re(\mathbf{a}_k^H\delta\mathbf{\Pi}_{2,T}\mathbf{a}_k^* e^{-i\phi_k}) = 0$$

deduced from identity $\tilde{\mathbf{a}}_k^H\delta\tilde{\mathbf{\Pi}}_T\tilde{\mathbf{a}}_k = 0$ (issued from $\tilde{\mathbf{\Pi}}\tilde{\mathbf{a}}_k = \mathbf{0}$ and $\delta\tilde{\mathbf{\Pi}}_T\tilde{\mathbf{a}}_k + \tilde{\mathbf{\Pi}}\delta\tilde{\mathbf{a}}_{k,T} = \mathbf{0}$), one can check that the first-order terms in $\delta\theta_{k,T}$ and $\delta r_{k,T}$ vanish and the following expression of $g_{3,T}(z_{k,T})$ of (C.4) is obtained after simple, but tedious algebra manipulations:

$$
\begin{aligned}
&g_{3,T}(z_{k,T}) = \\
&\frac{1}{2}\Big\{\Big(\mathrm{Tr}(\mathbf{\Pi}_1\mathbf{M}_k^{'}\mathbf{\Pi}_1\mathbf{M}_k^{'}) - \Re(\mathrm{Tr}(\mathbf{\Pi}_2^*\mathbf{M}_k^{'}\mathbf{\Pi}_2\mathbf{M}_k^{'*}))\Big) \\
&\big((\delta\theta_{k,T})^2 - (\delta r_{k,T})^2\big) + 2\Big(\mathrm{Tr}(\delta\mathbf{\Pi}_{1,T}\mathbf{M}_k\mathbf{\Pi}_1\mathbf{M}_k^{'}) \\
&-\Re(\mathrm{Tr}(\delta\mathbf{\Pi}_{2,T}^*\mathbf{M}_k\mathbf{\Pi}_2\mathbf{M}_k^{'*})) + \mathrm{Tr}(\mathbf{\Pi}_1\mathbf{M}_k\delta\mathbf{\Pi}_{1,T}\mathbf{M}_k^{'}) \\
&-\Re(\mathrm{Tr}(\mathbf{\Pi}_2^*\mathbf{M}_k\delta\mathbf{\Pi}_{2,T}\mathbf{M}_k^{'*}))\Big)\delta\theta_{k,T} + 2\left(\mathrm{Tr}(\mathbf{\Pi}_1\mathbf{M}_k\delta^2\mathbf{\Pi}_{1,T}\mathbf{M}_k)\right. \\
&\left.-\Re(\mathrm{Tr}(\mathbf{\Pi}_2^*\mathbf{M}_k\delta^2\mathbf{\Pi}_{2,T}\mathbf{M}_k^*))\right)\Big\} - 2i\Big\{\mathrm{Tr}(\delta\mathbf{\Pi}_{1,T}\mathbf{M}_k^{'}\mathbf{\Pi}_1\mathbf{M}_k) \\
&+\mathrm{Tr}(\mathbf{\Pi}_1\mathbf{M}_k^{'}\delta\mathbf{\Pi}_{1,T}\mathbf{M}_k) - \Re(\mathrm{Tr}(\delta\mathbf{\Pi}_{2,T}^*\mathbf{M}_k^{'}\mathbf{\Pi}_2\mathbf{M}_k^*) \\
&-\Re(\mathrm{Tr}(\mathbf{\Pi}_2^*\mathbf{M}_k^{'}\delta\mathbf{\Pi}_{2,T}\mathbf{M}_k^*) + \Big(\mathrm{Tr}(\mathbf{\Pi}_1\mathbf{M}_k^{'}\mathbf{\Pi}_1\mathbf{M}_k^{'}) \\
&-\Re(\mathrm{Tr}(\mathbf{\Pi}_2^*\mathbf{M}_k^{'}\mathbf{\Pi}_2\mathbf{M}_k^{'*}))\Big)\delta\theta_{k,T}\Big\}\delta r_{k,T} + o_2(\delta\theta_{k,T}, \delta r_{k,T}).
\end{aligned}
\tag{C.6}
$$

