Group Lifting Structures for Multirate Filter Banks, II: Linear Phase Filter Banks

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Abstract

The theory of group lifting structures is applied to linear phase lifting factorizations for the two nontrivial classes of two-channel linear phase perfect reconstruction filter banks, the whole- and half-sample symmetric classes. Group lifting structures defined for the reversible and irreversible classes of whole- and half-sample symmetric filter banks are shown to satisfy the hypotheses of the uniqueness theorem for group lifting structures. It follows that linear phase group lifting factorizations of whole- and half-sample symmetric filter banks are therefore independent of the factorization methods used to construct them. These results cover the specification of whole-sample symmetric filter banks in the ISO/IEC JPEG 2000 image coding standard.

Index Terms

Filter bank, wavelet, unique factorization, polyphase, lifting, linear phase filter, group.

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I. INTRODUCTION

This is the second paper on a new approach to lifting factorization for two-channel FIR perfect reconstruction filter banks. The first paper [1] introduced *group lifting structures* to parameterize universes of lifting factorizations for given classes of filter banks. While lifting factorizations are generally nonunique, it was proven in [1, Theorem 1] that, under suitable hypotheses, a filter bank has a *unique* irreducible lifting factorization within a given group lifting structure. The present paper applies [1, Theorem 1] to group lifting structures for linear phase lifting factorizations of whole-sample symmetric (WS) and halfsample symmetric (HS) filter banks. Existence of linear phase lifting factorizations for these classes was proven in [2], and group lifting structures were introduced in [1, Section IV] for both irreversible and reversible factorizations. Using [1, Theorem 1], we now prove uniqueness results for irreducible group lifting factorizations of these filter banks.

For WS filter banks, like those in the ISO/IEC JPEG 2000 standard [3, Annex F], [4, Annex G], our results imply there is only one way to factor them into alternating upper and lower triangular lifting matrices with HS lifting filters. Lifting HS filter banks from concentric, equal-length HS base filter banks with whole-sample antisymmetric (WA) lifting filters [2] is shown to be unique modulo one trivial degree of freedom ("unique modulo rescaling"). The key to [1, Theorem 1] is verifying a polyphase order-increasing hypothesis [1, Definition 10] by computing the order of lifted filter banks.

The remainder of the Introduction reviews key notation and concepts from [1], adding some useful computational tools. Section II proves a technical result, Lemma 2, that gives sufficient conditions for a group lifting structure to satisfy the order-increasing hypothesis. Section III uses Lemma 2 to prove that the reversible and irreversible WS group lifting structures satisfy the order-increasing property and generate unique group lifting factorizations. Analogous results for HS group lifting structures are proven in Section IV. Section V contains concluding remarks.

A. Filter Banks

Given an FIR filter,

$$F(z) = \sum_{n=a}^{b} f(n) \, z^{-n}$$

the support interval of f is defined [1, Definition 1] to be the largest closed interval of integers for which $f(a), f(b) \neq 0$:

$$\operatorname{supp_int}(F) \equiv \operatorname{supp_int}(f) \equiv [a, b] \subset \mathbb{Z}.$$
 (1)

If $supp_int(f) = [a, b]$ then the *order* of F is defined to be

$$\operatorname{order}(F) \equiv b - a \,. \tag{2}$$

Define the filter's support radius to be the integer

$$\operatorname{supp_rad}(f) \equiv \left\lfloor \frac{b-a+1}{2} \right\rfloor$$
 (3)

The polyphase vector representation of an FIR filter,

$$\boldsymbol{F}(z) \equiv \begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix} = \sum_{n=c}^d \boldsymbol{f}(n) \, z^{-n},\tag{4}$$

$$\boldsymbol{f}(n) \equiv \begin{bmatrix} f_0(n) \\ f_1(n) \end{bmatrix} \quad \text{with } \boldsymbol{f}(c), \, \boldsymbol{f}(d) \neq \boldsymbol{0},$$
(5)

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is defined in [1, Section II-A] based on the analysis polyphase-with-advance representation of the scalar filter [2],

$$F(z) = F_0(z^2) + zF_1(z^2)$$

The *polyphase* support interval of the filter with polyphase vector representation (5) is defined in [1] to be

$$\operatorname{supp_int}(f) \equiv [c, d].$$
 (6)

Define the *join* of two closed intervals, denoted with a \lor , to be the smallest closed interval containing their union:

$$[a,b] \lor [c,d] \equiv [\min(a,c), \max(b,d)] .$$
⁽⁷⁾

It is straightforward to verify that the polyphase support interval (6) is given by the join of the scalar support intervals for the two polyphase component filters:

$$\operatorname{supp_int}(f) = \operatorname{supp_int}(f_0) \lor \operatorname{supp_int}(f_1).$$
 (8)

Similarly, consider an FIR filter bank with polyphase matrix

$$\mathbf{H}(z) \equiv \begin{bmatrix} \mathbf{H}_0^T(z) \\ \mathbf{H}_1^T(z) \end{bmatrix} = \sum_{n=c}^d \mathbf{h}(n) \, z^{-n} \,, \tag{9}$$

$$\mathbf{h}(n) \equiv \begin{bmatrix} \mathbf{h}_0^T(n) \\ \mathbf{h}_1^T(n) \end{bmatrix} \quad \text{with } \mathbf{h}(c), \ \mathbf{h}(d) \neq \mathbf{0}.$$
(10)

The polyphase support interval of the filter bank (9, 10) is defined in [1, Section II-A] to be supp_int(h) \equiv [c, d]. Again, it is straightforward to verify that the polyphase support interval of the filter bank is equal to the join of the polyphase support intervals of the two polyphase filter vectors:

$$supp_int(\mathbf{h}) = supp_int(\mathbf{h}_0) \lor supp_int(\mathbf{h}_1).$$
(11)

The following lemma describes the support intervals that result from performing elementary operations on FIR filters.

Lemma 1: If F(z) and G(z) are scalar FIR filters then

$$supp_int(f * g) = supp_int(f) + supp_int(g).$$
(12)

(Note that (12) does *not* extend to matrix-vector or matrix-matrix convolution; the lack of such a simple formula for matrix-matrix convolution is responsible for many of the technical difficulties in this paper.) If $\mathbf{F}(z)$ is a matrix-, vector-, or scalar-valued FIR filter with supp_int(\mathbf{f}) = [a, b] then

$$\operatorname{supp_int}\left(\mathbf{F}(z^2)\right) = [2a, 2b].$$
(13)

If $\mathbf{G}(z)$ has the same dimensions as $\mathbf{F}(z)$ and $\operatorname{supp_int}(\mathbf{g}) \subset (a, b) = [a + 1, b - 1]$, where \subset denotes (non-proper) containment with possible equality, then

$$supp_int(\mathbf{f} + \mathbf{g}) = supp_int(\mathbf{f}).$$
(14)

B. Lifting Factorizations

Part I [1] works with partially factored lifting cascades,

$$\mathbf{H}(z) = \operatorname{diag}(1/K, K) \,\mathbf{S}_{N-1}(z) \cdots \mathbf{S}_0(z) \,\mathbf{B}(z) \,, \tag{15}$$

relative to a *base* filter bank, $\mathbf{B}(z)$. Cascades that are completely factored into lifting steps correspond to $\mathbf{B}(z) = \mathbf{I}$. The *update characteristic*, m_n , of $\mathbf{S}_n(z)$ [1, Definition 2] indicates whether a lifting matrix is upper triangular (i.e., a lowpass update) or lower triangular (a highpass update):

$$m_n = \begin{cases} 0 & \text{if } \mathbf{S}_n(z) \text{ is upper triangular,} \\ 1 & \text{if } \mathbf{S}_n(z) \text{ is lower triangular.} \end{cases}$$
(16)

A lifting cascade is *irreducible* [1, Definition 3] if the lifting steps strictly alternate between lower and upper triangular, implying that $m_{n+1} = 1 - m_n$ for irreducible liftings.

