Dynamic Spectrum Management with the Competitive Market Model

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***Abstract*— [1], [2] have shown that dynamic spectrum management (DSM) using the market competitive equilibrium (CE), which sets a price for transmission power on each channel, leads to better system performance in terms of the total data transmission rate (by reducing cross talk), than using the Nash equilibrium (NE). But how to achieve such a CE is an open problem. We show that the CE is the solution of a linear complementarity problem (LCP) and can be computed efficiently. We propose a decentralized tâtonnement process for adjusting the prices to achieve a CE. We show that under reasonable conditions, any tâtonnement process converges to the CE. The conditions are that users of a channel experience the same noise levels and that the cross-talk effects between users are low-rank and weak.**

***Index Terms*—Radio spectrum management, dynamic spectrum management (DSM), linear complementarity problem (LCP), competitive equilibrium**

I. Introduction

Dynamic spectrum management (DSM) is a technology to efficiently share the spectrum among the users in a communication system. DSM can be used in the digital subscriber line (DSL) systems to reduce cross-talk interference and improve total system throughput [3]–[5]. DSM is also a promising candidate for multiple access in cognitive radio [6]. In DSM, multiple users coexist in a channel, and this causes co-channel interference. The goal of DSM is to manage the power allocations in all the channels to maximize the sum of the data rates of all the users, subject to power constraints [3]. Unfortunately, this problem is non-convex and cannot be solved efficiently in polynomial time [5]. While we will use game-theoretic tools to find decentralized solutions to DSM, it is worth noting that [5] give a computationally tractable but centralized optimization formulation that is asymptotically optimal as the number of users becomes large.

Recently, the game-theoretic formulation of DSM has attracted interest in a variety of contexts including DSL [3], [4], [7] and wireless [8]. In the game-theoretic formulation, each user maximizes her data rate, the Shannon utility function, given knowledge of the other users' current power allocations. The Nash equilibrium (NE) of this competitive game has been well-studied, e.g. [3], [4], [7], [9], [10]. Under certain conditions the NE exists and is unique. One merit of the game theoretic formulation is that the user's problem can be solved efficiently because it is convex when holding the other users' power allocations fixed. However, the power allocation in a NE may not be socially optimal. Because of the non-cooperative nature of the NE, users tend to compete for "good" channels regardless of the interferences caused to others, to the detriment of overall system efficiency, when they may all be better off using different channels to avoid interference. This is an instance of the well-known "tragedy of the commons" from economics [11]. [1] presents a simple example demonstrating the inefficiency of the NE in DSM.

Y. Xie was supported by a Stanford Graduate Fellowship. Y. Ye was supported by Boeing and NSF grant DMS-0604513.

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Therefore we turn to the competitive market model for DSM described in [1]. (Taking a different approach to this problem, [12] analyze a generalization of the Nash Equilibrium that they call a "conjectural equilibrium".) In the competitive market model, each channel has a fictitious price per unit power, and each user purchases some power allocation in these channels, given her budget constraint, to maximize her data rate. The prices are determined by a central manager to keep the total power allocated in each channel to be below a spectral mask. A competitive equilibrium (CE) of a market model is a set of prices and the corresponding power allocations which maximizes all users' utility and clears the market, i.e., makes the total power allocated meet the spectral mask. While the CE has received a lot of recent attention in computer science, its application to resource management for communication systems appears rare. The existence of a CE for DSM was proven in [1]. Also, [2] showed that the CE achieves greater social utility (total transmission rate) than the NE, with properly assigned budgets to guarantee fairness among all users. It is worth noting that algorithm proposed in [2] to determine the budgets has low communication complexity because it only requires the data rate of each user rather than the complete channel state information. However, how to find a CE prices efficiently is an open problem. Traditionally, the prices are determined by distributed, auction-type algorithms called tâtonnement processes [11]. But it is not known whether such processes converge with the Shannon utility function.

This paper focuses on determining the CE of the competitive market model for DSM and makes three contributions. We first show that the CE is the solution of a linear complementarity problem (LCP) [13] despite the apparent nonlinearity of the problem. [4] showed a similar result for the NE. Secondly, we show that when the interference coefficients are user symmetric, then the problem is equivalent to finding KKT points of a quadratic program (QP), for which a fully polynomial-time approximation scheme (FPTAS) exists [14]. Lastly, we present decentralized tâtonnement processes to solve the CE, where the manager adjusts the prices based on the excess demand (the difference between the total power and the spectral mask). We prove under some low-rank conditions, the prices converge to the equilibrium prices (hence the tâtonnement processes converge to the CE).

The paper is organized as follows. The next section presents the problem formulation. Section III presents the LCP formulation and the FPTAS result, and Section IV is about decentralized price-adjustment tâtonnment processes. We conclude in Section VI. Technical proofs are in the appendix.

