

# Four-Group Decodable Space-Time Block Codes

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**Abstract**—Two new rate-one full-diversity space-time block codes (STBC) are proposed. They are characterized by the lowest decoding complexity among the known rate-one STBC, arising due to the complete separability of the transmitted symbols into four groups for maximum likelihood detection. The first and the second codes are delay-optimal if the number of transmit antennas is a power of 2 and even, respectively. The exact pair-wise error probability is derived to allow for the performance optimization of the two codes. Compared with existing low-decoding complexity STBC, the two new codes offer several advantages such as higher code rate, lower encoding/decoding delay and complexity, lower peak-to-average power ratio, and better performance.

**Index Terms**—Orthogonal designs, performance analysis, quasi-orthogonal space-time block codes, space-time block codes.

## I. INTRODUCTION

Space-time block codes (STBC<sup>1</sup>) have been extensively studied since they exploit the diversity and/or the capacity of multiple-input multiple-output (MIMO) channels. Among various STBC, orthogonal STBC (OSTBC) [1]–[3] offer the minimum decoding complexity and full diversity. However, they have low code rates when the number of transmit (Tx) antennas is more than 2 [3]. The rate of one symbol per channel use (pcu) only exists for 2 Tx antennas and the rate approaches 1/2 for a large number of Tx antennas [1]–[3].

To improve the low rate of OSTBC, several quasi-orthogonal STBC (QSTBC) have been proposed (see [4]–[7] and references therein). They allow joint maximum likelihood (ML) decoding of pairs of complex symbols. However, the rate-one QSTBC exist for 4 Tx antennas only and the code rate is smaller than 1 for more than 4 Tx antennas. Several rate-one STBC have been proposed (e.g. [8]–[10]), in which the transmitted symbols can be completely separated into

two groups for ML detection. However, for more than 4 Tx antennas, the decoding complexity of the rate-one STBC in [8]–[10] increases significantly compared with OSTBC and QSTBC.

In this paper, we propose two new rate-one STBC for any number of Tx antennas. Compared with the existing rate-one STBC, our new codes have lowest decoding complexity since the transmitted symbols can be decoupled into 4 groups (4Gp) for ML detection. The first code is called 4Gp-QSTBC. The second code is derived from semi-orthogonal algebraic space-time (SAST) codes [10] and thus called 4Gp-SAST codes. The first and the second codes are delay-optimal when the number of Tx antennas is a power of 2 and even, respectively. The equivalent transmit-receive signals are derived so that sphere decoders [11] can be applied for data detection. To achieve full-diversity, signal rotations are required for the two codes. The exact pair-wise error probability (PEP) of the two codes is derived to optimize the signal rotations.

We compare the main parameters of our new codes and several existing STBC for 6 and 8 Tx antennas in Table I. Clearly, the new codes offer several distinct advantages such as higher code rate, low decoding complexity, and lower encoding/decoding delay. The two new codes also have lower peak-to-average power ratio (PAPR) than OSTBC, QSTBC, and minimum decoding complexity (MDC) QSTBC [12]. Moreover, simulation results show that our new codes also yield significant SNR gains compared with the existing codes.

*Notation:* Superscripts  $\top$ ,  $*$ , and  $\dagger$  denote matrix transpose, conjugate, and transpose conjugate, respectively. The identity and all-zero square matrices of proper size are denoted by  $\mathbf{I}$  and  $\mathbf{0}$ . The diagonal matrix with elements of vector  $\mathbf{x}$  on the main diagonal is denoted by  $\text{diag}(\mathbf{x})$ .  $\|X\|_F$  stands for the Frobenius norm of matrix  $X$  and  $\otimes$  denotes Kronecker product [13]. A mean- $m$  and variance- $\sigma^2$  circularly complex Gaussian

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<sup>1</sup>The term "STBC" stands for space-time block code/codes/coding, depending on the context.

TABLE I

COMPARISON OF SEVERAL LOW COMPLEXITY STBC FOR 6 AND 8 ANTENNAS. THE NUMBERS IN THE PARENTHESES INDICATE THE CODES' PARAMETERS FOR 8 TX ANTENNAS.

Codes	Maximal rate	Delay	Real symbol decoding
OSTBC [3], [24]	2/3 (5/8)	30 (56)	1 or 2 (1 or 2)
CIOD [17]	6/7 (4/5)	14 (50)	2 (2)
MDC-QSTBC [12]	3/4 (3/4)	8 (8)	2 (2)
QSTBC [6]	3/4 (3/4)	8 (8)	4 (4)
2Gp-QSTBC [8]	1 (1)	8 (8)	8 (8)
SAST [10]	1 (1)	6 (8)	6 (8)
<b>4Gp-QSTBC (new)</b>	<b>1 (1)</b>	<b>8 (8)</b>	<b>4 (4)</b>
<b>4Gp-SAST (new)</b>	<b>1 (1)</b>	<b>6 (8)</b>	<b>3 (4)</b>

random variable is written by  $\mathcal{CN}(m, \sigma^2)$ .  $\Re(X)$  and  $\Im(X)$  denote the real and imaginary parts of  $X$ , respectively.

## II. SYSTEM MODEL AND PRELIMINARIES

### A. System Model

We consider data transmission over a MIMO quasi-static Rayleigh flat fading channel with  $M$  Tx and  $N$  receive (Rx) antennas [14]. The channel gain  $h_{mn}$  ( $m = 1, 2, \dots, M; n = 1, 2, \dots, N$ ) between the  $(m, n)$ -th Tx-Rx antenna pair is assumed  $\mathcal{CN}(0, 1)$  and remains constant over  $T$  time slots. We assume no spatial correlation at either Tx or Rx array. The receiver, but not the transmitter, completely knows the channel gains.

A  $T \times M$  STBC can be represented in a general dispersion form [14] as follows:

$$X = \sum_{k=1}^K (a_k A_k + b_k B_k) \quad (1)$$

where  $A_k$  and  $B_k$ , ( $k = 1, 2, \dots, K$ ) are  $T \times M$  constant matrices, commonly called dispersion matrices;  $a_k$  and  $b_k$  are the real and imaginary parts of the symbol  $s_k$ . We can use an equivalent form of STBC as

$$X = \sum_{l=1}^L c_l C_l \quad (2)$$

where  $L$  is the number (not necessarily even) of transmitted symbols,  $c_l$  are real-value transmitted symbols,  $C_l$  are dispersion matrices. The average energy of code matrices is constrained such that  $\mathcal{E}_X = \mathbb{E}[\|X\|_F^2] = T$ .