Many of the arguments in this paper involve finite induction and are based on the recursive formulation of lifting:

$$\mathbf{H}(z) = \operatorname{diag}(1/K, K) \mathbf{E}^{(N-1)}(z),
\mathbf{E}^{(n)}(z) = \mathbf{S}_n(z) \mathbf{E}^{(n-1)}(z) \quad \text{for } 0 \le n < N,
\mathbf{E}^{(-1)}(z) \equiv \mathbf{B}(z).$$
(17)

The scalar filters corresponding to an intermediate partial product of lifting matrices are given by [2, eqn. (9)],

$$\begin{bmatrix} E_0^{(n)}(z) \\ E_1^{(n)}(z) \end{bmatrix} = \mathbf{E}^{(n)}(z^2) \begin{bmatrix} 1 \\ z \end{bmatrix} = \mathbf{S}_n(z^2) \begin{bmatrix} E_0^{(n-1)}(z) \\ E_1^{(n-1)}(z) \end{bmatrix},$$
(18)

where $E_i^{(-1)}(z) \equiv B_i(z)$ for i = 0, 1. We make extensive use of the scalar version of this recursion, based on the update characteristic, m_n , of $\mathbf{S}_n(z)$. The filter updated by $\mathbf{S}_n(z)$ is $E_{m_n}^{(n)}(z)$, while $E_{1-m_n}^{(n)}(z)$ is *not* modified by $\mathbf{S}_n(z)$. Thus, (18) corresponds to the scalar lifting formulas

$$E_{m_n}^{(n)}(z) = E_{m_n}^{(n-1)}(z) + S_n(z^2) E_{1-m_n}^{(n-1)}(z), \qquad (19)$$

$$E_{1-m_n}^{(n)}(z) = E_{1-m_n}^{(n-1)}(z).$$
⁽²⁰⁾

C. Group Lifting Structures

A group lifting structure, \mathfrak{S} , is defined in [1, Definition 6] to be an ordered four-tuple,

$$\mathfrak{S} \equiv (\mathfrak{D}, \mathfrak{U}, \mathcal{L}, \mathfrak{B}). \tag{21}$$

 \mathcal{D} is an abelian group of diagonal gain scaling matrices, $\mathbf{D}_K \equiv \text{diag}(1/K, K)$. \mathcal{U} and \mathcal{L} are abelian groups of upper and lower triangular lifting matrices, and \mathfrak{B} is a set of base filter banks. The *lifting cascade group*, \mathcal{C} , is the nonabelian matrix group generated by \mathcal{U} and \mathcal{L} ,

$$\mathcal{C} \equiv \langle \mathcal{U} \cup \mathcal{L} \rangle = \left\{ \mathbf{S}_1 \cdots \mathbf{S}_k \, : \, k \ge 1, \; \mathbf{S}_i \in \mathcal{U} \cup \mathcal{L}
ight\} \, .$$

The universe of all lifted filter banks generated by \mathfrak{S} is

$$\mathcal{DCB} \equiv \left\{ \mathbf{DCB} : \mathbf{D} \in \mathcal{D}, \ \mathbf{C} \in \mathfrak{C}, \ \mathbf{B} \in \mathfrak{B} \right\}$$

The matrix $\mathbf{D}_K \in \mathcal{D}$ acts on transfer matrices via the inner automorphism γ_{κ} ,

$$\gamma_{\kappa} \mathbf{A}(z) \equiv \mathbf{D}_K \, \mathbf{A}(z) \, \mathbf{D}_K^{-1}. \tag{22}$$

A group \mathcal{G} of transfer matrices is \mathcal{D} -invariant [1, Definition 5] if $\gamma_{\kappa}\mathcal{G} = \mathcal{G}$ for all $\mathbf{D}_{K} \in \mathcal{D}$, and a group lifting structure (21) is \mathcal{D} -invariant if \mathcal{U} and \mathcal{L} , and therefore \mathcal{C} , are \mathcal{D} -invariant. \mathcal{D} -invariance is easy to verify in practice using [1, Lemma 2].

A lifting cascade (15) is called *strictly polyphase order-increasing* (or just *order-increasing*) [1, Definition 10] if the intermediate partial products (17) satisfy

order
$$\left(\mathbf{E}^{(n)}\right) >$$
order $\left(\mathbf{E}^{(n-1)}\right)$ for $0 \le n < N$.

A group lifting structure, \mathfrak{S} , is called order-increasing if every irreducible cascade in \mathfrak{CB} is order-increasing.

If \mathfrak{S} is an order-increasing, \mathcal{D} -invariant group lifting structure, the uniqueness theorem [1, Theorem 1] says that all irreducible factorizations of $\mathbf{H}(z)$ in \mathfrak{S} are "equivalent modulo rescaling." Specifically, given two such factorizations,

$$\mathbf{H}(z) = \mathbf{D}_K \mathbf{S}_{N-1}(z) \cdots \mathbf{S}_0(z) \mathbf{B}(z)$$
(23)

$$= \mathbf{D}_{K'} \mathbf{S}'_{N'-1}(z) \cdots \mathbf{S}'_0(z) \mathbf{B}'(z), \qquad (24)$$

the uniqueness theorem states that the number of lifting steps is the same (N' = N), with base filter banks related by

$$\mathbf{B}'(z) = \mathbf{D}_{\alpha} \mathbf{B}(z)$$
 for constant $\alpha = K/K'$, (25)

and lifting steps related by inner automorphisms,

$$\mathbf{S}'_i(z) = \gamma_\alpha \mathbf{S}_i(z), \quad i = 0, \dots, N-1.$$
(26)

We express this by saying that irreducible lifting factorizations in \mathfrak{S} are "unique modulo rescaling" [1, Definition 11].

The theorem can be strengthened to yield *unique* irreducible factorizations by normalizing a degree of freedom in the base filter banks. For instance, fixing a lowpass DC response like $B_0(1) = 1$ for all $\mathbf{B}(z) \in \mathfrak{B}$ implies $\alpha = 1$ by (25).

Our principal tool for verifying the order-increasing hypothesis of [1, Theorem 1] is developed in the next section.

II. SUFFICIENT CONDITIONS FOR ORDER-INCREASING GROUP LIFTING STRUCTURES

Since lifting only modifies one filter at a time, the order-increasing property hinges on the growth of the support intervals of the filters $E_{m_n}^{(n)}(z)$ being updated in each lifting step. A simplification occurs in the WS and HS cases because, as we shall show, the *polyphase* support interval of the updated polyphase filter vector, $E_{m_n}^{(n)}(z)$, always contains that of its complement, $E_{1-m_n}^{(n)}(z)$, making the polyphase order of $E^{(n)}(z)$ equal to the polyphase order of $E_{m_n}^{(n)}(z)$. This is convenient, but other order-increasing group lifting structures may exist that do not share this property.

Lemma 2: Let \mathfrak{S} be a group lifting structure satisfying the following two polyphase vector conditions. 1) For all $\mathbf{B}(z) \in \mathfrak{B}$, the polyphase support intervals (6) for the base polyphase filter vectors are equal:

$$supp_int(\boldsymbol{b}_0) = supp_int(\boldsymbol{b}_1) . \tag{27}$$

2) For all irreducible lifting cascades in $C\mathfrak{B}$, the polyphase support intervals (6) for the intermediate polyphase filter vectors satisfy the proper inclusions

$$\operatorname{supp_int}\left(\boldsymbol{e}_{1-m_n}^{(n)}\right) \subsetneq \operatorname{supp_int}\left(\boldsymbol{e}_{m_n}^{(n)}\right) \quad \text{for } n \ge 0.$$
(28)

It then follows that \mathfrak{S} is strictly polyphase order-increasing.

Proof: Let $\mathbf{S}_{N-1}(z) \cdots \mathbf{S}_0(z) \mathbf{B}(z) \in \mathfrak{CB}$ be irreducible. By (11) and (27) the base filter bank satisfies

$$supp_int(\mathbf{b}) = supp_int(\mathbf{b}_0) \lor supp_int(\mathbf{b}_1)$$
$$= supp_int(\mathbf{b}_0)$$
$$= supp_int(\mathbf{b}_1),$$

which implies

$$\operatorname{order}(\mathbf{B}) = \operatorname{order}(\mathbf{B}_0) = \operatorname{order}(\mathbf{B}_1).$$
 (29)

Similarly, (11) and (28) imply

$$\begin{split} ext{supp_int} \left(\mathbf{e}^{(n)}
ight) &= ext{supp_int} \left(oldsymbol{e}_0^{(n)}
ight) \lor ext{supp_int} \left(oldsymbol{e}_1^{(n)}
ight) \ &= ext{supp_int} \left(oldsymbol{e}_{m_n}^{(n)}
ight) \,, \end{split}$$

and therefore

order
$$\left(\mathbf{E}^{(n)}\right) = \operatorname{order}\left(\mathbf{E}^{(n)}_{m_n}\right)$$
. (30)

Note that (28) also implies

order
$$\left(\boldsymbol{E}_{m_n}^{(n)}\right)$$
 > order $\left(\boldsymbol{E}_{1-m_n}^{(n)}\right)$ for $n \ge 0.$ (31)

We now show that order $(\mathbf{E}^{(n)})$ is strictly increasing.