The notation in this paper is conventional. We use lower case, bold letters for vectors and capital, bold letters for matrices. $\boldsymbol{X} \geq 0$ and $\boldsymbol{x} \geq 0$ are elementwise inequalities while $\boldsymbol{X} \succeq 0$ and $\boldsymbol{X} \succ 0$ indicate that $\boldsymbol{X}$ is semi-positive definite and positive definite, respectively. In addition, $\boldsymbol{I}$ is the identity matrix; $\rho(\boldsymbol{X})$ is the spectral radius of $\boldsymbol{X}$; $\boldsymbol{X}^\dagger$ is the Moore-Penrose pseudoinverse of $\boldsymbol{X}$; and $(x)^+ := \max\{x, 0\}$.

II. Problem Formulation

Consider a communication system consisting of n users and m channels. Multiple users may use the same channel (at the same time) causing interference to each other. Suppose the power allocated by user i to channel j is $x_{ij} \geq 0$. The total power allocated by all the users in the jth channel is bounded above by the spectral mask c_j, $\sum_{i=1}^n x_{ij} \leq c_j$, for regulatory reasons. For example, in overlay cognitive radio [6], we may want to limit the interference experienced by the primary user due to transmissions from secondary users. (In that case we should actually scale the power allocations so that x_{ij} represents the power received by the primary user on

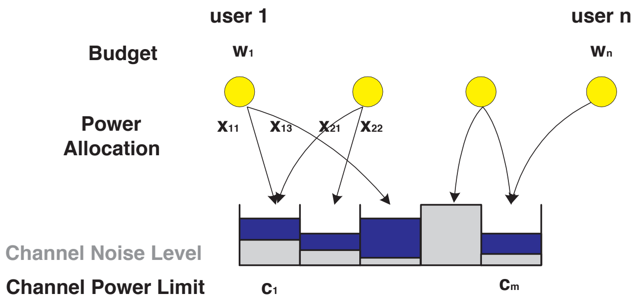

Fig. 1. Competitive spectrum market model.

channel j from user i. Such a scaling carries through the analysis cleanly.) To achieve an efficient allocation of spectrum we associate a price $p_j > 0$ with each channel j. For a given vector of prices, $\boldsymbol{p} = [p_1, \ldots, p_m]^\top$, each user i chooses the power allocation $\boldsymbol{x}_i = [x_{i1}, \ldots, x_{im}]^\top$ that maximizes her utility function subject to her budget w_i. ([2] discusses how to choose the users' budgets.) The spectrum manager adjusts the prices, so that eventually the "market clears": the demand in each channel, $\sum_{i=1}^n x_{ij}$, equals the supply, c_j.

In the weak-interference regime, user i's utility is her total data transmission rate across all the channels (Shannon utility):

$$u_i(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i) = \sum_{j=1}^{m} \log\left(1 + \frac{x_{ij}}{\sigma_{ij} + \sum_{k \neq i} a_{ik}^j x_{kj}}\right). \tag{1}$$

Here $\bar{\boldsymbol{x}}_i = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i-1}; \boldsymbol{x}_{i+1}, \ldots, \boldsymbol{x}_n]^\top$ is the power allocation of the other $n-1$ users; $\sigma_{ij} > 0$ is the noise level experienced by user i on channel j; and $a_{ik}^j \geq 0$ is the cross-talk coefficient for interference to user i on channel j from user $k \neq i$. The optimal power allocation $\boldsymbol{x}_i^*(\boldsymbol{p}, \bar{\boldsymbol{x}}_i)$ of user i, when she faces prices $\boldsymbol{p}$ and power allocations $\bar{\boldsymbol{x}}_i$ of the other users, is determined by the following convex optimization problem

$$\begin{aligned} \boldsymbol{x}_i^*(\boldsymbol{p}, \bar{\boldsymbol{x}}_i) = \arg\max_{\boldsymbol{x}_i} \ & u_i(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i) \\ s.t. \quad & \boldsymbol{p}^\top \boldsymbol{x}_i \leq w_i, \\ & \boldsymbol{x}_i \geq 0, \end{aligned} \tag{2}$$

which has a unique solution because it is strictly convex. Fig. 1 illustrates this competitive market model.

The competitive equilibrium (CE) [11] of this model is the vector of prices $\boldsymbol{p}^*$ and the corresponding user-optimal power allocations $\{x_{ij}^*\}$ so that the market clears, $\sum_{i=1}^n x_{ij} = c_j$. [1] proved the existence of a CE in this model. It can be easily shown that, given $\boldsymbol{p}$, each users' power allocation problem (2) has a water-filling solution

$$x_{ij}^* = \left(\frac{\nu_i}{p_j} - \sigma_{ij} - \sum_{k \neq i} a_{ik}^j x_{kj}^*\right)^+ \tag{3}$$

where the dual variable $\nu_i \geq 0$ is determined by the budget constraint $\boldsymbol{p}^\top \boldsymbol{x}_i \leq w_i$, which is tight at the CE [1].

