Minimax Approximation of Representation Coefficients From Generalized Samples

Tsvi G. Dvorkind*, Hagai Kirshner†,
Yonina C. Eldar††, *Member, IEEE,* and Moshe Porat†††, *Senior Member, IEEE*

***Abstract*— Many sources of information are of analog or continuous-time nature. However, digital signal processing applications rely on discrete data. We consider the problem of approximating L_2 inner products, *i.e.*, representation coefficients of a continuous-time signal, from its generalized samples. Taking a robust approach, we process these generalized samples in a minimax optimal sense. Specifically, for the worst possible signal, we find the best approximation of the desired representation coefficients by proper processing of the given sample sequence. We then extend our results to criteria which incorporate smoothness constraints on the unknown function. Finally, we compare our methods with the piecewise-constant approximation technique, commonly used for this problem, and discuss the possible improvements by the suggested schemes.**

EDICS Category: DSP-SAMP, DSP-RECO sampling, extrapolation, and interpolation

I. Introduction

SIGNAL processing applications are concerned mainly with digital data, although the origin of many sources of information is analog. This is the case for speech and audio, optics, radar, sonar, biomedical signals and more. In many cases, analysis of a continuous-time signal $\mathbf{x}(t)$ is obtained by evaluating L_2 inner-products $\langle \mathbf{w}_n(t), \mathbf{x}(t)\rangle_{L_2}$ for a set of predetermined analysis functions $\{\mathbf{w}_n(t)\}$. For example, one may calculate a Gabor [1] or wavelet [2] representation of a signal.

Typically, the analysis functions $\{\mathbf{w}_n(t)\}$ are analytically known. On the other hand, in many applications of digital signal processing, there is no knowledge of the continuous-time signal $\mathbf{x}(t)$, but only of its sample sequence. Our problem is to approximate the required L_2 inner-products, by proper processing of the available samples.

In some cases the sampled version of a signal is sufficient to calculate the original function. A well known example is the classical Whittaker-Shannon sampling theorem. See also [3], [4] for additional shift invariant settings. If the analog input can be determined from the sample sequence, then the required representation coefficients can be calculated as well. Our main focus here is on situations where the knowledge of the continuous-time function is incomplete, so that only approximations of the continuous-time inner products can be obtained. A well known example is the initialization problem in wavelet analysis. To initialize the pyramid algorithm [5] we need the representation coefficients of the continuous-time function $\mathbf{x}(t)$, in the initial scale. Unfortunately, these coefficients are typically unavailable, and we only have the samples of $\mathbf{x}(t)$, obtained at the output of an anti-aliasing filter. A common practice in wavelet analysis is to assume that the available samples are the required representation coefficients. This false assumption is also known in the literature as the 'wavelet crime' [6]. In [7] the authors address this problem by suggesting a digital filter to process the available sample sequence, prior to applying the pyramid algorithm. In fact, it can be shown that their result is compatible with a special case of our derivations, presented in Section IV-B.

A common approach to cope with incomplete knowledge of the continuous-time signal $\mathbf{x}(t)$ is to first interpolate the given samples using some synthesis functions. Then, the required L_2 inner-products can be performed using the approximation (see for example [8]). Unfortunately, the best choice of the synthesis functions is not always clear. See [9] for error analysis, when approximations of a function are performed in a shift invariant setup.

Yet another approach to approximate an L_2 inner-product is to perform numerical integration by a Riemann-type sum. Assuming ideal and uniform sampling, convergence analysis of such approximations was conducted in [10]. The ideal and uniform sampling case was also considered in [11], [12]. There, in order to approximate a single representation coefficient $\langle \mathbf{w}(t), \mathbf{x}(t)\rangle_{L_2}$, it was suggested to calculate an l_2 inner product $\sum_n b[n]\mathbf{x}(nT)$ instead. An upper bound for the approximation error was derived, and the sequence b was determined by minimizing that upper bound. In practice, however, ideal sampling is impossible to implement. A more practical model considers generalized samples [4], [13]–[18], which are represented as the inner products of the signal with a set of sampling functions $\{\mathbf{s}_n(t)\}$. Thus, the nth sample can be written as $c[n] = \langle \mathbf{s}_n(t), \mathbf{x}(t)\rangle_{L_2}$. This sampling model is general enough to describe any linear and bounded acquisition device (Riesz representation theorem [19], [20]).

In this paper we take an approach that is similar in spirit to the works in [7], [16] and generalize it according to [11]. Given the generalized samples, we approximate the desired representation coefficients $\{\langle \mathbf{w}_n(t), \mathbf{x}(t)\rangle_{L_2}\}$ in a minimax

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The authors are with the Department of Electrical Engineering Technion - Israel Institute of Technology, Fax: +(972) 4-8295757. * phone: + (972) 4-8294722, email: dvorkind@tx.technion.ac.il, † phone: + (972) 4-8294802, email: kirshner@tx.technion.ac.il, †† phone: + (972) 4-8293256, email: yonina@ee.technion.ac.il, ††† phone: + (972) 4-8294684, email: mp@ee.technion.ac.il

optimal sense. It turns out that the solutions to our robust objectives can also be interpreted as an interpolation of the given samples, followed by an application of the analysis functions $\{\mathbf{w}_n(t)\}$ to the interpolation $\hat{\mathbf{x}}(t)$. The nice thing is that the interpolation stage stems naturally from the setup of the problem, rather than being pre-specified arbitrarily. Additionally, the division of the algorithm into interpolation and analysis stages is more of conceptual rather than practical nature; both stages can be performed simultaneously, by digital processing of the available samples.

Our results extend [11] in several ways. First, by considering generalized samples our derivations are applicable to practical acquisition devices. Second, if there is prior knowledge that the generalized samples have been obtained from a smooth function, then we show how to incorporate that constraint into the proposed robust solution. Third, our derivations are applicable to a series of representation coefficients. Finally, we analyze the performance of the suggested approach, giving sufficient conditions for it to outperform piecewise-constant approximations.

The outline of the paper is as follows. In Section II we describe the notations and the mathematical preliminaries. Section III discusses situations where the required L_2 inner products can be evaluated exactly, and establishes a minimax approximation criterion when this is not the case. The minimax objective is solved in Section IV. In Section V we consider the problem of incorporating smoothness constraints. Specifically, if there is prior knowledge of the input to be smooth, then we show how to alter the minimax solution by recasting the problem in a proper Sobolev space [24], presenting [11] as a special setting of our derivations. Section VI discusses the relations between the errors due to the suggested minimax approach and approximations by a Riemann-type summation. We show the possible gain in performance by the proposed method and derive sufficient conditions for it to dominate the summation approach. Finally, in Section VII, we conclude with several simulations.

II. Notations and Mathematical Preliminaries

We denote continuous-time signals by bold lowercase letters, omitting the time dependence, when possible. The elements of a sequence $c \in l_2$ will be written with square brackets, *e.g.* $c[n]$. $\mathbf{X}^F(\omega) = \int \mathbf{x}(t)e^{-j\omega t}dt$ is the continuous-time Fourier transform of $\mathbf{x}$ and $C^f(\omega) = \sum_n c[n]e^{-j\omega n}$ is the (2π periodic) discrete-time Fourier transform (DTFT) of the sequence c. The operator $P_{\mathcal{A}}$ represents the orthogonal projection onto a closed subspace $\mathcal{A}$, and $\mathcal{A}^{\perp}$ is the orthogonal complement of $\mathcal{A}$. The *Moore-Penrose pseudo inverse* [21] and the adjoint of a bounded transformation T are written as $T^{\dagger}$ and T^*, respectively. $\Re$ stands for the real part.

Inner products and norms are denoted by $\langle a, b\rangle_{\mathcal{H}}$ and $\|a\|_{\mathcal{H}}$, respectively. Here, $\mathcal{H}$ stands for the Hilbert space involved. Usually, we will consider $\mathcal{H}$ to be L_2, l_2 or the order-one Sobolev space W_2^1, which will be discussed in detail in Section V. When the derivations are general enough to describe inner products and norms within any Hilbert space, we will omit the space subscript from the notations, *i.e.*, $\langle f, g\rangle$ or $\|f\|$. All inner products are linear with respect to the second argument. For example, $\langle \mathbf{x}, \mathbf{y}\rangle_{L_2} = \int_{-\infty}^{\infty} \overline{\mathbf{x}(t)}\mathbf{y}(t)dt$.

An easy way to describe linear combinations and inner products is by utilizing set transformations. A set transformation $V : l_2 \to \mathcal{H}$ corresponding to frame [22] vectors $\{\mathbf{v}_n(t)\}$ is defined by $Va = \sum_n a[n]\mathbf{v}_n(t)$ for all $a \in l_2$. From the definition of the adjoint, if $a = V^*\mathbf{y}$, then $a[n] = \langle \mathbf{v}_n, \mathbf{y}\rangle$.

We define by S (W) the set transformation corresponding to the vectors $\{\mathbf{s}_n\}$ ($\{\mathbf{w}_n\}$). Accordingly, the generalized samples $c[n] = \langle \mathbf{s}_n, \mathbf{x}\rangle_{L_2}$ can be written as $c = S^*\mathbf{x}$, and the desired representation coefficients $q[n] = \langle \mathbf{w}_n, \mathbf{x}\rangle_{L_2}$ by $q = W^*\mathbf{x}$. We define $\mathcal{S}$ to be the sampling space, which is the closure of $\mathrm{span}\{\mathbf{s}_n\}$. Similarly, $\mathcal{W}$ is the analysis space, obtained by the closure of $\mathrm{span}\{\mathbf{w}_n\}$.

To handle well posed problems, we assume that the sample sequence c and the desired representation coefficients q have finite energy, *i.e.*, $c, q \in l_2$. This will assure that for any bounded transformation $G : l_2 \to l_2$ applied to the generalized samples c, the error sequence $q - G(c)$ is in l_2 as well. Accordingly, criteria that consider the l_2 norm of the error sequence are well defined. One way to enforce $c, q \in l_2$ is to require that $\{\mathbf{s}_n\}$ and $\{\mathbf{w}_n\}$ form frames [22] for $\mathcal{S}$ and $\mathcal{W}$, respectively, which is an assumption made throughout this paper.

III. Problem Formulation

We are given the generalized samples of a continuous-time function $\mathbf{x}(t)$, modeled by

$$c[n] = \langle \mathbf{s}_n(t), \mathbf{x}(t)\rangle_{L_2}. \tag{1}$$

An example is an analog to digital converter which performs pre-filtering prior to sampling, as shown in Fig. 1. In such a setting, the sampling vectors $\{\mathbf{s}_n(t) = \mathbf{s}(t - nT)\}$ are shifted and mirrored versions of the impulse response of the pre-filter [13].

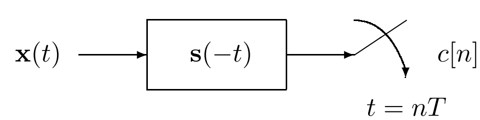


Fig. 1. Filtering with impulse response $\mathbf{s}(-t)$ followed by ideal sampling. The sampling vectors are $\{\mathbf{s}(t - nT)\}$.

We wish to evaluate a set of continuous-time inner products q defined by

$$q[n] = \langle \mathbf{w}_n, \mathbf{x}\rangle_{L_2}, \tag{2}$$

where the analysis functions $\{\mathbf{w}_n\}$ are analytically known. The input $\mathbf{x}$ is known only through its generalized samples c of (1). Our goal is to approximate the required representation coefficients q by proper processing of the sample sequence c.

A natural question to be first considered is whether there is an unavoidable error due to our partial knowledge of $\mathbf{x}(t)$, or can we evaluate **exactly** the required L_2 inner products, based on the generalized samples. The following theorem addresses this preliminary question.

Theorem 1: Let $\mathbf{x}$ be an arbitrary function, satisfying $c = S^*\mathbf{x}$. It is possible to obtain the coefficients $q = W^*\mathbf{x}$ by proper processing of the sample sequence c if and only if $\mathcal{W} \subseteq \mathcal{S}$. In that case, $q = W^*S(S^*S)^\dagger c$.

Proof: See Appendix I. □

In some cases, we may have additional prior knowledge on $\mathbf{x}$, such that not all signals in L_2 should be considered. By restricting our attention to a proper subgroup, it is possible to obtain a zero error, even if $\mathcal{W} \nsubseteq \mathcal{S}$. This is true whenever the knowledge of $\mathbf{x}$ allows us to determine a bijection (injective and surjective transformation) between $\mathbf{x}(t)$ and its samples. To illustrate this point, suppose that $\mathbf{x} \in \mathcal{A}$, where $\mathcal{A}$ is a closed subspace of L_2 satisfying the direct sum condition $L_2 = \mathcal{A} \oplus \mathcal{S}^\perp$ (*i.e.,* L_2 can be described by the sum set $\left\{\mathbf{a}+\mathbf{v};\ \mathbf{a} \in \mathcal{A}, \mathbf{v} \in \mathcal{S}^\perp\right\}$ with the property $\mathcal{A} \cap \mathcal{S}^\perp = \{0\}$). Then, we can perfectly reconstruct $\mathbf{x}$ from its generalized samples by

$$\mathbf{x} = A(S^*A)^\dagger c, \tag{3}$$

where A is any bounded set transformation with range $\mathcal{A}$ [16]. As a result, we can also perfectly evaluate the coefficients $q = W^*\mathbf{x}$ by

$$q = W^*A(S^*A)^\dagger c. \tag{4}$$

Another example in which a bijection between the signal and its generalized samples exists is the finite innovation case considered in [23].

