New Blind Block Synchronization for Transceivers Using Redundant Precoders

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***Abstract*—This paper studies the blind block synchronization problem in block transmission systems using linear redundant precoders (LRP). Two commonly used LRP systems, namely, zero padding (ZP) and cyclic prefix (CP) systems, are considered in this paper. In particular, the block synchronization problem in CP systems is a broader version of timing synchronization problem in the popular orthogonal frequency division multiplexing (OFDM) systems. The proposed algorithms exploit the rank deficiency property of the matrix composed of received blocks when the block synchronization is perfect and use a parameter called repetition index which can be chosen as any positive integer. Theoretical results suggest advantages in blind block synchronization performances when using a large repetition index. Furthermore, unlike previously reported algorithms, which require a large amount of received data, the proposed methods, with properly chosen repetition indices, guarantee correct block synchronization in absence of noise using only two received blocks in ZP systems and three in CP systems. Computer simulations are conducted to evaluate the performances of the proposed algorithms and compare them with previously reported algorithms. Simulation results not only verify the capability of the proposed algorithms to work with limited received data but also show significant improvements in the block synchronization error rate performance of the proposed algorithms over previously reported algorithms.**

***Index Terms*—Blind block synchronization, blind identification, cyclic prefix, frame synchronization, OFDM, repetition index, single-carrier cyclic prefix (SC-CP), zero padding.**

I. Introduction

BLIND channel estimation in block transmission systems using linear redundant filter bank precoders (LRP) has been studied extensively in the literature [1]–[5], [7], [13]–[15]. Besides a constant bandwidth overhead introduced in each block, a blind channel estimation method usually requires very little extra bandwidth to perform channel estimation. Most existing blind estimation methods for LRPs assume that block boundaries of the received streams are perfectly known to the receiver. In practical applications, however, this assumption is usually not true since no extra known samples are transmitted. The problem of blind recovery of block boundaries of the received signal is therefore important. However, up to date, the problem of blind block synchronization has not yet been given as much attention as the blind channel estimation problem has. In this paper, we consider the problem with two commonly used precoders, namely, the zero padding (ZP) and cyclic prefix (CP) precoders. For ZP systems, the first blind block synchronization algorithm was proposed by Scaglione *et al.* [1]. The blind synchronization algorithm uses the rank deficiency property of a matrix composed of received samples which was first used in a blind equalization algorithm also proposed in [1]. The rank deficiency property of the aforementioned matrix is valid at perfect block synchronization but is no longer valid when a nonzero timing mismatch is present. The algorithm proposed in [1] shows that block synchronization algorithms can be connected with existing blind channel estimation/equalization algorithms that exploit matrix null spaces.

In recent years, more advanced blind channel estimation algorithms were developed, and these suggest more opportunities to develop new blind channel synchronization algorithms that may possess new features. One of the most important features in recently reported blind channel estimation algorithms is the reduction of the amount of received data needed for an accurate channel estimate which is favorable in a time-varying environment such as a wireless channel. For ZP systems, Manton *et al.* first pointed out that blind channel estimation can be done with fewer received blocks by repeated use of each block [2], [3]. This concept was later generalized by Su and Vaidyanathan [4], [5] using a parameter called *repetition index*. The feature of using much less received data in the aforementioned blind channel estimation algorithms can also be properly transferred to blind synchronization algorithms if we adopt the concept of repetition index. The blind block synchronization algorithm for ZP systems proposed in this paper will explore this idea. Another novelty is that the proposed method for ZP systems is based on a subspace of dimension L rather than one as in [1] (where L is the channel order). This idea, combined with the repetition index, is shown to significantly improve the performance with sufficient amount of received data.

As more and more new communication standards adopt cyclic-prefix based systems such as orthogonal frequency division multiplexing (OFDM) and single carrier cyclic prefix (SC-CP) systems, the importance of studying CP systems is increasing. The blind block synchronization problem studied

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in this paper is a broader version of the timing synchronization or the symbol synchronization problem in OFDM systems. A number of blind block synchronization algorithms for OFDM systems have been developed [11], [16]–[19]. In particular, in [11], Negi and Cioffi proposed the first blind OFDM symbol synchronization for frequency selective channels. These previously reported methods, however, require a large amount of received data to obtain accurate statistics for successful block synchronization. Our approach to reduce the required amount of received data resorts to employing the idea of repetition index. As the idea of repetition index was recently extended to blind channel estimation in CP systems [10], we propose a new blind block synchronization algorithms in CP systems based on the foundation of [10]. Our proposed algorithm possesses two advantages over the previously reported methods: 1) In absence of noise, the proposed algorithm provides correct recovery of block boundaries using only three received blocks whereas all previously reported algorithms require the number of received blocks to be no less than the block size, and 2) when the noise is present, simulation results as reported in Section VI show that given the same amount of received data, the proposed algorithm has an obvious improvement in blind block synchronization error rate performance over the previously reported algorithm in [11].

The rest of the paper will be organized as follows. In Section II, the problems of interest, namely the blind block synchronization problems in ZP and CP systems, respectively, will be formulated. The notations used in the paper will also be defined. In Sections III and IV, the proposed blind block synchronization algorithms in ZP and CP systems, as well as their theoretical foundations, will be presented, respectively. In Section V, we study the practical issues of the proposed algorithms including analysis of computational complexity and system performance in the presence of noise. In Section VI, simulation results are provided to evaluate the system performances of the proposed algorithms and to compare them with those of previously reported algorithms. Finally, the conclusions are made in Section VII. Some parts of this paper have been presented in conferences: the proposed algorithm in ZP system was presented at ICASSP 2007 [12] and the proposed algorithm in CP systems was presented in ISCAS 2008 [6].

II. Problem Formulation

A. Notations

Boldfaced lower case letters represent column vectors. Boldfaced upper case letters are reserved for matrices. Superscripts *, ^{T}, and † as in a^*, $\mathbf{A}^T$, and $\mathbf{A}^\dagger$ denote the conjugate, transpose, and transpose-conjugate operations, respectively. All the vectors and matrices in this paper are complex-valued. $\mathbf{I}_n$ is the $n \times n$ identity matrix, and $\mathbf{0}_{m\times n}$ is the $m \times n$ zero matrix. If $\mathbf{v} = [v_1\ v_2 \cdots v_m]^T$ is an $m \times 1$ vector, we use $\mathcal{T}_n(\mathbf{v})$ to denote the $(m+n-1) \times n$ *full-banded Toeplitz matrix* [8] whose first column is $[\mathbf{v}^T\ \mathbf{0}_{1\times(n-1)}]^T$ and whose first row is $[v_1\ \mathbf{0}_{1\times(n-1)}]$. In figures, "$\uparrow N$" and "$\downarrow N$" denote the signal downsampler and upsampler, respectively [9].

Due to special properties of cyclic prefixes, we will use the following notation extensively in this paper. Suppose $\mathbf{y}$ is an $m \times 1$ column vector $\mathbf{y} = [y_1\ y_2 \cdots y_m]^T$. Then the notation $[\mathbf{y}]_{a:b}$ denotes the $(b-a+1) \times 1$ vector

$$[\mathbf{y}]_{a:b} = [y_a \quad y_{a+1} \quad \cdots \quad y_b]^T$$

if $1 \le a \le b \le m$. An extension of this definition to any arbitrary pair of integers a and b satisfying $a \le b$ is made by defining y_k as $y_{(k-1 \bmod m)+1}$ for any $k > m$ or $k < 1$. For example, if $\mathbf{y} = [y_1\ y_2\ y_3]^T$, $a = -1$, and $b = 7$, then $[\mathbf{y}]_{a:b}$ denotes the vector $[y_2\ y_3\ y_1\ y_2\ y_3\ y_1\ y_2\ y_3\ y_1]^T$. If $a > b$, then $[\mathbf{y}]_{a:b}$ denotes an empty vector.

B. Redundant Block Transmission Systems

Fig. 1 shows the structure of a block transmission system. The data samples, $s(n)$, are blocked into vectors $\mathbf{s}(n)$ of size M. Let L be a positive integer indicating the redundancy inserted in each block and assume $M > L$. The precoded vector, $\mathbf{u}(n)$, of size $P = M + L$, is obtained by multiplying a $P \times M$ matrix $\mathbf{R}$ with $\mathbf{s}(n)$. The vector sequence $\mathbf{u}(n)$ is then unblocked into a scalar sequence $u(n)$ before being sent over the channel. The channel is characterized as an finite-impulse-response (FIR) system with a maximum order L, i.e.,

$$H(z) = \sum_{k=0}^{L_0} h_k z^{-k}$$

where $L_0 \le L$. Assume h_0 and h_{L_0} are nonzero. Define $\mathbf{h}$ as the $(L+1)$-vector

$$[h_0 \quad h_1 \quad \cdots \quad h_L]^T$$

where the values of h_k are set to zeros for any k, $L_0 < k \le L$. The integer L_0 is called the effective channel order. The signal at the channel output is corrupted by an additive white Gaussian noise $e(n)$. At the receiver side, the received sample stream $y(n)$ is blocked into vectors $\mathbf{y}(n)$ of size P. An equalizer, characterized by an $M \times P$ matrix $\mathbf{E}$, is used to recover the data blocks $\mathbf{s}(n)$.

While the redundant precoder can be designed as any rank-M matrix $\mathbf{R}$, we consider specifically two commonly used classes of LRPs in this paper: the zero-padding (ZP) precoders and the cyclic prefixing (CP) precoders. A block transmission system using a ZP or CP precoder is called a ZP system or an CP system, respectively.

In a ZP system, the matrix $\mathbf{R}$ has the form of

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{\text{zp}} \\ \mathbf{0}_{L\times M} \end{bmatrix}$$

where $\mathbf{R}_{\text{zp}}$ is an $M \times M$ nonsingular matrix. Each precoded block $\mathbf{u}(n)$ is composed of a data part of length M followed by a zero block of length L. Due to trailing zero introduced in each block at the transmitter, each received block $\mathbf{y}(n)$ can be expressed as [1]

$$\mathbf{y}(n) = \mathcal{T}_M(\mathbf{h})\mathbf{R}_{\text{zp}}\mathbf{s}(n) + \mathbf{e}(n) \tag{1}$$

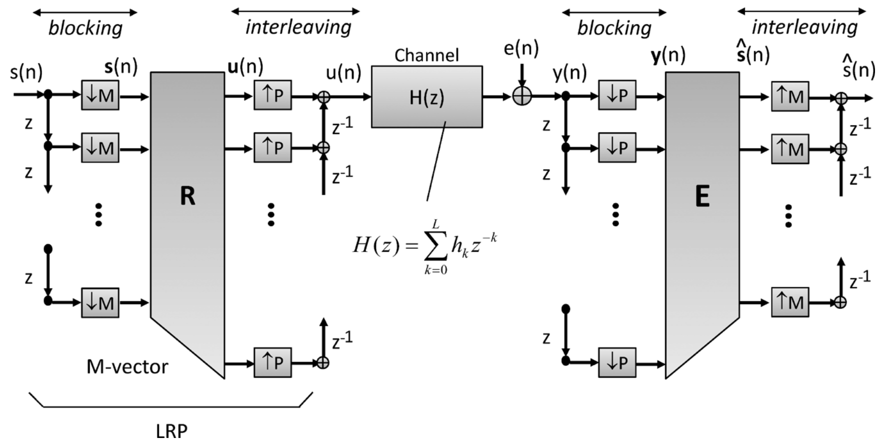

Fig. 1. Block transmission systems using linear redundant precoders.

where $\mathbf{e}(n)$ is the blocked version of $e(n)$. The notation $\mathcal{T}(\cdot)$ denotes a Toeplitz matrix and was defined in Section II-A. Note that $\mathbf{y}(n)$ depends only on $\mathbf{s}(n)$ and not on $\mathbf{s}(k)$ where $k \neq n$, so the interblock interference (IBI) is completely eliminated.

In a CP system, the precoder matrix $\mathbf{R}$ has the form of

$$\mathbf{R} = \left[\begin{array}{c|c} \mathbf{0}_{L\times(M-L)} & \mathbf{I}_L \\ \hline \multicolumn{2}{c}{\mathbf{I}_M} \end{array}\right] \cdot \mathbf{R}_{\rm cp}$$

where $\mathbf{R}_{\rm cp}$ is an $M \times M$ nonsingular matrix. Each precoded block $\mathbf{u}(n)$ is composed of a cyclic prefix $\mathbf{u}_{\rm cp}(n)$ of length L followed by the precoded data $\mathbf{u}_M(n) = \mathbf{R}_{\rm cp}\mathbf{s}(n)$ of length M. The cyclic prefix is a copy of the last L elements of the precoded data (i.e., $\mathbf{u}_{\rm cp}(n) = [\mathbf{u}_M(n)]_{M-L+1:M}$). Each received block can be expressed as

$$\mathbf{y}(n) = \begin{bmatrix} \mathbf{y}_{\rm cp}(n) \\ \mathbf{y}_M(n) \end{bmatrix} = \left[\begin{array}{c} \mathbf{H}_l\mathbf{u}_{\rm cp}(n) + \mathbf{H}_u\mathbf{u}_{\rm cp}(n-1) \\ \hline \mathbf{H}_{\rm cir}\mathbf{u}_M(n) \end{array}\right] \quad (2)$$

where $\mathbf{H}_{\rm cir}$ is an $M \times M$ circulant matrix [9] whose first column is $[h_0 \cdots h_L \ 0 \cdots 0]^T$

$$\mathbf{H}_l \triangleq \begin{bmatrix} h_0 & & \mathbf{0} \\ \vdots & \ddots & \\ h_{L-1} & \cdots & h_0 \end{bmatrix} \text{ and } \mathbf{H}_u \triangleq \begin{bmatrix} h_L & \cdots & h_1 \\ & \ddots & \vdots \\ \mathbf{0} & & h_L \end{bmatrix}$$

are $L \times L$ matrices. We can see that $\mathbf{y}_{\rm cp}(n)$, the CP part of $\mathbf{y}(n)$, contains IBI but $\mathbf{y}_M(n)$ is free from IBI. In particular, when $\mathbf{R}_{\rm cp}$ is chosen as the normalized inverse DFT matrix, the CP system is equivalent to the popular OFDM system.