Case: n = 0. Since the filter *not* being lifted is $E_{1-m_0}^{(0)}(z) = B_{1-m_0}(z)$, by (30), (31), and (29) we get

$$\operatorname{order}\left(\mathbf{E}^{(0)}\right) = \operatorname{order}\left(\mathbf{E}^{(0)}_{m_0}\right)$$

> $\operatorname{order}\left(\mathbf{E}^{(0)}_{1-m_0}\right) = \operatorname{order}\left(\mathbf{B}_{1-m_0}\right) = \operatorname{order}(\mathbf{B})$.

Case: n > 0. Since the cascade is irreducible by hypothesis, the filter not being lifted in step n must have been lifted in step n - 1:

$$\boldsymbol{E}_{1-m_n}^{(n)}(z) = \boldsymbol{E}_{m_{n-1}}^{(n-1)}(z).$$

Therefore,

order
$$\left(\mathbf{E}^{(n)}\right) = \operatorname{order}\left(\mathbf{E}^{(n)}_{m_n}\right)$$

> order $\left(\mathbf{E}^{(n)}_{1-m_n}\right) = \operatorname{order}\left(\mathbf{E}^{(n-1)}_{m_{n-1}}\right) = \operatorname{order}\left(\mathbf{E}^{(n-1)}\right).$

Remarks: The polyphase representation of the "causal lazy wavelet" filter bank [1, Example 7] does *not* satisfy (27). A scalar analogue of (28) became evident while computing linear phase lifting factorizations of both WS and HS filter banks, but the author found that (27) was the less-than-obvious key to unifying the WS and HS cases. The base filters,

$$B_0(z) = 1$$
 and $B_1(z) = z$, (32)

for the (anticausal) lazy wavelet filter bank $\mathbf{B}(z) = \mathbf{I}$ have equal lengths but are *not* concentric like, e.g., the Haar filters,

$$H_0(z) = (z+1)/2$$
 and $H_1(z) = z - 1$. (33)

An equal-length hypothesis for the base filters, though, is not enough by itself to imply the order-increasing property. Fortunately, (32) and (33) share a common feature that implies the order-increasing property, namely, the fact that in both cases the base filters have the same *polyphase* support intervals.

III. WHOLE-SAMPLE SYMMETRIC FILTER BANKS

We briefly review the definitions from [1, Section IV-A] of the group lifting structures for WS filter banks. For the irreversible WS group, W, the upper (respectively, lower) triangular lifting matrices, $\mathcal{U} \equiv v(\mathcal{P}_0)$ (respectively, $\mathcal{L} \equiv \lambda(\mathcal{P}_1)$), are defined by half-sample symmetric real Laurent polynomials:

$$\mathcal{P}_0 \equiv \left\{ S(z) \in \mathbb{R}[z, z^{-1}] : S(z^{-1}) = zS(z) \right\},$$
(34)

$$\mathcal{P}_1 \equiv \left\{ S(z) \in \mathbb{R}[z, z^{-1}] : S(z^{-1}) = z^{-1}S(z) \right\} .$$
(35)

The scaling matrices are $\mathcal{D} \equiv {\mathbf{D}_K : K \neq 0}$ and the base filter banks are trivial: $\mathfrak{B} \equiv {\mathbf{I}}$. The irreversible WS group lifting structure is $\mathfrak{S}_W \equiv (\mathcal{D}, \mathcal{U}, \mathcal{L}, \mathfrak{B})$, and its lifting cascade group is $\mathfrak{C}_W \equiv \langle \mathcal{U} \cup \mathcal{L} \rangle$.

In the reversible case there are no scaling matrices, $\mathcal{D}_r \equiv \{\mathbf{I}\}$, the set of base filter banks is still $\mathfrak{B}_r \equiv \{\mathbf{I}\}$, and the lifting filters (34), (35) are restricted to those with dyadic rational coefficients. The reversible WS group is *defined* by

$$\mathcal{W}_r \equiv \mathcal{C}_{\mathcal{W}_r} \equiv \langle \mathcal{U}_r \cup \mathcal{L}_r \rangle \,,$$

and its group lifting structure is $\mathfrak{S}_{W_r} \equiv (\mathfrak{D}_r, \mathfrak{U}_r, \mathfrak{L}_r, \mathfrak{B}_r).$

A. Main Uniqueness Result

Our main result on WS filter banks shows that \mathfrak{S}_{W} and \mathfrak{S}_{W_r} produce unique irreducible group lifting factorizations.

Theorem 1 (Uniqueness of WS group lifting factorizations): Let \mathfrak{S}_W and \mathfrak{S}_{W_r} be the group lifting structures defined in [1, Section IV-A]. Every filter bank in W has a unique irreducible lifting factorization in \mathfrak{S}_W and every filter bank in W_r has a unique irreducible lifting factorization in \mathfrak{S}_{W_r} .

Proof: Existence of irreducible WS group lifting factorizations for these groups was covered in [1, Section IV-A].

Uniqueness: As discussed above, $\mathfrak{B} = \{\mathbf{I}\}$ satisfies hypothesis (27) in Lemma 2 since

$$supp_int(b_i) = \{0\}, i = 0, 1.$$

Lemma 6 (Section III-D below) shows that irreducible lifting cascades in $\mathcal{C}_{\mathcal{W}}\mathfrak{B}$ satisfy (28), so Lemma 2 implies that $\mathfrak{S}_{\mathcal{W}}$ is order-increasing. \mathcal{D} -invariance of \mathcal{U} and \mathcal{L} follows from [1, Lemma 2], so [1, Theorem 1] applies to $\mathfrak{S}_{\mathcal{W}}$. Since $\mathfrak{B} = \{\mathbf{I}\}$, it follows that the degree of freedom, α , in the conclusion of [1, Theorem 1] must be unity: $\alpha = 1$. This means that irreducible group lifting factorizations in $\mathfrak{S}_{\mathcal{W}}$ are unique.

In the reversible case we have $\mathcal{C}_{W_r} < \mathcal{C}_{W}$ and $\mathfrak{B}_r = \mathfrak{B}$. The *reversible* WS group lifting factorizations are thus a subset of the *irreversible* factorizations, $\mathcal{C}_{W_r}\mathfrak{B}_r \subset \mathcal{D}\mathcal{C}_W\mathfrak{B}$, so uniqueness of irreducible factorizations in \mathfrak{S}_{W_r} follows from uniqueness of irreducible factorizations in \mathfrak{S}_W .

Remarks: While existence of cascade factorizations for arbitrary WS filter banks was asserted for the cascade structure in [5], Theorem 1 provides a more complete theory of both existence *and* uniqueness for factorizations of WS filter banks.

1) Applications: Theorem 1 implies that compliant lifting factorizations of user-defined WS filter banks in JPEG 2000 Part 2, which are specified in [4, Annex G] as irreducible WS group lifting factorizations, are *unique*.

Corollary 1: A delay-minimized unimodular WS filter bank can be specified in JPEG 2000 Part 2 Annex G syntax in one and only one way.

Example 1: In their book on JPEG 2000 [6, Section 6.4.4], Taubman and Marcellin note the relative simplicity of implementing lifting factorizations that only involve HS lifting filters of length two. (It is clear from [6, Figure 6.13] that this means *first-order* filters; i.e., symmetric filters with two consecutive terms, such as $c(1 + z^{-1})$.) They then claim [6, page 294]:

All two channel FIR subband transforms having odd length, symmetric filters with least dissimilar lengths (filter lengths differ by 2) may be factored into lifting steps of this form *[i.e., first-order HS lifting filters]*.

As an application of Theorem 1, we can construct counterexamples to this claim. Consider the following lifting cascade:

$$\mathbf{H}(z) = \begin{bmatrix} 1 & 1+z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z^2+z+1+z^{-1} & 1 \end{bmatrix}.$$
 (36)

By (18) the corresponding WS scalar filter bank is

$$H_0(z) = z^4 + 2z^2 + z + 3 + z^{-1} + 2z^{-2} + z^{-4}
 H_1(z) = z^4 + z^2 + z + 1 + z^{-2},$$
(37)

which has filters of least dissimilar lengths. Theorem 1 implies that (36) is its unique irreducible WS group lifting factorization, so it cannot have another such factorization using first-order HS filters. Generalizing this example, it follows from Lemma 5 (below) that an irreducible WS group lifting cascade corresponds to a WS filter bank with filters of least dissimilar lengths if and only if the *final* HS lifting filter is first-order.

The lemmas needed to verify hypothesis (28) of Lemma 2 occupy the remainder of Section III.