Since the different matrices $\mathbf{M}$ are composed of sums of rank-one matrices and that matrices $\mathbf{\Pi}_1$, $\delta\mathbf{\Pi}_{1,T}$ and $\delta^2\mathbf{\Pi}_{1,T}$, [resp. $\mathbf{\Pi}_2$, $\delta\mathbf{\Pi}_{2,T}$ and $\delta^2\mathbf{\Pi}_{2,T}$] are Hermitian [resp. complex symmetric] structured, one can check that the first four lines within the braces in (C.6) are real, while the last two lines in the second brace are purely imaginary. By setting the imaginary part of this expansion equal to zero ($z_{k,T}$ is a root of $g_{3,T}(z)$), (C.3) is found.

Remark: With different cost functions, we note the similarity of behavior of our algorithms 1 and 2, with the standard MUSIC and root-MUSIC algorithms analyzed in [13]: In the two cases, the asymptotic distributions of the DOA estimates given by the MUSIC and the associated root-MUSIC algorithm are identical. Furthermore the second-order terms in $\delta^2\mathbf{\Pi}_{1,T}$ and $\delta^2\mathbf{\Pi}_{2,T}$ are not used for the derivation of $\delta\theta_{k,T}$ (they would be used in the derivation of $\delta r_{k,T}$, which is not studied in this paper) for the two root-MUSIC algorithms.

Considering algorithm 1, the estimates $\theta_{k,T}$ and $\phi_{k,T}$ are solutions of the global minimization

$$(\theta_{k,T}, \phi_{k,T}) = \arg\min_{\theta,\phi} \tilde{g}_{1,T}(\theta, \phi)$$

with $\tilde{g}_{1,T}(\theta,\phi) = \tilde{\mathbf{a}}^H(\theta,\phi)\tilde{\mathbf{\Pi}}_T\tilde{\mathbf{a}}(\theta,\phi) = \mathrm{Tr}(\tilde{\mathbf{\Pi}}_T\mathbf{M}_{\theta,\phi})$ where $\mathbf{M}_{\theta,\phi} \stackrel{\text{def}}{=} \tilde{\mathbf{a}}(\theta,\phi)\tilde{\mathbf{a}}(\theta,\phi)^H$. Because $(\theta_{k,T}, \phi_{k,T})$ satisfies

$$\frac{\partial\tilde{g}_{1,T}(\theta,\phi)}{\partial\theta}_{|(\theta,\phi)=(\theta_{k,T},\phi_{k,T})=(\theta_k+\delta\theta_{k,T},\phi_k+\delta\phi_{k,T})} = 0$$

and

$$\frac{\partial\tilde{g}_{1,T}(\theta,\phi)}{\partial\phi}_{|(\theta,\phi)=(\theta_{k,T},\phi_{k,T})=(\theta_k+\delta\theta_{k,T},\phi_k+\delta\phi_{k,T})} = 0,$$

we straightforwardly obtain the following first-order perturbation expansion

$$
\begin{aligned}
\mathrm{Tr}(\tilde{\mathbf{\Pi}}\mathbf{M}_{\theta,\theta}^{''})\delta\theta_{k,T} + \mathrm{Tr}(\tilde{\mathbf{\Pi}}\mathbf{M}_{\theta,\phi}^{''})\delta\phi_{k,T} + \mathrm{Tr}(\delta\tilde{\mathbf{\Pi}}_T\mathbf{M}_\theta^{'}) &= 0 \\
\mathrm{Tr}(\tilde{\mathbf{\Pi}}\mathbf{M}_{\theta,\phi}^{''})\delta\theta_{k,T} + \mathrm{Tr}(\tilde{\mathbf{\Pi}}\mathbf{M}_{\phi,\phi}^{''})\delta\phi_{k,T} + \mathrm{Tr}(\delta\tilde{\mathbf{\Pi}}_T\mathbf{M}_\phi^{'}) &= 0
\end{aligned}
\tag{C.7}
$$