The received signals  $y_{tn}$  of the  $n$ th antenna at time  $t$  can be arranged in a matrix  $Y$  of size  $T \times N$ . Thus, one can represent the Tx-Rx signal relation as [14], [15]

$$Y = \sqrt{\rho} X H + Z \quad (3)$$

where  $H = [h_{mn}]$  is the channel matrix;  $Z = [z_{tn}]$  is the noise matrix of size  $T \times N$ , its elements  $z_{tn}$  are independently, identically distributed (i.i.d.)  $\mathcal{CN}(0, 1)$ . The Tx power is scaled by  $\rho$  so that the average signal-to-noise ratio (SNR) at each Rx antenna is  $\rho$ , independent of the number of Tx antennas.

Let the data vector be  $\mathbf{c} = [c_1 \ c_2 \ \dots \ c_L]^T$ . The ML decoding of STBC is to find the solution  $\hat{\mathbf{c}}$  so that:

$$\hat{\mathbf{c}} = \arg \min_{\mathbf{c}} \|Y - X H\|_F^2. \quad (4)$$

### B. Algebraic Constraints of QSTBC

The key idea of QSTBC is to divide the  $L$  (real) transmitted symbols embedded in a code matrix into  $\Gamma$  groups, so that the ML detection of the transmitted symbol vector can be decoupled into  $\Gamma$  sub-metrics, each metric involves the symbols of only one group [6], [8], [10], [16]. We provide a definition of STBC with this feature to unify the notation in this paper as follows.

*Definition 1:* A STBC is said to be  $\Gamma$ -group decodable STBC if the ML decoding metric (4) can be decoupled into a linear sum of  $\Gamma$  independent submetrics, each submetric

consists of the symbols from only one group. The  $\Gamma$ -group decodable STBC is denoted by  $\Gamma$ Gp-STBC for short.

In the most general case, we assume that there are  $\Gamma$  groups; each group is denoted by  $\Omega_i$  ( $i = 1, 2, \dots, \Gamma$ ) and has  $L_i$  symbols. Thus  $L = \sum_{i=1}^{\Gamma} L_i$ . Let  $\Theta_i$  be the set of indexes of symbols in the group  $\Omega_i$ .

Yuen *et al.* [16, Theorem 1] have shown a sufficient condition for a STBC to be  $\Gamma$ -group decodable. In fact, this condition is also necessary. We will state these results in the following theorem without proof for brevity.

*Theorem 1:* The necessary and sufficient conditions, so that a STBC is  $\Gamma$ -group decodable, are

$$C_p^\dagger C_q + C_q^\dagger C_p = \mathbf{0} \quad \forall p \in \Theta_i, \forall q \in \Theta_j, i \neq j. \quad (5)$$

Note that Theorem 1 covers [17, Theorem 9] (single-symbol decodable STBC) and can be shown similarly.

## III. FOUR-GROUP DECODABLE STBC DERIVED FROM QSTBC

### A. Encoding

In this section, we will study the new 4Gp-QSTBC. As we will see later, the general form of STBC in (1) is convenient for studying 4Gp-QSTBC; hence Theorem 1 can be restated as follows.

*Lemma 1 ([18]):* The necessary and sufficient conditions for a STBC in (1) to become  $\Gamma$ -group decodable are: (a)  $A_p^\dagger A_q + A_q^\dagger A_p = \mathbf{0}$ , (b)  $B_p^\dagger B_q + B_q^\dagger B_p = \mathbf{0}$ , and (c)  $A_p^\dagger B_q + B_q^\dagger A_p = \mathbf{0}$ ,  $\forall p \in \Theta_i, \forall q \in \Theta_j, 1 \leq i \neq j \leq \Gamma$ .

We next consider another sufficient condition so that a STBC is four-group decodable.

*Theorem 2:* Given a 4Gp-STBC for  $M$  Tx antennas with code length  $T$  and  $K$  sets of dispersion matrices ( $A_k, B_k; 1 \leq k \leq K$ ), a 4Gp-STBC with code length  $2T$  for  $2M$  Tx antennas, which consists of  $2K$  sets of dispersion matrices denoted as  $(\bar{A}_i, \bar{B}_i), 1 \leq i \leq 2K$ , can be constructed using the following mapping rules:

$$\begin{aligned} \bar{A}_{2k-1} &= \begin{bmatrix} A_k & \mathbf{0} \\ \mathbf{0} & A_k \end{bmatrix}, \quad \bar{A}_{2k} = \begin{bmatrix} B_k & \mathbf{0} \\ \mathbf{0} & B_k \end{bmatrix}, \\ \bar{B}_{2k-1} &= \begin{bmatrix} \mathbf{0} & A_k \\ A_k & \mathbf{0} \end{bmatrix}, \quad \bar{B}_{2k} = \begin{bmatrix} \mathbf{0} & B_k \\ B_k & \mathbf{0} \end{bmatrix}. \end{aligned} \quad (6)$$

*Proof:* Theorem 2 can be proved by showing that if the dispersion matrices  $(A_q, B_q)$  ( $1 \leq q \leq K$ ) satisfy Lemma 1 with  $(A_p, B_p)$  ( $1 \leq p \leq K$ ) where  $q \notin \Theta_p$ , then the dispersion matrices  $(\bar{A}_{2q-1}, \bar{B}_{2q-1}, \bar{A}_{2q}, \bar{B}_{2q})$  constructed from  $(A_q, B_q)$  using (6) will satisfy Theorem 2 with  $(\bar{A}_{2p-1}, \bar{B}_{2p-1}, \bar{A}_{2p}, \bar{B}_{2p})$  constructed from  $(A_p, B_p)$  using (6). The detailed proof is omitted here, as the steps are routine.  $\square$

The recursive construction of 4Gp-STBC specified in Theorem 2 suggests that we can start with the MDC-QSTBC for 4 Tx antennas proposed in [12] to construct 4Gp-STBC for 8, 16 Tx antennas and so on, because MDC-QSTBC is one of the STBC satisfying Lemma 1; the resulting STBC is thus called 4Gp-QSTBC. For practical interest, we will illustrate the encoding process of 4Gp-QSTBC for 8 Tx antennas from

the MDC-QSTBC for 4 Tx antennas [12]. The code matrix of MDC-QSTBC for 4 Tx antennas is

$$F_4 = \begin{bmatrix} a_1 + j a_3 & a_2 + j a_4 & b_1 + j b_3 & b_2 + j b_4 \\ -a_2 + j a_4 & a_1 - j a_3 & -b_2 + j b_4 & b_1 - j b_3 \\ b_1 + j b_3 & b_2 + j b_4 & a_1 + j a_3 & a_2 + j a_4 \\ -b_2 + j b_4 & b_1 - j b_3 & -a_2 + j a_4 & a_1 - j a_3 \end{bmatrix} \quad (7)$$

where  $j^2 = -1$ .