B. Polyphase Support Intervals

The relation between a filter's impulse response, e(k), and its polyphase components, $e_j(k)$, is given in [2, eqn. (10)]:

$$e_j(k) = e(2k - j), \quad j = 0, 1, \quad k \in \mathbb{Z}.$$
 (38)

In general, the support intervals (1) for the individual polyphase components are not completely determined by $supp_int(e)$ (see the proof of the following lemma). Nonetheless, the *polyphase* support interval, $supp_int(e)$, *is* completely determined by $supp_int(e)$. The following derivations do not depend on linear phase properties of the filters.

Lemma 3: Suppose E(z) is an odd-length FIR filter whose support interval (1) is centered at 0,

$$supp_int(e) = [-r, r], \qquad (39)$$

where $r \equiv \text{supp}_rad(e) \ge 0$ is the support radius (3). Then the *polyphase* support interval (6) of the filter is:

$$supp_int(e) = \begin{cases} [-r/2, r/2] & \text{if } r \text{ is even,} \\ [(-r+1)/2, (r+1)/2] & \text{if } r \text{ is odd.} \end{cases}$$

Proof: There are two cases.

Case: r Even. By (39) and (38) the even polyphase component is $e_0(k) = e(2k)$ for $-r \le 2k \le r$, and both e(-r) and e(r) are nonzero by hypothesis (39), so

$$supp_int(e_0) = [-r/2, r/2].$$
 (40)

The odd polyphase component is $e_1(k) = e(2k-1)$ for $-r+1 \le 2k-1 \le r-1$, but we don't know that e is nonzero at the endpoints since they lie in the interior of supp_int(e). For instance, Strang and Nguyen [7, Section 6.5] exhibit a family of dyadic WS wavelet filter banks, credited to Sweldens, whose lowpass filters have $h(\pm(r-1)) = 0$, and (37) provides another example. Thus, all we can conclude is that

$$supp_int(e_1) \subset [(-r+2)/2, r/2].$$
 (41)

Combining (40) and (41) using (8) yields

$$supp_int(e) = supp_int(e_0) \lor supp_int(e_1)$$
$$= [-r/2, r/2].$$
(42)

Case: r Odd. By (39) and (38) the even polyphase component is $e_0(k) = e(2k)$ for $-r + 1 \le 2k \le r - 1$, and e may vanish at either endpoint since they both lie in the interior of supp_int(e), so

$$supp_{int}(e_0) \subset [(-r+1)/2, (r-1)/2].$$
 (43)

The odd polyphase component is $e_1(k) = e(2k-1)$ for $-r \le 2k-1 \le r$, and both e(-r) and e(r) are nonzero so

$$supp_int(e_1) = [(-r+1)/2, (r+1)/2].$$
 (44)

Combining (43) and (44) using (8) yields

supp_int(e) = supp_int(e_0)
$$\lor$$
 supp_int(e_1)
= [(-r+1)/2, (r+1)/2]. (45)

Lemma 4: Suppose E(z) is an odd-length FIR filter whose support interval is centered at -1:

$$\operatorname{supp_int}(e) = [-r - 1, r - 1], \quad r \equiv \operatorname{supp_rad}(e) \ge 0.$$
(46)

Then the polyphase support interval of the filter is:

supp_int(e) =
$$\begin{cases} [-r/2, r/2] & \text{if } r \text{ is even,} \\ [(-r-1)/2, (r-1)/2] & \text{if } r \text{ is odd.} \end{cases}$$

Proof: There are two cases.

Case: r Even. The even polyphase component is $e_0(k) = e(2k)$ for $-r \le 2k \le r-2$, and e may vanish at either endpoint since both lie in the interior of supp_int(e), so

$$supp_{int}(e_0) \subset [-r/2, (r-2)/2].$$
 (47)

The odd polyphase component is $e_1(k) = e(2k-1)$ for $-r-1 \le 2k-1 \le r-1$, and $e_1(k) \ne 0$ at both endpoints so

$$supp_i(e_1) = [-r/2, r/2].$$
 (48)

Combining (47) and (48) using (8) yields

$$supp_i(e) = [-r/2, r/2].$$
 (49)

Case: r Odd. The even polyphase component is $e_0(k) = e(2k)$ for $-r - 1 \le 2k \le r - 1$, and e is nonzero at both endpoints so

$$supp_i(e_0) = [(-r-1)/2, (r-1)/2].$$
 (50)

The odd polyphase component is $e_1(k) = e(2k-1)$ for $-r \le 2k-1 \le r-2$, and e may vanish at either endpoint since they both lie in the interior of supp_int(e) so

$$supp_i(e_1) \subset [(-r+1)/2, (r-1)/2].$$
 (51)

Combining (50) and (51) using (8) yields

supp_int(
$$e$$
) = [($-r - 1$)/2, ($r - 1$)/2]. (52)

C. The WS Support Interval Formula

We now calculate the support intervals of the intermediate filters, $E_i^{(n)}(z)$, for irreducible lifting cascades in $\mathfrak{S}_{\mathcal{W}}$. Although stated for the case of interest—WS group lifting cascades—the next lemma only uses the support intervals for the filters and does not actually depend on the symmetry of either the intermediate WS filters or the HS lifting filters.

Lemma 5: Let $\mathbf{S}_{N-1}(z) \cdots \mathbf{S}_0(z) \in \mathcal{C}_W$ be an irreducible cascade with intermediate scalar filters $E_i^{(n)}(z)$, i = 0, 1. Let $r_i^{(n)}$ be the support radius of $e_i^{(n)}$, and let $t^{(n)} \ge 1$ be the support radius of the HS lifting filter $S_n(z)$. Then supp_int $\left(e_i^{(n)}\right)$ is centered at -i,

supp_int
$$\left(e_i^{(n)}\right) = \left[-r_i^{(n)} - i, r_i^{(n)} - i\right], \quad i = 0, 1,$$

where

$$r_{m_n}^{(n)} = r_{1-m_n}^{(n)} + 2t^{(n)} - 1 \quad \text{for } n \ge 0,$$
(53)

$$r_{1-m_n}^{(n)} = r_{m_n}^{(n-1)} + 2t^{(n-1)} - 1 \quad \text{for } n \ge 1,$$
(54)

with $r_{1-m_0}^{(0)} = r_{1-m_0}^{(-1)} = 0.$ *Proof:* Induction on *n*.

Case: n = 0, $m_0 = 0$. By (34) the lifting filter is centered at 1/2: supp_int(s_0) = $[-t^{(0)} + 1, t^{(0)}]$. By (19) the lifted intermediate filter is

$$E_0^{(0)}(z) = 1 + z \, S_0(z^2) \tag{55}$$

and its complement is $E_1^{(0)}(z) = B_1(z) = z$, which implies $r_1^{(0)} = 0$. Since $1 = z^0$ we have supp_int $(z^0) = \{0\}$. By (12) and (13)

supp_int
$$(z S_0(z^2)) = \{-1\} + [2(-t^{(0)} + 1), 2t^{(0)}]$$

= $[-2t^{(0)} + 1, 2t^{(0)} - 1].$

Thus, $t^{(0)} \ge 1$ implies

We conclude that $r_0^{(0)} =$

 $supp_i(z^0) \subset (-2t^{(0)} + 1, 2t^{(0)} - 1),$

which justifies applying formula (14) from Lemma 1 to (55):

$$\begin{aligned} \text{supp_int} \left(e_0^{(0)} \right) &= \text{supp_int} \left(z \, S_0(z^2) \right) \\ &= \left[-2t^{(0)} + 1, \, 2t^{(0)} - 1 \right]. \\ 2t^{(0)} - 1, \text{ which verifies (53) since } r_1^{(0)} = 0. \end{aligned}$$

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Case: n = 0, $m_0 = 1$. By (35) the lifting filter is centered at -1/2: supp_int $(s_0) = [-t^{(0)}, t^{(0)} - 1]$. By (19) the lifted intermediate filter is

$$E_1^{(0)}(z) = z + S_0(z^2)$$
(56)

and its complement is $E_0^{(0)}(z) = B_0(z) = 1$, which implies $r_0^{(0)} = 0$. Since supp_int $(z) = \{-1\}$ and supp_int $(S_0(z^2)) = [-2t^{(0)}, 2(t^{(0)} - 1)]$,

 $t^{(0)} \ge 1$ implies

$$supp_i(z) \subset (-2t^{(0)}, 2(t^{(0)} - 1)),$$

which justifies applying formula (14) to (56):

supp_int
$$(e_1^{(0)})$$
 = supp_int $(S_0(z^2))$
= $[-(2t^{(0)}-1)-1, (2t^{(0)}-1)-1].$

We conclude that $r_1^{(0)} = 2t^{(0)} - 1$, which verifies (53) since $r_0^{(0)} = 0$.