C. Blind Block Synchronization for LRP Systems

Fig. 2 illustrates the precoded sample stream $u(n)$ and the received sample stream $y(n)$ of a ZP and a CP system, respectively. The dashed lines shown in the precoded sample streams and received sample streams depict the block boundaries. While the block boundaries are easy to trace in precoded sample streams $u(n)$ by recognizing the zero part or the cyclic prefix part, there does not seem to exist a clear rule of thumb to determine by inspection the block boundaries in the received sample streams $y(n)$ in either ZP or CP systems, as the signal has been convolved with the frequency selective channel $H(z)$. Furthermore, there is additive channel noise. A more sophisticated method must be employed to recover the block boundaries in received signals.

When the block synchronization is perfect between the transmitter and the receiver, the nth received block $\mathbf{y}(n)$ is

$$\mathbf{y}(n) = [y(nP) \quad y(nP+1) \quad \cdots \quad y(nP+P-1)]^T.$$

Suppose the blocking was performed with an unknown timing mismatch $d \in [-P/2, P/2)$ between the transmitter and the receiver, then the samples collected in the nth block will be

$$\mathbf{y}^{(d)}(n) = [y(nP+d) \quad y(nP+d+1) \quad \cdots \quad y(nP+d+P-1)]^T.$$

The problem statement of this paper is explained as follows. In a ZP or CP system as described in Section II-B, given the received sample stream $y(n)$, with a possible unknown timing mismatch to the transmitter, how do we determine the optimal $d \in [-P/2, P/2)$ that represents the starting index of a received block without knowledge of $\mathbf{s}(n)$? Note that since the OFDM systems are a special case of CP systems, the block synchronization problem in CP systems is a broader version of the "timing synchronization" problem or the "symbol synchronization" problem in OFDM systems. Without loss of generality and for convenience of the presentation, we assume the "correct answer" is always $d = 0$. Furthermore, when the effective channel order L_0 is strictly smaller than the guard interval length L (i.e., trailing zeros or cyclic prefixes), we observe that

$$d = -L + L_0, -L + L_0 + 1, \ldots, 0 \quad (3)$$

can all be considered "correct answers" since we can think of the equivalent channel vector as

$$\mathbf{h}^{(d)} = \begin{bmatrix}\mathbf{0}_{1\times(-d)} & h_0 & \cdots & h_{L_0} & \mathbf{0}_{1\times(L-L_0+d)}\end{bmatrix}^T$$

in this case. No interblock interference will occur due to a timing-mismatch d, $-L + L_0 \leq d \leq 0$. If the redundancy is minimal, i.e., $L = L_0$, then the only choice is $d = 0$.

III. Proposed Algorithm for ZP Systems

The basic idea of the proposed blind block synchronization algorithm for ZP systems stems from (1). Notice that

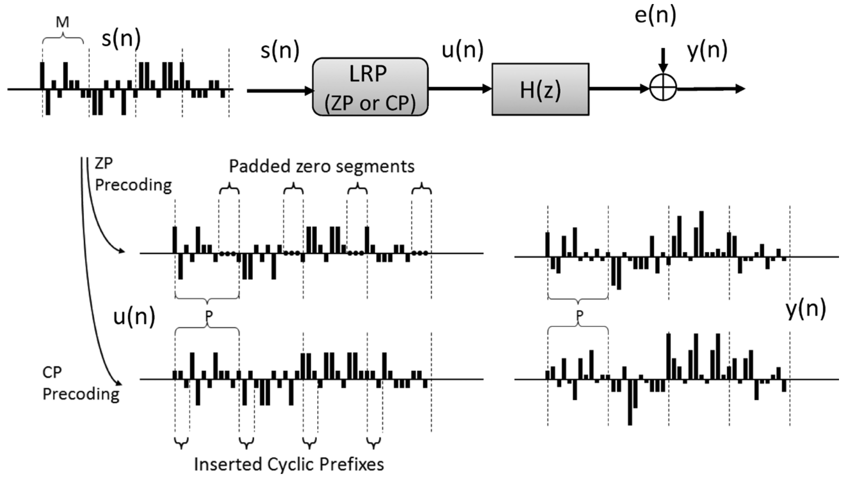

Fig. 2. Illustration of blind block synchronization problem in ZP and CP systems.

$\mathcal{T}_M(\mathbf{h})\mathbf{R}_{\text{zp}}$ is a $P \times M$ matrix whose rank is M. In absence of noise, each received block, $\mathbf{y}(n)$, must be a linear combination of columns of $\mathcal{T}_M(\mathbf{h})\mathbf{R}_{\text{zp}}$ when the block synchronization is perfect. In other words, a $P \times J$ matrix $\mathbf{Y}_J$ composed of J received blocks,

$$\mathbf{Y}_J \triangleq [\mathbf{y}(0) \quad \mathbf{y}(1) \quad \cdots \quad \mathbf{y}(J-1)] \tag{4}$$

must have rank M for some sufficiently large $J \geq M$. This implies that $\mathbf{Y}_J$ has a L-dimensional null space if the block synchronization is perfect. This property has been exploited in the blind channel estimation algorithm reported first in [1]. On the other hand, the matrix

$$\mathbf{Y}_J^{(d)} \triangleq \left[\mathbf{y}^{(d)}(0) \quad \mathbf{y}^{(d)}(1) \quad \cdots \quad \mathbf{y}^{(d)}(J-1)\right]$$

usually has a larger rank when $d \neq 0$ than when $d = 0$ (this will be verified later). The task of blind block synchronization can thus be completed by finding an optimal $d \in [-P/2, P/2)$ such that the rank deficiency of $\mathbf{Y}_J^{(d)}$ is L. In the presence of noise, the optimal d can be chosen such that the sum of the smallest L eigenvalues of $\mathbf{Y}_J^{(d)}\mathbf{Y}_J^{(d)\dagger}$ is minimized. It should be noted that this technique is different from the one reported in [1] and reviewed in Section III-B.

However, the blind block synchronization technique requires the condition that $J \geq M$ as the blind channel estimation algorithm in [1] needs it to satisfy certain full rank condition. This means the receiver has to accumulate at least MP data samples in order to determine the block boundary correctly. In a fast-varying environment such as a wireless channel, we usually do not have the luxury to collect so many samples since the channel status may have changed significantly during the data accumulation.

In order to reduce the amount of required data, we will use the idea of "repetition index," which first arose in a blind channel estimation problem [4]. The idea of repetition index is to repeatedly use each received block and has successfully reduced the number of received blocks needed for blind channel estimation problem in ZP and CP systems, as reported in [3], [4], and [10]. By properly applying this idea, we can develop blind block synchronization algorithms using less data. We shall present here the application of repetition index in blind block synchronization problems and the readers interested in the blind channel estimation algorithms are referred to [3], [4], and [10].

A. Derivation of the Proposed Algorithm

Consider the noise-free case first. It can be verified [4] that (1) is equivalent to

$$\mathcal{T}_Q(\mathbf{y}(n)) = \mathcal{T}_{M+Q-1}(\mathbf{h})\mathcal{T}_Q(\mathbf{u}_M(n)) \tag{5}$$

where Q is any positive integer and $\mathbf{u}_M(n)$ denotes $\mathbf{R}_{\text{zp}}\mathbf{s}(n)$ in the context of ZP systems. The notation $\mathcal{T}_n(\cdot)$ was defined in Section II-A. Note that when $Q = 1$, (5) reduces to (1). When $Q > 1$, (5) is similar to (1) in the sense that $\mathcal{T}_{M+Q-1}(\mathbf{h})$ is also a full-banded Toeplitz matrix, except that the size of $\mathcal{T}_{M+Q-1}(\mathbf{h})$ is larger than that of $\mathcal{T}_M(\mathbf{h})$ by $Q-1$. The parameter Q is called the repetition index since each received block is repeatedly used Q times. Note that $\mathcal{T}_Q(\mathbf{y}(n))$ is a $(P+Q-1) \times Q$ matrix and each column of $\mathcal{T}_Q(\mathbf{y}(n))$ is a linear combination of columns of $\mathcal{T}_{M+Q-1}(\mathbf{h})$. This property leads us to develop a new blind block synchronization algorithm as follows.

Suppose J received blocks are available at the receiver. Consider the $(P+Q-1) \times JQ$ matrix

$$\mathbf{Y}_{J,Q} = [\mathcal{T}_Q(\mathbf{y}(0)) \quad \mathcal{T}_Q(\mathbf{y}(1)) \quad \cdots \quad \mathcal{T}_Q(\mathbf{y}(J-1))] \tag{6}$$

It is readily verified that

$$\mathbf{Y}_{J,Q} = \mathcal{T}_{M+Q-1}(\mathbf{h})\mathbf{U}_{J,Q}$$

where

$$\mathbf{U}_{J,Q} = [\mathcal{T}_Q(\mathbf{u}_M(0)) \;\; \mathcal{T}_Q(\mathbf{u}_M(1)) \;\; \cdots \;\; \mathcal{T}_Q(\mathbf{u}_M(J-1))] \tag{7}$$

is a $(M+Q-1) \times JQ$ matrix. A necessary (but not sufficient) condition for $\mathbf{U}_{J,Q}$ to have full rank $M+Q-1$ is $JQ \geq M+Q-1$, or

$$J \geq 1 + \frac{M-1}{Q}. \tag{8}$$

Inequality (8) gives a lower bound on the required number of received blocks of the proposed blind synchronization algorithm with respect to the repetition index Q. When $\mathbf{U}_{J,Q}$ has full rank $M+Q-1$, the rank of $\mathbf{Y}_{J,Q}$ is also $M+Q-1$, and therefore the rank deficiency of $\mathbf{Y}_{J,Q}$ is exactly L. On the other hand, when a nonzero timing error $d \in [-P/2, P/2)$ is present, the matrix

$$\mathbf{Y}_{J,Q}^{(d)} = \left[\mathcal{T}_Q\left(\mathbf{y}^{(d)}(0)\right) \; \mathcal{T}_Q\left(\mathbf{y}^{(d)}(1)\right) \; \cdots \; \mathcal{T}_Q\left(\mathbf{y}^{(d)}(J-1)\right)\right] \tag{9}$$

usually has strictly less than L zero eigenvalues, as verified in the following theorem.

Theorem 1: Consider the noise-free situation. Assume each channel coefficient h_k, $0 \leq k \leq L$, is an independent complex random variable with nonzero variances. Suppose J is sufficiently large and the transmitted signal is selected such that $\mathbf{U}_{J,Q}$ defined in (7) has full rank for all Q. Then with probability one the following statement on the matrix $\mathbf{Y}_{J,Q}^{(d)}$ defined in (9) is true.

$$\begin{aligned} &\text{The number of zero eigenvalues of } \mathbf{Y}_{J,Q}^{(d)} \mathbf{Y}_{J,Q}^{(d)\dagger} \\ &= (P+Q-1) - \operatorname{rank}\left(\mathbf{Y}_{J,Q}^{(d)} \mathbf{Y}_{J,Q}^{(d)\dagger}\right) \\ &= \begin{cases} L, & \text{if } d = 0 \\ \max\{L - |d| - 2(Q-1), 0\}, & \text{if } d \neq 0. \end{cases} \end{aligned}$$

The notation of "probability one" used here, as well as in the other theorems throughout the paper, means the probability that the realization of channel coefficients h_k makes the statement true is unity. [21].

Proof: See the Appendix. ■

Theorem 1 gives the foundation of the proposed algorithm for ZP systems. We determine the estimated block boundaries by finding the optimal d so that the rank deficiency of $\mathbf{Y}_{J,Q}^{(d)}$ is exactly L. When the block synchronization is perfect (i.e., $d = 0$), the rank deficiency of $\mathbf{Y}_{J,Q}^{(d)}$ is exactly L. When the amount of timing error $|d|$ increases, this value decreases gradually to zero. In particular, if $Q = 1$, the rank deficiency of $\mathbf{Y}_{J,Q}^{(d)}$ is $\max\{L - |d|, 0\}$. The decrease in the rank deficiency of $\mathbf{Y}_{J,Q}^{(d)}$ when $|d|$ increases is relatively smooth. When $Q \geq 2$, the rank deficiency of $\mathbf{Y}_{J,Q}^{(d)}$ has an abrupt decrease when $|d|$ increases from 0 to 1. Furthermore, if $Q \geq (L+1)/2$, the rank deficiency of $\mathbf{Y}_{J,Q}^{(d)}$ goes immediately to zero whenever a nonzero timing error is present. This sharper decrease in rank deficiency of $\mathbf{Y}_{J,Q}^{(d)}$ demonstrates the advantage of using a larger repetition index Q for blind block synchronization, as will be discussed in Section V-B.