III. CE AS LCP

By applying the fact that $x = y^+$ is equivalent to $x \geq y \wedge x(x-y) = 0 \wedge x \geq 0$ to (3), we obtain the following nonlinear equations that characterize the CE:

$$\begin{aligned} x_{ij} &\geq \frac{\nu_i}{p_j} - \sigma_{ij} - \sum_{k \neq i} a_{ik}^j x_{kj} \quad \forall ij, \\ x_{ij}\left(x_{ij} - \frac{\nu_i}{p_j} + \sigma_{ij} + \sum_{k \neq i} a_{ik}^j x_{kj}\right) &= 0 \quad \forall ij, \\ \boldsymbol{p}^\top \boldsymbol{x}_i &= w_i \quad \forall i, \\ \sum_i x_{ij} &= c_j \quad \forall j, \\ x_{ij} &\geq 0 \quad \forall ij. \end{aligned} \tag{4}$$

Now we reformulate these equations as an LCP. Let the revenue of user i on channel j be $r_{ij} := x_{ij}p_j$. Define the vectors $\boldsymbol{r}_j := [r_{1j}, \ldots, r_{nj}]^\top$, $\boldsymbol{\sigma}_j := [\sigma_{1j}, \ldots, \sigma_{nj}]^\top$, $\boldsymbol{w} := [w_1, \ldots, w_n]^\top$, and $\boldsymbol{\nu} := [\nu_1, \ldots, \nu_n]^\top$. Also the define matrices $\boldsymbol{A}_j$ of cross-talk coefficients, $[\boldsymbol{A}_j]_{ik} = a_{ik}^j$ for $k \neq i$ with ones on the diagonal, $[\boldsymbol{A}_j]_{ii} = 1$. After rearranging terms and introducing the slack vectors $\boldsymbol{s}_j$, (4) becomes

$$\begin{aligned} \boldsymbol{A}_j \boldsymbol{r}_j + \boldsymbol{\sigma}_j p_j - \boldsymbol{\nu} - \boldsymbol{s}_j &= 0 \quad \forall j, \\ \sum_j \boldsymbol{r}_j &= \boldsymbol{w}, \\ \boldsymbol{1}^\top \boldsymbol{r}_j &= c_j p_j \quad \forall j, \\ r_{ij} s_{ij} &= 0 \quad \forall ij, \\ \boldsymbol{r}_j, \boldsymbol{s}_j &\geq \boldsymbol{0} \quad \forall j. \end{aligned} \tag{5}$$

We eliminate prices from the LCP by noting that the third line implies $p_j = (\boldsymbol{1}^\top \boldsymbol{r}_j)/c_j$:

$$\begin{aligned} \left(\boldsymbol{A}_j + \frac{1}{c_j}\boldsymbol{\sigma}_j \boldsymbol{1}^\top\right)\boldsymbol{r}_j - \boldsymbol{\nu} - \boldsymbol{s}_j &= 0 \quad \forall j, \\ \sum_j \boldsymbol{r}_j &= \boldsymbol{w}, \\ r_{ij} s_{ij} &= 0 \quad \forall ij, \\ \boldsymbol{r}_j, \boldsymbol{s}_j &\geq \boldsymbol{0} \quad \forall j. \end{aligned} \tag{6}$$

To see its LCP structure, consider an example with two channels, $m = 2$ and n users. Let $\boldsymbol{M}_j := \boldsymbol{A}_j + (\boldsymbol{\sigma}_j \boldsymbol{1}^\top)/c_j$. Then, (6) becomes

$$\begin{gathered} \begin{pmatrix} \boldsymbol{M}_1 & \boldsymbol{0} & -I \\ \boldsymbol{0} & \boldsymbol{M}_2 & -I \\ I & I & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{\nu} \end{pmatrix} = \begin{pmatrix} \boldsymbol{s}_1 \\ \boldsymbol{s}_2 \\ \boldsymbol{w} \end{pmatrix}, \\ \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \end{pmatrix} \geq \boldsymbol{0}, \quad \text{and} \quad \begin{pmatrix} \boldsymbol{s}_1 \\ \boldsymbol{s}_2 \end{pmatrix} \geq \boldsymbol{0}, \end{gathered} \tag{7}$$

where we look for a complementarity solution $\boldsymbol{r}_1^\top \boldsymbol{s}_1 + \boldsymbol{r}_2^\top \boldsymbol{s}_2 = 0$. If both $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are monotone matrices, that is, $\boldsymbol{M}_1 + \boldsymbol{M}_1^\top$ and $\boldsymbol{M}_2 + \boldsymbol{M}_2^\top$ are positive semidefinite, then the LCP matrix on the very left of (7) is also monotone. In that case an LCP solution can be computed in polynomial time [13]. Applying this fact and the fact that a KKT point of a QP can be computed by an FPTAS [14] to (6) leads to our first result.

Theorem 1: Consider the competitive equilibrium model for spectrum management.