The code matrix of 4Gp-QSTBC for 8 Tx antennas from  $F_4$  using mapping rules in (6) is given below:

$$F_8 = \begin{bmatrix} a_1 + j a_5 & a_3 + j a_7 & a_2 + j a_6 & a_4 + j a_8 & & & & \\ -a_3 + j a_7 & a_1 - j a_5 & -a_4 + j a_8 & a_2 - j a_6 & & & & \\ a_2 + j a_6 & a_4 + j a_8 & a_1 + j a_5 & a_3 + j a_7 & & & & \\ -a_4 + j a_8 & a_2 - j a_6 & -a_3 + j a_7 & a_1 - j a_5 & & & & \\ b_1 + j b_5 & b_3 + j b_7 & b_2 + j b_6 & b_4 + j b_8 & & & & \\ -b_3 + j b_7 & b_1 - j b_5 & -b_4 + j b_8 & b_2 - j b_6 & & & & \\ b_2 + j b_6 & b_4 + j b_8 & b_1 + j b_5 & b_3 + j b_7 & & & & \\ -b_4 + j b_8 & b_2 - j b_6 & -b_3 + j b_7 & b_1 - j b_5 & & & & \\ b_1 + j b_5 & b_3 + j b_7 & b_2 + j b_6 & b_4 + j b_8 & & & & \\ -b_3 + j b_7 & b_1 - j b_5 & -b_4 + j b_8 & b_2 - j b_6 & & & & \\ b_2 + j b_6 & b_4 + j b_8 & b_1 + j b_5 & b_3 + j b_7 & & & & \\ -b_4 + j b_8 & b_2 - j b_6 & -b_3 + j b_7 & b_1 - j b_5 & & & & \\ a_1 + j a_5 & a_3 + j a_7 & a_2 + j a_6 & a_4 + j a_8 & & & & \\ -a_3 + j a_7 & a_1 - j a_5 & -a_4 + j a_8 & a_2 - j a_6 & & & & \\ a_2 + j a_6 & a_4 + j a_8 & a_1 + j a_5 & a_3 + j a_7 & & & & \\ -a_4 + j a_8 & a_2 - j a_6 & -a_3 + j a_7 & a_1 - j a_5 & & & & \end{bmatrix} \quad (8)$$

The code rate of 4Gp-QSTBC for 8 Tx antennas is one symbol pcu. In general, by construction, the rate of 4Gp-QSTBC for  $2M$  Tx antennas is the same as the rate of MDC-QSTBC for  $M$  Tx antennas. The maximal rate of MDC-QSTBC is one symbol pcu [12], the maximal achievable rate of 4Gp-QSTBC is also one symbol pcu for  $2^m$  Tx antennas. If the number of Tx antennas is  $M < 2^m$  ( $m = 2, 3, \dots$ ), then  $(2^m - M)$  columns of the code matrix for  $2^m$  Tx antennas can be deleted to obtain the code for  $M$  antennas. Thus, *the maximum rate of 4Gp-QSTBC is one symbol pcu and it is achievable for any number of Tx antennas*. Additionally, the  $4 \times 4$  code matrix  $F_4$  is square. By recursive construction (6), the code matrices of 4Gp-QSTBC are also square for  $2^m$  Tx antennas; and therefore, 4Gp-QSTBC are delay optimal if the number of Tx antennas is  $2^m$  [17].

## B. Decoding

We know that the symbols  $s_1, s_2, s_3, s_4$  of  $F_4$  can be separately detected [12]. Therefore, from Theorem 2, the 4 groups of 8 symbols of  $F_8$  can be detected independently. These 4 groups are  $(s_1, s_2)$ ,  $(s_3, s_4)$ ,  $(s_5, s_6)$ , and  $(s_7, s_8)$ . The ML metric given in (4) can be derived to detect the 4 groups of symbols of  $F_8$ . However, to provide more insights into the decoding of 4Gp-QSTBC, we will derive an equivalent code and the equivalent channel of  $F_8$ . Furthermore, using the equivalent channel of  $F_8$ , we can use a sphere decoder [11] to reduce the complexity of the ML search.

The equivalent code of  $F_8$  is obtained by column permutations for the code matrix of  $F_8$  in (8): the order of

columns is changed to (1, 3, 5, 7, 2, 4, 6, 8). This order of permutations is also applied for the rows of  $F_8$ . Let  $x_1 = a_1 + j a_5, x_2 = a_2 + j a_6, x_3 = b_1 + j b_5, x_4 = b_2 + j b_6, x_5 = a_3 + j a_7, x_6 = a_4 + j a_8, x_7 = b_3 + j b_7, x_8 = b_4 + j b_8$  be the intermediate variables, we obtain a permutation-equivalent code of  $F_8$  below

$$D = \begin{bmatrix} \mathcal{D}_1 & \mathcal{D}_2 \\ -\mathcal{D}_2^* & \mathcal{D}_1^* \end{bmatrix} \quad (9)$$

where

$$\mathcal{D}_1 = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & x_3 & x_2 & x_1 \end{bmatrix}, \quad \mathcal{D}_2 = \begin{bmatrix} x_5 & x_6 & x_7 & x_8 \\ x_6 & x_5 & x_8 & x_7 \\ x_7 & x_8 & x_5 & x_6 \\ x_8 & x_7 & x_6 & x_5 \end{bmatrix} \quad (10)$$

The sub-matrices  $\mathcal{D}_1$  and  $\mathcal{D}_2$  have a special form called *block-circulant matrix with circulant blocks* [13].

We next show how to decode the code  $D$ . For simplicity, a single Rx antenna is considered. The generalization for multiple Rx antennas is straightforward. Assume that the Tx symbols are drawn from a constellation with unit average power, the Tx-Rx signal model in (3) for the case of STBC  $D$  follows

$$\mathbf{y} = \sqrt{\rho/8} D \mathbf{h} + \mathbf{z}. \quad (11)$$

Let  $\mathbf{x} = [x_1 \ x_2 \ \dots \ x_8]^\top$ ,  $\hat{\mathbf{y}} = [y_1 \ \dots \ y_4 \ y_5^* \ \dots \ y_8^*]^\top$ ,  $\hat{\mathbf{z}} = [z_1 \ \dots \ z_4 \ z_5^* \ \dots \ z_8^*]^\top$ , and

$$\mathcal{H}_1 = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 \\ h_2 & h_1 & h_4 & h_3 \\ h_3 & h_4 & h_1 & h_2 \\ h_4 & h_3 & h_2 & h_1 \end{bmatrix}, \quad \mathcal{H}_2 = \begin{bmatrix} h_5 & h_6 & h_7 & h_8 \\ h_6 & h_5 & h_8 & h_7 \\ h_7 & h_8 & h_5 & h_6 \\ h_8 & h_7 & h_6 & h_5 \end{bmatrix} \quad (12)$$