with $\mathbf{M}_\theta^{'} \stackrel{\text{def}}{=} \frac{\partial\mathbf{M}_{\theta,\phi}}{\partial\theta}$, $\mathbf{M}_\phi^{'} \stackrel{\text{def}}{=} \frac{\partial\mathbf{M}_{\theta,\phi}}{\partial\phi}$, $\mathbf{M}_{\theta,\theta}^{''} \stackrel{\text{def}}{=} \frac{\partial^2\mathbf{M}_{\theta,\phi}}{\partial\theta^2}$, $\mathbf{M}_{\theta,\phi}^{''} \stackrel{\text{def}}{=} \frac{\partial^2\mathbf{M}_{\theta,\phi}}{\partial\theta\partial\phi}$ and $\mathbf{M}_{\phi,\phi}^{''} \stackrel{\text{def}}{=} \frac{\partial^2\mathbf{M}_{\theta,\phi}}{\partial\phi^2}$ associated with the source k. Noting that $\mathrm{Tr}(\tilde{\mathbf{\Pi}}\mathbf{M}_{i,j}^{''}) = 2\Re(\tilde{\mathbf{a}'}_{i,k}^H\tilde{\mathbf{\Pi}}\tilde{\mathbf{a}}'_{j,k})$, $i,j = \theta,\phi$ which is denoted by $\alpha_{i,j}^{(k)}$, (C.7) gives

$$
\begin{aligned}
\delta\theta_{k,T} &= \frac{-1}{\alpha_{\theta,\theta}^{(k)}\alpha_{\phi,\phi}^{(k)} - (\alpha_{\theta,\phi}^{(k)})^2}\Big(\alpha_{\phi,\phi}^{(k)}\mathrm{Tr}(\delta\tilde{\mathbf{\Pi}}_T\mathbf{M}_\theta^{'}) \\
&\quad -\alpha_{\theta,\phi}^{(k)}\mathrm{Tr}(\delta\tilde{\mathbf{\Pi}}_T\mathbf{M}_\phi^{'})\Big) + o(\delta\tilde{\mathbf{\Pi}}_T),
\end{aligned}
$$

where $\mathrm{Tr}(\delta\tilde{\mathbf{\Pi}}_T\mathbf{M}_i^{'}) = (\tilde{\mathbf{a}'}_{i,k}^T \otimes \tilde{\mathbf{a}}_k^H + \tilde{\mathbf{a}}_k^T \otimes \tilde{\mathbf{a}'}_{i,k}^H)\mathrm{vec}(\delta\tilde{\mathbf{\Pi}}_T)$, $i = \theta,\phi$. Consequently the Jacobian matrix $\mathbf{D}_\Theta^{\mathrm{Alg}_1}$ of the mapping that associates Θ_T to $\tilde{\mathbf{\Pi}}_T$ is given by $\mathbf{D}_\Theta^{\mathrm{Alg}_1} = (\mathbf{d}_1, ..., \mathbf{d}_K)^T$ with

$$
\begin{aligned}
\mathbf{d}_k^T &= \frac{-1}{\alpha_{\theta,\theta}^{(k)}\alpha_{\phi,\phi}^{(k)} - (\alpha_{\theta,\phi}^{(k)})^2}\Big(\alpha_{\phi,\phi}^{(k)}(\tilde{\mathbf{a}'}_{\theta,k}^T \otimes \tilde{\mathbf{a}}_k^H + \tilde{\mathbf{a}}_k^T \otimes \tilde{\mathbf{a}'}_{\theta,k}^H) \\
&\quad -\alpha_{\theta,\phi}^{(k)}(\tilde{\mathbf{a}'}_{\phi,k}^T \otimes \tilde{\mathbf{a}}_k^H + \tilde{\mathbf{a}}_k^T \otimes \tilde{\mathbf{a}'}_{\phi,k}^H)\Big).
\end{aligned}
$$