1) Let w_i, c_j, σ_{ij} and a_{ik}^j be rational. Then, there exists a CE with rational entries, that is, the entries of the equilibrium point are rational values.
2) If the matrix $\boldsymbol{A}_j + (\boldsymbol{\sigma}_j \boldsymbol{1}^\top)/c_j$ is monotone for all j, then a CE can be computed in polynomial time.
3) If the matrix $\boldsymbol{A}_j + (\boldsymbol{\sigma}_j \boldsymbol{1}^\top)/c_j$ is symmetric (in particular, if $\boldsymbol{A}_j$ is symmetric and $\sigma_{1j} = \sigma_{ij}$ for all i) for all j, then the

competitive equilibria are the KKT points of the following QP

$$\begin{aligned}\underset{\boldsymbol{r}_1,\dots,\boldsymbol{r}_m}{\text{minimize}} \quad & \sum_j \frac{1}{2}\boldsymbol{r}_j^\top\left(\boldsymbol{A}_j + \frac{1}{c_j}\boldsymbol{\sigma}_j\mathbf{1}^\top\right)\boldsymbol{r}_j \\ s.t. \quad & \sum_j \boldsymbol{r}_j = \boldsymbol{w}, \quad \text{(with Lagrange multiplier } \boldsymbol{\nu}) \\ & \boldsymbol{r}_j \geq \mathbf{0}, \quad \forall j, \quad \text{(with Lagrange multiplier } \boldsymbol{s}_j).\end{aligned} \tag{8}$$

4) There is a FPTAS to compute a CE if the matrix $\boldsymbol{A}_j + (\boldsymbol{\sigma}_j\mathbf{1}^\top)/c_j$ is symmetric for all j.

Furthermore, assuming strict monotonicity (replacing "positive semidefinite" with "positive definite" in the definition of monotonicity) ensures that the CE is unique.

Corollary 2: There is a unique CE if the matrix $\boldsymbol{A}_j + (\boldsymbol{\sigma}_j\mathbf{1}^\top)/c_j$ is strictly monotone for all j,

For example, a symmetric and weak-interference condition, that is, for all j, $\sum_{k\neq i} a_{ik}^j < 1$ for all i and $\sum_{i\neq k} a_{ik}^j < 1$ for all k, will ensure that $\boldsymbol{A}_j$ is strictly monotone for all j. In addition, if we have equal noise: $\sigma_{1j} = \sigma_{ij}$, $\forall ij$, then $\boldsymbol{A}_j + (\boldsymbol{\sigma}_j\mathbf{1}^\top)/c_j$ will be strictly monotone for all j. It is reasonable to assume that the $\boldsymbol{A}_j$ are symmetric because most communication channels are reciprocal, including wireless and wired DSL channels. It is further reasonable to assume that the σ_{ij} are very small and equal because they are given by the specification to which the transmitters and receivers are built. Weak-interference is a standard assumption for DSL and is reasonable in some situations for wireless communication systems.

IV. Tâtonnement Process for Spectrum Management

In this section we present a decentralized algorithm for solving the CE. In the centralized approach as described above, the spectrum manager gathers all the parameters and then publishes the optimal power allocations. However, in the decentralized approach each user only sends her current power allocations and receives the channel prices from the manager. This reduces the communication between users and distributes the computational load to the users. The paper [15] provides a summary of the benefits of distributed algorithms over centralized ones.

Given the channel prices $\boldsymbol{p}$, the power allocations can be found by water filling. The key question is how to adjust the prices and to ensure that the process converges quickly to a CE. Tâtonnement processes [11] are a broad class of price-update rules that adjusts the price based on the excess demand: if the supply on channel j, c_j, exceeds the total demand, $\sum_i x_{ij}$, then decrease the price p_j (increase it if the demand falls short of supply). The users and the manager then alternate between updating their power allocations and the updating the channel prices, respectively, until the difference between demand and supply is small. The condition for the convergence of a tâtonnement process is known as the weak gross substitutability (WGS).

Theorem 3: 1) Suppose prices for each product j are adjusted continuously by

$$\frac{dp_j(t)}{dt} = f_j(y_j(\boldsymbol{p}(t))), \tag{9}$$

where $f_j(\cdot)$ is a sign preserving function (i.e., $\operatorname{sign} f_j(y) = \operatorname{sign} y$) and y_j is a measure of the excess of product j. Then $\boldsymbol{y} \to \mathbf{0}$ if weak gross substitutability holds, that is, $\partial_l y_j(\boldsymbol{p}) \geq 0$ for all $l \neq j$.

2) Suppose prices for each product j are adjusted discretely by

$$p_j^{t+1} = p_j^t + f_j(y_j(\boldsymbol{p}^t)), \tag{10}$$

where $f_j(\cdot)$ is a sign preserving function (i.e., $\operatorname{sign} f_j(y) = \operatorname{sign} y$) and y_j is a measure of the excess of product j. Then $\boldsymbol{y}^t \to \mathbf{0}$ if weak gross substitutability holds, that is, $\partial_l y_j(\boldsymbol{p}) \geq 0$ for all $l \neq j$.