Nevertheless, in the general case, the condition $\mathcal{W} \subseteq \mathcal{S}$ may not be satisfied, or there may be no prior knowledge of $\mathbf{x}(t)$. Thus, the coefficients $W^*\mathbf{x}$ cannot be computed exactly and instead must be approximated from the given samples c. A common practice is to perform Riemann-type sum approximations [10]:

$$\langle \mathbf{w}(t), \mathbf{x}(t)\rangle_{L_2} \approx T\sum_n c[n]\overline{\mathbf{w}(nT)}, \tag{5}$$

if one implicitly assumes that the generalized samples of $\mathbf{x}$ are close to the mean value of the input signal, within an interval of length T. Alternatively, we may approximate the continuous-time inner products by choosing a sequence d which minimizes the squared norm of the error vector $e = W^*\mathbf{x} - d$. Since $\mathbf{x}$ satisfies $c = S^*\mathbf{x}$, by decomposing $\mathbf{x}$ along $\mathcal{S}$ and $\mathcal{S}^\perp$ the error can be written as

$$e = W^*S(S^*S)^\dagger c + W^*P_{\mathcal{S}^\perp}\mathbf{x} - d, \tag{6}$$

where we used $P_{\mathcal{S}}\mathbf{x} = S(S^*S)^\dagger c$. This leads to the following objective

$$\min_d \left\|W^*S(S^*S)^\dagger c + W^*P_{\mathcal{S}^\perp}\mathbf{x} - d\right\|_{l_2}^2. \tag{7}$$

Unfortunately, the solution of (7) depends on $P_{\mathcal{S}^\perp}\mathbf{x}$, which is unknown. To eliminate the dependency on $\mathbf{x}$, we may instead consider a robust approach, where the sequence d is optimized for the worst possible input $\mathbf{x}$. Valid inputs must be consistent with the known samples, *i.e.,* must satisfy $c = S^*\mathbf{x}$. Additionally, if the norm of the input is unbounded, then so is the error. Therefore, to define a well posed problem, we assume that $\mathbf{x}$ is norm bounded by a positive constant L, and possible inputs are

$$\mathcal{D} = \left\{\mathbf{x};\ \|x\| \le L, c = S^*\mathbf{x}\right\}. \tag{8}$$

In order to assure a certain level of performance for each function from $\mathcal{D}$, we take a robust approach by considering the minimax objective

$$\min_d \max_{\mathbf{x}\in\mathcal{D}} \|W^*\mathbf{x} - d\|_{l_2}^2. \tag{9}$$

In the next sections, we derive a solution for d, and compare its performance with the piecewise-constant approximation approach given in (5).

IV. Minimax Approximation

The minimax problem of (9) is closely related to the generalized sampling problem considered in [16, Theorem 3]. Relying on results obtained in that context leads to the following theorem.

Theorem 2: Consider the problem

$$\min_d \max_{c=S^*\mathbf{x}, \|\mathbf{x}\|\le L} \|W^*\mathbf{x} - d\|_{l_2}^2,$$

where W and S are bounded set transformations with range $\mathcal{W}$ and $\mathcal{S}$, respectively. The (unique) solution is

$$d = W^*S(S^*S)^\dagger c. \tag{10}$$

Before going into the details of the proof, note that we have not specified the exact Hilbert space in which the bound $\|\mathbf{x}\| \le L$ and the inner products $S^*\mathbf{x}, W^*\mathbf{x}$ are calculated, since the derivations are general enough to be applicable to any Hilbert space. In Section V we will show how smoothness constraints can be incorporated by applying Theorem 2 to different Hilbert spaces. Additionally, the upper norm bound L is not expressed in the solution (10). Thus, one only has to be sure that the signal has a finite norm, while its exact value is irrelevant to the computation of d. The value of L will be used, however, in Section VI for analyzing the performance of the proposed algorithm.

Proof: First we note that any $\mathbf{x}$ in $\mathcal{D}$ of (8) is of the form $\mathbf{x} = S\left(S^*S\right)^\dagger c + \mathbf{v}$ for some $\mathbf{v} \in \mathcal{G}$ where

$$\mathcal{G} = \left\{\mathbf{v} \mid \mathbf{v} \in \mathcal{S}^\perp, \|\mathbf{v}\| \le L'\right\}, \tag{11}$$

and

$$L' = \sqrt{L^2 - \|S(S^*S)^\dagger c\|^2}. \tag{12}$$

Thus,

$$\begin{aligned}
&\max_{\mathbf{x}\in\mathcal{D}} \|W^*\mathbf{x} - d\|_{l_2}^2 = \qquad (13)\\
&= \max_{\mathbf{v}\in\mathcal{G}} \left\|W^*S(S^*S)^\dagger c - d + W^*\mathbf{v}\right\|_{l_2}^2\\
&= \max_{\mathbf{v}\in\mathcal{G}} \|a_d + W^*\mathbf{v}\|_{l_2}^2\\
&= \max_{\mathbf{v}\in\mathcal{G}} \left(\|a_d\|_{l_2}^2 + 2\Re\left\{\langle a_d, W^*\mathbf{v}\rangle_{l_2}\right\} + \|W^*\mathbf{v}\|_{l_2}^2\right),
\end{aligned}$$

where we defined $a_d = W^*S(S^*S)^\dagger c - d$. As a result, the maximum in (13) is achieved when

$$\Re\left\{\langle a_d, W^*\mathbf{v}\rangle_{l_2}\right\} = |\langle a_d, W^*\mathbf{v}\rangle_{l_2}|. \tag{14}$$

Indeed, let $\mathbf{v} \in \mathcal{G}$ be the vector for which the maximum is achieved. If $\langle a_d, W^*\mathbf{v}\rangle_{l_2} = 0$ then (14) is trivially true. Otherwise, we can define

$$\mathbf{v}_2 = \frac{\langle W^*\mathbf{v}, a_d\rangle_{l_2}}{|\langle W^*\mathbf{v}, a_d\rangle_{l_2}|}\mathbf{v}. \tag{15}$$

Clearly, $\|\mathbf{v}\| = \|\mathbf{v}_2\|$ and $\mathbf{v}_2 \in \mathcal{G}$. In addition, $\|W^*\mathbf{v}\|_{l_2} = \|W^*\mathbf{v}_2\|_{l_2}$ and $\langle a_d, W^*\mathbf{v}_2\rangle_{l_2} = |\langle a_d, W^*\mathbf{v}\rangle_{l_2}|$ so that the objective in (13) at $\mathbf{v}_2$ is larger than the objective at $\mathbf{v}$ unless (14) is satisfied.

Combining (14) and (13), our problem becomes

$$\min_d \max_{\mathbf{v}\in\mathcal{G}} \left(\|a_d\|_{l_2}^2 + 2\,|\langle a_d, W^*\mathbf{v}\rangle_{l_2}| + \|W^*\mathbf{v}\|_{l_2}^2 \right). \quad (16)$$

Denoting the optimal objective value by A, and replacing the order of minimization and maximization, we get a lower bound

$$\begin{aligned} A &\geq \max_{\mathbf{v}\in\mathcal{G}} \min_d \left(\|a_d\|_{l_2}^2 + 2\,|\langle a_d, W^*\mathbf{v}\rangle_{l_2}| + \|W^*\mathbf{v}\|_{l_2}^2 \right) \\ &= \max_{\mathbf{v}\in\mathcal{G}} \|W^*\mathbf{v}\|_{l_2}^2, \end{aligned} \quad (17)$$

where we used the fact that $\|a_d\|_{l_2}^2 + 2\,|\langle a_d, W^*\mathbf{v}\rangle_{l_2}| \geq 0$, with equality for d of (10). Thus, for any choice of d,

$$\min_d \max_{\mathbf{v}\in\mathcal{G}} \|a_d - W^*\mathbf{v}\|_{l_2}^2 \geq \max_{\mathbf{v}\in\mathcal{G}} \|W^*\mathbf{v}\|_{l_2}^2. \quad (18)$$

The proof then follows from the fact that d given by (10) achieves the lower bound (18). Uniqueness of d follows from (16), as the optimal solution must satisfy $a_d = 0$. □

Note that (10) resembles the solution of the Wiener-Hopf equations, where the Gramian matrix of the autocorrelations is first inverted (pseudo-inverted), and the cross-correlation Gramian matrix is then applied. Another interesting interpretation of (10) is obtained by noticing that $P_{\mathcal{S}}\mathbf{x} = S(S^*S)^\dagger c$. This leads to the following corollary.

Corollary 1: The solution (10) can be written as

$$d = W^* P_{\mathcal{S}}\mathbf{x}. \quad (19)$$

This means that our robust approach first approximates the signal by its orthogonal projection onto the sampling space, and then applies the analysis functions $\{\mathbf{w}_n\}$. Thus, we can also conclude that the suggested approximation method results in zero error if $\mathcal{W} \subseteq \mathcal{S}$ or if the prior knowledge $\mathbf{x} \in \mathcal{S}$ exists. In fact, by identifying A of (4) with S, the solutions indeed coincide. Interestingly, $P_{\mathcal{S}}\mathbf{x}$ is the minimax approximation of $\mathbf{x}$ over the set $\mathcal{D}$ of (8), as incorporated in the following proposition.

Proposition 1: The unique solution of

$$\min_{\hat{\mathbf{x}}} \max_{\mathbf{x}\in\mathcal{D}} \|\mathbf{x} - \hat{\mathbf{x}}\|^2,$$

with $\mathcal{D}$ of (8) is $\hat{\mathbf{x}} = P_{\mathcal{S}}\mathbf{x}$.

Proof: Projecting $\mathbf{x} - \hat{\mathbf{x}}$ onto $\mathcal{S}$ and $\mathcal{S}^\perp$ we have

$$\|\mathbf{x} - \hat{\mathbf{x}}\|^2 = \left\|S(S^*S)^\dagger c - P_{\mathcal{S}}\hat{\mathbf{x}}\right\|^2 + \|P_{\mathcal{S}^\perp}\mathbf{x} - P_{\mathcal{S}^\perp}\hat{\mathbf{x}}\|^2. \quad (20)$$

The maximization is then

$$\begin{aligned} &\max_{\mathbf{x}\in\mathcal{D}} \|\mathbf{x} - \hat{\mathbf{x}}\|^2 = \|e_{\mathcal{S}}(\hat{\mathbf{x}})\|^2 + \max_{\mathbf{x}\in\mathcal{D}} \|P_{\mathcal{S}^\perp}\mathbf{x} - P_{\mathcal{S}^\perp}\hat{\mathbf{x}}\|^2 \\ &= \|e_{\mathcal{S}}(\hat{\mathbf{x}})\|^2 + \\ &+ \max_{\mathbf{x}\in\mathcal{D}} \left(\|P_{\mathcal{S}^\perp}\mathbf{x}\|^2 - 2\Re\langle P_{\mathcal{S}^\perp}\mathbf{x}, P_{\mathcal{S}^\perp}\hat{\mathbf{x}}\rangle + \|P_{\mathcal{S}^\perp}\hat{\mathbf{x}}\|^2 \right), \end{aligned} \quad (21)$$

where $e_{\mathcal{S}}(\hat{\mathbf{x}}) = S(S^*S)^\dagger c - P_{\mathcal{S}}\hat{\mathbf{x}}$. Similarly to the proof of Theorem 2, we can replace $-2\Re\langle P_{\mathcal{S}^\perp}\mathbf{x}, P_{\mathcal{S}^\perp}\hat{\mathbf{x}}\rangle$ by its absolute value. The minimax objective is then lower bounded by

$$\begin{aligned} &\min_{\hat{\mathbf{x}}} \Bigg(\|e_{\mathcal{S}}(\hat{\mathbf{x}})\|^2 + \\ &\max_{\mathbf{x}\in\mathcal{D}} \left(\|P_{\mathcal{S}^\perp}\mathbf{x}\|^2 + 2\,|\langle P_{\mathcal{S}^\perp}\mathbf{x}, P_{\mathcal{S}^\perp}\hat{\mathbf{x}}\rangle| + \|P_{\mathcal{S}^\perp}\hat{\mathbf{x}}\|^2 \right) \Bigg) \\ &\geq \max_{\mathbf{x}\in\mathcal{D}} \|P_{\mathcal{S}^\perp}\mathbf{x}\|^2, \end{aligned} \quad (22)$$

where we used the fact that for all $\hat{\mathbf{x}}$ we must have $\|e_{\mathcal{S}}(\hat{\mathbf{x}})\|^2 \geq 0$ and $2\,|\langle P_{\mathcal{S}^\perp}\mathbf{x}, P_{\mathcal{S}^\perp}\hat{\mathbf{x}}\rangle| + \|P_{\mathcal{S}^\perp}\hat{\mathbf{x}}\|^2 \geq 0$. The proof then follows by noticing that $\hat{\mathbf{x}} = P_{\mathcal{S}}\mathbf{x}$ is the minimizer which achieves this lower bound. Furthermore, it is unique, since from (22) the optimal solution must satisfy $P_{\mathcal{S}^\perp}\hat{\mathbf{x}} = 0$ and $e_{\mathcal{S}}(\hat{\mathbf{x}}) = 0$. □

We conclude that the problem of approximating the representation coefficients in a minimax sense could be split into two parts; first obtaining the minimax approximation of $\mathbf{x}$ itself, then applying the analysis operator W^* to that approximation.