In the case where noise is present, the calculation of rank deficiency of $\mathbf{Y}_{J,Q}^{(d)} \mathbf{Y}_{J,Q}^{(d)\dagger}$ is replaced by eigen-decomposition of $\mathbf{Y}_{J,Q}^{(d)} \mathbf{Y}_{J,Q}^{(d)\dagger}$. We now present the proposed blind block synchronization algorithm in ZP systems as follows.

Algorithm 1:

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1) Choose the repetition index $Q \geq 1$ and the number of received blocks $J \geq 2$ so that inequality (8) is satisfied.
2) Collect $(J+1)P$ consecutive received samples and form the matrix $\mathbf{Y}_{J,Q}^{(d)}$ as defined in (9) for each $d \in [-P/2, P/2)$.
3) Perform eigen-decomposition on the matrix $\mathbf{Y}_{J,Q}^{(d)} \mathbf{Y}_{J,Q}^{(d)\dagger}$ for each d and take the L smallest eigenvalues $\sigma_{L,(d)}^2 \geq \sigma_{L-1,(d)}^2 \geq \cdots \geq \sigma_{2,(d)}^2 \geq \sigma_{1,(d)}^2 \geq 0$.
4) Calculate the cost function

$$\lambda_{\mathrm{zp},Q}(d) := \sum_{k=1}^{L} \sigma_{k,(d)}^2 \tag{10}$$

and decide the estimated timing mismatch

$$\hat{d} = \arg \min_{-\frac{P}{2} \leq d < \frac{P}{2}} \lambda_{\mathrm{zp},Q}(d).$$

■

The use of a large Q has two major advantages. First of all, it requires less received data as suggested in inequality (8). If Q is selected sufficiently large (e.g., $Q = M-1$), J can be as small as 2. Secondly, the robustness of block boundary detection is potentially improved due to a smaller value of rank deficiency of $\mathbf{Y}_{J,Q}^{(d)}$ when $d \neq 0$ if a larger repetition index Q is used, as we have learned in Theorem 1. This will be discussed more in detail in Section V-B.

B. Comparison With an Earlier Algorithm

We review here a blind block synchronization algorithm proposed earlier by Scaglione, Giannakis, and Barbarossa in [1] (which we call the SGB method from now on) and compare it with Algorithm 1. Suppose J consecutive blocks are collected at the receiver with a timing mismatch of d samples. Let $\mathbf{J}_n$ denote an $n \times n$ square shift matrix

$$\mathbf{J}_n = \begin{bmatrix} \mathbf{0}^T & 0 \\ \mathbf{I}_{n-1} & \mathbf{0} \end{bmatrix}$$

and consider the $P \times JL$ matrix

$$\mathcal{Y}_{\mathrm{SGB}}^{(d)} \triangleq \left[\mathbf{Y}_J^{(d)} \quad \mathbf{J}_P \mathbf{Y}_J^{(d)} \quad \cdots \quad \mathbf{J}_P^{L-1} \mathbf{Y}_J^{(d)}\right]. \tag{11}$$

The following claim has been proved (as Theorem 4 in [1]) regarding the rank of $\mathcal{Y}_{\mathrm{SGB}}^{(d)} \mathcal{Y}_{\mathrm{SGB}}^{(d)\dagger}$. The effective channel length L_0 was defined in Section II-B.

Claim: Consider the noise-free situation and assume $L_0 = L$. Then $\mathcal{Y}_{\mathrm{SGB}}^{(d)} \mathcal{Y}_{\mathrm{SGB}}^{(d)\dagger}$ has full rank P when $d \neq 0$ and has rank $P-1$ when $d = 0$. ■

The block synchronization problem can thus be solved by finding the only d which makes the matrix $\mathcal{Y}_{\mathrm{SGB}}^{(d)} \mathcal{Y}_{\mathrm{SGB}}^{(d)\dagger}$ rank deficient. In practice when the noise is present, the cost function can be defined as

$$\lambda_{\mathrm{SGB}}(d) \triangleq \min\left\{\text{eigenvalues of } \mathcal{Y}_{\mathrm{SGB}}^{(d)} \mathcal{Y}_{\mathrm{SGB}}^{(d)\dagger}\right\}. \tag{12}$$

The optimal d can be chosen as

$$\hat{d} = \arg \min_{-\frac{P}{2} \leq d < \frac{P}{2}} \lambda_{\mathrm{SGB}}(d).$$

The matrix $\mathcal{Y}_{\mathrm{SGB}}^{(d)}$ defined in (11) was first proposed in [1] for blind direct channel equalization and was used in blind block synchronization. Note that when $d=0$, the matrix $\mathcal{Y}_{\rm SGB}^{(d)}$ happens to be a truncation of $\mathbf{Y}_{J,L}^{(d)}$ after a proper permutation of columns:

$$\mathcal{Y}_{\rm SGB}^{(0)} = \left[\mathbf{I}_P \quad \mathbf{0}_{P\times(L-1)}\right]\mathbf{Y}_{J,L}^{(0)}\mathcal{P} = \mathcal{H}\mathbf{U}_{J,L}\mathcal{P}$$

where $\mathcal{P}$ is a $JL\times JL$ permutation matrix, $\mathbf{U}_{J,L}$ is as defined in (7) with $Q=L$, and $\mathcal{H}$ is a $P\times(P-1)$ matrix defined as $[\mathbf{I}_P\ \mathbf{0}_{P\times(L-1)}]\mathcal{T}_{P-1}(\mathbf{h})$ (i.e., dropping the last $L-1$ rows of $\mathcal{T}_{P-1}(\mathbf{h})$). The SGB method exploits the property that the rank deficiency $\mathcal{Y}_{\rm SGB}^{(0)}$ is unity when $d=0$. In order to use this property properly, the matrix $\mathbf{U}_{J,L}$ must have full rank. This implies that $J\ge 1+(M-1)/L$, which is equivalent to the requirement of Algorithm 1 with $Q=L$. Of course the SGB method was not developed from the point of view of a repetition index, but the fact that $\mathcal{Y}_{\rm SGB}^{(d)}$ happens to be a truncation of $\mathbf{Y}_{J,L}^{(d)}$ suggests a potential performance degradation of the SGB algorithm from Algorithm 1 with $Q=L$. Indeed, this will be verified in the simulation results presented in Section VI.

IV. Proposed Algorithm for CP Systems

In this section we consider the blind block synchronization problem in CP systems. The block synchronization problem in CP systems is a broader version of the so-called "timing-synchronization" or "OFDM symbol synchronization." Here we will tackle this problem without knowledge of the transmitted blocks and exploit a rank deficiency property that has been observed in an existing blind channel estimation algorithm for CP systems [10]. Unlike in ZP systems, where each received block is free from IBI, a received block in CP systems, as indicated in (2), contains IBI in some part of it. This makes it difficult to express the received block as a linear combination of less than P linearly independent vectors, as we did in ZP systems. To overcome this problem, a concept of "composite block" composed of elements from two consecutive received blocks is employed, as described below.

A. Derivation of the Proposed Algorithm

The proposed approach to the blind block synchronization problem is derived from the blind channel estimation algorithm proposed in [10]. We first consider the situation where the noise is absent. Define a "composite block" whose elements are chosen from two consecutive received blocks:

$$\bar{\mathbf{y}}(n) = \left[\mathbf{y}_M(n-1)^T \quad \mathbf{y}_{\rm cp}(n)^T \quad \mathbf{y}_M(n)^T\right]^T.$$

It can be verified that [15]

$$\bar{\mathbf{y}}(n) = \tilde{\mathbf{H}}\tilde{\mathbf{u}}(n) \tag{13}$$

where

$$\tilde{\mathbf{H}} = \left[\begin{array}{c|c} \mathbf{H}_{\rm cir} & \mathbf{0}_{M\times M} \\ \hline [\mathbf{0}_{L\times(M-L)}|\mathbf{H}_u] & [\mathbf{0}_{L\times(M-L)}|\mathbf{H}_l] \\ \hline \mathbf{0}_{M\times M} & \mathbf{H}_{\rm cir} \end{array}\right]$$

$\tilde{\mathbf{u}}(n) = [\mathbf{u}_M(n-1)^T\ \mathbf{u}_M(n)^T]^T$, and $\mathbf{u}_M(n)$ denotes $\mathbf{R}_{\rm cp}\mathbf{s}(n)$ in the context of CP systems. Note that here $\tilde{\mathbf{H}}$ has a size of $(2M+L)\times 2M$. This means each composite block, $\bar{\mathbf{y}}(n)$, of size $2M+L$, is a linear combination of $2M$ columns of $\tilde{\mathbf{H}}$, and is always limited to a $2M$-dimension subspace. This special property, however, is no longer true when the block synchronization is not correct (this will be verified later). This observation constitutes the basic idea of the proposed method for blind block synchronization.

Furthermore, employing the idea of repetition index, each received composite block $\bar{\mathbf{y}}(n)$ can be reformulated into a Q-column matrix $\bar{\mathbf{Y}}_Q(n)$ as defined below.

$$\bar{\mathbf{Y}}_Q(n) = [\bar{\mathbf{y}}_{0,Q-1}(n) \quad \bar{\mathbf{y}}_{1,Q-2}(n) \quad \cdots \quad \bar{\mathbf{y}}_{Q-1,0}(n)]$$

where each column is a $(2M+L+Q-1)$-vector defined as

$$\bar{\mathbf{y}}_{kl}(n) = \begin{bmatrix} [\mathbf{y}_M(n-1)]_{-k+1:M} \\ \mathbf{y}_{\rm cp}(n) \\ [\mathbf{y}_M(n)]_{1:M+l} \end{bmatrix}$$

for $k,l=0,1,\ldots,Q-1$. When block synchronization between the transmitter and the receiver is perfect, it can be shown that [10]

$$\bar{\mathbf{Y}}_Q(n) = \bar{\mathbf{H}}_Q\bar{\mathbf{U}}_Q(n) \tag{14}$$

where $\bar{\mathbf{H}}_Q$ and $\bar{\mathbf{U}}_Q(n)$ are defined as follows:

$$\bar{\mathbf{H}}_Q = \begin{bmatrix} \mathbf{H}_{\rm cir} & & \mathbf{0}_{M\times(M+Q-1)} \\ \mathbf{0}_{(L+Q-1)\times(M-L)} & \mathcal{H}_{L+Q-1} & \mathbf{0}_{(L+Q-1)\times(M-L)} \\ \mathbf{0}_{M\times(M+Q-1)} & & \mathbf{H}_{\rm cir2} \end{bmatrix} \tag{15}$$

where $\mathbf{H}_{\rm cir2}$ is obtained by moving the first L rows of $\mathbf{H}_{\rm cir}$ to the bottom and $\mathcal{H}_{L+Q-1}$ is a $(L+Q-1)\times(2L+Q-1)$ Toeplitz matrix whose first row is $[h_L\cdots h_0\ 0\cdots 0]$ and whose first column is $[h_L\ 0\cdots 0]^T$. In (14)

$$\bar{\mathbf{U}}_Q(n) \triangleq [\bar{\mathbf{u}}_{0,Q-1}(n) \quad \bar{\mathbf{u}}_{1,Q-2}(n) \quad \cdots \quad \bar{\mathbf{u}}_{Q-1,0}(n)] \tag{16}$$

where

$$\bar{\mathbf{u}}_{kl}(n) \triangleq \begin{bmatrix} [\mathbf{u}_M(n-1)]_{-k+1:M} \\ [\mathbf{u}_M(n)]_{-L+1:M+l-L} \end{bmatrix}.$$

Note that $\bar{\mathbf{H}}_Q$ is a tall matrix with a size $(2M+Q+L-1)\times(2M+Q-1)$. So each column of $\bar{\mathbf{Y}}_Q(n)$ is limited to a $(2M+Q-1)$-dimension subspace. Also note that when $Q=1$, (14) reduces to (13). Now, consider J consecutive received blocks $\mathbf{y}(n)$, $n=0,1,\ldots,J-1$ and the $(2M+L+Q-1)\times(J-1)Q$ matrix

$$\bar{\mathbf{Y}}_{J,Q} = \left[\bar{\mathbf{Y}}_Q(1) \quad \bar{\mathbf{Y}}_Q(2) \quad \cdots \quad \bar{\mathbf{Y}}_Q(J-1)\right]. \tag{17}$$