Proof: Part 1 is Theorem 4.1 of [16] (also found in [11]) while part 2 is proved by [17]. ■

With some conditions, we can prove WGS for our competitive market model. For algebraic simplicity we use excess revenue instead of excess demand (this is without loss of generality since for each j the factor p_j could easily be incorporated into $f_j(y)$).

Theorem 4: For each channel j define $y_j(\boldsymbol{p}) := p_j\left(\sum_i x_{ij}^* - c_j\right)$. Assume the following conditions

1) symmetric, weak-interference condition: $\sum_{k\neq i} a_{ik}^j < 1$ and $\sum_{k\neq i} a_{ki}^j < 1 \ \forall j$;
2) low-rank condition: the matrices of cross-talk coefficients can be written as $\boldsymbol{A}_j = \boldsymbol{D}_j + \boldsymbol{a}_j\boldsymbol{b}_j^\top \ \forall j$ where $\boldsymbol{D}_j$ diagonal, $\boldsymbol{D}_j, \boldsymbol{a}_j, \boldsymbol{b_j} \geq 0$, and $\boldsymbol{a}_j, \boldsymbol{b_j}$ in the range of $\boldsymbol{D}_j$; and
3) equal noise condition: $\sigma_{ij} = \sigma_j \ \forall ij$.

Then our spectrum model satisfies WGS, i.e., $\partial_l y_j(\boldsymbol{p}) \geq 0$ for all $l \neq j$, so that both continuous and discrete tâtonnement price-adjustment processes converge.

Condition 2 is a sensible approximation of the cross-talk coefficients and the coefficients $[\boldsymbol{a}_j]_i$ and $[\boldsymbol{b}_j]_i$ can be interpreted as the isolation level of the receiver and transmitter of user i, respectively. We remark that [1], [2] also use condition 2) from Theorem 4 and the assumption that $a_{ik}^j = a_i^j \leq 1$ for all ijk to show that the equilibrium set is convex. For two channels, $m = 2$ weaker conditions suffice:

Theorem 5: If $m = 2$ and the weak-interference condition holds for $\boldsymbol{A}_1^\top$ and $\boldsymbol{A}_2^\top$, then WGS holds and tâtonnement processes converge.

V. Numerical Examples

We present three examples with $n = 10$ users and fewer channels than users ($m = 6$), an equal number of channels and users ($m = 10$), and more channels than users ($m = 14$) channels, respectively. The cross-talk coefficients are independent random samples from the uniform distribution on $[0, 1/(n-1)]$, ensuring that the weak interference condition is satisfied. The noise levels satisfy the equal noise condition, and the σ_j are independent random samples from the uniform distribution on $[0, 1]$. For all i and j, $w_i = 1$ and $c_j = 1$. Fig. 2 shows how channel prices with a decentralized tâtonnment process converge to the CE prices calculated with the LCP in (6). After 100 iterations of the tâtonnement process, the relative difference between each user's utility and their utility at the CE is less than 10^{-3}. For these examples, we compare a modified CE to the NE (where each user's total transmission power is limited to 1). To not favor the CE we scale each users' power allocations in the CE to match the NE's limit on the total transmission power per user. Thus the modified CE obeys the power constraints imposed on the NE and its performance is no better than that of the true CE. We find that the average user's utility at this modified CE is higher than at the NE by 5%, 6%, and 2%, respectively. (We calculate the NE with the LCP in [4].) [2] has more comparisons of the CE and the NE.

VI. Conclusions

We considered a competitive market model for dynamic spectrum management of a communication system. We showed that the problem of finding the competitive equilibrium can be formulated as a linear complementarity problem (LCP) and solved efficiently. Besides the centralized LCP solution, we also proposed decentralized tâtonnement processes for adjusting prices. We proved these processes convergence to the CE under certain conditions. In our model,

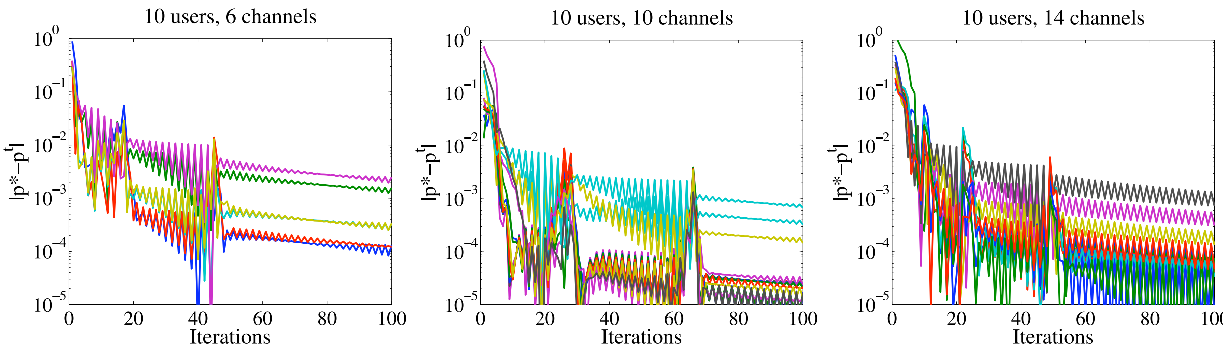

Fig. 2. Convergence under the tâtonnement process of the channel prices to the CE, $\left|\boldsymbol{p}^* - \boldsymbol{p}^t\right|$, for three examples.