We have an equivalent expression of (11) as

$$\hat{\mathbf{y}} = \sqrt{\frac{\rho}{8}} \underbrace{\begin{bmatrix} \mathcal{H}_1 & \mathcal{H}_2 \\ \mathcal{H}_2^* & -\mathcal{H}_1^* \end{bmatrix}}_{\hat{\mathcal{H}}} \mathbf{x} + \hat{\mathbf{z}}. \quad (13)$$

Note that  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are block-circulant matrices with circulant-blocks [13]. Thus, they are commutative and so do  $\mathcal{H}_1^*$  and  $\mathcal{H}_2^*$ . We can multiply both sides of (13) with  $\hat{\mathcal{H}}^\dagger$  to get

$$\hat{\mathcal{H}}^\dagger \hat{\mathbf{y}} = \sqrt{\frac{\rho}{8}} \begin{bmatrix} \mathcal{H}_1^* \mathcal{H}_1 + \mathcal{H}_2^* \mathcal{H}_2 & \mathbf{0} \\ \mathbf{0} & \mathcal{H}_1^* \mathcal{H}_1 + \mathcal{H}_2^* \mathcal{H}_2 \end{bmatrix} \mathbf{x} + \hat{\mathcal{H}}^\dagger \hat{\mathbf{z}}. \quad (14)$$

It can be shown that the noise elements of vector  $\hat{\mathbf{z}}$  are correlated with covariance matrix  $\hat{\mathcal{H}}^\dagger \hat{\mathcal{H}}$ . Thus this noise vector can be whitened by multiplying both side of (14) with the matrix  $(\hat{\mathcal{H}}^\dagger \hat{\mathcal{H}})^{-1/2}$ . Let  $\hat{\mathcal{H}} = \mathcal{H}_1^* \mathcal{H}_1 + \mathcal{H}_2^* \mathcal{H}_2$ . After the noise whitening step, (14) is equivalent to the following equations

$$\hat{\mathcal{H}}^{-1/2} \hat{\mathbf{y}}_i = \sqrt{\frac{\rho}{8}} \hat{\mathcal{H}}^{1/2} \mathbf{x}_i + \hat{\mathbf{z}}_i, \quad (i = 1, 2), \quad (15)$$

where  $\bar{\mathbf{y}}_i = [\bar{y}_{4i-3} \ \bar{y}_{4i-2} \ \bar{y}_{4i-1} \ \bar{y}_{4i}]^\top$ ,  $\mathbf{x}_i = [x_{4i-3} \ x_{4i-2} \ x_{4i-1} \ x_{4i}]^\top$ , the noise vectors  $\bar{\mathbf{z}}_i = \hat{\mathcal{H}}^{-1/2} [\bar{z}_{4i-3} \ \bar{z}_{4i-2} \ \bar{z}_{4i-1} \ \bar{z}_{4i}]^\top$  are uncorrelated and have elements  $\sim \mathcal{CN}(0, 1)$ .

At this point, the decoding of the 8 transmitted symbols of the code  $D$  can be readily decoupled into 2 groups. However, since the code is a 4Gp-STBC, we can further decompose them into 4 groups in the following.

Denote the  $2 \times 2$  (real) discrete Fourier transform (DFT) matrix by  $\mathcal{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ . The block-circulant matrices  $\mathcal{H}_1$  and  $\mathcal{H}_2$  can be diagonalized by a (real) unitary matrix  $\Theta = \frac{1}{2} \mathcal{F}_2 \otimes \mathcal{F}_2$  [13, Theorem 5.8.2, p. 185]. Note that  $\Theta^\dagger = \Theta$ , therefore,  $\mathcal{H}_1 = \Theta \Lambda_1 \Theta$  and  $\mathcal{H}_2 = \Theta \Lambda_2 \Theta$ , where  $\Lambda_1$  and  $\Lambda_2$  are diagonal matrices, with eigenvalues of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  in the main diagonal, respectively. Thus,  $\hat{\mathcal{H}} = \Theta(\Lambda_1^\dagger \Lambda_1 + \Lambda_2^\dagger \Lambda_2) \Theta$ , and also  $\hat{\mathcal{H}}^{1/2} = \Theta(\Lambda_1^\dagger \Lambda_1 + \Lambda_2^\dagger \Lambda_2)^{1/2} \Theta$ . Since  $\hat{\mathcal{H}}^{1/2}$  is a real matrix, (15) becomes

$$\hat{\mathcal{H}}^{-1/2} \Re(\bar{\mathbf{y}}_i) = \sqrt{\rho/8} \hat{\mathcal{H}}^{1/2} \Re(\mathbf{x}_i) + \Re(\bar{\mathbf{z}}_i), \quad i = 1, 2, \quad (16a)$$

$$\hat{\mathcal{H}}^{-1/2} \Im(\bar{\mathbf{y}}_i) = \sqrt{\rho/8} \hat{\mathcal{H}}^{1/2} \Im(\mathbf{x}_i) + \Im(\bar{\mathbf{z}}_i), \quad i = 1, 2 \quad (16b)$$

Note that  $\Re(\mathbf{x}_1) = [a_1 \ a_2 \ b_1 \ b_2]^\top := \mathbf{d}_1$ , i.e.  $\Re(\mathbf{x}_1)$  is only dependent on the complex symbols  $s_1$  and  $s_2$ . Similarly,  $\Re(\mathbf{x}_2)$ ,  $\Im(\mathbf{x}_1)$ , and  $\Im(\mathbf{x}_2)$  depend on  $(s_3, s_4)$ ,  $(s_5, s_6)$ , and  $(s_7, s_8)$ , respectively.

Eq. (16) shows that the decoding of 8 transmitted symbols of STBC  $D$  is separated into the decoding of 4 groups, each with two symbols (thus the search space size has been reduced from  $Q^8$  to  $4Q^2$  where  $Q$  is the transmit constellation size). A sphere decoder [11] can also be used to reduce the complexity of the ML search for each group. The matrix  $\hat{\mathcal{H}}^{1/2}$  can be considered as the *equivalent channel* of the 4Gp-QSTBC  $D$ .

### C. Performance Analysis

In (16), the PEP of the four transmit symbol vectors are the same. We thus need to consider the PEP of one of the vectors  $\mathbf{d}_1 = \Re(\mathbf{x}_1) = [a_1 \ a_2 \ b_1 \ b_2]^\top$ . For notational simplicity, the subindex 1 of  $\mathbf{d}_1$  is dropped. Additionally, we can introduce redundancy on the signal space by using a  $4 \times 4$  real unitary rotation  $R$  to the data vector  $[a_1 \ a_2 \ b_1 \ b_2]^\top$ . Thus the data vector  $\mathbf{d} = R[a_1 \ a_2 \ b_1 \ b_2]^\top$ .