A. Element-Wise Optimality

In Theorem 2, we approximate a set of representation coefficients $W^*\mathbf{x}$, by optimizing (in a minimax sense) the squared norm of the error sequence $W^*\mathbf{x} - d$. Instead, one may consider alternative objectives which combine the entries of the error sequence in an l_1 norm *i.e.*, $\|W^*\mathbf{x} - d\|_{l_1}$ or the l_∞ norm $\|W^*\mathbf{x} - d\|_{l_\infty}$. We now show that the suggested solution (10) is optimal according to all the above criteria, as it is minimax optimal element-wise.

Theorem 3: The sequence $d = W^*S(S^*S)^\dagger c$ is a solution of $\min_d \max_{c=S^*\mathbf{x}, \|\mathbf{x}\|\leq L} \|W^*\mathbf{x} - d\|_{l_p}$, for any natural p.

Proof: First assume that a single representation coefficient is to be approximated in a minimax sense

$$\min_{\tilde{d}} \max_{c=S^*\mathbf{x}, \|\mathbf{x}\|\leq L} \left|\langle \mathbf{w}, \mathbf{x}\rangle - \tilde{d}\right|^2, \quad (23)$$

where $\tilde{d}$ is a scalar. Degenerating the result (10) of Theorem 2 by letting W be the set transformation of the single function $\mathbf{w}(t)$, the solution of (23) is

$$\tilde{d} = \langle \mathbf{w}, S(S^*S)^\dagger c\rangle.$$

Since (10) satisfies $d[n] = \langle \mathbf{w}_n, S(S^*S)^\dagger c\rangle$ for each entry of the vector d, we conclude that it is element-wise optimal, implying that the solution will not change if we combine the individual errors in l_1, l_∞ or any l_p norm. □

B. The Shift Invariant Case

The approximation (10) was derived for general sampling and analysis subspaces. An interesting special case of this setup is when in addition, $\mathcal{S}$ and $\mathcal{W}$ are real shift invariant (SI) subspaces, each spanned by shifts of length T of some fixed generating function [13], [16]. In this case, as we will show, the approximation sequence d can be obtained by discrete-time filtering of the sample sequence c.

Let $\mathbf{s}(t)$ and $\mathbf{w}(t)$ be the real generators of $\mathcal{S}$ and $\mathcal{W}$, respectively. Then, the SI subspaces are

$$\mathcal{S} = \left\{ \mathbf{f}(t) \mid \mathbf{f}(t) = \sum_n a[n]\mathbf{s}(t - nT), a \in \ell_2 \right\};$$
$$\mathcal{W} = \left\{ \mathbf{f}(t) \mid \mathbf{f}(t) = \sum_n a[n]\mathbf{w}(t - nT), a \in \ell_2 \right\}. \quad (24)$$

In this SI case, the samples $c[n]$, which are given by

$$c[n] = \int \mathbf{s}(t - nT)\mathbf{x}(t)dt = \mathbf{x}(t) * \mathbf{s}(-t)|_{t=nT}, \quad (25)$$

correspond to ideal sampling at times $t = nT$ of the output of a filter with an impulse response $\mathbf{s}(-t)$, with $\mathbf{x}(t)$ as its input (see Fig. 1). Here $\mathbf{g}(t) * \mathbf{z}(t)$ denotes continuous-time convolution between the signals $\mathbf{g}(t)$ and $\mathbf{z}(t)$, and $\mathbf{y}(t)|_{t=nT} = \mathbf{y}(nT)$.

To ensure that the functions $\{\mathbf{s}(t - nT)\}$ and $\{\mathbf{w}(t - nT)\}$ form frames for $\mathcal{S}$ and $\mathcal{W}$ respectively, a simple condition can be verified in the frequency domain [22]:

$$\begin{aligned} \alpha \le R^f_{\mathbf{W},\mathbf{W}}(\omega) \le \beta, \quad \omega \in \mathcal{I}_{\mathcal{W}}; \\ \gamma \le R^f_{\mathbf{S},\mathbf{S}}(\omega) \le \eta, \quad \omega \in \mathcal{I}_{\mathcal{S}}, \end{aligned} \quad (26)$$

for some $0 < \alpha \le \beta < \infty$ and $0 < \gamma \le \eta < \infty$. Here we denote,

$$R^f_{\mathbf{A},\mathbf{B}}(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \overline{\mathbf{A}^F} \left(\frac{\omega + 2\pi k}{T} \right) \mathbf{B}^F \left(\frac{\omega + 2\pi k}{T} \right), \quad (27)$$

where $\mathbf{W}^F(\omega), \mathbf{S}^F(\omega)$ are the continuous-time Fourier transforms of the generators $\mathbf{s}(t), \mathbf{w}(t)$, and $\mathcal{I}_{\mathcal{W}}, \mathcal{I}_{\mathcal{S}}$ are the set of frequencies ω for which $R^f_{\mathbf{W},\mathbf{W}}(\omega) \neq 0$ and $R^f_{\mathbf{S},\mathbf{S}}(\omega) \neq 0$, respectively.

Letting S and W be the set transformations of $\{\mathbf{s}(t - nT)\}$ and $\{\mathbf{w}(t - nT)\}$ respectively, it is easy to see that W^*Sa is equivalent to filtering the sequence $\{a[n]\}$ by a discrete-time LTI filter having the frequency response $R^f_{\mathbf{W},\mathbf{S}}(\omega)$. Similarly, the pseudo-inverse operator $(S^*S)^\dagger$ takes the form of applying a filter with DTFT $1/R^f_{\mathbf{S},\mathbf{S}}(\omega)$ for $\omega \in \mathcal{I}_{\mathcal{S}}$ and zero otherwise. Combining the above, we have that the sequence $d = W^*S(S^*S)^\dagger c$ can be obtained by filtering the sample sequence c with a digital filter

$$G^f(\omega) = \begin{cases} \frac{R^f_{\mathbf{W},\mathbf{S}}(\omega)}{R^f_{\mathbf{S},\mathbf{S}}(\omega)}, & \omega \in \mathcal{I}_{\mathcal{S}}; \\ 0, & \omega \notin \mathcal{I}_{\mathcal{S}}. \end{cases} \quad (28)$$

We point out that by a proper choice of the sampling and analysis functions, the filter (28) is compatible with the solution for the 'wavelet crime' problem obtained in [7].

V. Imposing Smoothness by Sobolev Spaces

The objective in Theorem 2 considers functions within the set $\mathcal{D} = \{\mathbf{x} \mid S^*\mathbf{x} = c, \ \|\mathbf{x}\| \le L\}$. However, sometimes we have prior knowledge that the input signal is 'smooth'. If we could restrict the set of possible inputs to include only smooth functions, then the performance of the robust objective may be improved.

Sobolev spaces are natural candidates to describe smoothness. For simplicity, our main discussion will concern the Sobolev space of order one [24].

Definition 1: The Sobolev space of order one W^1_2 is the Hilbert space of functions which have a finite L_2 norm, and so is their first derivative. A possible choice of inner product in this space is

$$\langle \mathbf{a}, \mathbf{b} \rangle_{W^1_2} = \langle \mathbf{a}, \mathbf{b} \rangle_{L_2} + \langle \mathbf{a}^{(1)}, \mathbf{b}^{(1)} \rangle_{L_2}, \quad (29)$$

where $\mathbf{a}^{(1)}$ and $\mathbf{b}^{(1)}$ stand for the first derivative of $\mathbf{a}$ and $\mathbf{b}$, respectively.

If we have prior knowledge that the input $\mathbf{x}$ and its first derivative are of finite energy (which in particular implies that $\mathbf{x}$ is continuous), we may consider the set of possible inputs to be

$$\tilde{\mathcal{D}} = \left\{ \mathbf{x} \mid S^*\mathbf{x} = c, \ \|\mathbf{x}\|_{W^1_2} \le \tilde{L} \right\}, \quad (30)$$

where $\tilde{L}$ is an upper bound on the W^1_2 norm of $\mathbf{x}$. This leads to the following minimax objective:

$$\min_d \max_{\mathbf{x} \in \tilde{\mathcal{D}}} \|W^*\mathbf{x} - d\|^2_{l_2}. \quad (31)$$

To solve (31) we may use Theorem 2 as its derivations are general enough to be applicable to any Hilbert space. Note however that objective (31) contains mixed inner products and norms; $S^*\mathbf{x}$ and $W^*\mathbf{x}$ describe L_2 inner products, while $\|\mathbf{x}\|_{W^1_2} \le \tilde{L}$ is a Sobolev norm constraint. Hence, we will first recast the whole problem into the order one Sobolev space, and then apply the results of Theorem 2.

To this end, note that W^1_2 inner products (29) can be compactly written in the Fourier domain by

$$\langle \mathbf{s}, \mathbf{x} \rangle_{W^1_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\mathbf{S}^F}(\omega)\mathbf{X}^F(\omega) \left(1 + \omega^2\right) d\omega, \quad (32)$$

where $\mathbf{S}^F(\omega)$ and $\mathbf{X}^F(\omega)$ are the Fourier transforms of $\mathbf{s}(t)$ and $\mathbf{x}(t)$, respectively. As introduced in [11], we can use (32) to rewrite L_2 inner products as W^1_2 inner products. Specifically, for any $\mathbf{a} \in L_2, \mathbf{b} \in W^1_2$

$$\langle \mathbf{a}(t), \mathbf{b}(t) \rangle_{L_2} = \langle \mathbf{a}(t) * \mathbf{u}(t), \mathbf{b}(t) \rangle_{W^1_2}, \quad (33)$$

where $*$ stands for the convolution operation and

$$\mathbf{u}(t) = \frac{1}{2} e^{-|t|} \quad (34)$$

is the inverse Fourier transform of $1/(1 + \omega^2)$.

Using (33), we can replace the L_2 inner products $W^*\mathbf{x}$ and $S^*\mathbf{x}$ using their Sobolev counterparts, which leads to the following theorem.

Theorem 4: Consider the problem

$$\min_d \max_{x \in \tilde{\mathcal{D}}} \|W^*\mathbf{x} - d\|^2_{l_2}, \quad (35)$$

where W and S are bounded set transformations with range $\mathcal{W}$ and $\mathcal{S}$, respectively and $\tilde{\mathcal{D}}$ is given by (30). The (unique) solution is

$$d = \tilde{W}^*\tilde{S}(\tilde{S}^*\tilde{S})^\dagger c, \quad (36)$$

where the inner products described by (36) are computed in the order one Sobolev space, $\tilde{S}, \tilde{W}$ are the set transformations of $\{\tilde{\mathbf{s}}_n = \mathbf{s}_n * \mathbf{u}\}$ and $\{\tilde{\mathbf{w}}_n = \mathbf{w}_n * \mathbf{u}\}$ respectively, and the function $\mathbf{u}(t)$ is given by (34).