each user's budget constraint implicitly limits their total transmission power. We plan to extend this model by incorporating explicit limits on the transmission power of each user and by relaxing the weak-interference assumption and the low-rank assumptions on the matrices $\boldsymbol{A}_j$ of cross-talk coefficients.

Appendix

Proof of Theorem 4: Let $[\boldsymbol{r}_1^*(\boldsymbol{p}), \ldots, \boldsymbol{r}_m^*(\boldsymbol{p})]$ be the solution to (5). We rewrite $y_j(\boldsymbol{p}) = \mathbf{1}^\top \boldsymbol{r}_j^*(\boldsymbol{p}) - p_j c_j$. For $j \neq l$, we will show that both the left and right hand limits are $\partial_l y_j(\boldsymbol{p}) = \mathbf{1}^\top \partial_l \boldsymbol{r}_j^*(\boldsymbol{p}) \geq 0$. Let us look at the left hand limit (the right hand limit will be similar). Then there is a small open interval $(t-\epsilon, t)$ in which the active set of the LCP is constant. Let the set S_j be the active set of channel j, $S_j := \{i : s_{ij} = 0\}$ and $\boldsymbol{I}_j$ the $n \times n$ matrix so that $[\boldsymbol{I}_j]_{il} := 1$ if $i = l \in S_j$ and 0 otherwise. Note that $\boldsymbol{I}_j \boldsymbol{s}_j = \mathbf{0}$ and $r_{ij} = 0$ for $i \notin S_j$, thus $\boldsymbol{I}_j \boldsymbol{r}_j = \boldsymbol{r}_j$. Thus the first equation in (5) becomes

$$\boldsymbol{I}_j \boldsymbol{A}_j \boldsymbol{I}_j \boldsymbol{r}_j = \boldsymbol{I}_j \boldsymbol{\nu} - p_j \boldsymbol{I}_j \boldsymbol{\sigma}_j. \tag{11}$$

Defining $\bar{\boldsymbol{A}}_j := \boldsymbol{I}_j \boldsymbol{A}_j \boldsymbol{I}_j$ it follows that $\bar{\boldsymbol{A}}_j{}^\dagger \boldsymbol{I}_j = \bar{\boldsymbol{A}}_j{}^\dagger$ and that one solution is

$$\boldsymbol{r}_j = \bar{\boldsymbol{A}}_j{}^\dagger \boldsymbol{\nu} - p_j \bar{\boldsymbol{A}}_j{}^\dagger \boldsymbol{\sigma}_j. \tag{12}$$

Then the budget constraint (the last equation in (5)) gives us

$$\sum_k \bar{\boldsymbol{A}}_k^\dagger \boldsymbol{\nu} - p_k \bar{\boldsymbol{A}}_k^\dagger \boldsymbol{\sigma}_k = \boldsymbol{w}. \tag{13}$$

Thus one solution for $\boldsymbol{\nu}$ is

$$\boldsymbol{\nu} = \left(\sum_k \bar{\boldsymbol{A}}_k^\dagger\right)^\dagger \left(\boldsymbol{w} + \sum_k p_k \bar{\boldsymbol{A}}_k^\dagger \boldsymbol{\sigma}_k\right). \tag{14}$$

Thus for $j \neq l$,

$$\frac{\partial y_j}{\partial p_l} = \frac{\partial}{\partial p_l} \mathbf{1}^\top \boldsymbol{r}_j = \mathbf{1}^\top \bar{\boldsymbol{A}}_j^\dagger \frac{\partial \boldsymbol{\nu}}{\partial p_l}, \tag{15}$$

$$\frac{\partial y_j}{\partial p_l} = \mathbf{1}^\top \bar{\boldsymbol{A}}_j^\dagger \left(\sum_k \bar{\boldsymbol{A}}_k^\dagger\right)^\dagger \bar{\boldsymbol{A}}_l^\dagger \boldsymbol{\sigma}_l. \tag{16}$$