Case: $n > 0, m_n = 0$. The case $m_n = 0$ means that

$$E_1^{(n)}(z) = E_1^{(n-1)}(z)$$

and the induction hypothesis implies that supp_int $(e_1^{(n)})$ is centered at -1. The cascade is assumed irreducible so $E_1^{(n-1)}(z)$ must have been lifted in step n-1, and the induction hypothesis therefore implies

$$r_1^{(n)} = r_1^{(n-1)} = r_0^{(n-1)} + 2t^{(n-1)} - 1,$$
(57)

which verifies (54). By (19) the lifted intermediate filter is

$$E_0^{(n)}(z) = E_0^{(n-1)}(z) + S_n(z^2) E_1^{(n-1)}(z).$$
(58)

 $S_n(z)$ is centered at 1/2 so we have

$$supp_int(s_n) = [-t^{(n)} + 1, t^{(n)}]$$

and (12) and (13) give

$$supp_int\left(S_n(z^2) E_1^{(n-1)}(z)\right) = \left[2(-t^{(n)}+1), 2t^{(n)}\right] + \left[-r_1^{(n-1)} - 1, r_1^{(n-1)} - 1\right] = \left[-r_1^{(n-1)} - 2t^{(n)} + 1, r_1^{(n-1)} + 2t^{(n)} - 1\right].$$
(59)

Substitution of (57) into (59) gives

supp_int
$$\left(S_n(z^2) E_1^{(n-1)}(z)\right)$$

= $\left[-r_0^{(n-1)} - 2(t^{(n-1)} + t^{(n)} - 1), r_0^{(n-1)} + 2(t^{(n-1)} + t^{(n)} - 1)\right]$.

The induction hypothesis says that supp_int $(e_0^{(n-1)})$ is centered at 0, with $t^{(n-1)} \ge 1$ and $t^{(n)} \ge 1$, so

$$\begin{split} \text{supp_int} \left(e_0^{(n-1)} \right) &= \left[-r_0^{(n-1)}, \, r_0^{(n-1)} \right] \\ &\subset \left(-r_0^{(n-1)} - 2(t^{(n-1)} + t^{(n)} - 1) \right), \\ &\qquad r_0^{(n-1)} + 2(t^{(n-1)} + t^{(n)} - 1) \right) \,. \end{split}$$

This justifies applying (14) to (58):

$$\operatorname{supp_int}\left(e_{0}^{(n)}\right) = \operatorname{supp_int}\left(S_{n}(z^{2}) E_{1}^{(n-1)}(z)\right) \,.$$

Formula (59) implies that supp_int $\left(e_0^{(n)}\right)$ is centered at 0 and, since $r_1^{(n)} = r_1^{(n-1)}$, it verifies (53): $r_0^{(n)} = r_1^{(n)} + 2t^{(n)} - 1$.

Case:
$$n > 0, m_n = 1$$
. The case $m_n = 1$ means that

$$E_0^{(n)}(z) = E_0^{(n-1)}(z) ,$$

and the induction hypothesis implies that supp_int $\left(e_0^{(n)}\right)$ is centered at 0. The cascade is assumed irreducible so $E_0^{(n-1)}(z)$ must have been lifted in step n-1, and the induction hypothesis therefore implies

$$r_0^{(n)} = r_0^{(n-1)} = r_1^{(n-1)} + 2t^{(n-1)} - 1,$$
(60)

which verifies (54). By (19) the lifted intermediate filter is

$$E_1^{(n)}(z) = E_1^{(n-1)}(z) + S_n(z^2) E_0^{(n-1)}(z).$$
(61)

 $S_n(z)$ is centered at -1/2 so we have

$$supp_int(s_n) = [-t^{(n)}, t^{(n)} - 1]$$

and (12) and (13) give

$$supp_{int} \left(S_n(z^2) E_0^{(n-1)}(z) \right)$$

= $[-2t^{(n)}, 2(t^{(n)} - 1)] + [-r_0^{(n-1)}, r_0^{(n-1)}]$
= $\left[-r_0^{(n-1)} - 2t^{(n)}, r_0^{(n-1)} + 2t^{(n)} - 2 \right].$ (62)

Substitution of (60) into (62) gives

$$\begin{split} \text{supp_int} & \left(S_n(z^2) \, E_0^{(n-1)}(z) \right) \\ & = \left[-r_1^{(n-1)} - 2(t^{(n-1)} + t^{(n)}) + 1 \,, \right. \\ & \left. r_1^{(n-1)} + 2(t^{(n-1)} + t^{(n)}) - 3 \right] \,. \end{split}$$

The induction hypothesis says that supp_int $\left(e_1^{(n-1)}\right)$ is centered at -1, with $t^{(n-1)} \ge 1$ and $t^{(n)} \ge 1$, so

$$\begin{split} \text{supp_int} \left(e_1^{(n-1)} \right) &= \left[-r_1^{(n-1)} - 1, \, r_1^{(n-1)} - 1 \right] \\ &\subset \left(-r_1^{(n-1)} - 2(t^{(n-1)} + t^{(n)}) + 1 \,, \right. \\ &\left. r_1^{(n-1)} + 2(t^{(n-1)} + t^{(n)}) - 3 \right) \end{split}$$

which justifies applying (14) to (61):

$$\begin{aligned} & \text{supp_int}\left(e_1^{(n)}\right) = \text{supp_int}\left(S_n(z^2) \, E_0^{(n-1)}(z)\right) \,. \end{aligned}$$

Formula (62) implies that supp_int $\left(e_1^{(n)}\right)$ is centered at -1 and, since $r_0^{(n)} = r_0^{(n-1)}$, it verifies (53):
 $r_1^{(n)} = r_0^{(n)} + 2t^{(n)} - 1 \,. \end{aligned}$

D. The Support-Covering Property

Now we can prove that $\mathfrak{S}_{\mathcal{W}}$ satisfies the polyphase support-covering hypothesis (28) in Lemma 2. Lemma 6: Let $\mathbf{S}_{N-1}(z) \cdots \mathbf{S}_0(z) \in \mathbb{C}_{\mathcal{W}}$ be irreducible, with intermediate filters $E_i^{(n)}(z)$, i = 0, 1. Then

$$\operatorname{supp_int}\left(\boldsymbol{e}_{1-m_{n}}^{(n)}\right) \subsetneq \operatorname{supp_int}\left(\boldsymbol{e}_{m_{n}}^{(n)}\right) \quad \text{for } n \ge 0.$$
(63)

Proof: Let $r_i^{(n)}$ be the support radius of $e_i^{(n)}$, i = 0, 1. By Lemma 5 it follows that $r_{m_n}^{(n)} + r_{1-m_n}^{(n)}$ is odd, so $r_0^{(n)}$ and $r_1^{(n)}$ always have *opposite* parities. *Case:* $r_0^{(n)}$ *even,* $r_1^{(n)}$ *odd.* By Lemma 3 and Lemma 4 the polyphase support intervals are:

supp_int
$$\left(e_0^{(n)} \right) = \left[-r_0^{(n)}/2, r_0^{(n)}/2 \right],$$
 (64)

supp_int
$$\left(\boldsymbol{e}_{1}^{(n)} \right) = \left[-\left(r_{1}^{(n)} + 1 \right) / 2, \left(r_{1}^{(n)} - 1 \right) / 2 \right].$$
 (65)

If $m_n = 0$ then applying Lemma 5 to (64) gives

supp_int
$$\left(\boldsymbol{e}_{0}^{(n)} \right)$$

= $\left[-\left(r_{1}^{(n)} + 2t^{(n)} - 1 \right) / 2, \left(r_{1}^{(n)} + 2t^{(n)} - 1 \right) / 2 \right]$.