From (16a), the PEP of the pair  $\mathbf{d}$  and  $\bar{\mathbf{d}}$  can be expressed by the Gaussian tail function as [19]

$$\begin{aligned} P(\mathbf{d} \rightarrow \bar{\mathbf{d}} | \hat{\mathcal{H}}) &= Q \left( \sqrt{\frac{\rho \|\hat{\mathcal{H}}^{1/2} R \delta\|_{\mathbb{F}}^2}{8 \cdot 4N_0}} \right) \\ &= Q \left( \sqrt{\frac{\rho \left[ \delta^\top R^\top \Theta^\dagger (\Lambda_1^\dagger \Lambda_1 + \Lambda_2^\dagger \Lambda_2) \Theta R \delta \right]}{16}} \right). \end{aligned} \quad (17)$$

where  $\delta = \mathbf{d} - \bar{\mathbf{d}}$ ,  $N_0 = 1/2$  is the variance of the elements of the white noise vector  $\Re(\mathbf{z}_1)$  in (16a).

Remember that  $\Lambda_1$  is a diagonal matrix with eigenvalues of  $\mathcal{H}_1$  on the main diagonal. Let  $\lambda_{i,j}$  ( $i = 1, 2; j = 1, 2, 3, 4$ ) be

the eigenvalues of  $\mathcal{H}_i$ . Then  $\Lambda_i = \text{diag}(\lambda_{i,1}, \lambda_{i,2}, \lambda_{i,3}, \lambda_{i,4})$ . Let  $\beta = \Theta R \delta$ , we have

$$P(\mathbf{d} \rightarrow \bar{\mathbf{d}} | \hat{\mathcal{H}}) = Q \left( \sqrt{\frac{\rho (\sum_{i=1}^2 \sum_{j=1}^4 \beta_j^2 |\lambda_{i,j}|^2)}{16}} \right). \quad (18)$$

To derive a closed form of (18), we need to evaluate the distribution of  $\lambda_{i,j}$ . The eigenvectors of  $\mathcal{H}_1$  is the columns of the matrix  $\Theta = \frac{1}{2} \mathcal{F}_2 \otimes \mathcal{F}_2$ . Thus, the eigenvalues of  $\mathcal{H}_1$  are:  $[\lambda_{1,1} \ \lambda_{1,2} \ \lambda_{1,3} \ \lambda_{1,4}]^\top = (\mathcal{F}_2 \otimes \mathcal{F}_2) [h_1 \ h_2 \ h_3 \ h_4]^\top$ . Since  $h_j \sim \mathcal{CN}(0, 1)$  for ( $j = 1, \dots, 4$ ), thus  $\lambda_{1,j} \sim \mathcal{CN}(0, 4)$  and so do  $\lambda_{2,j}$ .

We now use the Craig's formula [20] to derive the conditional PEP in (18).

$$\begin{aligned} P(\mathbf{d} \rightarrow \bar{\mathbf{d}} | \hat{\mathcal{H}}) &= Q \left( \sqrt{\frac{\rho (\sum_{i=1}^2 \sum_{j=1}^4 \beta_j^2 |\lambda_{i,j}|^2)}{16}} \right) \\ &= \frac{1}{\pi} \int_0^{\pi/2} \exp \left( \frac{-\rho (\sum_{i=1}^2 \sum_{j=1}^4 \beta_j^2 |\lambda_{i,j}|^2)}{32 \sin^2 \alpha} \right) d\alpha. \end{aligned} \quad (19)$$

Applying a method based on the moment generating function [19], we obtain the unconditional PEP as:

$$P(\mathbf{d} \rightarrow \bar{\mathbf{d}}) = \frac{1}{\pi} \int_0^{\pi/2} \left[ \prod_{i=1}^4 \left( 1 + \frac{\rho \beta_i^2}{8 \sin^2 \alpha} \right) \right]^{-2} d\alpha. \quad (20)$$

If  $\beta_i \neq 0 \forall i = 1, \dots, 4$ , then  $1 + \frac{\rho \beta_i^2}{8 \sin^2 \alpha} \approx \frac{\rho \beta_i^2}{8 \sin^2 \alpha}$  at high SNR, the approximation of the exact PEP in (20) is

$$\begin{aligned} P(\mathbf{d} \rightarrow \bar{\mathbf{d}}) &\approx \left( \frac{2^{24} \rho^{-8}}{\pi} \int_0^{\pi/2} (\sin \alpha)^{16} d\alpha \right) \prod_{i=1}^4 |\beta_i|^{-4} \\ &= \frac{2^7 16! \rho^{-8}}{8! 8!} \prod_{i=1}^4 |\beta_i|^{-4}. \end{aligned} \quad (21)$$

The exponent of SNR in (21) is -8. This indicates that the maximum diversity order of 4Gp-QSTBC is 8 and it is achievable if the product distance  $\prod_{i=1}^4 \beta_i$  (see [21] and references therein) is nonzero for all possible data vectors. Furthermore, at high SNR, the asymptotic PEP becomes very tight to the exact PEP. Recall that  $\beta = \Theta R(\mathbf{d} - \bar{\mathbf{d}})$ ; thus, the product matrix  $\Theta R$  is the combined rotation matrix for data vector  $\mathbf{d}$ . Since  $\Theta$  is a constant matrix, we can optimize the matrix  $R$  so that the minimum product distance  $d_{p,\min} = \min_{\mathbf{d} \neq \bar{\mathbf{d}}} \prod_{k=1}^4 |\beta_k|$ , where  $\beta = [\Theta R(\mathbf{d} - \bar{\mathbf{d}})]$  is nonzero and maximized.

If the complex signals are drawn from QAM, the (real) elements of  $\mathbf{d}$  are in the set  $\{\pm 1, \pm 3, \pm 5, \dots\}$ . The best known rotations for QAM in terms of maximizing the minimum product distance are provided in [21], [22]. Denoting the rotation matrix in [21], [22] by  $R_{BOV}$ , the signal rotation for our 4Gp-QSTBC is given by

$$R = \Theta R_{BOV}. \quad (22)$$

Simulations show that the above vector signal rotation perform better than the symbol-wise rotation proposed in [18] (details omitted for brevity). We have presented important properties of 4Gp-QSTBC. In the next section, we will investigate 4Gp-SAST codes.