Before stating the proof, note that (36) describes W_2^1 inner products. In practice, this means that the i,jth element of the matrix $\tilde{W}^*\tilde{S}$ is

$$\langle \tilde{\mathbf{w}}_i, \tilde{\mathbf{s}}_j\rangle_{W_2^1} = \langle \mathbf{w}_i, \tilde{\mathbf{s}}_j\rangle_{L_2} = \langle \mathbf{w}_i, \mathbf{s}_j * \mathbf{u}\rangle_{L_2}, \tag{37}$$

with $\mathbf{u}$ as in (34). Similarly, the i,jth element of $\tilde{S}^*\tilde{S}$ is computed by

$$\langle \tilde{\mathbf{s}}_i, \tilde{\mathbf{s}}_j\rangle_{W_2^1} = \langle \mathbf{s}_i, \mathbf{s}_j * \mathbf{u}\rangle_{L_2}. \tag{38}$$

Proof: Using (33), we can rewrite (35) as

$$\min_{d} \max_{c=\tilde{S}^*\mathbf{x}, \|\mathbf{x}\|_{W_2^1} \le \tilde{L}} \left\| \tilde{W}^*\mathbf{x} - d \right\|_{l_2}^2,$$

where $\tilde{S}$ and $\tilde{W}$ are the set transformation of $\{\tilde{\mathbf{s}}_n = \mathbf{s}_n * \mathbf{u}\}$ and $\{\tilde{\mathbf{w}}_n = \mathbf{w}_n * \mathbf{u}\}$, respectively, and $\mathbf{u}(t)$ is given by (34). Since the derivations of Theorem 2 apply to any Hilbert space, the solution has the same form as in (10), resulting in (36). $\square$

The result of Theorem (4) can be interpreted in several ways. Rewriting $P_{\widetilde{\mathcal{S}}}\mathbf{x} = \tilde{S}(\tilde{S}^*\tilde{S})^\dagger \tilde{S}^*\mathbf{x} = \tilde{S}(\tilde{S}^*\tilde{S})^\dagger c$ and using (37), we obtain the following corollary.

Corollary 2: The solution (36) can be written as

$$d = W^* P_{\widetilde{\mathcal{S}}}\mathbf{x}, \tag{39}$$

where $\widetilde{\mathcal{S}}$ is the closure of $\operatorname{span}\{\tilde{\mathbf{s}}_n\}$ and $P_{\widetilde{\mathcal{S}}}\mathbf{x}$ stands for the orthogonal projection of $\mathbf{x}$, in the W_2^1 sense, onto $\widetilde{\mathcal{S}}$. The operator W^* describes the usual L_2 inner products with the analysis functions.

Note that we implicitly assume that the possibly infinite sum involved in the computation of $P_{\widetilde{\mathcal{S}}}\mathbf{x}$ is well defined. This can be assured if the functions $\{\tilde{\mathbf{s}}_n\}$ form a frame for the closure of their span. We address this question in Appendix II.

Another interesting interpretation of Theorem 4 is evident by rewriting all the inner products in the L_2 space. Combining (38) with (39) we obtain the following corollary.

Corollary 3: The solution (36) can be written as

$$d = W^*\tilde{S}(S^*\tilde{S})^\dagger S^*\mathbf{x} = W^* E_{\widetilde{\mathcal{S}},\mathcal{S}^\perp}\mathbf{x}, \tag{40}$$

where $E_{\widetilde{\mathcal{S}},\mathcal{S}^\perp}$ stands for the oblique projection operator [16], [25], in the L_2 sense, with a range space $\widetilde{\mathcal{S}}$ and a null space $\mathcal{S}^\perp$.

In analogy to Proposition IV it can be shown that $P_{\widetilde{\mathcal{S}}}\mathbf{x} = E_{\widetilde{\mathcal{S}},\mathcal{S}^\perp}\mathbf{x}$ is the unique solution of

$$\arg\min_{\hat{\mathbf{x}}} \max_{\mathbf{x}\in\widetilde{\mathcal{D}}} \|\mathbf{x} - \hat{\mathbf{x}}\|_{W_2^1}^2.$$

Corollary 3 implies that the problem of Theorem 4 could be split into two parts; first obtaining the Sobolev minimax approximation of $\mathbf{x}$ itself (which is an oblique projection in the L_2 space), and only then applying the analysis operator W^* to that approximation.

In this section we have considered the Sobolev space of order one. It is possible to extend the derivations to higher order Sobolev spaces, if a sufficient degree of smoothness is known to be present. The order r Sobolev space W_2^r is composed of finite energy functions with r finite energy derivatives; W_2^r inner products can be written as $\langle \mathbf{a}, \mathbf{b}\rangle_{W_2^r} = \sum_{k=0}^r \langle \mathbf{a}^{(\mathbf{k})}, \mathbf{b}^{(\mathbf{k})}\rangle_{L_2}$, where the (k) superscript stands for the kth derivative. Thus, we can obtain similar results, which only require the replacement of the function $\mathbf{u}(t)$ of (34) with the inverse Fourier transform of $1/(1+\omega^2+\omega^4\ldots+\omega^{2r})$.

As a concluding remark, we note that our solution takes the form of applying the analysis functions $\{\mathbf{w}_n\}$ to $\hat{\mathbf{x}} = P_{\widetilde{\mathcal{S}}}\mathbf{x} = E_{\widetilde{\mathcal{S}},\mathcal{S}^\perp}\mathbf{x}$, which is the minimax approximation of $\mathbf{x}$ within the space $\widetilde{\mathcal{S}}$. This space is determined by the sampling functions $\{\mathbf{s}_n\}$ and the smoothness constraint (manifested by $\mathbf{u}$). Thus, we have obtained a nice counterpart to methods that **arbitrarily** choose the interpolation space.

A. Smoothness and the Shift Invariant Case

Consider the result of Theorem 4, while assuming the special case where $\mathcal{S}$ and $\mathcal{W}$ are real SI subspaces, as stated in (24). Then, the sample sequence c may be processed by a digital filter in order to obtain the minimax approximation (36). Let $\mathbf{u}(t)$, as given by (34), be the inverse Fourier transform of $1/(1+\omega^2)$ and define $\tilde{\mathbf{s}} = \mathbf{s} * \mathbf{u}$. In this SI setup with smoothness constraint, the frequency response of the optimal filter then takes the form

$$G^f(\omega) = \begin{cases} \frac{R^f_{\mathbf{W},\tilde{\mathbf{S}}}(\omega)}{R^f_{\mathbf{S},\tilde{\mathbf{S}}}(\omega)}, & \omega \in \mathcal{I}_{\mathcal{S}}; \\ 0, & \omega \notin \mathcal{I}_{\mathcal{S}}, \end{cases} \tag{41}$$

where $\tilde{\mathbf{S}}^F(\omega) = \mathbf{S}^F(\omega)/(1+\omega^2)$ is the Fourier transform of $\tilde{\mathbf{s}}(t)$, $\mathcal{I}_{\mathcal{S}}$ defines the support of $R^f_{\mathbf{S},\mathbf{S}}(\omega)$, and $R^f_{\mathbf{W},\tilde{\mathbf{S}}}(\omega), R^f_{\mathbf{S},\tilde{\mathbf{S}}}(\omega)$ are defined according to (27).

B. Extension of the Ideal Sampling Results

In this section, we show how Theorem 4 extends the results of [11], [12]. In these works it was assumed that a single representation coefficient $\langle \mathbf{w}(t), \mathbf{x}(t)\rangle_{L_2}$ is to be approximated by linearly processing the ideal sample sequence $\{\mathbf{x}(nT)\}$ of some function $\mathbf{x} \in W_2^1$. Denoting by S_T the ideal sampling operator

$$S_T\mathbf{x} = \{\mathbf{x}(nT)\}, \tag{42}$$

the processing is performed by calculating an l_2 inner product $\langle b, S_T\mathbf{x}\rangle_{l_2}$ with some sequence b. Then, the approximation error was upper bounded by

$$|\langle \mathbf{w}(t), \mathbf{x}(t)\rangle_{L_2} - \langle b, S_T\mathbf{x}\rangle_{l_2}| \le B\|\mathbf{x}\|_{W_2^1},$$

where B is a constant that depends on $T, b, \mathbf{w}$ and the function $\mathbf{u}(t)$ of (34). Finally, B was minimized with respect to the processing sequence b.

Reinterpreting the derivations in [11], the approximation problem of [11, Thr. 3] can be restated as a minimax objective:

$$\min_{b} \max_{\|\mathbf{x}\|_{W_2^1} \le \tilde{L}} |\langle \mathbf{w}, \mathbf{x}\rangle_{L_2} - \langle b, S_T\mathbf{x}\rangle_{l_2}|, \tag{43}$$

where $\tilde{L}$ is some (finite) upper bound on the W_2^1 norm of $\mathbf{x}$. In [11] it is found that the optimal sequence b satisfies

$$\sum_n \mathbf{u}(t-nT)b[n] = P_{\mathcal{U}}(\mathbf{w} * \mathbf{u}), \tag{44}$$

where $P_{\mathcal{U}}$ is the orthogonal projection, in the W_2^1 sense, onto $\mathcal{U}$, which is the closure of $\text{span}\{\mathbf{u}(t-nT)\}$.

We now show that this result of [11] is a special case of Theorem 4. First, define U to be the set transformation of the function set $\{\mathbf{u}(t-nT)\}$. It is not hard to show that on W_2^1, U is the adjoint of the ideal sampling operator S_T, *i.e.*, using operator notations

$$S_T\mathbf{x} = U^*\mathbf{x}. \tag{45}$$

We note that U (as well as $U^* = S_T$) is a well defined bounded operator in W_2^1 [9, Appendix C]. Additionally, the single representation coefficient $\langle \mathbf{w}, \mathbf{x}\rangle_{L_2}$ can be written as the order one Sobolev inner product $\tilde{W}^*\mathbf{x}$, with $\tilde{W}$ being the set transform of $\tilde{\mathbf{w}}(t) = \mathbf{w}(t) * \mathbf{u}(t)$. Identifying $\tilde{S}$ with U, we have from (36)

$$\begin{aligned} d &= \tilde{W}^*U(U^*U)^\dagger c = \langle \tilde{\mathbf{w}}, U(U^*U)^\dagger c\rangle_{W_2^1} \\ &= \langle (U^*U)^\dagger U^*\tilde{\mathbf{w}}, c\rangle_{l_2} = \langle b, c\rangle_{l_2}, \end{aligned} \tag{46}$$

where we denote $b = (U^*U)^\dagger U^*\tilde{\mathbf{w}}$. As a result, Ub is exactly the orthogonal projection of $\tilde{\mathbf{w}} = \mathbf{w} * \mathbf{u}$ onto the space $\mathcal{U}$, which is compatible with (44).

VI. Error Analysis

In this section we investigate the error resulting from the minimax method. We then derive sufficient conditions for our method to outperform the sum approximation (5). Although we use the S and W operators (as opposed to their Sobolev counterparts $\tilde{S}, \tilde{W}$), all derivations are applicable to Sobolev spaces by considering the appropriate inner products.

Let

$$e_{mx} = W^*\mathbf{x} - d \tag{47}$$

be the error sequence due to the minimax approach, where d is given by (10). Using (19) we can express the error as

$$e_{mx} = W^*P_{\mathcal{S}^\perp}\mathbf{x}. \tag{48}$$

Define e_{sum} to be the error sequence due to the sum approximation method (5). The nth element of e_{sum} satisfies

$$e_{sum}[n] = \langle \mathbf{w}_n, \mathbf{x}\rangle_{L_2} - T\langle S_T\mathbf{w}_n, c\rangle_{l_2}, \tag{49}$$

where S_T is the ideal sampling operator (42). Note that e_{mx} and e_{sum} depend on the input signal $\mathbf{x}$. However, to simplify the exposition, we omit this dependence from the notations.

We now examine the conditions which will assure that $\|e_{mx}\|_{l_2}^2 \le \|e_{sum}\|_{l_2}^2$ for all possible inputs. In the following lemma we first introduce tight bounds for the difference $\|e_{sum}\|_{l_2}^2 - \|e_{mx}\|_{l_2}^2$. Clearly, if the difference is positive, then the minimax method is preferable to the sum approximation method, and vice versa.