The equal noise condition and Lemma 8 then prove the claim. ∎

Proof of Theorem 5: Following the proof of Theorem 4 we need to show that $\partial y_2 / \partial p_1$ given by (16) is nonnegative:

$$\frac{\partial y_2}{\partial p_1} = \mathbf{1}^\top \bar{\boldsymbol{A}}_2^\dagger \left(\bar{\boldsymbol{A}}_1^\dagger + \bar{\boldsymbol{A}}_2^\dagger\right)^\dagger \bar{\boldsymbol{A}}_1^\dagger \boldsymbol{\sigma}_1 = \mathbf{1}^\top \left(\bar{\boldsymbol{A}}_2 + \bar{\boldsymbol{A}}_1\right)^\dagger \boldsymbol{\sigma}_1. \tag{17}$$

Since $0.5(\bar{\boldsymbol{A}}_2 + \bar{\boldsymbol{A}}_1)^\top$ is a channel matrix obeying weak interference we can apply Lemma 7 to show that $\mathbf{1}^\top \left(\bar{\boldsymbol{A}}_2 + \bar{\boldsymbol{A}}_1\right)^\dagger$ is a nonnegative vector. The fact that $\boldsymbol{\sigma}_1 \geq 0$ completes the proof. ∎

The following lemmas are needed in the above proofs.

Lemma 6: For $i = 1, \ldots, m$, let $\boldsymbol{A}_j = \boldsymbol{D}_j + \boldsymbol{a}_j \boldsymbol{b}_j^\top$ where $\boldsymbol{D}_j$ diagonal, $\boldsymbol{D}_j, \boldsymbol{a}_j, \boldsymbol{b}_j \geq 0$, and $\boldsymbol{a}_j, \boldsymbol{b}_j$ in the range of $\boldsymbol{D}_j$. If for each i there exists j such that $[\boldsymbol{D}_j]_{ii} > 0$, then, $\left(\sum_j \boldsymbol{A}_j^\dagger\right)^{-1}$ exists and is nonnegative.

Proof: Applying the Sherman-Morrison formula to the range of $\boldsymbol{A}_j$ we obtain $\boldsymbol{A}_j^\dagger = \boldsymbol{D}_j^\dagger - \boldsymbol{B}_j$, where $\boldsymbol{B}_j := (\boldsymbol{D}_j^\dagger \boldsymbol{a}_j \boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger)/(1 + \boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger \boldsymbol{a}_j)$. Since $\boldsymbol{D}_j \geq 0$, $\boldsymbol{D}_j^\dagger \geq 0$. Therefore, $\boldsymbol{a}_j, \boldsymbol{b}_j \geq 0$ implies $\boldsymbol{B}_j \geq 0$. Thus $\sum_j \boldsymbol{A}_j^\dagger$ can be written as

$$\sum_j \boldsymbol{A}_j^\dagger = \boldsymbol{D} - \boldsymbol{B}, \tag{18}$$

where $\boldsymbol{D} := \sum_j \boldsymbol{D}_j^\dagger$ and $\boldsymbol{B} := \sum_j \boldsymbol{B}_j$. Since $\boldsymbol{D} \succ 0$, $\boldsymbol{D}^{-1}$ exists and we may define $\boldsymbol{C} := \boldsymbol{D}^{-1/2} \boldsymbol{B} \boldsymbol{D}^{-1/2}$. Note that $\boldsymbol{D} \geq 0$, $\boldsymbol{B} \geq 0$, and $\boldsymbol{D}$ diagonal. Thus $\boldsymbol{D}^{-1/2} \geq 0$ and $\boldsymbol{C} \geq 0$. Note that for any $\boldsymbol{x} \neq \mathbf{0}$,

$$\left|\boldsymbol{x}^\top \boldsymbol{C} \boldsymbol{x}\right| = \left|\boldsymbol{x} \boldsymbol{D}^{-1/2} \boldsymbol{B} \boldsymbol{D}^{-1/2} \boldsymbol{x}\right| \leq \sum_j \left|\boldsymbol{x} \boldsymbol{D}^{-1/2} \boldsymbol{B}_j \boldsymbol{D}^{-1/2} \boldsymbol{x}\right| \tag{19}$$

$$= \sum_j \left| \frac{\boldsymbol{x} \boldsymbol{D}^{-1/2} \boldsymbol{D}_j^\dagger \boldsymbol{a}_j \boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger \boldsymbol{D}^{-1/2} \boldsymbol{x}}{1 + \boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger \boldsymbol{a}_j} \right| \tag{20}$$

$$\leq \sum_j \frac{\rho((\boldsymbol{D}_j^\dagger)^{1/2} \boldsymbol{a}_j \boldsymbol{b}_j^\top (\boldsymbol{D}_j^\dagger)^{1/2}) \left\|(\boldsymbol{D}_j^\dagger)^{1/2} \boldsymbol{D}^{-1/2} \boldsymbol{x}\right\|_2^2}{1 + \boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger \boldsymbol{a}_j} \tag{21}$$

$$= \sum_j \frac{(\boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger \boldsymbol{a}_j)(\boldsymbol{x}^\top \boldsymbol{D}^{-1/2} \boldsymbol{D}_j^\dagger \boldsymbol{D}^{-1/2} \boldsymbol{x})}{1 + \boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger \boldsymbol{a}_j} \tag{22}$$