#### IV. FOUR-GROUP DECODABLE STBC DERIVED FROM SAST CODES

##### A. Encoding

The SAST code matrix is constructed for  $M = 2\bar{M}$  Tx antennas using circulant blocks. Two length- $\bar{M}$  data vectors  $\mathbf{s}_1 = [s_1 \ s_2 \ \dots \ s_{\bar{M}}]^\top$  and  $\mathbf{s}_2 = [s_{\bar{M}+1} \ s_{\bar{M}+2} \ \dots \ s_{2\bar{M}}]^\top$  are used to generate two  $\bar{M}$ -by- $\bar{M}$  circulant matrices [13]. Note that the first row of circulant matrix  $\mathcal{C}(\mathbf{x})$  copies the row vector  $\mathbf{x}$ ; the  $i$ th row is obtained by circular shift  $(i-1)$  times to the right the vector  $\mathbf{x}$ . The SAST code matrix is constructed as

$$\mathcal{S} = \begin{bmatrix} \mathcal{C}(\mathbf{s}_1^\top) & \mathcal{C}(\mathbf{s}_2^\top) \\ -\mathcal{C}^\dagger(\mathbf{s}_2^\top) & \mathcal{C}^\dagger(\mathbf{s}_1^\top) \end{bmatrix}. \quad (23)$$

By construction, 4Gp-SAST codes have rate of one symbol pccu; the code matrices for an even number of Tx antennas are square; thus 4Gp-SAST codes are delay-optimal for even number of Tx antennas.

##### B. Decoder of 4Gp-SAST codes

Similar to 4Gp-QSTBC, the decoding of 4Gp-SAST codes requires two steps. First, the two data vectors  $\mathbf{s}_1$  and  $\mathbf{s}_2$  are decoupled [10]; then, the real and imaginary parts of vectors  $\mathbf{s}_1$  and  $\mathbf{s}_2$  are separated. We provide the detail decoder with only one Rx antenna as generalization for multiple Rx antennas can be easily done.

We introduce another type of circulant matrix called left circulant, denoted by  $\mathcal{C}_L(\mathbf{x})$ , where the  $i$ th row is obtained by circular shifts  $(i-1)$  times to the left for the row vector  $\mathbf{x}$ .

Let us define a permutation  $\Pi$  on an arbitrary  $M \times M$  matrix  $X$  such that, the  $(M-i+2)$ th row is permuted with the  $i$ th row for  $i = 2, 3, \dots, \lceil \frac{M}{2} \rceil$ , where  $\lceil \cdot \rceil$  is the ceiling function. One can verify that

$$\Pi(\mathcal{C}_L(\mathbf{x})) = \mathcal{C}(\mathbf{x}). \quad (24)$$

Let  $\mathbf{y} = [\mathbf{y}_1^\top \ \mathbf{y}_2^\top]^\top$ ,  $\mathbf{y}_1 = [y_1 \ y_2 \ \dots \ y_{\bar{M}}]^\top$ ,  $\mathbf{y}_2 = [y_{\bar{M}+1} \ y_{\bar{M}+2} \ \dots \ y_M]^\top$ ,  $\mathbf{h} = [\mathbf{h}_1^\top \ \mathbf{h}_2^\top]^\top$ ,  $\mathbf{h}_1 = [h_1 \ h_2 \ \dots \ h_{\bar{M}}]^\top$ ,  $\mathbf{h}_2 = [h_{\bar{M}+1} \ h_{\bar{M}+2} \ \dots \ h_{2\bar{M}}]^\top$ ,  $\mathbf{z} = [\mathbf{z}_1^\top \ \mathbf{z}_2^\top]^\top$ ,  $\mathbf{z}_1 = [z_1 \ z_2 \ \dots \ z_{\bar{M}}]^\top$ ,  $\mathbf{z}_2 = [z_{\bar{M}+1} \ z_{\bar{M}+2} \ \dots \ z_{2\bar{M}}]^\top$ . We can write the Tx-Rx signal relation as

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \sqrt{\frac{\rho}{M}} \begin{bmatrix} \mathcal{C}(\mathbf{s}_1) & \mathcal{C}(\mathbf{s}_2) \\ -\mathcal{C}^\dagger(\mathbf{s}_2) & \mathcal{C}^\dagger(\mathbf{s}_1) \end{bmatrix} \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix}. \quad (25)$$

An equivalent form of (25) is

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2^* \end{bmatrix} = \sqrt{\frac{\rho}{M}} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2^* \end{bmatrix} \quad (26)$$

where  $X_1 = \mathcal{C}_L(\mathbf{h}_1^\top)$ ,  $X_2 = \mathcal{C}_L(\mathbf{h}_2^\top)$ ,  $X_3 = \mathcal{C}^\dagger(\mathbf{h}_2^\top)$ ,  $X_4 = -\mathcal{C}^\dagger(\mathbf{h}_1^\top)$ .

Applying permutation  $\Pi$  in (24) for the column matrix  $\mathbf{y}_1$ , we obtain

$$\begin{aligned} \begin{bmatrix} \bar{\mathbf{y}}_1 \\ \bar{\mathbf{y}}_2 \end{bmatrix} &\triangleq \begin{bmatrix} \Pi(\mathbf{y}_1) \\ \mathbf{y}_2^* \end{bmatrix} \\ &= \sqrt{\frac{\rho}{M}} \begin{bmatrix} \Pi(X_1) & \Pi(X_2) \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{bmatrix} + \begin{bmatrix} \Pi(\mathbf{z}_1) \\ \mathbf{z}_2^* \end{bmatrix} \\ &= \sqrt{\frac{\rho}{M}} \underbrace{\begin{bmatrix} H_1 & H_2 \\ H_2^\dagger & -H_1^\dagger \end{bmatrix}}_{\mathcal{H}} \begin{bmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{z}}_1 \\ \bar{\mathbf{z}}_2 \end{bmatrix} \end{aligned} \quad (27)$$

where  $H_1 = \mathcal{C}(\mathbf{h}_1^\top)$ ,  $H_2 = \mathcal{C}(\mathbf{h}_2^\top)$ ,  $\bar{\mathbf{z}}_1 = \Pi(\mathbf{z}_1)$ ,  $\bar{\mathbf{z}}_2 = \mathbf{z}_2^*$ . The elements of  $\bar{\mathbf{z}}_1$  and  $\bar{\mathbf{z}}_2$  are  $\sim \mathcal{CN}(0, 1)$ , as elements of  $\mathbf{z}_1$  and  $\mathbf{z}_2$ . We now multiply  $\mathcal{H}^\dagger$  with both sides of (27). Let  $\hat{\mathcal{H}} = H_1^\dagger H_1 + H_2^\dagger H_2$ , we get

$$\begin{aligned} \begin{bmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \end{bmatrix} &= \mathcal{H}^\dagger \begin{bmatrix} \bar{\mathbf{y}}_1 \\ \bar{\mathbf{y}}_2 \end{bmatrix} = \sqrt{\frac{\rho}{M}} \begin{bmatrix} \hat{\mathcal{H}} & \mathbf{0}_{\bar{M}} \\ \mathbf{0}_{\bar{M}} & \hat{\mathcal{H}} \end{bmatrix} \begin{bmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{bmatrix} + \mathcal{H}^\dagger \begin{bmatrix} \bar{\mathbf{z}}_1 \\ \bar{\mathbf{z}}_2 \end{bmatrix} \\ &= \sqrt{\frac{\rho}{M}} \begin{bmatrix} \hat{\mathcal{H}} & \mathbf{0}_{\bar{M}} \\ \mathbf{0}_{\bar{M}} & \hat{\mathcal{H}} \end{bmatrix} \begin{bmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \hat{\mathbf{z}}_1 \\ \hat{\mathbf{z}}_2 \end{bmatrix}}_{\hat{\mathbf{z}}}. \end{aligned} \quad (28)$$