Lemma 1: Let $\mathbf{x} \in \mathcal{D} = \left\{\mathbf{x};\ \|\mathbf{x}\|_{L_2} \le L, c = S^*\mathbf{x}\right\}$. Then

$$B_L \le \|e_{sum}\|_{l_2}^2 - \|e_{mx}\|_{l_2}^2 \le B_H, \tag{50}$$

where the bounds are tight. Here,

$$\begin{aligned} B_L &= \|a\|_{l_2}^2 - 2\left|\langle a, e_{mx}\rangle_{l_2}\right|, \\ B_H &= \|a\|_{l_2}^2 + 2\left|\langle a, e_{mx}\rangle_{l_2}\right|, \end{aligned} \tag{51}$$

and

$$a[n] = \langle \mathbf{w}_n, S(S^*S)^\dagger c\rangle_{L_2} - T\langle S_T\mathbf{w}_n, c\rangle_{l_2}. \tag{52}$$

Proof: Using (47) and (49) we can relate the two error sequences by

$$e_{sum} = a + e_{mx}, \tag{53}$$

with a given by (52). Note that since the sample sequence c is available, and so are T, S and W, the sequence a is known as well. Furthermore, $a \in l_2$. The latter is evident by rewriting $a = W^*\left(S(S^*S)^\dagger - TS_T^*\right)c$. Since $c \in l_2$, it is sufficient to show that $W^*\left(S(S^*S)^\dagger - TS_T^*\right)$ is a bounded operator. Indeed, since S_T is bounded on W_2^1 [9, Appendix C], so is S_T^*. Additionally, $S(S^*S)^\dagger$ and W^* are bounded due to the frame assumptions. Taking the squared norm of both sides of (53) and rearranging terms, we get

$$\|e_{sum}\|_{l_2}^2 - \|e_{mx}\|_{l_2}^2 = \|a\|_{l_2}^2 + 2\Re\left\{\langle a, e_{mx}\rangle_{l_2}\right\}.$$

The bounds (51) then follow from

$$-\left|\langle a, e_{mx}\rangle_{l_2}\right| \le \Re\left\{\langle a, e_{mx}\rangle_{l_2}\right\} \le \left|\langle a, e_{mx}\rangle_{l_2}\right|.$$

We now show that the bounds are tight. Assume to the contrary that for all $\mathbf{x} \in \mathcal{D}$,

$$\Re\left\{\langle a, e_{mx}\rangle_{l_2}\right\} < \left|\langle a, e_{mx}\rangle_{l_2}\right|.$$

Define $\mathbf{x}_2 = S(S^*S)^\dagger c + \frac{\langle W^*P_{\mathcal{S}^\perp}\mathbf{x}, a\rangle_{l_2}}{\left|\langle W^*P_{\mathcal{S}^\perp}\mathbf{x}, a\rangle_{l_2}\right|}P_{\mathcal{S}^\perp}\mathbf{x}$ using some $\mathbf{x} \in \mathcal{D}$. Clearly $\mathbf{x}_2 \in \mathcal{D}$. However, $e_{mx} = W^*P_{\mathcal{S}^\perp}\mathbf{x}_2$ satisfies

$$\Re\left\{\langle a, W^*P_{\mathcal{S}^\perp}\mathbf{x}_2\rangle_{l_2}\right\} = \left|\langle a, W^*P_{\mathcal{S}^\perp}\mathbf{x}_2\rangle_{l_2}\right|,$$

thus contradicting our initial assumption. The proof of tightness for the lower bound is similar. $\square$

Since the tight upper bound B_H is nonnegative for all choices of e_{mx}, we conclude that the sum approximation method **cannot** outperform the proposed minimax approach, for **all** possible inputs. On the other hand, in some cases, it is possible to have better performance by the minimax approach, for all possible inputs. To assure this, the lower bound B_L must be positive. In the following lemma, we provide a tight upper bound on $\|e_{mx}\|_{l_2}$ assuming that the set $\{\mathbf{w}_n(t)\}$ is orthonormal. Using this bound, we then state a sufficient condition for the minimax method to outperform the standard sum approximation approach for all $\mathbf{x} \in \mathcal{D}$.

Lemma 2: Let $\{\mathbf{w}_n(t)\}$ be an orthonormal set, and let $\mathbf{x} \in \mathcal{D}$. Then

$$\|e_{mx}\|_{l_2} \le B_{mx} = \sqrt{1 - \cos^2(\mathcal{W}, \mathcal{S})}L', \tag{54}$$

where

$$L' = \sqrt{L^2 - \|S^*(S^*S)^\dagger c\|^2} \tag{55}$$

is the norm of $P_{\mathcal{S}^\perp}\mathbf{x}$ and $\cos(\mathcal{W}, \mathcal{S}) = \inf_{\mathbf{y}\in\mathcal{W}, \|\mathbf{y}\|=1}\|P_{\mathcal{S}}\mathbf{y}\|$.

Before giving the proof, we mention that $\cos(\mathcal{W}, \mathcal{S})$ is related to the largest angle [13], [16] between the spaces. An explicit expression for $\cos(\mathcal{W}, \mathcal{S})$ in the case of SI spaces is given in [13].

Proof: From the definition of e_{mx},

$$\begin{aligned}\|e_{mx}\|_{l_2}^2 &= \langle W^* P_{\mathcal{S}^\perp}\mathbf{x}, W^* P_{\mathcal{S}^\perp}\mathbf{x}\rangle_{l_2} \\ &= \langle P_{\mathcal{S}^\perp}\mathbf{x}, WW^* P_{\mathcal{S}^\perp}\mathbf{x}\rangle \\ &= \langle P_{\mathcal{S}^\perp}\mathbf{x}, P_{\mathcal{W}} P_{\mathcal{S}^\perp}\mathbf{x}\rangle \\ &= \langle P_{\mathcal{W}} P_{\mathcal{S}^\perp}\mathbf{x}, P_{\mathcal{W}} P_{\mathcal{S}^\perp}\mathbf{x}\rangle \\ &= \|P_{\mathcal{W}} P_{\mathcal{S}^\perp}\mathbf{x}\|^2, \end{aligned} \tag{56}$$

where we utilized the orthonormality of the analysis set $\{\mathbf{w}_n(t)\}$ to write $WW^* = P_{\mathcal{W}}$. For any $\mathbf{x} \in \mathcal{D}$ we have $\|P_{\mathcal{S}^\perp}\mathbf{x}\| \leq L'$, where L' is given by (55). Thus we can bound

$$\|P_{\mathcal{W}} P_{\mathcal{S}^\perp}\mathbf{x}\| \leq \sin\left(\mathcal{S}^\perp, \mathcal{W}^\perp\right) L',$$

where $\sin\left(\mathcal{S}^\perp, \mathcal{W}^\perp\right) = \sup_{\mathbf{y}\in\mathcal{S}^\perp, \|\mathbf{y}\|=1} \|P_{\mathcal{W}}\mathbf{y}\|$. From [13], [27] $\sin\left(\mathcal{S}^\perp, \mathcal{W}^\perp\right) = \sqrt{1-\cos^2\left(\mathcal{W}, \mathcal{S}\right)}$ and the proof follows. □

Corollary 4: Let $\{\mathbf{w}_n(t)\}$ be an orthonormal set. A sufficient condition for the minimax method to outperform the sum approach for all $\mathbf{x} \in \mathcal{D}$ is $\|a\|_{l_2} \geq 2B_{mx}$, where a and B_{mx} are given by (52) and (54), respectively.

Proof: Using Lemma 1, the Cauchy-Schwartz inequality and Lemma 2, we have

$$B_L \geq \|a\|_{l_2}^2 - 2\|a\|_{l_2}\|e_{mx}\|_{l_2} \geq \|a\|_{l_2}^2 - 2\|a\|_{l_2} B_{mx},$$

from which the proof follows. □

The error analysis is summarized in Figure 2.

$\|e_{sum}\| - \|e_{mx}\|$ (Lemma 1)

B_L $\quad$ $B_H \geq 0$

$\|e_{mx}\| \leq B_{mx}$ (Lemma 2)

$B_{mx} \leq \|a\|_{l_2}/2 \Rightarrow B_L \geq 0$ (Corollary4)

Fig. 2. Regions of $\|e_{sum}\| - \|e_{mx}\|$ for the case where $\{\mathbf{w}_n(t)\}$ is an orthonormal set. If the maximal norm of the minimax error (54) is smaller than $\|a\|_{l_2}/2$, then the minimax approach is superior to the sum method, for all possible inputs.

Another interesting case, which is easy to evaluate, is when a single representation coefficient $\langle\mathbf{w}, \mathbf{x}\rangle$ is to be approximated. In this setting, e_{mx}, e_{sum} and a are all scalars. It can then be shown that the minimax method and the sum approximation approach are tightly upper bounded by

$$\begin{aligned}|e_{mx}| &\leq B_{mx} = L'\|P_{\mathcal{S}^\perp}\mathbf{w}\| \\ |e_{sum}| &\leq B_{sum} = |a| + B_{mx},\end{aligned} \tag{57}$$

where the input that achieves both upper bounds is

$$\mathbf{x}_{worst} = S(S^*S)^\dagger c + \frac{aL'}{|a|\,\|P_{\mathcal{S}^\perp}\mathbf{w}\|} P_{\mathcal{S}^\perp}\mathbf{w}. \tag{58}$$

A sufficient condition for the minimax method to outperform the sum approach becomes

$$|a| \geq 2L'\|P_{\mathcal{S}^\perp}\mathbf{w}\|. \tag{59}$$

The proof is provided in Appendix III.

To conclude, when the spaces $\mathcal{W}$ and $\mathcal{S}$ are *close* (such that $\cos\left(\mathcal{W}, \mathcal{S}\right)$ is close to one), or when most of the signal's energy lies within the sampling space $\mathcal{S}$ (such that L' is small), then the minimax method will outperform the standard approach. Similarly, for large sampling intervals T, $\|a\|_{l_2}$ can become large enough, assuring better performance by the minimax method.

VII. Simulations

In this section we simulate an example of approximating a single representation coefficient $\langle\mathbf{w}, \mathbf{x}\rangle_{L_2}$. The analysis function $\mathbf{w}(t)$ is a modulated and normalized Gaussian

$$\mathbf{w}(t) = \alpha e^{-t^2/2}\cos(4\pi t), \tag{60}$$

with α chosen such that $\|\mathbf{w}\|_{L_2} = 1$. The input $\mathbf{x}$ is set to be

$$\mathbf{x}(t) = e^{-50t^2} - e^{-50(t-0.75)^2}, \tag{61}$$

i.e., it is composed of two Gaussians, synchronized with the analysis function $\mathbf{w}(t)$ (see Fig. 3). For this example $\langle\mathbf{w}, \mathbf{x}\rangle_{L_2} \approx 0.2$. We will consider two separate sampling schemes; ZOH (zero-order-hold) and RC (resistor-capacitor). In both schemes we approximate the single representation coefficient $\langle\mathbf{w}, \mathbf{x}\rangle_{L_2}$ based on the generalized samples.

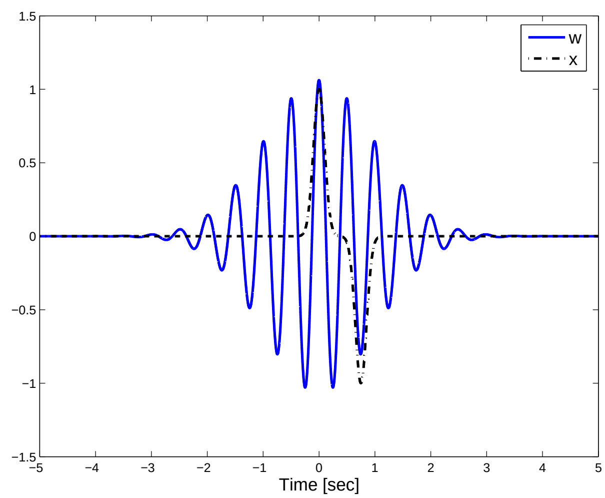


Fig. 3. The analysis function $\mathbf{w}(t)$ and the input signal $\mathbf{x}(t)$.

A. ZOH Sampling

Assume that the generalized samples of $\mathbf{x}(t)$ are obtained by averaging the value of $\mathbf{x}(t)$ within a small interval of length Δ, *i.e.*,

$$c[n] = \frac{1}{\Delta}\int_{nT}^{nT+\Delta} \mathbf{x}(t)dt. \tag{62}$$

In this setting, the nth sampling vector $\mathbf{s}_n(t)$ of (1) is

$$\mathbf{s}_n(t) = \begin{cases} 1/\Delta, & t \in [nT, nT+\Delta]; \\ 0, & \text{otherwise.}\end{cases} \tag{63}$$

By processing the generalized samples $\{c[n] = \langle\mathbf{s}_n, \mathbf{x}\rangle_{L_2}\}$ using the transformation (10), we obtain the minimax approximation of $\langle\mathbf{w}, \mathbf{x}\rangle_{L_2}$. The approximation can be obtained in the

L_2 space, or transformed into a proper Sobolev space using (36), when smoothness is of concern. Note that the input signal of the example (61) indeed satisfies $\mathbf{x} \in W_2^1$. Subsequently, as we will show, the minimax solution with the smoothness constraint outperforms the standard minimax method.