It is readily verified that

$$\bar{\mathbf{Y}}_{J,Q} = \bar{\mathbf{H}}_Q\bar{\mathbf{U}}_{J,Q}$$

where

$$\bar{\mathbf{U}}_{J,Q} = \left[\bar{\mathbf{U}}_Q(1) \quad \bar{\mathbf{U}}_Q(1) \quad \cdots \quad \bar{\mathbf{U}}_Q(J-1)\right] \tag{18}$$

is a $(2M+Q-1)\times Q(J-1)$ matrix. Suppose J is sufficiently large so that $\bar{\mathbf{U}}_{J,Q}$ has full rank $2M+Q-1$. Then the rank of $\mathbf{Y}_{J,Q}$ is exactly $2M+Q-1$, i.e., $\mathbf{Y}_{J,Q}\mathbf{Y}_{J,Q}^{\dagger}$ has exactly L zero eigenvalues. This property, however, is no longer true when the block synchronization is not perfect. When a timing mismatch d is present, the matrix in (17) becomes

$$\bar{\mathbf{Y}}_{J,Q}^{(d)} = \begin{bmatrix}\bar{\mathbf{Y}}_{Q}^{(d)}(1) & \bar{\mathbf{Y}}_{Q}(2) & \cdots & \bar{\mathbf{Y}}_{Q}^{(d)}(J-1)\end{bmatrix} \tag{19}$$

where

$$\bar{\mathbf{Y}}_{Q}^{(d)}(n) = \begin{bmatrix}\bar{\mathbf{y}}_{0,Q-1}^{(d)}(n) & \bar{\mathbf{y}}_{1,Q-2}^{(d)}(n) & \cdots & \bar{\mathbf{y}}_{Q-1,0}^{(d)}(n)\end{bmatrix}$$

$$\bar{\mathbf{y}}_{kl}^{(d)}(n) = \begin{bmatrix}\left[\mathbf{y}_{M}^{(d)}(n-1)\right]_{-k+1:M}\\ \mathbf{y}_{\mathrm{cp}}^{(d)}(n)\\ \left[\mathbf{y}_{M}^{(d)}(n)\right]_{1:M+l}\end{bmatrix}$$

$$\mathbf{y}_{\mathrm{cp}}^{(d)}(n) \triangleq \left[\mathbf{y}^{(d)}(n)\right]_{1:L}$$

and

$$\mathbf{y}_{M}^{(d)}(n) \triangleq \left[\mathbf{y}^{(d)}(n)\right]_{L+1:P}.$$

The rank deficiency of the matrix $\bar{\mathbf{Y}}_{J,Q}^{(d)}$ is the key to the proposed blind block synchronization algorithm. The following theorem presents the theoretical foundation of the proposed algorithm for CP systems.

Theorem 2: Consider the noise-free situation. Assume each channel coefficient h_k, $0\le k\le L$, is an independent complex random variable with a nonzero variance. Suppose J is sufficiently large and the transmitted signal is selected such that $\bar{\mathbf{U}}_{J,Q}$ defined in (18) has full rank for all Q. Then with probability one the following statement on the matrix $\bar{\mathbf{Y}}_{J,Q}^{(d)}$ defined in (19) is true.

$$\begin{aligned}&\text{The number of zero eigenvalues of } \bar{\mathbf{Y}}_{J,Q}^{(d)}\bar{\mathbf{Y}}_{J,Q}^{(d)\dagger}\\ &\quad= (2M+L+Q-1) - \operatorname{rank}\left(\bar{\mathbf{Y}}_{J,Q}^{(d)}\bar{\mathbf{Y}}_{J,Q}^{(d)\dagger}\right)\\ &\quad= \begin{cases} L, & \text{if } d=0\\ \max\{L-|d|-2(Q-1),0\}, & \text{if } d\neq 0.\end{cases}\end{aligned}$$

Proof: See the Appendix. ■

The behavior of the rank deficiency of $\bar{\mathbf{Y}}_{J,Q}^{(d)}$ is exactly equal to that of $\mathbf{Y}_{J,Q}^{(d)}$ in ZP systems as presented in Theorem 1, even though the sizes of $\bar{\mathbf{Y}}_{J,Q}^{(d)}$ and $\mathbf{Y}_{J,Q}^{(d)}$ are completely different. Similar comments can therefore be made as follows. When $Q=1$, the rank deficiency of $\bar{\mathbf{Y}}_{J,Q}^{(d)}$ is $\max\{L-|d|,0\}$ and when $Q\ge (L+1)/2$, the rank deficiency of $\bar{\mathbf{Y}}_{J,Q}^{(d)}$ is $L\cdot\delta[d]$ where $\delta[\cdot]$ denotes the discrete Delta function. An advantage is present for using a larger Q: the reduction in the rank deficiency of $\bar{\mathbf{Y}}_{J,Q}^{(d)}$ when $d\neq 0$ is more significant. This potentially improves the accuracy of blind block synchronization performance.

We should note that a necessary (but not sufficient) condition for $\bar{\mathbf{U}}_{J,Q}$ to have full rank is [10]

$$J \ge 2 + \frac{2M-1}{Q}. \tag{20}$$

Although (20) is not sufficient, the probability that $\bar{\mathbf{U}}_{J,Q}$ has full rank is usually very high in the simulation shown in Section VI. Inequality (20) also shows that, when the repetition index Q is chosen sufficiently large (e.g., $Q=2M-1$), the proposed algorithm can work with only *three* received blocks in absence of noise! In practical applications, the parameter J can be chosen as large as the channel coefficients remain roughly constant during J received blocks.

In the presence of noise, the optimal d can be chosen as the one which minimizes the sum of the smallest L eigenvalues of $\bar{\mathbf{Y}}_{J,Q}^{(d)}\bar{\mathbf{Y}}_{J,Q}^{(d)\dagger}$. The proposed algorithm can be summarized as follows.

Algorithm 2:

---

1) Choose the repetition index $Q\ge 1$ and the number of received blocks $J\ge 3$ so that (20) is satisfied.
2) Collect $(J+1)P$ consecutive received samples and form the matrix $\bar{\mathbf{Y}}_{J,Q}^{(d)}$ as in (19) for each $d\in[-P/2,P/2)$.
3) Perform eigen-decomposition on the matrix $\bar{\mathbf{Y}}_{J,Q}^{(d)}\bar{\mathbf{Y}}_{J,Q}^{(d)\dagger}$ and take the L smallest eigenvalues $\sigma_{L,(d)}^2\ge\sigma_{L-1,(d)}^2\ge\cdots\ge\sigma_{2,(d)}^2\ge\sigma_{1,(d)}^2\ge 0$.
4) Calculate the cost function $\lambda_{\mathrm{cp},Q}(d) := \sum_{k=1}^{L}\sigma_{k,(d)}^2$, and decide the estimated timing mismatch $\hat{d} = \arg\min_{(P/2)\le d<(P/2)}\lambda_{\mathrm{cp},Q}(d)$.

■

B. Comparisons With a Previously Reported Algorithm

In [11], a block synchronization algorithm was proposed by Negi and Cioffi based on the estimated rank of the autocorrelation matrix of received blocks. The basic idea of the Negi–Cioffi algorithm is to use the $(M+L-L_0)\times J$ matrix

$$\mathcal{Y}_{\mathrm{NC}}^{(d)} = \begin{bmatrix}\left[\mathbf{y}_{\mathrm{cp}}^{(d)}(0)\right]_{L_0+1:L} & \cdots & \left[\mathbf{y}_{\mathrm{cp}}^{(d)}(J-1)\right]_{L_0+1:L}\\ \hline \mathbf{y}_{M}^{(d)}(0) & \cdots & \mathbf{y}_{M}^{(d)}(J-1)\end{bmatrix}.$$

Define

$$\mathbf{u}^{(d)}(n) = [u(nP+d) \quad u(nP+d+1) \quad \cdots \quad u(nP+d+P-1)]^T.$$

Then it can be verified that

$$\mathcal{Y}_{\mathrm{NC}}^{(d)} = \mathcal{H}_{L_0}\mathbf{U}_{\mathrm{NC}}^{(d)}$$

where $\mathcal{H}_{L_0}$ is a $(P-L_0)\times P$ Toeplitz matrix whose first row is $[h_{L_0}\cdots h_0\ 0\cdots 0]$ and whose first column is $[h_{L_0}\ 0\cdots 0]^T$, and

$$\mathbf{U}_{\mathrm{NC}}^{(d)} = \begin{bmatrix}\mathbf{u}^{(d)}(0) & \mathbf{u}^{(d)}(1) & \cdots & \mathbf{u}^{(d)}(J-1)\end{bmatrix}$$

is a $P\times J$ matrix. When $d=0$, $\mathbf{U}_{\mathrm{NC}}^{(d)}$ has rank M and the matrix $\mathcal{Y}_{\mathrm{NC}}^{(d)}$ has a $L-L_0$ rank deficiency. When $d\neq 0$, the rank of $\mathbf{U}_{\mathrm{NC}}^{(d)}$ is larger than M and the rank deficiency of $\mathcal{Y}_{\mathrm{NC}}^{(d)}$ will not exceed $L-L_0$. This provides a way to determine the block boundaries by finding the d which gives $\mathcal{Y}_{\mathrm{NC}}^{(d)}$ the smallest

TABLE I
COMPUTATIONAL COMPLEXITIES OF BLIND BLOCK SYNCHRONIZATION ALGORITHMS

System	Algorithm	Complexity
ZP	Proposed algorithm 1	$\mathcal{O}(P(P+Q-1)^3)$
	SGB algorithm [1]	$\mathcal{O}(P^4)$
CP	Proposed algorithm 2	$\mathcal{O}(P(2P+Q-1)^3)$
	Negi-Cioffi algorithm [11]	$\mathcal{O}(P(P-L_0)^3)$

rank. In order to make the method work, $L-L_0$ must be a positive integer, which implies that the effective channel order L_0 must be strictly less than the cyclic prefix length L. In our proposed Algorithm 2, $L-L_0$ does not have to be positive. Another difference between the Negi–Cioffi algorithm and Algorithm 2 resides in the matrix $\mathbf{U}_{\mathrm{NC}}^{(d)}$. In order to make $\mathbf{U}_{\mathrm{NC}}^{(d)}$ rank M, a condition $J \geq M$ must be satisfied, which means the minimum number of received blocks is equal to the block size. As a comparison with (20), we find that the required number of received blocks for Algorithm 2 is always smaller than that of the Negi–Cioffi algorithm whenever the repetition index Q is chosen greater than 2.

In [11], the optimal d is chosen by estimating the rank of $\mathcal{Y}_{\mathrm{NC}}^{(d)}$ using a minimum description length (MDL) criterion, assuming the channel noise variance is known. However, our proposed algorithm does not assume known channel noise variance. In order to make a fair comparison between Algorithm 2 and the Negi–Cioffi algorithm, in our simulations presented in Section VI, we will slightly modify the optimal d decision procedure in the Negi–Cioffi algorithm by using the following cost function:

$$\lambda_{\mathrm{NC}}(d) \triangleq \sum_{k=1}^{L-L_0} \sigma_{k,(d)}^2 \tag{21}$$

where $\sigma_{k,(d)}^2$ is the kth smallest eigenvalue of $\mathcal{Y}_{\mathrm{NC}}^{(d)}\mathcal{Y}_{\mathrm{NC}}^{(d)\dagger}$. The optimal d is chosen as the one which minimizes the value of $\lambda_{\mathrm{NC}}(d)$.

V. PRACTICAL CONSIDERATIONS

A. Complexity Analysis

The computational complexity of the proposed algorithms is mostly accounted for by the eigendecomposition of the matrices composed of received signals. In Algorithm 1, we use the matrix $\mathbf{Y}_{J,Q}^{(d)}\mathbf{Y}_{J,Q}^{(d)\dagger}$ which has a size $(P+Q-1)\times(P+Q-1)$. The eigen-decomposition is performed on P matrices $\mathbf{Y}_{J,Q}^{(d)}\mathbf{Y}_{J,Q}^{(d)\dagger}$ for $d \in [-P/2, P/2)$. Since the complexity for computing the eigen-decomposition of an $n\times n$ matrix is $\mathcal{O}(n^3)$ [20], the computational complexity for Algorithm 1 is $\mathcal{O}(P(P+Q-1)^3)$. Similarly, the complexity of Algorithm 2 can be found to be $\mathcal{O}(P(2P+Q-1)^3)$ since the size of the matrix $\bar{\mathbf{Y}}_{J,Q}^{(d)}\bar{\mathbf{Y}}_{J,Q}^{(d)\dagger}$ is $(2P+Q-1)\times(2P+Q-1)$. The previously reported algorithms mentioned in Sections III-B and IV-B also depend on eigendecomposition of certain matrices. The computational complexity thereof can be determined in a similar way. A comparison of computational complexities of the proposed algorithms and previously reported algorithms is summarized in Table I.

From Table I, we find that the proposed algorithms have slightly higher complexity than the SGB method and the Negi–Cioffi method. However, as long as Q is small compared to P, the increase of computational complexity is insignificant.