Because $\tilde{\mathbf{\Pi}}_T$ is asymptotically Gaussian distributed with first covariance $\mathbf{C}_{\tilde{\Pi}}$, Θ_T is also asymptotically Gaussian distributed thanks to the standard theorem of continuity (see e.g., [18, p. 122]) with covariance $\mathbf{C}_\Theta = \mathbf{D}_\Theta^{\mathrm{Alg}_1}\mathbf{C}_{\tilde{\Pi}}(\mathbf{D}_\Theta^{\mathrm{Alg}_1})^H$. Using, the expression (V.1) of $\mathbf{C}_{\tilde{\Pi}}$, the expressions (V.4) and (V.5) of $\mathbf{C}_\Theta$ are straightforwardly deduced after simple but tedious algebra manipulations thanks to the identities $(\mathbf{a} \otimes \mathbf{b})\mathbf{K}_{2M} = (\mathbf{b} \otimes \mathbf{a})$ for all $2M \times 1$ vectors $\mathbf{a}$, $\mathbf{b}$, $\tilde{\mathbf{\Pi}}\tilde{\mathbf{a}} = \mathbf{0}$ and $\tilde{\mathbf{a}}^H\mathbf{J}\tilde{\mathbf{\Pi}}^* = \mathbf{0}^T$. In the case of a single source, $\tilde{\mathbf{U}} = \frac{1}{2\|\mathbf{a}\|^2}\left(\frac{\sigma_n^2}{\sigma_1^2} + \frac{1}{2\|\mathbf{a}\|^2}\frac{\sigma_n^4}{\sigma_1^4}\right)\left(\frac{\tilde{\mathbf{a}}}{\sqrt{2}\|\mathbf{a}\|}\right)\left(\frac{\tilde{\mathbf{a}}}{\sqrt{2}\|\mathbf{a}\|}\right)^H$ and (V.6) is straightforwardly deduced as well.

IV. Appendix: Proof of theorem 7

Because $\theta_{k,T}$ satisfies $\frac{\partial g_{5,T}(\theta)}{\partial \theta}_{|\theta=\theta_{k,T}=\theta_k+\delta\theta_{k,T}} = 0$, we straightforwardly obtain the following first-order perturbation expansion thanks to $\tilde{\mathbf{\Pi}}_T = \tilde{\mathbf{\Pi}} + \delta\tilde{\mathbf{\Pi}}_T$, where we have used the identity $\mathrm{Tr}(\mathbf{ABCD}) = \mathrm{vec}^T(\mathbf{A}^T)(\mathbf{D}^T \otimes \mathbf{B})\mathrm{vec}(\mathbf{C})$ [19, th. 7.17]

$$\begin{aligned}\delta\theta_{k,T} &= \frac{\mathrm{vec}^H(\mathbf{W})(\bar{\mathbf{A}}_k^T \otimes \bar{\mathbf{A}}'^H_k + \bar{\mathbf{A}}'^T_k \otimes \bar{\mathbf{A}}_k^H)\mathrm{vec}(\delta\tilde{\mathbf{\Pi}}_T)}{2\mathrm{Tr}(\mathbf{W}\bar{\mathbf{A}}'^H_k \tilde{\mathbf{\Pi}}\bar{\mathbf{A}}'_k)} \\ &+ o(\delta\tilde{\mathbf{\Pi}}_T). \qquad \text{(D.1)}\end{aligned}$$