Since $t^{(n)} \ge 1$, comparison with (65) yields

$$\begin{split} &-\left(r_1^{(n)} + 2t^{(n)} - 1\right)/2 \leq -\left(r_1^{(n)} + 1\right)/2 \quad \text{and} \\ &\left(r_1^{(n)} + 2t^{(n)} - 1\right)/2 > \left(r_1^{(n)} - 1\right)/2 \,, \end{split}$$

which imply supp_int $(e_1^{(n)}) \subsetneq$ supp_int $(e_0^{(n)})$. If $m_n = 1$ then applying Lemma 5 to (65) gives

supp_int
$$(e_1^{(n)})$$

= $\left[-\left(r_0^{(n)} + 2t^{(n)}\right)/2, \left(r_0^{(n)} + 2t^{(n)} - 2\right)/2 \right]$

Since $t^{(n)} \ge 1$, comparison with (64) yields

$$\begin{split} &-\left(r_0^{(n)}+2t^{(n)}\right)/2<-r_0^{(n)}/2 \quad \text{and} \\ &\left(r_0^{(n)}+2t^{(n)}-2\right)/2\geq r_0^{(n)}/2\,, \end{split}$$

which imply supp_int $(e_0^{(n)}) \subseteq$ supp_int $(e_1^{(n)})$. *Case:* $r_0^{(n)}$ odd, $r_1^{(n)}$ even. By Lemma 3 and Lemma 4 the polyphase support intervals are:

$$supp_int\left(e_0^{(n)}\right) = \left[-\left(r_0^{(n)} - 1\right)/2, \left(r_0^{(n)} + 1\right)/2\right], \tag{66}$$

supp_int
$$\left(e_1^{(n)} \right) = \left[-r_1^{(n)}/2, r_1^{(n)}/2 \right]$$
. (67)

If $m_n = 0$ then applying Lemma 5 to (66) gives

supp_int
$$\left(e_0^{(n)} \right)$$

= $\left[-\left(r_1^{(n)} + 2t^{(n)} - 2 \right) / 2, \left(r_1^{(n)} + 2t^{(n)} \right) / 2 \right]$.

Since $t^{(n)} \ge 1$, comparison with (67) yields

$$\begin{split} - \left(r_1^{(n)} + 2t^{(n)} - 2 \right) / 2 &\leq -r_1^{(n)} / 2 \quad \text{and} \\ \left(r_1^{(n)} + 2t^{(n)} \right) / 2 &> r_1^{(n)} / 2 \,, \end{split}$$

which implies supp_int $(e_1^{(n)}) \subseteq$ supp_int $(e_0^{(n)})$. If $m_n = 1$ then applying Lemma 5 to (67) gives

supp_int
$$\left(e_1^{(n)}\right)$$

= $\left[-\left(r_0^{(n)} + 2t^{(n)} - 1\right)/2, \left(r_0^{(n)} + 2t^{(n)} - 1\right)/2\right].$

Since $t^{(n)} \ge 1$, comparison with (66) yields

$$-\left(r_{0}^{(n)}+2t^{(n)}-1\right)/2 < -\left(r_{0}^{(n)}-1\right)/2 \text{ and} \\ \left(r_{0}^{(n)}+2t^{(n)}-1\right)/2 \geq \left(r_{0}^{(n)}+1\right)/2,$$

which implies supp_int $\left(e_{0}^{(n)}
ight) \varsubsetneq$ supp_int $\left(e_{1}^{(n)}
ight)$.

IV. HALF-SAMPLE SYMMETRIC FILTER BANKS

We briefly review the definitions from [1, Section IV-B] of the group lifting structures for the unimodular HS class, \mathfrak{H} . In the irreversible case, the scaling matrices are $\mathfrak{D} \equiv {\mathbf{D}_K : K \neq 0}$. The groups of upper (respectively, lower) triangular lifting matrices, $\mathcal{U} \equiv v(\mathcal{P})$ (respectively, $\mathcal{L} \equiv \lambda(\mathcal{P})$), are defined by the same group of whole-sample antisymmetric real Laurent polynomials:

$$\mathcal{P} \equiv \left\{ S(z) \in \mathbb{R}[z, z^{-1}] : S(z^{-1}) = -S(z) \right\} .$$
(68)

The set of concentric equal-length base HS filter banks defined in [1, Section IV-B] is

$$\mathfrak{B}_{\mathfrak{H}} \equiv \{ \mathbf{B} \in \mathfrak{H} : \operatorname{order}(B_0) = \operatorname{order}(B_1) \}.$$
(69)

The lifting cascade group for the irreversible HS class is $C_{\mathfrak{H}} \equiv \langle \mathfrak{U} \cup \mathcal{L} \rangle$ and the irreversible HS group lifting structure is $\mathfrak{S}_{\mathfrak{H}} \equiv (\mathfrak{D}, \mathfrak{U}, \mathcal{L}, \mathfrak{B}_{\mathfrak{H}})$.

There are no scaling matrices in the reversible case, $\mathcal{D}_r \equiv \{\mathbf{I}\}$, and the lifting filters (68) are restricted to those having dyadic rational coefficients. The set of base filter banks, $\mathfrak{B}_{\mathfrak{H}_r}$, for the reversible HS class is defined in [1, Section IV-B] to be the set of all matrices in $\mathfrak{B}_{\mathfrak{H}}$ that have dyadic lifting factorizations with no scaling matrices. Note that there are no other requirements (e.g., symmetry) for the dyadic lifting filters that factor reversible, concentric, equal-length HS base filter banks. The reversible HS class is *defined* to be

 $\mathfrak{H}_r \equiv \mathfrak{C}_{\mathfrak{H}_r} \mathfrak{B}_{\mathfrak{H}_r}$ where $\mathfrak{C}_{\mathfrak{H}_r} \equiv \langle \mathfrak{U}_r \cup \mathfrak{L}_r \rangle$,

and its group lifting structure is $\mathfrak{S}_{\mathfrak{H}_r} \equiv (\mathfrak{D}_r, \mathfrak{U}_r, \mathfrak{L}_r, \mathfrak{B}_{\mathfrak{H}_r}).$

A. Main Uniqueness Result

Our main result on HS filter banks shows that irreducible group lifting factorizations in $\mathfrak{S}_{\mathfrak{H}}$ are "almost" unique and that irreducible lifting factorizations in $\mathfrak{S}_{\mathfrak{H}_r}$ are unique.

Theorem 2 (Uniqueness of HS group lifting factorizations): Let $\mathfrak{S}_{\mathfrak{H}_r}$ be the group lifting structures defined in [1, Section IV-B]. Every filter bank in \mathfrak{H} has an irreducible group lifting factorization in $\mathfrak{S}_{\mathfrak{H}}$ that is unique modulo rescaling. Every filter bank in \mathfrak{H}_r has a unique irreducible group lifting factorization factorization in $\mathfrak{S}_{\mathfrak{H}_r}$.

Proof: Existence of irreducible group lifting factorizations for these classes was discussed in [1, Section IV-B].

Uniqueness: Let $\mathbf{B}(z) \in \mathfrak{B}_{\mathfrak{H}}$. By the definition of \mathfrak{H} [1, Definition 9], both impulse responses are symmetric about -1/2 so (69) implies that b_0 and b_1 have the same scalar support intervals. By Lemma 7 (Section IV-B below), b_0 and b_1 have the same *polyphase* support intervals, verifying (27). Lemma 9 (Section IV-D below) shows that irreducible lifting cascades in $\mathfrak{C}_{\mathfrak{H}}\mathfrak{B}_{\mathfrak{H}}$ satisfy (28), so Lemma 2 implies that $\mathfrak{S}_{\mathfrak{H}}$ is order-increasing. \mathfrak{D} -invariance follows from [1, Lemma 2], so [1, Theorem 1] implies that irreducible lifting factorizations in $\mathfrak{S}_{\mathfrak{H}}$ are unique modulo rescaling.

In the reversible case, $\mathcal{C}_{\mathfrak{H}_r} < \mathcal{C}_{\mathfrak{H}}$ and $\mathfrak{B}_{\mathfrak{H}_r} \subset \mathfrak{B}_{\mathfrak{H}}$. The *reversible* HS group lifting factorizations thus form a subset of the *irreversible* factorizations and are therefore unique modulo rescaling. However, given two irreducible factorizations of the same reversible filter bank,

$$\mathbf{S}_{N-1}(z)\cdots\mathbf{S}_0(z)\mathbf{B}(z)=\mathbf{S}'_{N-1}(z)\cdots\mathbf{S}'_0(z)\mathbf{B}'(z)\,,$$

we have K' = K = 1 since there are no scaling operations. Thus, $\mathbf{B}'(z) = \mathbf{B}(z)$ and $\mathbf{S}'_i(z) = \mathbf{S}_i(z)$ for all *i*, implying uniqueness of reversible HS group lifting factorizations.

Remarks: As in the WS case, Theorem 2 provides a more complete theory of both existence *and* uniqueness for cascade factorizations of HS filter banks than [5].

1) Applications: The JPEG 2000 standard does not contain source coding specifications for HS filter banks as a class, in part because of the complications inherent in their lifting factorizations. Instead, JPEG 2000 Part 2 [4, Annex H] contains specifications for lifting factorizations of *arbitrary* user-defined FIR filter banks, with no assumptions about the symmetries of either the filters or their lifting factors. Part 2 Annex H does contain a number of examples of HS filter banks, however, and we can use Theorem 2 to answer some questions about lifting factorization of HS filter banks that arose on the ISO committee during the writing of Part 2.