B. System Performance in the Presence of Noise

In this subsection, we provide a qualitative analysis on the performance of both proposed algorithms in the presence of noise. The block synchronization error rate of Algorithms 1 and 2 can be bounded by the following inequality:

$$P_{\mathrm{err}} \leq \sum_{\substack{d\in(-P/2,P/2] \\ d\neq 0}} P\left(\lambda(d) < \lambda(0)\right)$$

where $\lambda(d)$ is the sum of the L smallest eigenvalues of $\mathbf{Y}_{J,Q}^{(d)}\mathbf{Y}_{J,Q}^{(d)\dagger}$ or $\bar{\mathbf{Y}}_{J,Q}^{(d)}\bar{\mathbf{Y}}_{J,Q}^{(d)\dagger}$ as defined as in (10) or (21), depending on whether Algorithm 1 or Algorithm 2 is of interest. Under a given noise level N_0, the value $P(\lambda(d) < \lambda(0))$ depends on the value $\lambda(d)$ in absence of noise. Moreover, the value $\lambda(d)$ in absence of noise depends on how many nonzero eigenvalues it is composed of. In Theorems 1 and 2, we learn that the number of zero eigenvalues of $\mathbf{Y}_{J,Q}^{(d)}\mathbf{Y}_{J,Q}^{(d)\dagger}$ or $\bar{\mathbf{Y}}_{J,Q}^{(d)}\bar{\mathbf{Y}}_{J,Q}^{(d)\dagger}$ is L when $d=0$ and it decreases to $L-2Q+1$ when $|d|=1$ (if $L\geq 2Q-1$). So $\lambda(d)$ with $d=\pm 1$ is obtained by adding up $2Q-1$ nonzero eigenvalues of $\mathbf{Y}_{J,Q}^{(d)}\mathbf{Y}_{J,Q}^{(d)\dagger}$. When $Q=1$, $\lambda(1)$ is obtained by a single nonzero eigenvalue, which can be very close to zero with a certain probability. When $Q=2$, $\lambda(1)$ is obtained by adding three nonzero eigenvalues, and the probability that all of them are very small will drop significantly. Similarly, when $|d|\geq 2$, the value $\lambda(d)$ is obtained by adding up $\max\{2Q-2+|d|, L\}$ nonzero eigenvalues. This means it is still more likely that $P(\lambda(d) < \lambda(0))$ has a smaller value when Q is a larger number. Therefore, we can say $P(\lambda(d) < \lambda(0))$ tends to have a smaller value when Q is a larger number. In summary, when Q is chosen as a larger number, the block synchronization error rate P_{err} tends to be smaller. These observations will be further confirmed in Section VI.

The above discussions also give us a guidance on how to choose parameters J and Q in practical applications. The parameter J is chosen according to the channel mobility, that is, J should be chosen so that the channel coefficients remain constant during $(J+1)P$ consecutive received samples. The repetition index Q should be chosen sufficiently large so that inequality (8) is satisfied. In addition, it should also be chosen so that $Q\geq 2$ since this avoids the possibility that $\lambda(1)$ and $\lambda(-1)$ are composed with a single nonzero eigenvalue and therefore greatly reduces the probabilities $P(\lambda(d) < \lambda(0))$, $|d|=1$, compared to the case $Q=1$.

VI. SIMULATION RESULTS AND DISCUSSIONS

In this section, we conduct simulations to evaluate the performances of Algorithms 1 and 2 and compare each of them with well known algorithms. In all simulations, the number of data samples per block is chosen as $M=16$ and the length of guard intervals per block is $L=4$ (which implies $P=20$). The constellation of data samples is QPSK. In the plots, $E_s = E[|s_k(n)|^2]$ and $N_0 = E[|e(n)|^2]$.

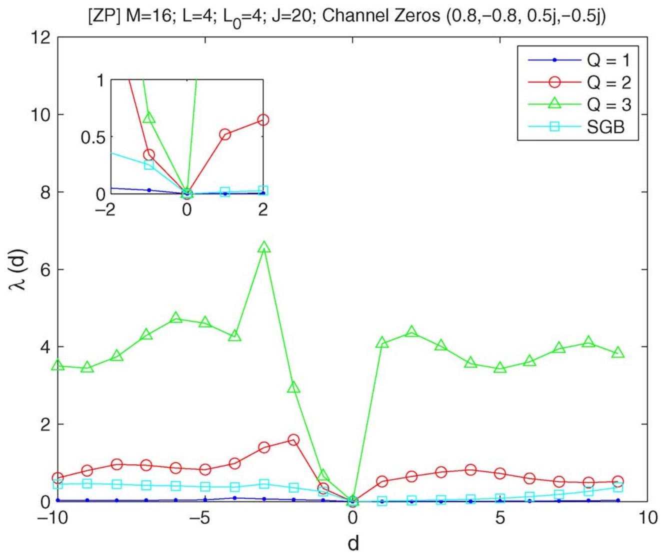

Fig. 3. Function $\lambda(d)$ versus time mismatch d for a channel with zeros at (0.8, -0.8, $0.5j$, $-0.5j$) in absence of noise.

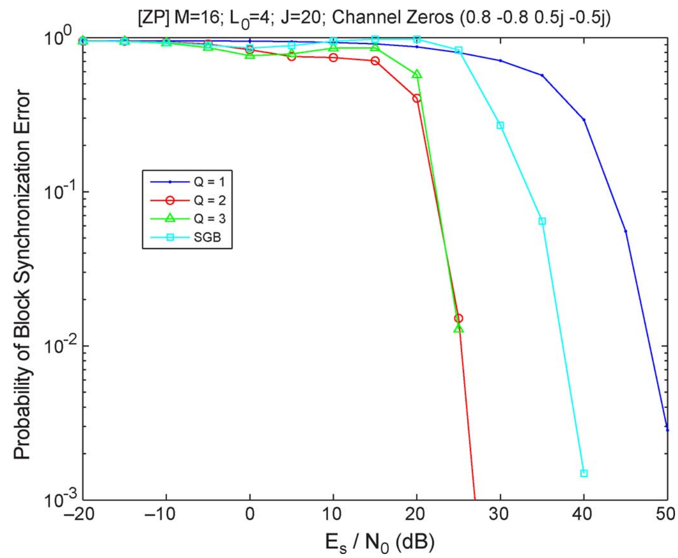

Fig. 4. Blind block synchronization error rate performance for a channel with zeros at (0.8, -0.8, $0.5j$, $-0.5j$) when $J = 20$ in ZP systems.

A. *Simulation Results for Zero Padding Systems*

We first present simulation results for zero padding systems. The precoder is chosen as $\mathbf{R}_{\rm zp} = \mathbf{I}_M$. We test Algorithm 1 with $Q = 1, 2, 3$ and compare the performances with that of the SGB algorithm proposed in [1]. Simulations are first conducted with two different fixed fourth order FIR channels. Channel 1 has zero locations at (0.8, -0.8, $0.5j$, $-0.5j$) and is a minimum-phase system. Channel 2 has zero locations at (1.2, -0.9, $0.7j$, -0.7). As suggested in (8), the number of received blocks must be at least $J \geq M = 16$. We choose $J = 20$, that is, $(J+1)P = 420$ consecutive received samples $y(n)$ are available for blind block synchronization. For each blind synchronization attempt, 420 samples $y(n)$ are collected and the cost functions $\lambda(d)$ as defined in (10) and (12) (i.e., $\lambda_{{\rm zp},Q}(d)$ and $\lambda_{\rm SGB}(d)$ for Algorithm 1 and the SGB method, respectively) are evaluated for each $d \in [-P/2, P/2)$. A successful block synchronization is declared when $\lambda(d)$ gives the minimum value at $d = 0$. Over 50 000 block synchronization attempts are conducted in the simulations in different E_s/N_0 levels ranging from -20 to 50 dB, and the block synchronization error rates are calculated accordingly. We also calculated the average values of $\lambda(d)$ over all block synchronization attempts in a noise-free environment. Fig. 3 shows the average value of the cost functions $\lambda(d)$ versus the timing mismatch $d \in [-P/2, P/2)$ with Channel 1. For a clearer view of the values of $\lambda(d)$ in the neighborhood of $d = 0$, a close-up window is put at the top of each plot. As expected, $\lambda(d) = 0$ when $d = 0$ and is nonzero otherwise for all curves. The robustness of a particular algorithm against noise perturbation with respect to a specific Q may be roughly evaluated by looking at the values $\lambda(-1)$ and $\lambda(1)$. When an additive noise is present, the values of $\lambda(d)$ will change and an algorithm may mistakenly decide the optimal timing mismatch as $d = -1$ or $d = 1$. Therefore, larger values of $\lambda(-1)$ and $\lambda(1)$, in Fig. 3, represent a larger margin against noise perturbation and suggest a better performance for a particular algorithm.

As we can see in Fig. 3, the SGB method has a good $\lambda(-1)$ but a relatively small $\lambda(1)$. Algorithm 1 with $Q = 2$ or 3 has a much better $\lambda(1)$ but both $\lambda(-1)$ and $\lambda(1)$ are poor with $Q = 1$. These observations are consistent with the discussions in Section V-B. When $Q = 2$, $\lambda(1)$ is obtained by adding three nonzero eigenvalues, and the probability that all of them are very small is much smaller than the case $Q = 1$. Simulations under a noisy environment, as shown in Fig. 4, confirm the expectation that the algorithms (for various Q) perform better for larger values of $\lambda(-1)$ and $\lambda(1)$. Clearly, Algorithm 1 with $Q = 2$ has a significant gain (more than 10 dB!) over the SGB algorithm. Increasing Q from 2 to 3, however, does not further improve the performance. Algorithm 1 with $Q = 1$, however, requires a 50-dB E_s/N_0 ratio to achieve a satisfactory error rate, which is considered infeasible in practice.

We now show the simulation results for Channel 2, which contains channel zeros both inside and outside the unit circle and so is neither a minimum phase nor a maximum phase system. In the noiseless environment, the plot of averaged values of $\lambda(d)$ versus timing mismatch d is shown in Fig. 5. We observe that the SGB method has a much larger $\lambda(1)$ than it does with Channel 1, a minimum phase system. Yet Algorithm 1 with $Q \geq 2$ possesses even larger $\lambda(-1)$ and $\lambda(1)$ than the SGB algorithm. In Fig. 6, we show the plot in the presence of noise with $E_s/N_0 = 10$ dB. We observe that values of $\lambda(0)$ are still clearly smaller than $\lambda(1)$ and $\lambda(-1)$ when $Q = 2$ or $Q = 3$. But this is not the case for Algorithm 1 with $Q = 1$ or the SGB method. This suggests an advantage in block synchronization error rate for Algorithm 1 with $Q \geq 2$ in the presence of noise. This advantage is also shown in Fig. 7, where we observe that Algorithm 1 with $Q = 2$ or $Q = 3$ has a roughly 5-dB gain over the SGB method. Algorithm 1 with $Q = 1$ still has the poorest performance in this plot. From Figs. 4 and 7 for Channels 1 and 2, respectively, we find that the block synchronization error rate performance depends not only on the algorithms, but also on the channels. Minimum phase channels appear to be less favorable

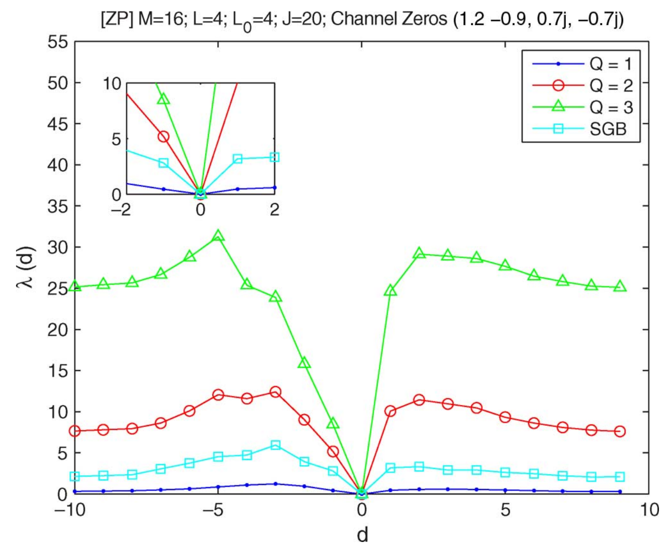

Fig. 5. Function $\lambda(d)$ versus time mismatch d for a channel with zeros at (1.2, -0.9, $0.7j$, $-0.7j$) in absence of noise when $J = 20$.

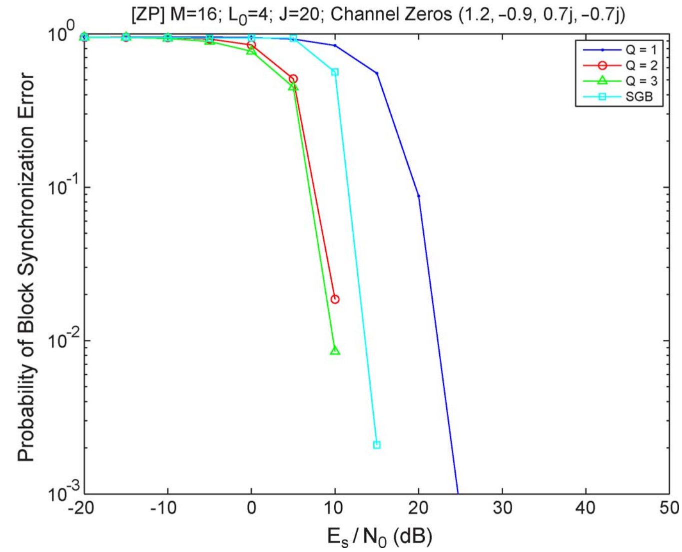

Fig. 7. Blind block synchronization error rate performance for a channel with zeros at (1.2, -0.9, $0.7j$, $-0.7j$) when $J = 20$ in ZP systems.