And because $\tilde{\mathbf{\Pi}}_T$ is asymptotically Gaussian distributed, Θ_T is also asymptotically Gaussian distributed thanks to the standard theorem of continuity (see e.g., [18, p. 122]) with covariance:

$$(\mathbf{C}_\Theta)_{k,l} = \frac{1}{\beta_k\beta_l}\mathrm{vec}^H(\mathbf{W})\mathbf{M}_k\mathbf{C}_{\tilde{\Pi}}\mathbf{M}_l^H\mathrm{vec}(\mathbf{W}),$$

with $\beta_k \stackrel{\text{def}}{=} 2\mathrm{Tr}(\mathbf{W}\bar{\mathbf{A}}'^H_k\tilde{\mathbf{\Pi}}\bar{\mathbf{A}}'_k)$ and $\mathbf{M}_k \stackrel{\text{def}}{=} \bar{\mathbf{A}}_k^T \otimes \bar{\mathbf{A}}'^H_k + \bar{\mathbf{A}}'^T_k \otimes \bar{\mathbf{A}}_k^H$. Using the alternative expression $\frac{1}{2}\mathbf{L}_M(\tilde{\mathbf{\Pi}}^* \otimes \tilde{\mathbf{U}} + \tilde{\mathbf{U}}^* \otimes \tilde{\mathbf{\Pi}})\mathbf{L}_M$ of $\mathbf{C}_{\tilde{\Pi}}$ given by (V.1) where $\mathbf{L}_M \stackrel{\text{def}}{=} (\mathbf{I}_{4M^2} + \mathbf{K}_{2M}(\mathbf{J}_M \otimes \mathbf{J}_M))$, we obtain thanks to straightforward algebra manipulations

$$\begin{aligned}\mathbf{M}_k\mathbf{C}_{\tilde{\Pi}}\mathbf{M}_l^H = \frac{1}{2}\mathbf{L}_4\Big((\bar{\mathbf{A}}_k^T\tilde{\mathbf{U}}^*\bar{\mathbf{A}}_l^*) \otimes (\bar{\mathbf{A}}'^H_k\tilde{\mathbf{\Pi}}\bar{\mathbf{A}}'_l) \\ +(\bar{\mathbf{A}}'^T_k\tilde{\mathbf{\Pi}}^*\bar{\mathbf{A}}^{*\prime}_l) \otimes (\bar{\mathbf{A}}_k^H\tilde{\mathbf{U}}\bar{\mathbf{A}}_l)\Big)\mathbf{L}_4,\end{aligned}$$

and because $\mathbf{L}_4\mathrm{vec}(\mathbf{W}) = \begin{pmatrix} w_{1,1}+w_{2,2} \\ 2w_{1,2}^* \\ 2w_{1,2} \\ w_{1,1}+w_{2,2}\end{pmatrix}$ and $\beta = 2(w_{1,1}+w_{2,2})\mathbf{a}'^H_k\mathbf{\Pi}_1\mathbf{a}'_k$, expression (V.11) is proved.

Expressing the matrix $\mathbf{C} \stackrel{\text{def}}{=} (\bar{\mathbf{A}}_k^T\tilde{\mathbf{U}}^*\bar{\mathbf{A}}_l^*) \otimes (\bar{\mathbf{A}}'^H_k\tilde{\mathbf{\Pi}}\bar{\mathbf{A}}'_l) + (\bar{\mathbf{A}}'^T_k\tilde{\mathbf{\Pi}}^*\bar{\mathbf{A}}'^*_l) \otimes (\bar{\mathbf{A}}_k^H\tilde{\mathbf{U}}\bar{\mathbf{A}}_l)$ of (V.11) as a function of $\mathbf{a}_k$, $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$, we obtain after simple but tedious algebra manipulations $\mathbf{C} = \begin{pmatrix} 2\alpha & \beta & \beta^* & 0 \\ \beta^* & 2\alpha & 0 & \beta^* \\ \beta & 0 & 2\alpha & \beta \\ 0 & \beta & \beta^* & 2\alpha \end{pmatrix}$ with $\alpha \stackrel{\text{def}}{=} (\mathbf{a}_k^H\mathbf{U}_1\mathbf{a}_k)(\mathbf{a}'^H_k\mathbf{\Pi}_1\mathbf{a}'_k)$ and $\beta \stackrel{\text{def}}{=} (\mathbf{a}_k^H\mathbf{U}_2\mathbf{a}_k^*)(\mathbf{a}'^H_k\mathbf{\Pi}_1\mathbf{a}'_k)$. Consequently (V.11) becomes