$$\leq \lambda \sum_j \boldsymbol{x}^\top \boldsymbol{D}^{-1/2} \boldsymbol{D}_j^\dagger \boldsymbol{D}^{-1/2} \boldsymbol{x} \tag{23}$$

where $\lambda = \max_j (\boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger \boldsymbol{a}_j)/(1 + \boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger \boldsymbol{a}_j)$. Since $\boldsymbol{D}_j, \boldsymbol{a}, \boldsymbol{b} \geq 0$, $\lambda \geq 0$ and since $\boldsymbol{a}$ and $\boldsymbol{b}$ are in the range of $\boldsymbol{D}_j$, $\lambda < 1$. Therefore, for any $\boldsymbol{x} \neq \mathbf{0}$,

$$\left|\boldsymbol{x}^\top \boldsymbol{C} \boldsymbol{x}\right| < \sum_j \boldsymbol{x}^\top \boldsymbol{D}^{-1/2} \boldsymbol{D}_j^\dagger \boldsymbol{D}^{-1/2} \boldsymbol{x}$$

$$= \boldsymbol{x}^\top \boldsymbol{D}^{-1/2} \boldsymbol{D} \boldsymbol{D}^{-1/2} \boldsymbol{x} = \boldsymbol{x}^\top \boldsymbol{x}. \tag{24}$$

Hence, $\rho(\boldsymbol{C}) < 1$ and thus $(\boldsymbol{I} - \boldsymbol{C})^{-1} = \sum_{k=0}^\infty \boldsymbol{C}^k \geq 0$. Therefore, $(\sum_j \boldsymbol{A}_j^\dagger)^{-1} = (\boldsymbol{D} - \boldsymbol{B})^{-1} = \boldsymbol{D}^{-1/2} (\boldsymbol{I} - \boldsymbol{C})^{-1} \boldsymbol{D}^{-1/2} \geq 0$. ∎

Lemma 7: If $\boldsymbol{A}$ is a channel matrix satisfying the weak-interference assumption, then $\boldsymbol{A}^{-1} \mathbf{1} \geq 0$.

Proof: Since $\boldsymbol{A}$ is a channel matrix we can write $\boldsymbol{A} = \boldsymbol{I} + \boldsymbol{B}$ for some $\boldsymbol{B} \geq 0$. Hence

$$\begin{aligned} \boldsymbol{A}^{-1} \mathbf{1} &= (\boldsymbol{I} + \boldsymbol{B})^{-1} \mathbf{1} = (\boldsymbol{I} + \boldsymbol{B})^{-1} (\boldsymbol{I} - \boldsymbol{B})^{-1} (\boldsymbol{I} - \boldsymbol{B}) \mathbf{1} \\ &= \left(\boldsymbol{I} - \boldsymbol{B}^2\right)^{-1} (\boldsymbol{I} - \boldsymbol{B}) \mathbf{1}. \end{aligned} \tag{25}$$

The weak interference assumption implies that $\rho(\boldsymbol{B}) < 1$. Hence $\left(\boldsymbol{I} - \boldsymbol{B}^2\right)^{-1}$ exists and equals $\sum_{k=0}^{\infty} \boldsymbol{B}^{2k} \geq 0$. In addition, $(\boldsymbol{I} - \boldsymbol{B})\mathbf{1} > 0$, due to the weak interference assumption. Thus $A^{-1}\mathbf{1} \geq 0$. ∎

Lemma 8: Assume conditions 1)–3) of Theorem 4 hold. For each j consider a set S_j and construct $\bar{\boldsymbol{A}}_j$ so that $[\bar{\boldsymbol{A}}_j]_{il} := [\boldsymbol{A}_j]_{il}$ if $i, l \in S_j$ and 0 otherwise. Then

$$\mathbf{1}^\top \bar{\boldsymbol{A}}_j^\dagger \left(\sum_k \bar{\boldsymbol{A}}_k^\dagger \right)^\dagger \bar{\boldsymbol{A}}_l{}^\dagger \mathbf{1} \geq 0 \quad \forall j, l. \tag{26}$$

Proof: Applying Lemma 7 to the range of $\bar{\boldsymbol{A}}_l$ implies that $\bar{\boldsymbol{A}}_l^\dagger \mathbf{1} \geq 0$. Similarly for $\bar{\boldsymbol{A}}_k$. Applying Lemma 6 to the union of the ranges of $\boldsymbol{D}_j$ shows that $\left(\sum_k \bar{\boldsymbol{A}}_k^\dagger\right)^\dagger \geq 0$. This proves the claim because the product of nonnegative vectors and a nonnegative matrix is nonnegative. ∎

Acknowledgment

The authors thank Erick Delage, Yichuan Ding, and Ramesh Johari for their helpful comments and discussions.

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