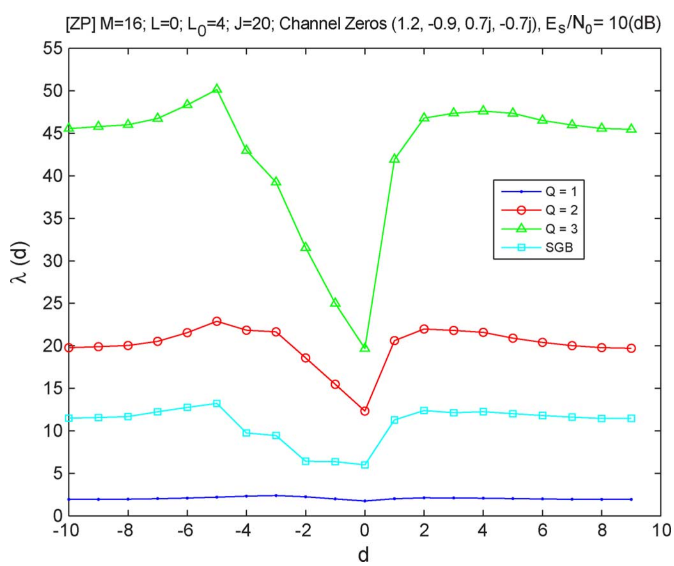

Fig. 6. Function $\lambda(d)$versus time mismatch d for a channel with zeros at (1.2, -0.9, $0.7j$, $-0.7j$) when $E_s/N_0 = 10$ dB and $J = 20$.

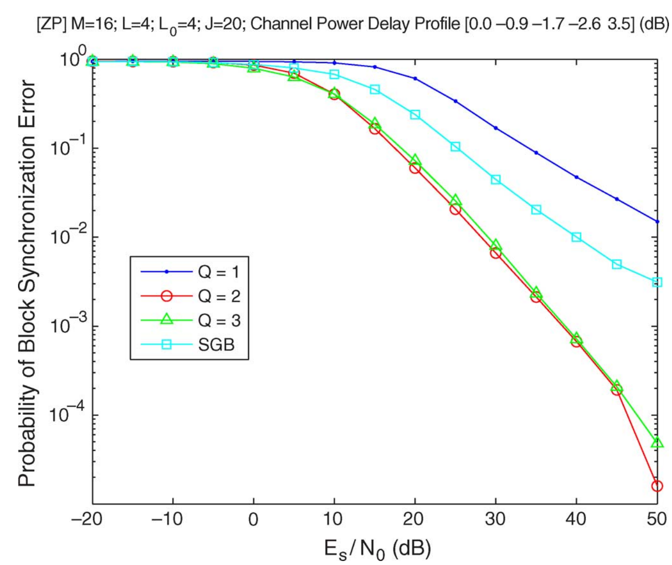

Fig. 8. Blind block synchronization error rate performance for a Rayleigh random channel with $J = 20$ in ZP systems.

for blind block synchronization in general than other types of channels.

Simulation results presented so far are based on fixed channels. We now try the comparison in a fourth order Rayleigh random channel with a power delay profile $[0.0 - 0.9 - 1.7 - 2.6 - 3.5]$ (dB). Over 4000 independent realizations of the channel are used in the simulation and for each channel realization four block synchronization attempts are performed (i.e., four different sets of data samples $s(n)$ are used). Fig. 8 shows the average block synchronization error rate performances for all cases. As we can see, Algorithm 1 with $Q = 2$ has a roughly 10-dB gain over the SGB algorithm. Increasing Q from 2 to 3 does not significantly improve the system performance.

Finally in this subsection, we demonstrate the capability of Algorithm 1 to conduct blind block synchronization with extremely limited received data. We choose J ranging from 2 to 5 and the repetition index Q is properly chosen so that inequality (8) is satisfied. Fig. 9 shows the simulation plot. As discussed in Section III-B, the SGB algorithm is similar to Algorithm 1 with $Q = L = 4$ except for some omissions of data. As shown in the plot, when $J = 5$, the SGB method indeed has a much worse performance than Algorithm 1 with $Q = 4$. Furthermore, when $J = 4$, the SGB algorithm fails while Algorithm 1 continues to work even when J is as small as 2. Note that when $J = 2$, the number of available consecutive received samples is only $(J + 1)P = 60$. Even though the block synchronization error rate performance is satisfactory only when the E_s/N_0 value is very high, Algorithm 1 is presumably the first one to perform

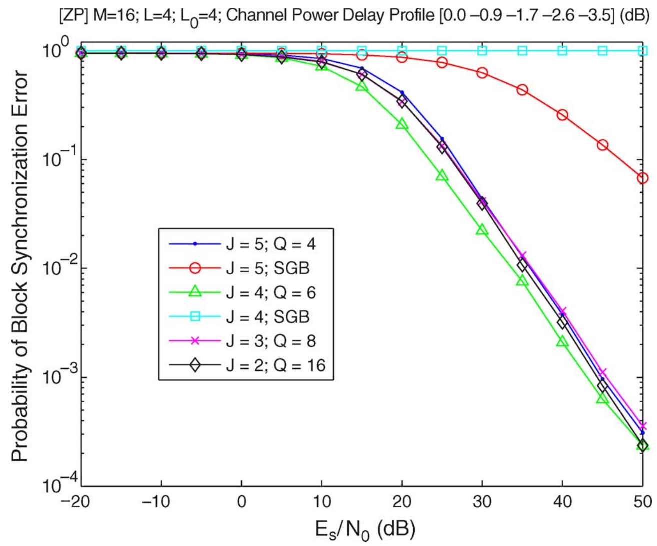

Fig. 9. Blind block synchronization error rate performance for a Rayleigh random channel with small number of blocks in ZP systems.

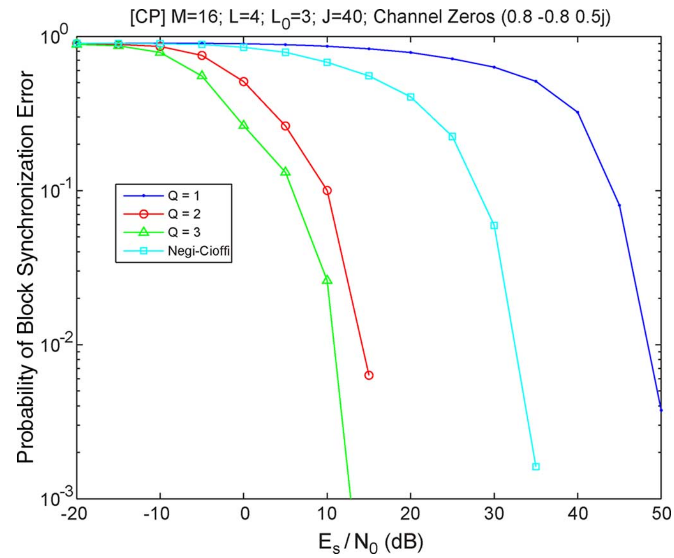

Fig. 10. Blind block synchronization error rate performance for a channel with zeros at $(0.8, -0.8, 0.5j)$ when $J = 40$ in CP systems.

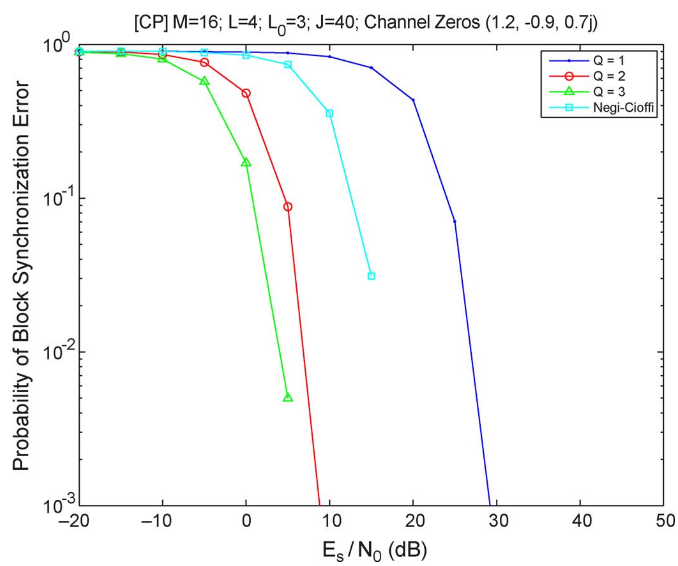

Fig. 11. Blind block synchronization error rate performance for a channel with zeros at $(1.2, -0.9, 0.7j)$ when $J = 40$ in CP systems.

blind block synchronization properly with such limited received data.

B. Simulation Results for Cyclic Prefix Systems

We now present simulation results for cyclic prefix systems. The precoder is chosen as $\mathbf{R}_{\rm cp} = \mathbf{I}_M$. We test Algorithm 2 with $Q = 1, 2, 3$ and compare the performances with that of the Negi-Cioffi algorithm [11]. As suggested in inequality (20), J must be chosen as at least $2M + 1 = 37$ for Algorithm 2 with $Q = 1$ to work. We choose $J = 40$, that is, $(J+1)P = 820$ consecutive received samples $y(n)$ are available for blind block synchronization.

We first test the algorithm with third-order channels (i.e., $L_0 = 3$). Note that a cyclic prefix of length $L = 4$ allows a maximum channel order to be four to avoid interblock interference. The reason why we chose only third-order channels is for proper comparison with the Negi–Cioffi algorithm due to its restriction (see Section IV-B). In the simulations, we use the cost function defined in (21) for the Negi–Cioffi algorithm.

Simulations are first conducted with two different fixed third order FIR channels. Channel 3 has zero locations at $(0.8, -0.8, 0.5j)$ and is a minimum-phase system. Channel 4 has zero locations at $(1.2, -0.9, 0.7j)$. The block synchronization is declared successful when the cost function $\lambda(d)$ has a minimum value at either $d = 0$ or $d = -1$ [see (3)]. Fig. 10 shows the simulation results with Channel 3. We see that when $Q = 1$, Algorithm 2 works properly in the high-SNR region, but with a rather unsatisfactory performance. Algorithm 2 with $Q = 2$ has a much better performance and has a 20-dB gain over the Negi–Cioffi algorithm. Increasing Q from 2 to 3 further improves the performance. For the simulation results with Channel 4, the block synchronization error rate performance is shown in Fig. 11 and the plot of averaged values of $\lambda(d)$ at the noise level of E_s/N_0 is shown in Fig. 12. We observe that, for Algorithm 2 with $Q = 2$ and $Q = 3$, the values $\lambda(d)$ with $d = 0$ and $d = -1$ are clearly smaller than those with $d = 1$ and $d = -2$. Recall that $d = 0$ and $d = -1$ can both be considered correct answers since $L - L_0 = 1$. The distinction of smaller values of $\lambda(d)$ with $d = 0$ and $d = -1$ does not exist in the case for $Q = 1$ or the Negi–Cioffi algorithm. This explains the advantage of Algorithm 2 with $Q = 2$ and $Q = 3$ in the presence of noise observed in Fig. 11.

We now perform the simulation in a third-order Rayleigh random channel with a power delay profile $[0.0 - 0.9 - 1.7 - 2.6]$ dB and the results are shown in Fig. 13. Algorithm 2 with $Q = 2$ has a 10-dB gain over the Negi–Cioffi algorithm. Increasing Q from 2 to 3 does not significantly improve the system performance.

Recall that we chose $L_0 = 3 < L$ in previous simulations due to a restriction of the Negi–Cioffi algorithm. Now we conduct simulations with a fourth order Rayleigh channel to verify that Algorithm 2 also works in the situation where $L = L_0$. As

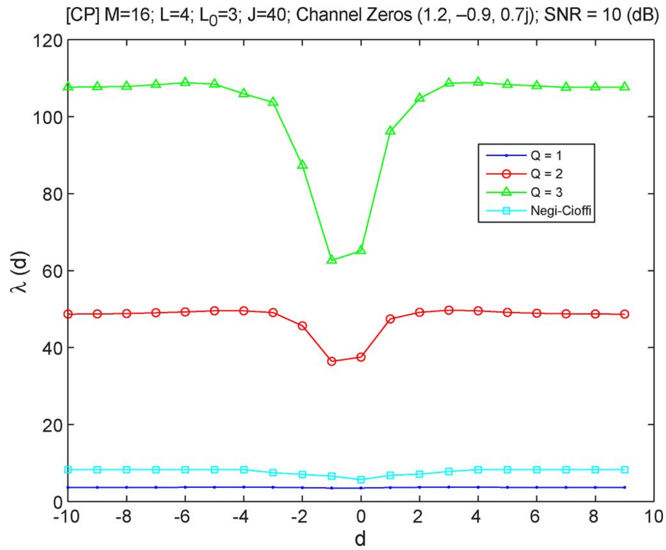

Fig. 12. Function $\lambda(d)$ versus time mismatch d for a channel with zeros at (1.2, $-0.9, 0.7j, -0.7j$) when $E_s/N_0 = 10$ dB and $J = 40$ in CP systems.