First let us address the filter bank normalization specifications in JPEG 2000 Part 2. We mentioned in Section I-C that [1, Theorem 1] yields unique irreducible factorizations if one normalizes a degree of freedom in the base filter banks, e.g., by fixing a common nonzero value for $B_0(1)$. As explained in [8],

JPEG 2000 Part 2 requires the normalization $H_0(1) = 1$, a similar constraint. The JPEG 2000 constraint does *not* imply uniqueness, however. Let

$$\mathbf{H}(z) = \mathbf{D}_K \,\mathbf{S}_{N-1}(z) \cdots \mathbf{S}_0(z) \,\mathbf{B}(z) \tag{70}$$

satisfy $H_0(1) = 1$, and pick $K' \neq K$. Define $\alpha \equiv K/K'$ so that $\mathbf{D}_K = \mathbf{D}_{K'}\mathbf{D}_{\alpha}$ and substitute in (70). By [1, eqn. (28)],

$$\mathbf{H}(z) = \mathbf{D}_{K'} \gamma_{\alpha} \mathbf{S}_{N-1}(z) \cdots \gamma_{\alpha} \mathbf{S}_0(z) \mathbf{B}'(z) ,$$

where $\mathbf{B}'(z) \equiv \mathbf{D}_{\alpha} \mathbf{B}(z)$. This gain transfer between the scaling and base matrices leaves $\mathbf{H}(z)$ unchanged, and therefore does *not* violate the constraint $H_0(1) = 1$. According to Theorem 2, though, this is the *only* way in which two irreducible HS group lifting factorizations of $\mathbf{H}(z)$ can differ.

As described in [1, Section I-A], the JPEG 2000 example filter bank in [4, Annex H.4.1.2.1] contains a lifting factorization of an irreversible 6-tap/10-tap HS wavelet filter bank, $\mathbf{H}_{6,10}(z)$, that is lifted from a concentric 6-tap/6-tap HS base filter bank, $\mathbf{B}_{6.6}(z)$, by a second-order WA lifting step:

$$\mathbf{H}_{6,10}(z) = \mathbf{S}_0(z) \,\mathbf{B}_{6,6}(z) \,. \tag{71}$$

Interestingly, although $\mathbf{H}_{6,10}(z)$ is irreversible, the scaling factor is trivial (K = 1), which is why the scaling matrix has been omitted from (71). Factorization (71) is unique modulo rescaling by Theorem 2, which answers in the negative the question of whether $\mathbf{H}_{6,10}(z)$ can be lifted from the Haar filter bank by a clever choice of WA lifting filters. The same answer holds for the question of whether the 10-tap/18-tap HS filter bank in [4, Annex H.4.1.2.2], which is lifted from a 10-tap/10-tap HS base filter bank by a fourth-order WA lifting filter, can similarly be lifted from the Haar using WA lifting filters.

Theorem 2 also applies to the group lifting structure \mathfrak{S}_{ELASF} for the ELASF family of reversible HS filter banks lifted from the Haar, as defined by M. Adams [9], [10]. It was shown in [1, Example 6] that the ELASF filter banks form a subset of the reversible HS class, \mathfrak{H}_r , and that every group lifting factorization in \mathfrak{S}_{ELASF} is a factorization in $\mathfrak{S}_{\mathfrak{H}_r}$. Thus, irreducible group lifting factorizations in \mathfrak{S}_{ELASF} are unique.

The lemmas needed to verify hypothesis (28) of Lemma 2 occupy the remainder of Section IV.

B. Polyphase Support Intervals

Polyphase support intervals for FIR filters centered at -1/2 are sufficient to cover HS group lifting structures. The formulas do not depend on linear phase properties of the filters.

Lemma 7: Suppose E(z) is an even-length FIR filter whose support interval is centered at -1/2:

$$\operatorname{supp_int}(e) = [-r, r-1] \quad \text{for } r \equiv \operatorname{supp_rad}(e) \ge 1.$$
 (72)

Then the polyphase support interval of the filter is:

supp_int(e) =
$$\begin{cases} [-r/2, r/2] & \text{if } r \text{ is even,} \\ [-(r-1)/2, (r-1)/2] & \text{if } r \text{ is odd.} \end{cases}$$

Proof: There are two cases.

Case: r Even. The even polyphase component is $e_0(k) = e(2k)$ for $-r \le 2k \le r-2$, and $e(-r) \ne 0$ while e(r-2) may be zero, so

$$supp_i(e_0) = [-r/2, n]$$
 (73)

for some n with $-r/2 \le n \le (r-2)/2$.

The odd component is $e_1(k) = e(2k-1)$ for $-r+1 \le 2k-1 \le r-1$, and $e(r-1) \ne 0$ while e(-r+1) may be zero, so

$$supp_int(e_1) = [n, r/2]$$
(74)

for some n with $(-r+2)/2 \le n \le r/2$. Combining (73) and (74) using (8) yields

$$supp_int(e) = supp_int(e_0) \lor supp_int(e_1)$$
$$= [-r/2, r/2].$$
(75)

Case: r *Odd.* The even polyphase component is $e_0(k) = e(2k)$ for $-r+1 \le 2k \le r-1$, and $e(r-1) \ne 0$ while e(-r+1) may be zero, so

$$supp_int(e_0) = [n, (r-1)/2]$$
 (76)

for some *n* with $(-r+1)/2 \le n \le (r-1)/2$.

The odd component is $e_1(k) = e(2k-1)$ for $-r \le 2k-1 \le r-2$, and $e(-r) \ne 0$ while e(r-2) may be zero, so

$$supp_int(e_1) = [(-r+1)/2, n]$$
 (77)

for some n with $(-r+1)/2 \le n \le (r-1)/2$. Combining (76) and (77) using (8) yields

$$supp_i(e) = [-(r-1)/2, (r-1)/2].$$
 (78)

C. The HS Support Interval Formula

As in the WS case, the HS support interval formula does not depend on linear phase properties of the filters.

Lemma 8: Let $\mathbf{S}_{N-1}(z) \cdots \mathbf{S}_0(z) \mathbf{B}(z) \in \mathcal{C}_5 \mathfrak{B}_5$ be irreducible, with intermediate scalar filters $E_i^{(n)}(z)$, i = 0, 1. Let $r_i^{(n)}$ be the support radius of $e_i^{(n)}$, and let $t^{(n)} \ge 1$ be the support radius of the WA lifting filter $S_n(z)$. Then supp_int $\left(e_i^{(n)}\right)$ is centered at -1/2,

supp_int
$$(e_i^{(n)}) = [-r_i^{(n)}, r_i^{(n)} - 1], \quad i = 0, 1,$$

where

$$r_{m_n}^{(n)} = r_{1-m_n}^{(n)} + 2t^{(n)} \quad \text{for } n \ge 0,$$
(79)

$$r_{1-m_n}^{(n)} = r_{m_n}^{(n-1)} + 2t^{(n-1)} \quad \text{for } n \ge 1,$$
(80)

with $r_{1-m_0}^{(0)} = r^{(-1)}$, the common support radius of both base filters, b_i .

Proof. Induction on n. In all cases, the WA lifting filter is centered at 0: supp_int $(s_n) = [-t^{(n)}, t^{(n)}]$. The concentric equal-length base filter bank, $\mathbf{B}(z) \in \mathfrak{B}_{\mathfrak{H}}$, has both filters centered at -1/2:

supp_int
$$(b_i) = [-r^{(-1)}, r^{(-1)} - 1], \quad i = 0, 1.$$
 (81)

Case: n = 0. By (19) the lifted intermediate filter is

$$E_{m_0}^{(0)}(z) = B_{m_0}(z) + S_0(z^2) B_{1-m_0}(z)$$
(82)

and its complement is $E_{1-m_0}^{(0)}(z) = B_{1-m_0}(z)$, implying $r_{1-m_0}^{(0)} = r^{(-1)}$. By (12) and (13), supp int $(S_0(z^2) B_{1-m_0}(z))$

upp_int
$$(S_0(z^2) B_{1-m_0}(z))$$

= $[-2t^{(0)}, 2t^{(0)}] + [-r^{(-1)}, r^{(-1)} - 1]$
= $[-2t^{(0)} - r^{(-1)}, 2t^{(0)} + r^{(-1)} - 1]$.