Interpreting the minimax solutions as the application of the analysis operator W^* to the approximates $P_{\mathcal{S}}\mathbf{x}$ and $E_{\widetilde{\mathcal{S}},\mathcal{S}^\perp}\mathbf{x}$ ((19) and (40) respectively), it is interesting to observe the signal approximations. Fig. 4 depicts the generator functions $s_0(t) \in \mathcal{S}$ and $\tilde{s}_0(t) \in \widetilde{\mathcal{S}}$ for $\Delta = 0.05$. In Fig. 5 we plot a

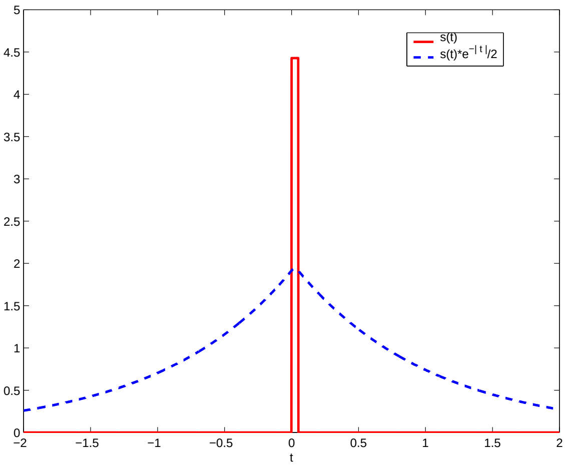


Fig. 4. *ZOH sampling with $\Delta = 0.05$. Shown are $\mathbf{s}_0(t)$ (63) and $\tilde{\mathbf{s}}_0(t) = s_0(t) * e^{-|t|}/2$, which are the generators for the shift invariant spaces $\mathcal{S}$ and $\widetilde{\mathcal{S}}$, respectively. For presentation purposes the plots are scaled.*

section of $\mathbf{x}$ with its projections onto the appropriate sampling spaces. The parameters T and Δ were set to 0.1[sec] and 0.05[sec], respectively. In this example, the space $\widetilde{\mathcal{S}}$ captures most of the signal's energy. Indeed, as can be seen in Fig. 5, the approximation $E_{\widetilde{\mathcal{S}},\mathcal{S}^\perp}\mathbf{x}$ is very close to the original input.

For comparison, we also processed the samples using the standard sum approach (5). In Fig. 6 we present the errors for the input (61) using several choices of T. The minimax solution is optimized for the worst possible input within the considered set, which is different than (61). As a result, for some sampling intervals, the suggested robust solutions are better, while for others they are outperformed by the sum approximation.

It is also of interest to examine the signals that cause the highest value of the cost function. In Fig. 7 we plot these worst inputs. In both cases, the worst possible input is calculated according to (58), and is given by a projection of $\mathbf{x}$ onto the sampling space, and a vector in $\mathcal{S}^\perp$, which has the smallest angle with the analysis function $\mathbf{w}(t)$. As can be seen in Fig. 7(a), the worst possible input in the set $\mathcal{D} = \left\{\mathbf{x} \mid \|\mathbf{x}\|_{L_2} \le L, c = S^*\mathbf{x}\right\}$ is a highly non smooth function. This input is indeed possible in the L_2 space, but it is not likely to appear if we know the signal to be smooth. If we consider only order one Sobolev functions, the worst input is a smooth function and is much closer to the original input, as depicted in Fig. 7(b). The exceptionally good results of Fig. 7(b) are due to the fact that for this example, most

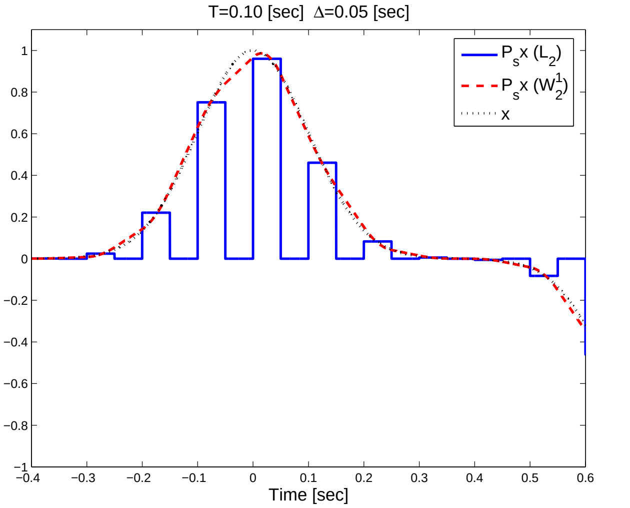


Fig. 5. A section of $\mathbf{x}$ and its approximation in the sampling space; The L_2 orthogonal projection onto $\mathcal{S}$ yields rectangular pulses. The oblique projection onto $\mathcal{S}$ yields a smooth function, which is very close to the original input.

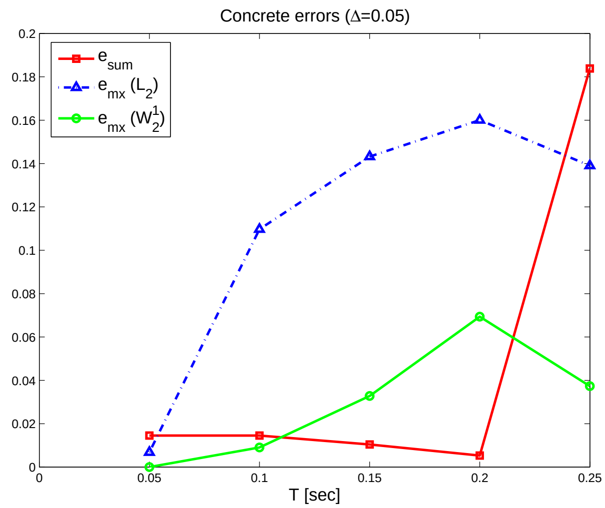


Fig. 6. The errors as a function of T for the specific input (61).

of the signals energy lies within the space $\widetilde{\mathcal{S}}$ (alternatively, L' is small). As a result, the approximation $P_{\widetilde{\mathcal{S}}}\mathbf{x} = E_{\widetilde{\mathcal{S}},\mathcal{S}^\perp}\mathbf{x}$ describes well the original input.

Note that in all cases, the worst inputs look the same for the acquisition device, as they both produce the same generalized samples. To illustrate the last point, in Fig. 8 we plot a section of $\mathbf{x}$ and the worst possible inputs (for the L_2 and the W_2^1 sets). In addition, we present the orthogonal projection $P_{\mathcal{S}}\mathbf{x}$, in the L_2 sense, which is composed of rectangular pulses describing the integration zone due to the sampling functions (63). As can be seen, all signals yield the same generalized samples, as they all have the same area within the rectangular pulses.

In Fig. 9 we plot the upper bounds of the performance for the different approximation methods. The upper two curves are due to (57). If in addition the input is known to be smooth, then we can perform all the inner products and norms in the order one Sobolev space. As a result, the value of the upper bound B_{mx} changes, and so does B_{sum} (the lower two curves of Fig.

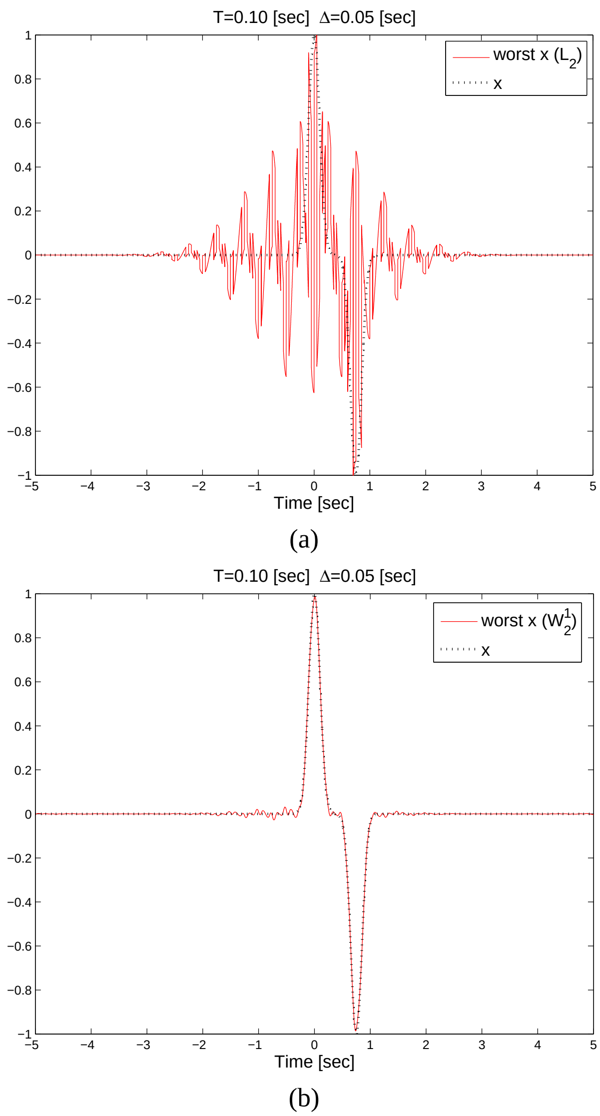


Fig. 7. The original input and the worst possible counterpart in (a) L_2, (b) Sobolev space of order one.

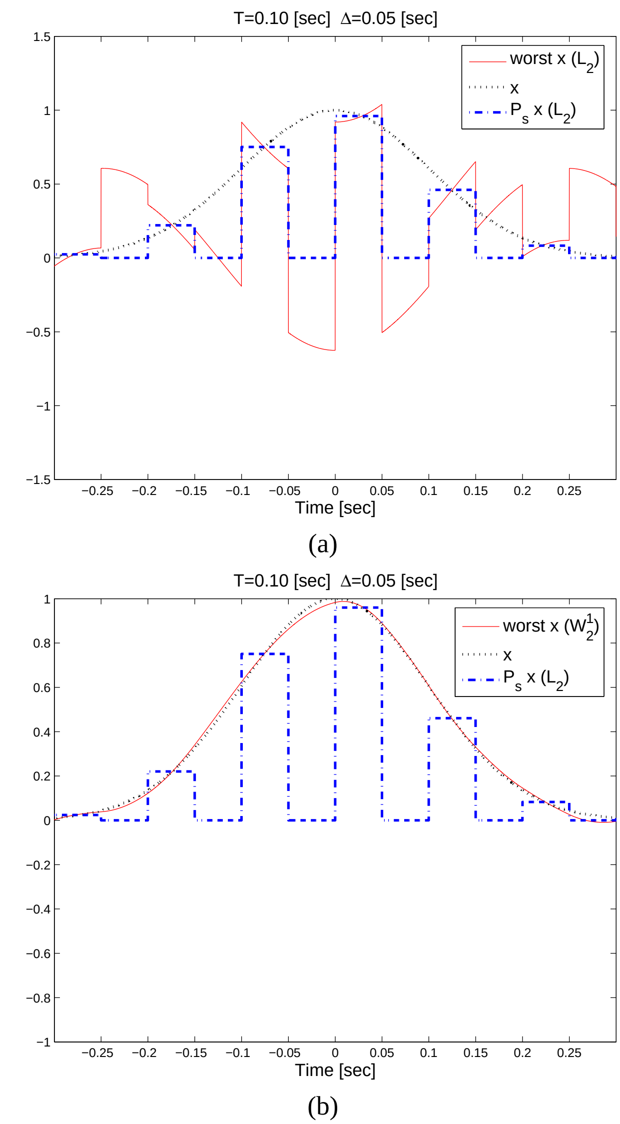


Fig. 8. A section of $\mathbf{x}$ and the worst possible counterpart in (a) L_2, (b) Sobolev space of order one. Both are plotted against $P_{\mathcal{S}}\mathbf{x}$ to describe the integration zones.

9). The upper bounds are obtained by the worst possible inputs plotted in Fig. 7. Specifically, the signal of Fig. 7(b) achieves the lower two curves of Fig. 9 (with the lowest curve for the minimax method with the smoothness constraints, and the one above it for the sum approach). Similarly, when smoothness is not of concern, the signal of Fig. 7(a) achieves the top two error bound curves of Fig. 9 (with the higher curve for the sum approximation).

B. RC Sampling

As an additional example, suppose that the acquisition device is a low pass RC circuit, followed by an ideal sampler with interval T (Fig. 10). Here, the frequency response of the acquisition filter is given by $1/(1 + j\omega RC)$, and the nth sampling vector is shifted and mirrored version of the impulse response

$$\mathbf{s}_n(t) = \begin{cases} (RC)^{-1} e^{\frac{t-nT}{RC}}, & t \le nT; \\ 0, & \text{otherwise}. \end{cases} \tag{64}$$

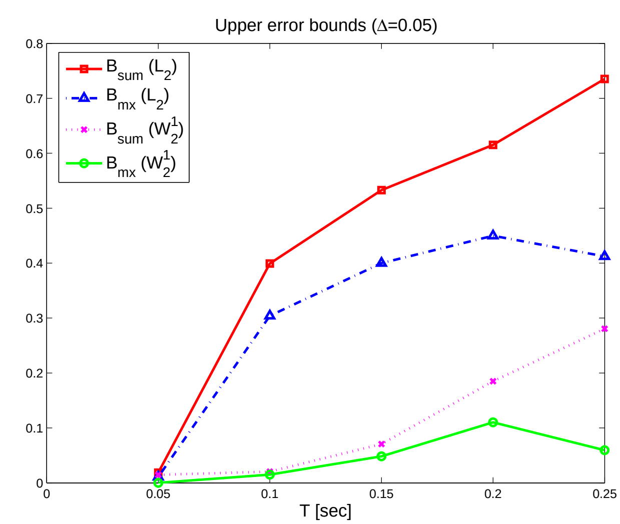


Fig. 9. Upper error bounds according to equation (57). The sampling functions are given by (63).