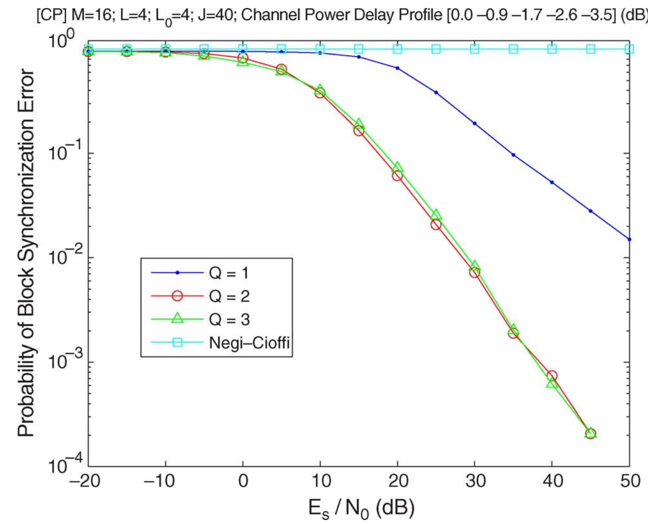

Fig. 14. Blind block synchronization error rate performance for a fourth-order Rayleigh random channel with $J = 40$ in CP systems.

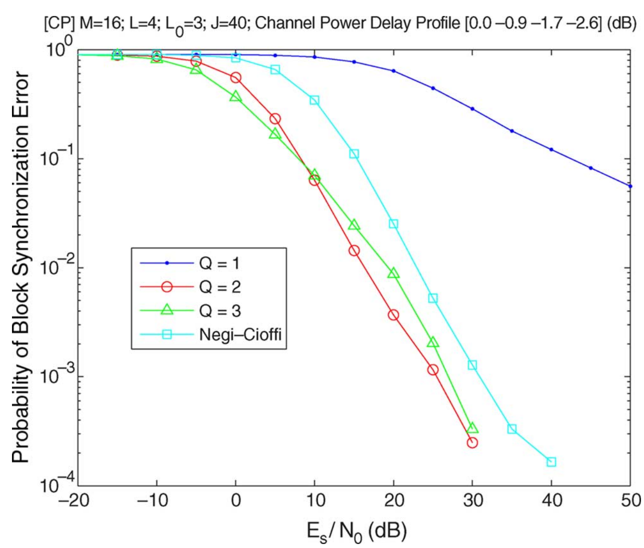

Fig. 13. Blind block synchronization error rate performance for a third-order Rayleigh random channel with $J = 40$ in CP systems.

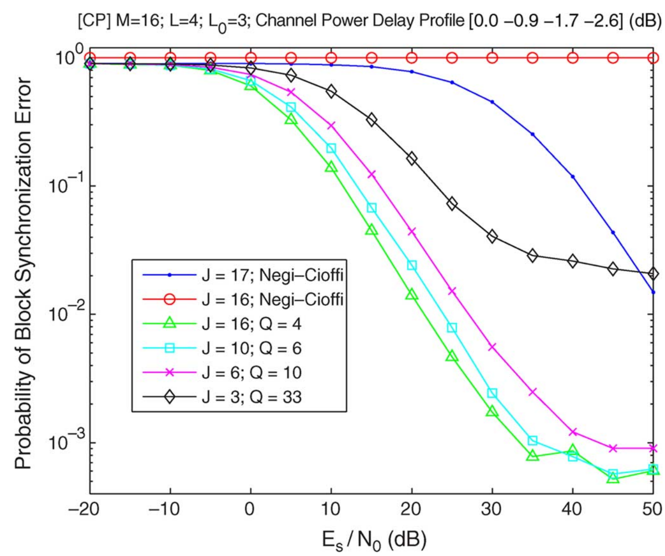

Fig. 15. Blind block synchronization error rate performance for a third-order Rayleigh random channel in CP systems when J is small.

shown in Fig. 14, Algorithm 2 works fine with all Q while the Negi–Cioffi algorithm fails.

Finally in this subsection, we demonstrate the capability of Algorithm 2 to conduct blind block synchronization with small amount of received data. We choose J ranging from 3 to 16 and the repetition index Q is properly chosen so that inequality (20) is satisfied. We also compare the performances with those of the Negi–Cioffi algorithm with $J = 16$ and 17. Fig. 15 shows the simulation plot. As discussed in Section IV-B, the Negi–Cioffi algorithm requires at least $M = 16$ blocks to work properly. From the simulation plot, we see that it does not even work with $J = 16$. When $J = 17$, the Negi–Cioffi algorithm appears to work, but with a somewhat poor performance. On the other hand, Algorithm 2 with $Q = 4$ already works when $J = 16$ with a fairly satisfactory performance. As the parameter J decreases, the performance of Algorithm 2 (with a properly chosen Q) degrades slowly. Even when $J = 3$ (which implies the number of available consecutive received sample is only $(J+1)P = 80$), Algorithm 2 still possesses a much better performance than the Negi–Cioffi algorithm with $J = 17$.

VII. Conclusion

In this paper we proposed two algorithms for blind block synchronization in zero-padding (ZP) systems and in cyclic prefix (CP) systems, respectively. Both algorithms use a parameter called repetition index (Q) which can be chosen as any positive integer. The CP algorithm can be directly applied to blind symbol synchronization problem in the popular orthogonal frequency division multiplexing (OFDM) systems. Theoretical results prove the validity of the proposed algorithms in the noiseless case and suggest that the algorithms would have a better performance when the repetition index is larger in the noisy case. The proposed algorithms are capable of blindly recovering the block boundaries using much less received data than previously reported algorithms. This feature makes the proposed algorithms more favorable in an environment of fast-varying channels. Simulation results of the proposed algorithm not only demonstrate the capability to work properly with limited amount of received data but also reveal significant improvement in block synchronization error rate performance over previously reported algorithms.

In the future, performance evaluation of the proposed algorithms for time-varying channels will be important for a more realistic scenario. A theoretical analysis of the system performance is also of interest.

APPENDIX

Proof of Theorem 1: We first consider the case $d = 0$.

$$\mathbf{y}(n) = \mathcal{H}\begin{bmatrix}\mathbf{0}_{L\times 1}\\ \hline \mathbf{u}(n)\\ \hline \mathbf{0}_{L\times 1}\end{bmatrix} = \mathcal{T}_M(\mathbf{h})\mathbf{u}(n)$$

where $\mathcal{H}$ is the $P\times(P+L)$ Toeplitz matrix whose first column is $[h_L, 0, \dots, 0]^T$ and whose first row is $[h_L, \dots, h_0, 0, \dots, 0]$.

When $d = 0$, it can be shown that

$$\mathcal{T}_Q\left(\mathbf{y}^{(0)}(n)\right) = \mathcal{T}_{M+Q-1}(\mathbf{h})\mathcal{T}_Q\left(\mathbf{u}^{(0)}(n)\right).$$

With probability one, for sufficiently large J, the matrix

$$\mathbf{Y}_{J,Q}^{(d)} = \mathcal{T}_{M+Q-1}(\mathbf{h})\mathbf{U}_Q^{(J)}$$

has rank $M+Q-1$. This implies $\mathbf{Y}_{J,Q}^{(d)}\mathbf{Y}_{J,Q}^{(d)\dagger}$ has exactly L zero eigenvalues.

When $d \neq 0$, we assume $d > 0$ with loss of generality since the following arguments can be extended to the case when $-P/2 \leq d < 0$ due to symmetry. When $d > 0$, we have

$$\mathbf{y}^{(d)}(n) = \mathcal{H}\mathbf{u}^{(d)}(n)$$

where

$$\mathbf{u}^{(d)}(n) = \begin{bmatrix}\mathbf{0}_{(L-d)\times 1}\\ \hline \mathbf{u}(n)\\ \hline \mathbf{0}_{L\times 1}\\ \hline [\mathbf{u}(n+1)]_{1:d}\end{bmatrix}$$

when $0 < d \leq L$ and

$$\mathbf{u}^{(d)}(n) = \begin{bmatrix}[\mathbf{u}(n)]_{d-L+1:M}\\ \hline \mathbf{0}_{L\times 1}\\ \hline [\mathbf{u}(n+1)]_{1:d}\end{bmatrix}$$

when $L < d \leq P/2$.

The qth column of $\mathcal{T}_Q(\mathbf{y}^{(d)}(n))$ can be written as

$$\begin{bmatrix}\mathbf{0}_{(q-1)\times 1}\\ \mathbf{y}^{(d)}(n)\\ \mathbf{0}_{(Q-q)\times 1}\end{bmatrix} = \mathcal{H}_Q\begin{bmatrix}v_{q-1}^{(n)}\\ \vdots\\ v_1^{(n)}\\ \mathbf{u}^{(d)}(n)\\ x_1^{(n)}\\ \vdots\\ x_{Q-q}^{(n)}\end{bmatrix}$$

where $\mathcal{H}_Q$ is a $(P+L+Q-1)\times(P+Q-1)$ Toeplitz matrix whose first column is $[h_L, 0, \dots, 0]^T$ and whose first row is $[h_L, \dots, h_0, 0, \dots, 0]$ and sequences $\{x_k^{(n)}\}$ and $\{v_k^{(n)}\}$ are defined recursively as follows:

$$x_l^{(n)} \triangleq -\frac{1}{h_0}\left[\sum_{k=1}^{L} h_k x_{l-k}^{(n)}\right], \quad l = 1, \dots, Q-1$$

$$v_l^{(n)} \triangleq -\frac{1}{h_L}\left[\sum_{k=0}^{L-1} h_k v_{l-L+k}^{(n)}\right], \quad l = 1, \dots, Q-1$$

where the initial values of the sequences are defined as

$$\left[x_{-L+1}^{(n)}, x_{-L+2}^{(n)}, \cdots, x_0^{(n)}\right]^T \triangleq \left[\mathbf{u}^{(d)}(n)\right]_{P:P+L}$$

$$\left[v_0^{(n)}, v_{-1}^{(n)}, \cdots, v_{-L+1}^{(n)}\right]^T \triangleq \left[\mathbf{u}^{(d)}(n)\right]_{1:L}.$$

So

$$\mathcal{T}_Q\left(\mathbf{y}^{(d)}(n)\right) = \mathcal{H}_Q\mathbf{U}_Q^{(d)}(n)$$

where

$$\mathbf{U}_Q^{(d)}(n) = \mathcal{T}_Q\left(\mathbf{u}^{(d)}(n)\right) + \mathbf{V}(n)$$

and $\mathbf{V}(n)$ is a $(P+L+Q-1)\times Q$ Toeplitz matrix whose first column is $[\mathbf{0}_{1\times(P+L)}\; x_1 \cdots x_{Q-1}]^T$ and whose first row is $[0\; v_1 \cdots v_{Q-1}]$.

Define

$$\mathbf{U}_{J,Q}^{(d)} = \left[\mathbf{U}_Q^{(d)}(0) \quad \mathbf{U}_Q^{(d)}(1) \quad \cdots \quad \mathbf{U}_Q^{(d)}(J-1)\right]$$

and denote $n(L,Q,d)$ as the number of zero rows in $\mathbf{U}_{J,Q}^{(d)}$. When $Q \leq L+1$, the zero block $\mathbf{0}_{L\times 1}$ in $\mathbf{u}^{(d)}(n)$ accounts for $(L-Q-1)$ zero rows in $\mathbf{U}_{J,Q}^{(d)}$. Furthermore, if $0 < d \leq L-Q+1$, the zero block $\mathbf{0}_{(L-d)\times 1}$ in $\mathbf{u}^{(d)}(n)$ accounts for $(L-Q-1-d)$ zero rows in $\mathbf{U}_{J,Q}^{(d)}$. If $L-Q-1 < d < P/2$, the zero block $\mathbf{0}_{(L-d)\times 1}$ in $\mathbf{u}^{(d)}(n)$ does not account for any zero rows in $\mathbf{U}_{J,Q}^{(d)}$ (when $d > L$, $\mathbf{0}_{(L-d)\times L}$ does not even exist). The above arguments can be extended to the case when $-P/2 \leq d < 0$ due to symmetry. So, when $Q \leq L+1$,

$$n(L,Q,d) = \underbrace{(L-Q+1)}_{\text{from } \mathbf{0}_{L\times 1}} + \underbrace{\max\{L-Q+1-|d|, 0\}}_{\text{from } \mathbf{0}_{(L-d)\times 1}}.$$

When $Q > L+1$, neither blocks $\mathbf{0}_{L\times 1}$ nor $\mathbf{0}_{(L-d)\times 1}$ in $\mathbf{u}^{(d)}(n)$ account for any zero rows. So

$$n(L,Q,d) = 0$$

when $Q > L+1$.