The covariance matrix of the additive noise vector  $\hat{\mathbf{z}}$  is  $E[\hat{\mathbf{z}}\hat{\mathbf{z}}^\dagger] = \begin{bmatrix} \hat{\mathcal{H}} & \mathbf{0}_{\bar{M}} \\ \mathbf{0}_{\bar{M}} & \hat{\mathcal{H}} \end{bmatrix}$ . Therefore, the noise vectors  $\hat{\mathbf{z}}_1$  and  $\hat{\mathbf{z}}_2$  are uncorrelated and have the same covariance matrix  $\hat{\mathcal{H}}$ . Thus  $\mathbf{s}_1$  and  $\mathbf{s}_2$  can be decoded separately using  $\hat{\mathbf{y}}_i = \hat{\mathcal{H}}\mathbf{s}_i + \hat{\mathbf{z}}_i$ ,  $i = 1, 2$ . The noise vectors  $\hat{\mathbf{z}}_1$  and  $\hat{\mathbf{z}}_2$  can be whitened by the same whitening matrix  $\hat{\mathcal{H}}^{-1/2}$ . The equivalent equations for Tx-Rx signals are

$$\hat{\mathcal{H}}^{-1/2}\hat{\mathbf{y}}_i = \sqrt{\rho/\bar{M}}\hat{\mathcal{H}}^{1/2}\mathbf{s}_i + \hat{\mathcal{H}}^{-1/2}\hat{\mathbf{z}}_i, \quad i = 1, 2. \quad (29)$$

At this point, the decoding of SAST codes becomes the detection of 2 group of complex symbols  $\mathbf{s}_i$  ( $i = 1, 2$ ); this is similar to the detection of 4Gp-QSTBC in (15). Our next step is to separate the real and imaginary parts of vectors  $\mathbf{s}_i$  to obtain 4 groups of symbols for data detection.

Recall that  $\hat{\mathcal{H}} = H_1^\dagger H_1 + H_2^\dagger H_2$ , and both  $H_1$  and  $H_2$  are circulant. Hence,  $\hat{\mathcal{H}}$  is also circulant [13]. Let  $\Lambda_i = [\lambda_{i,1} \ \lambda_{i,2} \ \dots \ \lambda_{i,m}]$  be the  $m$  eigenvalues of  $H_i$  ( $i = 1, 2$ ). We can diagonalize  $H_i$  by DFT matrix as  $H_i = \mathcal{F}^\dagger \Lambda_i \mathcal{F}$ . Thus  $\hat{\mathcal{H}} = \mathcal{F}^\dagger (\Lambda_1^\dagger \Lambda_1 + \Lambda_2^\dagger \Lambda_2) \mathcal{F}$ . Let  $\Lambda_1^\dagger \Lambda_1 + \Lambda_2^\dagger \Lambda_2 = \Lambda$ , then  $\Lambda$  has real and non-negative entries in the main diagonal and  $\hat{\mathcal{H}}^{1/2} = \mathcal{F}^\dagger \Lambda^{1/2} \mathcal{F}$  and  $\hat{\mathcal{H}}^{-1/2} = \mathcal{F}^\dagger \Lambda^{-1/2} \mathcal{F}$ .

We assume that  $\mathbf{s}_i$  is pre-multiplied (or rotated) by an IDFT matrix  $\mathcal{F}^\dagger$  of proper size. Substituting  $\mathbf{s}_i$  by  $\mathcal{F}^\dagger \mathbf{s}_i$  and multiplying both sides of (29) with the DFT matrix  $\mathcal{F}$ , we obtain

$$\begin{aligned} \Lambda^{-1/2} \mathcal{F} \hat{\mathbf{y}}_i &= \sqrt{\rho/\bar{M}} \mathcal{F} \hat{\mathcal{H}}^{1/2} \mathcal{F}^\dagger \mathbf{s}_i + \Lambda^{-1/2} \mathcal{F} \hat{\mathbf{z}}_i \\ &= \sqrt{\rho/\bar{M}} \Lambda^{1/2} \mathbf{s}_i + \underbrace{\Lambda^{-1/2} \mathcal{F} \hat{\mathbf{z}}_i}_{\hat{\mathbf{z}}_i}. \end{aligned} \quad (30)$$

Since  $\Lambda^{1/2}$  is a real matrix, the real and imaginary parts of  $\mathbf{s}_i$  ( $i = 1, 2$ ) can now be separated for detection.

$$\Lambda^{-1/2} \Re(\mathcal{F} \hat{\mathbf{y}}_i) = \sqrt{\rho/\bar{M}} \Lambda^{1/2} \Re(\mathbf{s}_i) + \Re(\hat{\mathbf{z}}_i), \quad (31a)$$

$$\Lambda^{-1/2} \Im(\mathcal{F} \hat{\mathbf{y}}_i) = \sqrt{\rho/\bar{M}} \Lambda^{1/2} \Im(\mathbf{s}_i) + \Im(\hat{\mathbf{z}}_i). \quad (31b)$$

We finish deriving the general decoder for 4Gp-SAST codes. Using (31), one can use a sphere decoder to detect the transmitted symbols. The *equivalent channel* of 4Gp-SAST codes is  $\Lambda^{1/2}$ .

### C. Performance Analysis

Note that the eigenvalues of  $m \times m$  matrices  $H_1$  and  $H_2$  can be found easily using unnormalized DFT of the channel vectors  $\mathbf{h}_1$  and  $\mathbf{h}_2$  [13]. Therefore, the eigenvalues of  $H_1$  and  $H_2$  have distribution  $\sim \mathcal{CN}(0, m)$ .

Similar to the case of 4Gp-QSTBC, we can introduce a real orthogonal transformation  $R$  to the data vectors  $\Re(\mathbf{s}_i)$  and  $\Im(\mathbf{s}_i)$  ( $i = 1, 2$ ) to improve the performance of 4Gp-SAST codes. Thus the actual signal rotation of 4Gp-SAST codes is  $\mathcal{F}^\dagger R$ .