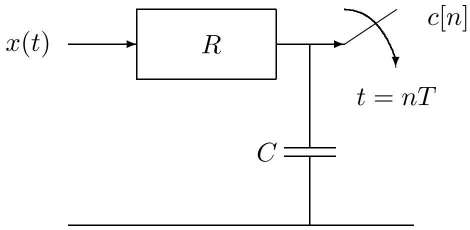
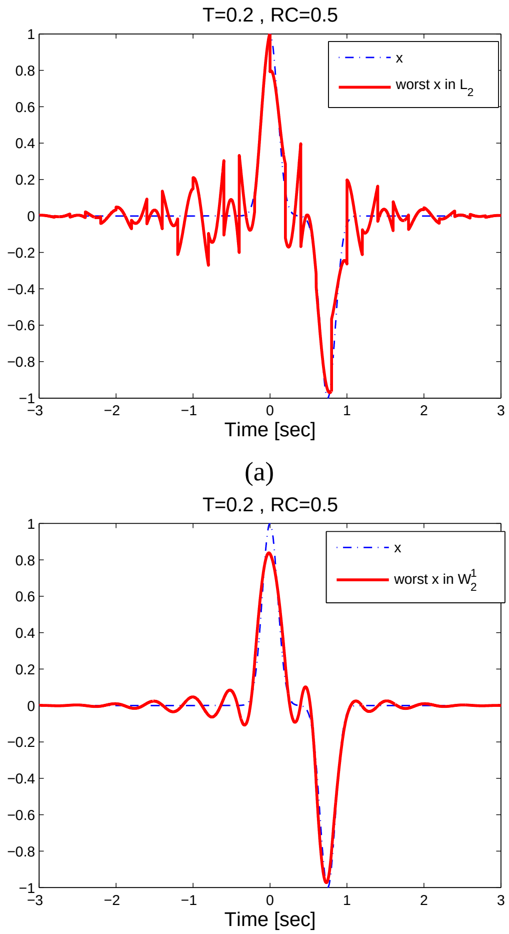
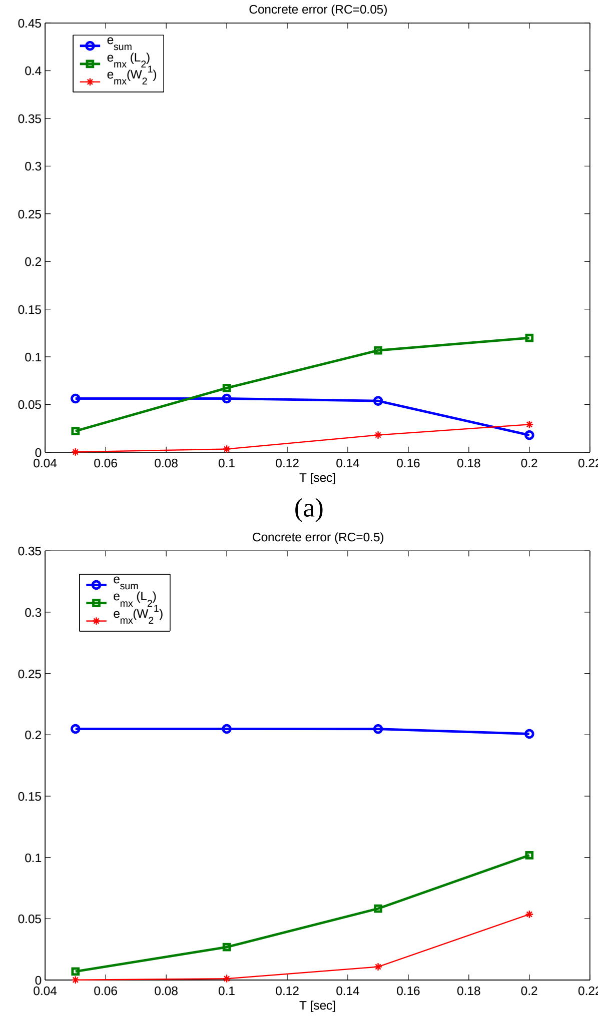


Fig. 10. An RC circuit, followed by ideal sampler, serves as the acquisition device.



(b)

Fig. 11. The original input and the worst possible counterpart in (a) L_2, (b) Sobolev space of order one. The sampling functions are given by (64).



(b)

Fig. 12. Concrete approximation errors for the input (61) processed by the RC circuit. (a) $RC = 0.05$, (b) $RC = 0.5$.

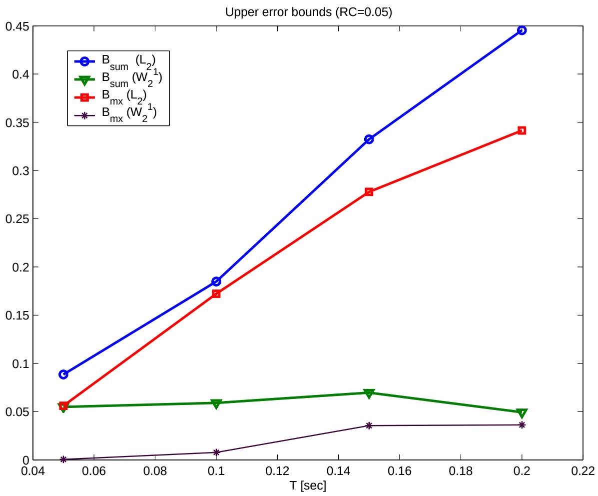
Fig. 11 is similar to Fig. 7, when using the RC circuit sampling function (64) with $RC = 0.5$. Here as well, the sampling functions posses discontinuities, giving rise to a non-smooth worst-case function, as shown in Fig. 11(a). When we expect the input to be smooth, the minimax objective with the smoothness constraint can be used. For such a criterion, the worst-case input function behaves accordingly (Fig. 11(b)).

Fig. 12 shows the approximation error for the input $\mathbf{x}(t)$ of (61). Since $\mathbf{x}(t)$ is a smooth function, imposing the smoothness constraint indeed improves the performance of the minimax methods. Here as well, the proposed robust criteria do not always outperform the Rieman sum approximation (Fig. 12(a)). However, by considering the worst possible input, the superiority of the minimax methods is guaranteed. In Fig. 13 we show the upper error bounds for several values of T and RC. As expected, the robust approaches outperform the sum counterpart. Additionally, when we restrict the set of possible inputs to order one Sobolev functions, the worst case errors are smaller. As with the previous simulation, the presented error bounds are tight. For example, the worst case inputs of Fig. 11 achieve the error bounds of Figure 13(b).

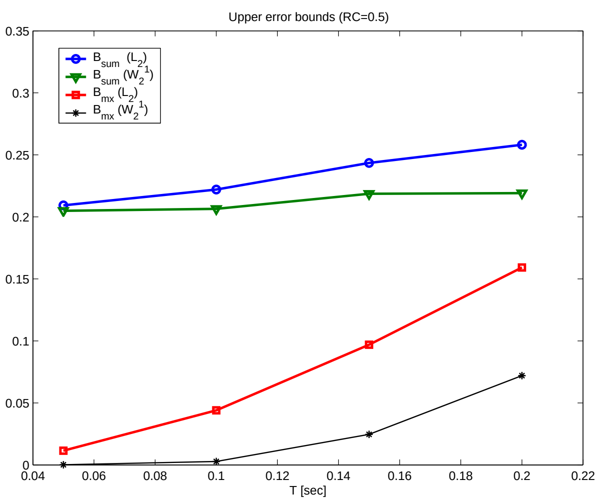
As a final remark, note that the worst-case signal (58) depends on the sampling and analysis functions. Therefore, when either of them is non-smooth, the worst-case function might be non-smooth as well, being the sum of functions with discontinuities. As a result, if we have prior knowledge that the input $\mathbf{x}(t)$ is smooth, it is recommended to implement the minimax solution with the smoothness constraint.

VIII. Summary

A minimax approach has been introduced for approximating inner-product calculations within the continuous-time domain,



(a)



(b)

Fig. 13. Upper error bounds. The sampling functions are given by (64) with (a) $RC = 0.05$, (b) $RC = 0.5$.

while having the generalized samples of the signal as the only available data. We have shown that if the input signal is known to be a smooth function, then a smoothness constraint can be incorporated into the minimax criterion. The latter was achieved by recasting the problem into a proper Sobolev space. A comparison of our proposed robust methods with a piecewise-constant approximation has been presented. Error bounds for the different methods were derived, showing the possible improvement by the minimax methods. The derivations presented herein extend recent results concerning the ideal sampling case, allowing for practical acquisition devices to be incorporated.

Appendix I
Proof of Theorem 1

In this appendix we show that for a general $\mathbf{x} \in \mathcal{H}$, satisfying $c = S^*\mathbf{x}$, it is possible to obtain the required inner products $q = W^*\mathbf{x}$ if and only if $\mathcal{W} \subseteq \mathcal{S}$. The proof of this claim is similar to the proof of a sampling problem, considered in [16, Sec. 3]. For completeness, we detail the derivations below.

Assume $\mathcal{W} \subseteq \mathcal{S}$ and let $d = Gc$ where

$$G = W^* S \left(S^* S\right)^{\dagger}. \tag{65}$$

We now show that $d = W^*\mathbf{x}$. Indeed, since for any function $\mathbf{f}$, $W^*\mathbf{f} = W^* P_{\mathcal{W}}\mathbf{f}$ we have $Gc = W^* P_{\mathcal{W}} S \left(S^* S\right)^{\dagger} c$. Substituting $c = S^*\mathbf{x}$,

$$d = Gc = W^* P_{\mathcal{W}} P_{\mathcal{S}}\mathbf{x} = W^* P_{\mathcal{W}}\mathbf{x} = W^*\mathbf{x}, \tag{66}$$

where we used the fact that $P_{\mathcal{W}} P_{\mathcal{S}} = P_{\mathcal{W}}$ since $\mathcal{W} \subseteq \mathcal{S}$.

Now, assume that $\mathcal{W} \nsubseteq \mathcal{S}$ and suppose that there exits a transformation $d = G(c)$ achieving $d = W^*\mathbf{x}$. Consider the signal $\mathbf{x}$ defined by $\mathbf{x} = \mathbf{x}_{\mathcal{S}^\perp} + \mathbf{x}_{\mathcal{W}}$ where $\mathbf{x}_{\mathcal{S}^\perp}$ is in $\mathcal{S}^\perp$ but not in $\mathcal{W}^\perp$ (such a function always exists since $\mathcal{W} \nsubseteq \mathcal{S}$) and $\mathbf{x}_{\mathcal{W}} \in \mathcal{W}$. For this choice, $c = S^*\mathbf{x} = S^*\mathbf{x}_{\mathcal{W}}$ but $W^*\mathbf{x} - W^*\mathbf{x}_{\mathcal{W}} = W^*\mathbf{x}_{\mathcal{S}^\perp} \neq 0$. Since we assumed $W^*\mathbf{x} = G(S^*\mathbf{x})$ and $W^*\mathbf{x}_{\mathcal{W}} = G(S^*\mathbf{x}_{\mathcal{W}})$ we also have

$$W^*\mathbf{x}_{\mathcal{S}^\perp} = G(S^*\mathbf{x}) - G(S^*\mathbf{x}_{\mathcal{W}}) = 0, \tag{67}$$

which implies that $\mathbf{x}_{\mathcal{S}^\perp} \in \mathcal{W}^\perp$, contradicting our assumption. $\square$

Appendix II
Frame Condition in the Sobolev Space

In this appendix we address the following question: Assuming that the sampling functions $\{\mathbf{s}_n(t)\}$ constitute a frame for $\mathcal{S} \subseteq L_2$, do the modified functions $\{\tilde{\mathbf{s}}_n(t) = \mathbf{s}_n(t) * \mathbf{u}(t)\}$ constitute a frame for $\tilde{\mathcal{S}} \subseteq W_2^1$? As we will show, this is not always the case, but we give a sufficient condition for this to hold.

Since the sampling functions $\{\mathbf{s}_n(t)\}$ form a frame for the closure of their span, there exist $0 < A, B < \infty$ such that $\forall \mathbf{x} \in \mathcal{S}$

$$A \left\|\mathbf{x}\right\|_{L_2}^2 \leq \sum_n \left|\langle \mathbf{s}_n, \mathbf{x}\rangle_{L_2}\right|^2 \leq B \left\|\mathbf{x}\right\|_{L_2}^2. \tag{68}$$

Defining the modified functions to be $\{\tilde{\mathbf{s}}_n(t) = \mathbf{s}_n(t) * \mathbf{u}(t)\}$, where $\mathbf{u}(t)$ is given by (34), and setting $\tilde{\mathcal{S}} \subseteq W_2^1$ to be the closure of $\operatorname{span}\{\tilde{\mathbf{s}}_n\}$, we wish to examine whether for all $\mathbf{y} \in \tilde{\mathcal{S}}$ there are constants $0 < \tilde{A}, \tilde{B} < \infty$ such that

$$\tilde{A} \left\|\mathbf{y}\right\|_{W_2^1}^2 \leq \sum_n \left|\langle \tilde{\mathbf{s}}_n, \mathbf{y}\rangle_{W_2^1}\right|^2 \leq \tilde{B} \left\|\mathbf{y}\right\|_{W_2^1}^2. \tag{69}$$

First note that if the number of sampling functions is finite, then (69) always holds, as any finite set of functions which spans $\tilde{\mathcal{S}}$ is a frame for $\tilde{\mathcal{S}}$. However, in the infinite dimensional case, this is no longer true.