By (81), $t^{(0)} \ge 1$ implies

$$\operatorname{supp_int}(b_{m_0}) \subset (-2t^{(0)} - r^{(-1)}, 2t^{(0)} + r^{(-1)} - 1),$$

which justifies applying formula (14) from Lemma 1 to (82):

supp_int
$$\left(e_{m_0}^{(0)}\right)$$
 = supp_int $\left(S_0(z^2) B_{1-m_0}(z)\right)$
= $\left[-2t^{(0)} - r^{(-1)}, 2t^{(0)} + r^{(-1)} - 1\right]$

This shows that supp_int $\left(e_{m_0}^{(0)}\right)$ is centered at -1/2 with

supp_rad
$$\left(e_{m_0}^{(0)} \right) = r^{(-1)} + 2t^{(0)}$$

verifying (79).

Case: n > 0. The complement of the lifted filter is

$$E_{1-m_n}^{(n)}(z) = E_{1-m_n}^{(n-1)}(z) = E_{m_{n-1}}^{(n-1)}(z)$$

by irreducibility. Use the induction hypothesis to apply (79) to $E_{m_{n-1}}^{(n-1)}(z)$ and note that $m_n = 1 - m_{n-1}$:

$$r_{1-m_n}^{(n)} = r_{1-m_n}^{(n-1)} = r_{m_n-1}^{(n-1)} = r_{m_n}^{(n-1)} + 2t^{(n-1)},$$
(83)

which verifies (80). By (19) the lifted intermediate filter is

$$E_{m_n}^{(n)}(z) = E_{m_n}^{(n-1)}(z) + S_n(z^2) E_{1-m_n}^{(n-1)}(z).$$
(84)

supp_int $\left(e_{1-m_n}^{(n-1)}\right)$ is centered at -1/2 by induction, so (12) and (13) give

$$supp_int \left(S_n(z^2) E_{1-m_n}^{(n-1)}(z) \right) \\ = \left[-2t^{(n)}, 2t^{(n)} \right] + \left[-r_{1-m_n}^{(n-1)}, r_{1-m_n}^{(n-1)} - 1 \right] \\ = \left[-2t^{(n)} - r_{1-m_n}^{(n-1)}, 2t^{(n)} + r_{1-m_n}^{(n-1)} - 1 \right].$$

$$(85)$$

Substitution of (83) into (85) gives

supp_int
$$\left(S_n(z^2) E_{1-m_n}^{(n-1)}(z)\right)$$

= $\left[-r_{m_n}^{(n-1)} - 2(t^{(n-1)} + t^{(n)}), r_{m_n}^{(n-1)} + 2(t^{(n-1)} + t^{(n)}) - 1\right].$

The induction hypothesis says that supp_int $\left(e_{m_n}^{(n-1)}\right)$ is centered at -1/2, with $t^{(n-1)} \ge 1$ and $t^{(n)} \ge 1$, so

supp_int
$$\left(e_{m_n}^{(n-1)}\right) = \left[-r_{m_n}^{(n-1)}, r_{m_n}^{(n-1)} - 1\right]$$

 $\subset \left(-r_{m_n}^{(n-1)} - 2(t^{(n-1)} + t^{(n)}), r_{m_n}^{(n-1)} + 2(t^{(n-1)} + t^{(n)}) - 1\right)$

This justifies applying (14) to (84):

supp_int
$$\left(e_{m_n}^{(n)}\right)$$
 = supp_int $\left(S_n(z^2) E_{1-m_n}^{(n-1)}(z)\right)$.

Formula (85) shows that supp_int $\left(e_{m_n}^{(n)}\right)$ is centered at -1/2, and substituting $r_{1-m_n}^{(n)} = r_{1-m_n}^{(n-1)}$ into (85) verifies (79):

$$r_{m_n}^{(n)} = r_{1-m_n}^{(n)} + 2t^{(n)}$$
.

D. The Support-Covering Property

We now prove that $\mathfrak{S}_{\mathfrak{H}}$ satisfies the polyphase support-covering hypothesis (28) in Lemma 2.

Lemma 9: Let $\mathbf{S}_{N-1}(z) \cdots \mathbf{S}_0(z) \mathbf{B}(z) \in \mathcal{C}_{\mathfrak{H}} \mathfrak{B}_{\mathfrak{H}}$ be irreducible, with intermediate filters $E_i^{(n)}(z), i =$ 0, 1. Then

$$\operatorname{supp_int}\left(\boldsymbol{e}_{1-m_{n}}^{(n)}\right) \subsetneq \operatorname{supp_int}\left(\boldsymbol{e}_{m_{n}}^{(n)}\right) \quad \text{for } n \ge 0.$$
(86)

Proof: Let $r_i^{(n)}$ be the support radius of $e_i^{(n)}$, i = 0, 1. By Lemma 8 it follows that $r_{m_n}^{(n)} + r_{1-m_n}^{(n)}$ is even, so $r_0^{(n)}$ and $r_1^{(n)}$ always have equal parities. Case: $r_0^{(n)}$, $r_1^{(n)}$ even. Regardless of the value of m_n , Lemma 7 implies

$$supp_{int} \left(e_{m_n}^{(n)} \right) = \left[-r_{m_n}^{(n)}/2, r_{m_n}^{(n)}/2 \right],$$
(87)

supp_int
$$\left(e_{1-m_n}^{(n)} \right) = \left[-r_{1-m_n}^{(n)}/2, r_{1-m_n}^{(n)}/2 \right]$$
 (88)

Applying Lemma 8 to (87) gives

supp_int
$$\left(\boldsymbol{e}_{m_n}^{(n)} \right)$$

= $\left[- \left(r_{1-m_n}^{(n)} + 2t^{(n)} \right) / 2, \left(r_{1-m_n}^{(n)} + 2t^{(n)} \right) / 2 \right].$

Since $t^{(n)} \ge 1$, comparison with (88) shows that

$$\operatorname{supp_int}\left(oldsymbol{e}_{1-m_n}^{(n)}
ight) \varsubsetneq \operatorname{supp_int}\left(oldsymbol{e}_{m_n}^{(n)}
ight)$$
 .

Case: $r_0^{(n)}$, $r_1^{(n)}$ odd. Regardless of the value of m_n , Lemma 7 implies

supp_int
$$\left(\boldsymbol{e}_{m_n}^{(n)} \right)$$

= $\left[-\left(r_{m_n}^{(n)} - 1 \right) / 2, \left(r_{m_n}^{(n)} - 1 \right) / 2 \right],$ (89)

supp_int
$$\left(e_{1-m_n}^{(n)} \right)$$

= $\left[-\left(r_{1-m_n}^{(n)} - 1 \right) / 2, \left(r_{1-m_n}^{(n)} - 1 \right) / 2 \right]$. (90)

Applying Lemma 8 to (89) gives

supp_int
$$\left(e_{m_n}^{(n)} \right)$$

= $\left[-\left(r_{1-m_n}^{(n)} + 2t^{(n)} - 1 \right) / 2, \left(r_{1-m_n}^{(n)} + 2t^{(n)} - 1 \right) / 2 \right].$

Since $t^{(n)} \ge 1$, comparison with (90) shows that

$$\operatorname{supp_int}\left(\boldsymbol{e}_{1-m_n}^{(n)}\right) \subsetneq \operatorname{supp_int}\left(\boldsymbol{e}_{m_n}^{(n)}\right)$$
.

X 7	Covermente
ν.	CONCLUSIONS

The theory of group lifting structures developed in [1] has been used to prove uniqueness of linear phase lifting factorizations for the two nontrivial classes of linear phase two-channel perfect reconstruction filter banks, the WS and HS classes. A key technical result, Lemma 2, provides sufficient conditions for inferring that a group lifting structure satisfies the critical polyphase order-increasing property defined in [1]. In the WS case, Theorem 1 states that both reversible and irreversible WS filter banks have unique irreducible group lifting factorizations using HS lifting filters. Theorem 1 implies, e.g., that left matrix factorizations of WS filter banks into HS lifting filters are identical to right matrix factorizations. The scope of Theorem 1 covers the specifications for WS filter banks in Part 2 Annex G of the ISO/IEC JPEG 2000 standard.

In the HS case, Theorem 2 covers liftings from equal-length HS base filter banks using WA lifting filters. In the irreversible case, the WA lifting filters and the equal-length HS base filter banks are unique modulo rescaling. This implies, for instance, that 6-tap/10-tap and 10-tap/18-tap HS filter banks specified in JPEG 2000 Part 2 Annex H *cannot* be lifted from the Haar filter bank using WA lifting filters. Group lifting factorizations of reversible HS filter banks are unique, including M. Adams' ELASF class of reversible HS filter banks lifted from the Haar. A follow-on paper will characterize the structure of the groups associated with group lifting structures that satisfy the hypotheses of the uniqueness theorem.

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