Since the PEP of vectors  $\Re(\mathbf{s}_i)$  and  $\Im(\mathbf{s}_i)$  ( $i = 1, 2$ ) are the same, we only calculate the PEP of the vector  $\Re(\mathbf{s}_1)$ . Let  $\mathbf{d} = \Re(\mathbf{s}_1)$ . The PEP of distinct vectors  $\mathbf{d}$  and  $\bar{\mathbf{d}}$  can be calculated in a similar manner to that of 4Gp-QSTBC in Section III-C. details are omitted for brevity. The PEP of 4Gp-SAST codes is given below.

$$P(\mathbf{d} \rightarrow \bar{\mathbf{d}}) = \frac{1}{\pi} \int_0^{\pi/2} \left[ \prod_{i=1}^m \left( 1 + \frac{\rho \beta_i^2}{8 \sin^2 \alpha} \right) \right]^{-2} d\alpha \quad (32)$$

where  $[\beta_1 \ \beta_2 \ \dots \ \beta_m]^\top = R(\mathbf{d} - \bar{\mathbf{d}})$ . One can find the asymptotic PEP of 4Gp-SAST codes at high SNR in a similar fashion to the case of 4Gp-QSTBC in (21) as follows.

$$\begin{aligned} P(\mathbf{d} \rightarrow \bar{\mathbf{d}}) &\approx \left( \frac{2^{6m} \rho^{-2m}}{\pi} \int_0^{\pi/2} (\sin \alpha)^{16} d\alpha \right) \prod_{i=1}^m \beta_i^{-4} \\ &= \frac{2^{6m} \rho^{-2m}}{2^{17}} \frac{16!}{8!8!} \prod_{i=1}^m \beta_i^{-4}. \end{aligned} \quad (33)$$

Thus, if the product distance  $\prod_{i=1}^m \beta_i$  is nonzero, 4Gp-SAST codes will achieve full-diversity. Similar to 4Gp-QSTBC, with QAM, the signal rotations  $R_{BOV}$  in [21], [22] can be used to minimize the worst-case PEP.

*Remark:* It is interesting to recognize that, the optimal rotation matrices of 4Gp-QSTBC ( $R = \Theta R_{BOV}$ ) and 4Gp-SAST codes ( $R = \mathcal{F} R_{BOV}$ ) have a similar formula. The precoding matrices  $\Theta$  and  $\mathcal{F}$  are added to diagonalize the channels of the two codes. Thus each real symbol is equivalently transmitted in a separate channel, but full diversity is not achievable. The real rotation matrix  $R_{BOV}$  is applied to the data vectors so that the real symbols are spread over all the channels, and thus full diversity is achievable.

## V. SIMULATION RESULTS

Simulation results are presented in Fig. 1 to compare the performances of 4Gp-QSTBC and 4Gp-SAST codes with OSTBC, MDC-QSTBC [12], QSTBC [6], and SAST codes [10] for 6 Tx and 1 Rx antennas. To produce the desired bit rates, two 8QAM constellations are used. The first constellation is rectangular, denoted by 8QAM-R, and has signal points  $\{\pm 1 \pm j, \pm 3 \pm j\}$ . The other constellation, denoted by 8QAM-S, has the best minimum Euclidean distance; its geometrical shape is depicted in [6, Fig. 2(c)].

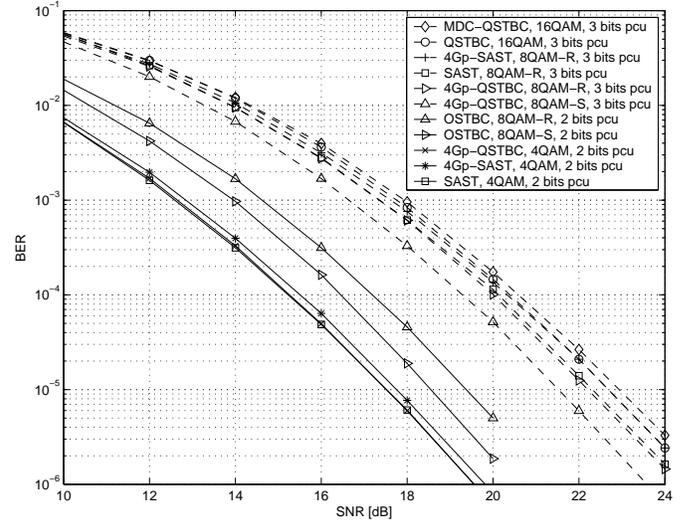


Fig. 1. Performances of 4Gp-QSTBC and 4Gp-SAST codes compared with OSTBC, MDC-QSTBC, QSTBC and SAST codes, 6 Tx and 1 Rx antennas, 2 and 3 bits pcu.

We compare the performance of our new codes with OSTBC and SAST codes for a spectral efficiency of 2 bits pcu. To get this bit rate, 8QAM signals are combined with rate-2/3 OSTBC, while 4QAM is used for the SAST, 4Gp-QSTBC and 4Gp-SAST codes. Two columns (4 and 8) of 4Gp-QSTBC for 8 Tx antennas is deleted to create the code for 6 Tx antennas. From Fig. 1, 4Gp-SAST codes gains 0.8 and 1.6 dB over OSTBC with 8QAM-S and 8QAM-R, respectively, while the decoding complexity slightly increases (see Table I). The performance improvement of 4Gp-QSTBC is even better, 1 dB compared with OSTBC (using 8QAM-S) and 0.2 dB compared with 4Gp-SAST codes. Note that for 6 antennas, the decoding complexity of 4Gp-QSTBC is slightly higher than that of 4Gp-SAST codes (see Table I).

In Fig. 1, the performance of 4Gp-QSTBC and 4Gp-SAST codes with 3 bits pcu is also compared with that of the rate-3/4 QSTBC and MDC-QSTBC (using 16QAM). 4Gp-SAST code yields a 0.3 dB improvement over MDC-QSTBC and performs the same as QSTBC. Specifically, 4Gp-QSTBC using 8QAM-S performs much better than the QSTBC; it produces a 1.2 dB gain over QSTBC with the same decoding complexity.

Further simulations for 5 and 8 Tx antennas also confirm that 4Gp-QSTBC and 4Gp-SAST codes perform better than OSTBC, MDC-QSTBC, QSTBC, and SAST codes. Due to the lack of space, we omit the details.

## VI. CONCLUSIONS

We have presented two new rate-one STBC with four-group decoding, called 4Gp-QSTBC and 4Gp-SAST codes. They offer the lowest decoding complexity compared with the existing rate-one STBC. Their closed-form PEP are derived, enabling the optimization of signal rotations. Compared with other existing low decoding complexity STBC (such as OSTBC, MDC-QSTBC, CIOD, and QSTBC), our newly designed STBC have several additional advantages including higher code rate, better BER performance, lower encoding/decoding

delay, and lower peak-to-average power ratio (PAPR) because zero-amplitude symbols are avoided in the code matrices. Recent results in [23] present a flexible design of multi-group STBC. However, the code rate is still limited by 1 symbol p.c.u. Thus, the systematic design of high-rate multi-group STBC is still an open research problem.

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