We first show that the upper bound in (69) is always satisfied with $\tilde{B} = B$. To see this, let $\mathbf{y} \in \tilde{\mathcal{S}}$. Recalling that for any $\mathbf{y} \in W_2^1$, we have $\langle \mathbf{s}_n, \mathbf{y}\rangle_{L_2} = \langle \tilde{\mathbf{s}}_n, \mathbf{y}\rangle_{W_2^1}$, we can rewrite the middle term of (69) as $\sum_n \left|\langle \mathbf{s}_n, \mathbf{y}\rangle_{L_2}\right|^2$. Since $\{\mathbf{s}_n\}$ is a frame for $\mathcal{S}$,

$$A \left\|P_{\mathcal{S}}\mathbf{y}\right\|_{L_2}^2 \leq \sum_n \left|\langle \mathbf{s}_n, \mathbf{y}\rangle_{L_2}\right|^2 \leq B \left\|P_{\mathcal{S}}\mathbf{y}\right\|_{L_2}^2.$$

Using

$$B \left\|P_{\mathcal{S}}\mathbf{y}\right\|_{L_2}^2 \leq B \left\|\mathbf{y}\right\|_{L_2}^2 \leq B \left\|\mathbf{y}\right\|_{W_2^1}^2,$$

we conclude that with $\tilde{B} = B$, the upper bound in (69) is always satisfied.

Unfortunately, satisfying the lower bound of (69) is not always possible. As an example, consider the case where for each n, $s_n(t)$ has the Fourier transform

$$\mathbf{S}_n^F(\omega) = \begin{cases} 1 & n < \omega \leq n+1, \\ 0 & \text{otherwise.} \end{cases}$$

This is an orthonormal set of sampling functions, and hence it is a tight frame for $\mathcal{S}$ with frame bounds $A = B = 1$. However, there is no strictly positive lower bound $\tilde{A}$ satisfying the left hand side inequality of (69); notice that for the choice $y(t) = \tilde{s}_n(t)$

$$\|\mathbf{y}\|_{W_2^1} = \frac{1}{2\pi} \int_n^{n+1} \frac{1}{1+\omega^2} d\omega$$

. Defining $b_n = \|\mathbf{y}\|_{W_2^1}$ we have from the orthonormality of the expansion,

$$\sum_n \left| \langle \tilde{\mathbf{s}}_n, \mathbf{y} \rangle_{W_2^1} \right|^2 = b_n^2. \tag{70}$$

Recalling (69), its left hand side should satisfy $\tilde{A} b_n \leq b_n^2$ for this particular example. However, by increasing n we can construct a sequence $\{b_n\}$, which is strictly positive, and converges to zero. Thus, for this example we must have $\tilde{A} = 0$.

Nevertheless, assuming that the sampling functions have a shift invariant structure, *i.e.*, that for each n, $\mathbf{s}_n(t) = \mathbf{s}(t - nT)$ (and naturally also $\tilde{\mathbf{s}}_n(t) = \tilde{\mathbf{s}}(t - nT)$), we can state a sufficient condition that will assure the existence of a strictly positive lower bound $\tilde{A}$.

Proposition 2: Let $\{s(t - nT)\}$ be a frame for $\mathcal{S}$. If the partial sums

$$\hat{R}_{\mathbf{S},\mathbf{S}}^f(K, \omega) = \frac{1}{T} \sum_{k=-K}^{K} \left| \mathbf{S}^F \left(\frac{\omega + 2\pi k}{T} \right) \right|^2 \tag{71}$$

converge uniformly, then $\{\tilde{\mathbf{s}}(t - nT)\}$ is a frame for $\widetilde{\mathcal{S}}$.

Proof: Relying on known results for the shift invariant setup [22], the frame condition for $\widetilde{\mathcal{S}}$ is satisfied if

$$0 < \tilde{A} \leq R_{\tilde{\mathbf{S}},\mathbf{S}}^f(\omega) \leq \tilde{B} < \infty, \quad \omega \in \mathcal{I}_{\mathcal{S}}, \tag{72}$$

where

$$R_{\tilde{\mathbf{S}},\mathbf{S}}^f(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \left| \mathbf{S}^F \left(\frac{\omega + 2\pi k}{T} \right) \right|^2 \frac{1}{1 + \left(\frac{\omega + 2\pi k}{T}\right)^2}, \tag{73}$$

is the DTFT of the correlation function $r_{\tilde{\mathbf{s}},\mathbf{s}}[n] = \langle \tilde{\mathbf{s}}(t - nT), \mathbf{s}(t) \rangle_{L_2}$ and $\mathcal{I}_{\mathcal{S}}$ are the set of frequencies ω for which $R_{\tilde{\mathbf{S}},\mathbf{S}}^f(\omega) \neq 0$. We now show that if the infinite sum $\frac{1}{T}\sum_{k=-\infty}^{\infty} \left| \mathbf{S}^F \left(\frac{\omega + 2\pi k}{T} \right) \right|^2$ converges uniformly, then the lower bound in (72) is satisfied. Indeed, choose $\epsilon = A/2$, where A is the lower frame bound of $\mathcal{S}$. Then, there is an index K_ϵ, such that the partial sum $\hat{R}_{\mathbf{S},\mathbf{S}}^f(K_\epsilon, \omega) = \frac{1}{T}\sum_{k=-K_\epsilon}^{K_\epsilon} \left| \mathbf{S}^F \left(\frac{\omega + 2\pi k}{T} \right) \right|^2$ satisfies

$$R_{\mathbf{S},\mathbf{S}}^f(\omega) - \hat{R}_{\mathbf{S},\mathbf{S}}^f(K_\epsilon, \omega) < \epsilon \tag{74}$$

for all $\omega \in \mathcal{I}_{\mathcal{S}}$. Therefore,

$$\begin{aligned} R_{\tilde{\mathbf{S}},\mathbf{S}}^f(\omega) &\geq \frac{1}{T} \sum_{k=-K_\epsilon}^{K_\epsilon} \left| \mathbf{S}^F \left(\frac{\omega + 2\pi k}{T} \right) \right|^2 \frac{1}{1 + \left(\frac{\omega + 2\pi k}{T}\right)^2} \\ &\geq \frac{1}{T} \sum_{k=-K_\epsilon}^{K_\epsilon} \left| \mathbf{S}^F \left(\frac{\omega + 2\pi k}{T} \right) \right|^2 D(K_\epsilon) \\ &= \hat{R}_{\mathbf{S},\mathbf{S}}^f(K_\epsilon, \omega) D(K_\epsilon), \end{aligned}$$

where we define the strictly positive constant $D(K_\epsilon) = \left(1 + \left(\frac{2\pi + 2\pi K_\epsilon}{T}\right)^2\right)^{-1}$. Combining with (74),

$$R_{\tilde{\mathbf{S}},\mathbf{S}}^f(\omega) \geq (R_{\mathbf{S},\mathbf{S}}^f(\omega) - \epsilon) D(K_\epsilon) \geq \frac{A}{2} D(K_\epsilon),$$

where we used $\epsilon = A/2$ and the frame bound $R_{\mathbf{S},\mathbf{S}}^f(\omega) > A$, $\forall \omega \in I_{\mathcal{S}}$ of $\mathcal{S}$ in the last inequality. □

APPENDIX III
ERROR BOUNDS FOR THE SCALAR CASE

In this appendix we prove (57), (58) and (59).

To prove (57), note that for a single representation coefficient we have

$$\begin{aligned} |e_{mx}| &= |\langle \mathbf{w}, P_{\mathcal{S}^\perp} x \rangle| \\ &= |\langle P_{\mathcal{S}^\perp} \mathbf{w}, P_{\mathcal{S}^\perp} x \rangle| \\ &\leq \|P_{\mathcal{S}^\perp} \mathbf{w}\| L', \end{aligned} \tag{75}$$

where we used the Cauchy-Schwartz inequality and the norm constraint $\|P_{\mathcal{S}^\perp} x\| \leq L'$, with L' given by (55). The bound is tight, since

$$\mathbf{x} = S(S^*S)^\dagger c + \frac{P_{\mathcal{S}^\perp} \mathbf{w}}{\|P_{\mathcal{S}^\perp} \mathbf{w}\|} L' \tag{76}$$

is a valid input which achieves (75) with equality. Similarly, we can bound the error due to the sum method. Using (53),

$$|e_{sum}| \leq |a| + |e_{mx}| \leq |a| + \|P_{\mathcal{S}^\perp} \mathbf{w}\| L'. \tag{77}$$

This upper bound is obtained by setting $\mathbf{x} = \mathbf{x}_{worst}$ as in (58). In fact, the signal $\mathbf{x}_{worst}$ of (58) also achieves the uper bound in (75). Thus, there is a valid input which makes both the sum and the minimax methods to operate as worst as possible.

To prove (59), we must find a sufficient condition that ensures that the lower bound B_L of (51) is positive. Using (51) and (75) we have that $B_L \geq |a|^2 - 2|a| \|P_{\mathcal{S}^\perp} \mathbf{w}\| L'$ from which (59) follows.

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Tsvi G. Dvorkind received his B.Sc. degree in computer engineering in 2000 and the M.Sc. degree in electrical engineering in 2003, both summa cum laude and both from the Technion - Israeli Institute of Technology, Israel. He is now at the Technion, pursuing his Ph.D. degree in electrical engineering.

From 1998 to 2000 he worked at the Electro-Optics Research & Development company at the Technion, and during 2000-2001 at the Jigami corporation. Starting from 2001 he is a research assistant and a project supervisor at the Signal and Image Processing Lab (SIPL), faculty of electrical engineering, Technion. His research interests include speech enhancement and acoustical localization, general parameter estimation problems, and sampling theory.

Hagai Kirshner received the B.Sc. (summa cum laude) and M.Sc. degrees both in electrical engineering from the Technion – Israel institute of Technology, Haifa, Israel in 1997 and 2005 respectively. He is currently pursuing the Ph.D. degree in electrical engineering at the Technion. From 1997 to 2004 he was a system engineer within the Technology Division, IDF. His research interests include sampling theory and signal representation approaches to image processing and for visual and auditory systems.

Yonina C. Eldar received the B.Sc. degree in Physics in 1995 and the B.Sc. degree in Electrical Engineering in 1996 both from Tel-Aviv University (TAU), Tel-Aviv, Israel, and the Ph.D. degree in Electrical Engineering and Computer Science in 2001 from the Massachusetts Institute of Technology (MIT), Cambridge.

From January 2002 to July 2002 she was a Postdoctoral Fellow at the Digital Signal Processing Group at MIT. She is currently an Associate Professor in the Department of Electrical Engineering at the Technion - Israel Institute of Technology, Haifa, Israel. She is also a Research Affiliate with the Research Laboratory of Electronics at MIT.

From 1992 through 1996 she was in the program for outstanding students in TAU. In 1998 she held the Rosenblith Fellowship for study in Electrical Engineering at MIT, and in 2000 she held an IBM Research Fellowship. She was a Horev Fellow in the Leaders in Science and Technology program at the Technion, and an Alon Fellow. In 2004, she was awarded the Wolf Foundation Krill Prize for Excellence in Scientific Research, and in 2005 the Andre and Bella Meyer Lectureship. She is a member of the IEEE Signal Processing Theory and Methods technical committee, and Co-chair of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing, 2007. She is also an Associate Editor for the IEEE Transactions on Signal Processing and the EURASIP Juornal on Signal Proceeing, and on the Editorial Board of Foundations and Trends in Signal Processing.

Moshe Porat received his B.Sc. (summa cum laude) and D.Sc. degrees both in Electrical Engineering from the Technion – Israel institute of Technology, Haifa, Israel in 1982 and 1987 respectively. Since 1988 he has been on the faculty of Electrical Engineering, Technion. During 1991-1992 he was with the Signal Processing department at AT&T Bell laboratories, Murray Hill, NJ, on leave from the Technion. Dr. Porat has also held visiting positions at several universities, including the University of Auckland, New Zealand, where he spent a sabbatical year in 2001-2002 as Associate Professor at the Department of Electrical and Electronic Engineering. His professional interests are in the area of signal processing, specializing in localized representations of signals for image processing and computer vision.