$$(\mathbf{C}_\Theta)_{k,k} = \frac{1}{2\alpha_k^2}\, 4[\alpha(1+|z|^2) + 2\Re(\beta z)],$$

which is minimum for $z_{opt} = -\frac{\beta^*}{\alpha} = -\frac{\mathbf{a}_k^T\mathbf{U}_2^*\mathbf{a}_k}{\mathbf{a}_k^H\mathbf{U}_1\mathbf{a}_k}$

The associated minimum value of $(\mathbf{C}_\Theta)_{k,k}$ is

$$\min_z (\mathbf{C}_\Theta)_{k,k} = \frac{1}{2\alpha_k^2}\frac{4}{\alpha}(\alpha^2 - |\beta|^2) = \frac{(\mathbf{a}_k^H\mathbf{U}_1\mathbf{a}_k)^2 - |\mathbf{a}_k^H\mathbf{U}_2\mathbf{a}_k^*|^2}{2(\mathbf{a}'^H_k\mathbf{\Pi}_1\mathbf{a}'_k)(\mathbf{a}_k^H\mathbf{U}_1\mathbf{a}_k)} \qquad \text{(D.2)}$$

and the proof is completed.

We note that replacement of $\mathbf{W}$ by an arbitrary consistent estimate that satisfies $\mathbf{W}_T = \mathbf{W} + O(\mathbf{R}_{\tilde{y}} - \mathbf{R}_{\tilde{y},T})$ has no effect on the asymptotic variance of the weighted MUSIC estimates because the first order perturbation (D.1) is preserved.

V. Appendix: Proof of corollary 1

For the single source case, we obtain from the expressions of $\tilde{\mathbf{U}}$ given at the end of Appendix D

$$\begin{aligned}\mathbf{U}_1 &= \frac{\sigma_n^2}{\sigma_1^2\|\mathbf{a}_1\|^4(1-\rho_1^2)}\left(1 + \frac{\sigma_n^2}{\sigma_1^2\|\mathbf{a}_1\|^2}\frac{1+\rho_1^2}{1-\rho_1^2}\right)\mathbf{a}_1\mathbf{a}_1^H, \\ \mathbf{U}_2 &= -\frac{\rho_1\sigma_n^2}{\sigma_1^2\|\mathbf{a}_1\|^4(1-\rho_1^2)}\left(1 + \frac{\sigma_n^2}{\sigma_1^2\|\mathbf{a}_1\|^2}\frac{2}{1-\rho_1^2}\right)e^{i\phi_1}\mathbf{a}_1\mathbf{a}_1^T.\end{aligned}$$

Then, using these values in expression (D.2) of $\min_z(\mathbf{C}_\Theta)_{k,k}$ we obtain after tedious but simple algebra manipulations

$$\begin{aligned}\min_z C_{\theta_1} &= \frac{1}{\alpha_1}\frac{(\mathbf{a}_1^H\mathbf{U}_1\mathbf{a}_1)^2 - |\mathbf{a}_1^H\mathbf{U}_2\mathbf{a}_1^*|^2}{(\mathbf{a}_1^H\mathbf{U}_1\mathbf{a}_1)} \\ &= \frac{1}{\alpha_1}\left[\frac{2r_1^{-1} + \|\mathbf{a}_1\|^{-2}r_1^{-2} + \|\mathbf{a}_1\|^2 - \|\mathbf{a}_1\|^2\rho_1^2}{\|\mathbf{a}_1\|^2 r_1 + 1 + (1-\|\mathbf{a}_1\|^2 r_1)\rho_1^2}\right]\end{aligned}$$

where $r_1 \stackrel{\text{def}}{=} \frac{\sigma_1^2}{\sigma_n^2}$, which is the expression of the non-circular Gaussian Cramer Rao bound proved in [4].