Now $\mathbf{Y}_{J,Q}^{(d)} = \mathcal{H}_Q \mathbf{U}_{J,Q}^{(d)} = \mathcal{H}'_Q \mathbf{U}'^{(d)}_{J,Q}$ where $\mathbf{U}'^{(d)}_{J,Q}$ is obtained by eliminating the $n(L,Q,d)$ zeros rows in $\mathbf{U}_{J,Q}^{(d)}$ and $\mathcal{H}'_Q$ is obtained by eliminating the corresponding columns in $\mathcal{H}_Q$. We are interested in the difference of the number of rows and the number of columns of $\mathcal{H}'_Q$. Denote this value as $n(\mathcal{H}'_Q)$. This value represents the column rank deficiency of $\mathcal{H}'_Q$ if $n(\mathcal{H}'_Q) \geq 0$. It is readily verified that $n(\mathcal{H}'_Q) = n(L,Q,d) - L$ and so the column rank deficiency of $\mathcal{H}'_Q$ is $\max\{n(\mathcal{H}'_Q), 0\}$. Due to the random nature of $\mathbf{u}(n)$, $\{x_k\}$, and $\{v_k\}$, with probability one, there exists a sufficiently large J such that $\mathbf{U}'^{(d)}_{J,Q}$ has full rank $(P+L+Q-1-n(L,Q,d))$. The rank deficiency of $\mathbf{Y}_{J,Q}^{(d)}\mathbf{Y}_{J,Q}^{(d)\dagger}$ is $\max\{n(L,Q,d) - L, 0\}$. When $Q > L+1$, $n(L,Q,d) = 0$, so the rank deficiency of $\mathbf{Y}_{J,Q}^{(d)}\mathbf{Y}_{J,Q}^{(d)\dagger}$ is zero. When $Q \leq L+1$ and when $|d| \geq L - Q + 1$, $n(L,Q,d) = L - Q + 1$, so the rank deficiency of $\mathbf{Y}_{J,Q}^{(d)}\mathbf{Y}_{J,Q}^{(d)\dagger}$ is $\max\{-Q+1, 0\} = 0$. When $Q \leq L+1$ and when $|d| \leq L - Q + 1$, $n(L,Q,d) = 2L - 2Q + 2 - |d|$, so the rank deficiency of $\mathbf{Y}_{J,Q}^{(d)}\mathbf{Y}_{J,Q}^{(d)\dagger}$ is $\max\{2L - 2Q + 2 - |d| - L, 0\} = L - |d| - 2(Q-1)$.

In summary, with probability one, the number of zero eigenvalues of $\mathbf{Y}_{J,Q}^{(d)}\mathbf{Y}_{J,Q}^{(d)\dagger}$ is $\max\{L - |d| - 2(Q-1), 0\}$ when $d \neq 0$. This completes the proof. ∎

Proof of Theorem 2: We first consider the case $d = 0$.

$$\bar{\mathbf{y}}(n) = \mathcal{H}\begin{bmatrix} \mathbf{u}_{\mathrm{cp}}(n-1) \\ \hline \mathbf{u}_M(n-1) \\ \hline \mathbf{u}'_M(n) \\ \hline \mathbf{u}_{\mathrm{cp}}(n) \end{bmatrix} = \bar{\mathbf{H}}\bar{\mathbf{u}}(n)$$

where $\mathcal{H}$ is the $(2M+L)\times(2M+2L)$ Toeplitz matrix whose first column is $[h_L, 0, \ldots, 0]^T$ and whose first row is $[h_L, \cdots, h_0, 0, \ldots, 0]$, and $\mathbf{u}'_M(n)$ is a permutation of $\mathbf{u}_M(n)$ defined as $\mathbf{u}'_M(n) = [\mathbf{u}_M(n)]_{-L+1:M-L}$.

With probability one there exists a sufficiently large J such that

$$\mathbf{Y}_{J,Q}^{(d)} = \bar{\mathbf{H}}_Q \mathbf{U}_{J,Q}$$

where $\bar{\mathbf{H}}_Q$ is a $(2M+L+Q-1)\times(2M+Q-1)$ matrix which has full column rank $(2M+Q-1)$ with probability one and $\mathbf{U}_{J,Q}$ has full row rank. The rank of $\mathbf{Y}_{J,Q}^{(d)}$ is thus exactly equal to $2M+Q-1$ since $\bar{\mathbf{H}}$ has full column rank $2M+Q-1$ and $\mathbf{U}_{J,Q}^{(0)}$ has full row rank $2M+Q-1$. This implies $\mathbf{Y}_{J,Q}^{(d)}\mathbf{Y}_{J,Q}^{(d)\dagger}$ has exactly $(2M+L+Q-1)-(2M+Q-1) = L$ zero eigenvalues.

When $d \neq 0$, we have

$$\bar{\mathbf{y}}^{(d)}(n) = \mathcal{H}\bar{\mathbf{u}}^{(d)}(n)$$

where

$$\bar{\mathbf{u}}^{(d)}(n) = \begin{bmatrix} [\mathbf{u}_{\mathrm{cp}}(n-1)]_{d+1:L} \\ \hline \mathbf{u}_M(n-1) \\ \mathbf{u}'_M(n) \\ \mathbf{u}_{\mathrm{cp}}(n) \\ \hline [\mathbf{u}_{\mathrm{cp}}(n+1)]_{1:d} \end{bmatrix}$$

when $0 < d \leq L$ and

$$\bar{\mathbf{u}}^{(d)}(n) = \begin{bmatrix} [\mathbf{u}_M(n-1)]_{(d-L+1):M} \\ \hline \mathbf{u}'_M(n) \\ \mathbf{u}_{\mathrm{cp}}(n) \\ \hline \mathbf{u}_{\mathrm{cp}}(n+1) \\ [\mathbf{u}_M(n+1)]_{1:d-L} \end{bmatrix}$$

when $L < d \leq P/2$.

Now, from the definition of $\bar{\mathbf{Y}}_Q^{(d)}(n)$, the qth column of $\bar{\mathbf{Y}}_Q^{(d)}(n)$ can be expressed as

$$\begin{bmatrix} \left[\mathbf{y}_M^{(d)}(n-1)\right]_{M-q+1:M} \\ \hline \bar{\mathbf{y}}^{(d)}(n) \\ \hline \left[\mathbf{y}_M^{(d)}(n)\right]_{1:Q-q} \end{bmatrix} = \mathcal{H}_{2Q}\begin{bmatrix} v_{q-1} \\ \vdots \\ v_1 \\ \hline \bar{\mathbf{u}}^{(d)}(n) \\ \hline x_1 \\ \vdots \\ x_{Q-q} \end{bmatrix}, \quad q = 1, 2, \ldots, Q$$

where $\mathcal{H}_{2Q}$ is a $(2M+L+Q-1)\times(2P+Q-1)$ Toeplitz matrix whose first column is $[h_L\ 0\cdots 0]^T$ and whose first row is $[h_L \cdots h_0\ 0 \cdots 0]$ and sequences $\{x_k\}$ and $\{v_k\}$ are defined as follows:

$$[x_{-L+1}, x_{-L+2}, \cdots, x_0]^T \triangleq \left[\bar{\mathbf{u}}^{(d)}(n)\right]_{2P-L+1:2P}$$
$$[v_0, v_{-1}, \cdots, v_{-L+1}]^T \triangleq \left[\bar{\mathbf{u}}^{(d)}(n)\right]_{1:L}$$
$$x_l \triangleq \frac{1}{h_0}\left[\left[\mathbf{y}_M^{(d)}(n)\right]_l - \sum_{k=1}^{L} h_k x_{l-k}\right]$$

and

$$v_l \triangleq \frac{1}{h_L}\left[\left[\mathbf{y}_M^{(d)}(n-1)\right]_{M+1-l} - \sum_{k=0}^{L-1} h_k v_{l-L+k}\right]$$

for $l = 1, \ldots, Q-1$. So

$$\bar{\mathbf{Y}}_Q^{(d)}(n) = \mathcal{H}_{2Q}\bar{\mathbf{U}}_Q^{(d)}(n)$$

where

$$\bar{\mathbf{U}}_Q^{(d)}(n) = \mathcal{T}_Q\left(\bar{\mathbf{u}}^{(d)}(n)\right) + \mathbf{V}$$

and $\mathbf{V}$ is a $(2P+Q-1)\times Q$ Toeplitz matrix whose first column is $[\mathbf{0}_{1\times 2P}\ x_1 \cdots x_{Q-1}]^T$ and whose first row is $[0\ v_1 \cdots v_{Q-1}]$.

Denote $n(L,Q,d)$ as the number of pairs of identical rows in $\bar{\mathbf{U}}_Q^{(d)}(n)$. When $Q \le L+1$, the segment $\mathbf{u}_{\text{cp}}(n)$ in $\bar{\mathbf{u}}^{(d)}(n)$ accounts for $(L-Q-1)$ pairs of identical rows in $\bar{\mathbf{U}}_Q^{(d)}(n)$ (Recall that $\mathbf{u}_{\text{cp}}(n) = [\bar{\mathbf{u}}'_M(n)]_{1:L}$.) Furthermore, if $0 < d \le L-Q+1$, the segment $[\mathbf{u}_{\text{cp}}(n-1)]_{d+1:L}$ accounts for $(L-Q-1-d)$ pairs of identical rows in $\bar{\mathbf{U}}_Q^{(d)}(n)$. If $L-Q-1 < d < P/2$, on the other hand, the segment $[\mathbf{u}_{\text{cp}}(n-1)]_{d+1:L}$ will not account for any pairs of identical rows (when $d > L$, this segment does not even exist). The above arguments can be extended to the case when $-P/2 \le d < 0$ due to symmetry. So, when $Q \le L+1$,

$$n(L,Q,d) = \underbrace{(L-Q+1)}_{\text{from } \mathbf{u}_{\text{cp}}(n)} + \underbrace{\max\{L-Q+1-|d|,0\}}_{\text{from}[\mathbf{u}_{\text{cp}}(n-1)]_{d+1:L}}.$$

When $Q > L+1$, neither segments $\mathbf{u}_{\text{cp}}(n)$ nor $[\mathbf{u}_{\text{cp}}(n-1)]_{d+1:L}$ in $\bar{\mathbf{u}}^{(d)}(n)$ account for any pairs of identical rows. So

$$n(L,Q,d) = 0$$

when $Q > L+1$.

Now $\bar{\mathbf{Y}}_{J,Q}^{(d)} = \mathcal{H}_{2Q}\bar{\mathbf{U}}_{J,Q}^{(d)} = \mathcal{H}'_{2Q}\bar{\mathbf{U}}'^{(d)}_{J,Q}$ where $\bar{\mathbf{U}}'^{(d)}_{J,Q}$ is obtained by eliminating the $n(L,Q,d)$ duplicated rows in $\mathbf{U}_{J,Q}^{(d)}$ and $\mathcal{H}'_{2Q}$ is obtained by merging the corresponding column pairs of $\mathcal{H}_{2Q}$. We are interested in the value of the number of rows of $\mathcal{H}'_{2Q}$ minus the number of columns of $\mathcal{H}'_{2Q}$. Denote this value as $n(\mathcal{H}'_{2Q})$. This value represents the column rank deficiency of $\mathcal{H}'_{2Q}$ if $n(\mathcal{H}'_{2Q}) \ge 0$. It is readily verified that $n(\mathcal{H}'_{2Q}) = n(L,Q,d) - L$ and so the column rank deficiency of $\mathcal{H}'_{2Q}$ is $\max\{n(\mathcal{H}'_{2Q}),0\}$. Due to the random nature of $\mathbf{u}_M(n)$, $\{x_k\}$, and $\{v_k\}$, with probability one, there exists a sufficiently large J such that $\bar{\mathbf{U}}'^{(d)}_{J,Q}$ has full rank $(2P+Q-1-n(L,Q,d))$. The rank deficiency of $\bar{\mathbf{Y}}_{J,Q}^{(d)}\bar{\mathbf{Y}}_{J,Q}^{(d)\dagger}$ is $\max\{n(L,Q,d)-L,0\}$. When $Q > L+1$, $n(L,Q,d) = 0$, so the rank deficiency of $\bar{\mathbf{Y}}_{J,Q}^{(d)}\bar{\mathbf{Y}}_{J,Q}^{(d)\dagger}$ is zero. When $Q \le L+1$ and when $|d| \ge L-Q+1$, $n(L,Q,d) = L-Q+1$, so the rank deficiency of $\bar{\mathbf{Y}}_{J,Q}^{(d)}\bar{\mathbf{Y}}_{J,Q}^{(d)\dagger}$ is $\max\{-Q+1,0\} = 0$. When $Q \le L+1$ and when $|d| \le L-Q+1$, $n(L,Q,d) = 2L-2Q+2-|d|$, so the rank deficiency of $\bar{\mathbf{Y}}_{J,Q}^{(d)}\bar{\mathbf{Y}}_{J,Q}^{(d)\dagger}$ is $\max\{2L-2Q+2-|d|-L,0\} = L-|d|-2(Q-1)$.

In summary, with probability one, the number of zero eigenvalues of $\bar{\mathbf{Y}}_{J,Q}^{(d)}\bar{\mathbf{Y}}_{J,Q}^{(d)\dagger}$ is $\max\{L-|d|-2(Q-1),0\}$ when $d \ne 0$. This completes the proof. ∎

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