References

[1] P. Stoica and A. Nehorai, "MUSIC, maximum likelihood, and Cramer-Rao bound," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. 37, no. 5, pp. 720-741, May 1989.

[2] P. Stoica and A. Nehorai, "MUSIC, maximum likelihood, and Cramer-Rao bound: Further results and comparisons," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. 38, no. 12, pp. 2140-2150, December 1990.

[3] J.P. Delmas, "Asymptotically minimum variance second-order estimation for non-circular signals with application to DOA estimation," *IEEE Trans. Signal Processing*, vol. 52, no. 5, pp. 1235-1241, May 2004.

[4] J.P. Delmas, H. Abeida, "Stochastic Cramer-Rao bound for non-circular signals with application to DOA estimation," *IEEE Trans. Signal Processing*, vol. 52, no. 11, pp. 3192-3199, November 2004.

[5] P. Gounon, C. Adnet and J. Galy, "Localisation angulaire de signaux non circulaires," *Traitement du Signal*, vol. 15, no. 1 pp. 17-23, 1998.

[6] P. Chargé, Y. Wang and J. Saillard, "A non-circular sources direction finding method using polynomial rooting," *Signal Processing*, vol. 81, pp. 1765-1770, 2001.

[7] M. Haardt and F. Römer, "Enhancements of unitary esprit for non-circular sources" *Proc. ICASSP, Montreal*, vol.2, p.101-104, May 2004.

[8] A. Zoubir, P. Chargé, Y. Wang, "Non circular sources localization with ESPRIT", *Proc. 6th European Conf. Wireless Technology, Munich*, October 2003.

[9] D. B Williams and D. H. Johnson, "On Resolving $2M-1$ Narrow-Band Signals with an M Sensor Uniform Linear Array", *IEEE Trans. Signal Processing*, vol.40, no.3, p.707-711, March 1992.

[10] P. Stoica and A. Eriksson, "MUSIC estimation of real-valued sine-wave frequencies," *Signal Processing*, vol. 42, pp. 139-146, 1995.

[11] P. Stoica, A. Eriksson and T. Söderström, "Optimally weighted MUSIC for frequency estimation," *SIAM J. Matrix Anal. Appl.*, vol. 16, no. 3, pp. 811-827, July 1995.

[12] J.P. Delmas, "Asymptotic performance of second-order algorithms," *IEEE Trans. Signal Processing*, vol. 50, no. 1, pp. 49-57, January 2002.

[13] H. Krim, P. Forster and G. Proakis, "Operator approach to performance analysis of root-MUSIC and root-min-norm," *IEEE Trans. Signal Processing*, vol. 40, no. 7, pp. 1687-1696, July 1992.

[14] B. Porat and B. Friedlander, "Performance analysis of parameter estimation algorithms based on high-order moments," *International Journal of Adaptive Control and Signal Processing*, vol. 3, pp. 191-229, 1989.

[15] J.F. Cardoso and E. Moulines, "Invariance of subspace based estimator," *IEEE Trans. Signal Processing*, vol. 48, no. 9, pp. 2495-2505, Sep. 2000.

[16] B. Friedlander and B. Porat, "Asymptotically optimal estimation of MA and ARMA parameters of non-Gaussian processes from high-order moments," *IEEE Trans. Automatic Control*, vol. 35, no. 1, pp. 27-35, January 1990.

[17] T. Kato, *Perturbation Theory for Linear Operators*, Springer Berlin, 1995.

[18] R.J. Serfling, *Approximation Theorems of Mathematical Statistics*, John Wiley and Sons, 1980.

[19] J.R. Schott, *Matrix Analysis for Statistics*, John Wiley and Sons, 1980.

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