Competitive Spectrum Management with Incomplete Information

Yair Noam, Amir Leshem, *Senior Member, IEEE,* and Hagit Messer, *Fellow, IEEE,*

Abstract

An important issue in wireless communication is the interaction between selfish and independent wireless communication systems in the same frequency band. Due to the selfish nature of each system, this interaction is well modeled as a strategic game where each player (system) behaves to maximize its own utility. This paper studies an interference interaction (game) where each system (player) has *incomplete information* about the other player's channel conditions. Using partial information, players choose between frequency division multiplexing (FDM) and full spread (FS) of their transmitted power. An important notion in game theory is the Nash equilibrium (NE) which represents a steady point in the game; that is, each player can only lose by unilaterally deviating from it. A trivial Nash equilibrium point in this game is where players mutually choose FS and interfere with each other. This point may lead to poor spectrum utilization from a global network point of view and even for each user individually.

In this paper, we provide a closed form expression for a non pure-FS ϵ-Nash equilibrium point; i.e., an equilibrium point where players choose FDM for some channel realizations and FS for the others. To reach this point, the only instantaneous channel state information (CSI) required by each user is its own interference-to-signal ratio. We show that operating in this non pure-FS ϵ-Nash equilibrium point increases each user's throughput and therefore improves the spectrum utilization, and demonstrate that this performance gain can be substantial. Finally, important insights are provided into the behaviour of selfish and rational wireless users as a function of the channel parameters such as fading probabilities, the interference-to-signal ratio.

Index Terms

Dynamic Spectrum Access, Bayesian Games, Interference Channel, FDM, Nash Equilibrium, incomplete Channel State information.

I. INTRODUCTION

Wireless communication has become increasingly popular in recent years since more and more communication systems share the same band. Consider for example an urban area with wireless local access networks (LAN), bluetooth systems, cordless phone, etc. These systems create interference which results in major performance loss. This is why, interference mitigation is such an important issue [e.g. 1–9].

In wireless networks, interference can be high and the channel is time varying [see e.g. 10]. Furthermore, users may be independent of each other and selfish in the sense that each one is only interested in maximizing its own utility. Thus, non cooperative game theory is an appropriate tool to analyze such interactions. An important notion in game theory is the Nash equilibrium (NE) which represents a steady point the game; that is, the NE point is a strategy profile which is the best response of each player given that the others do not deviate from it. As such, it can be self imposed on network users who are selfish in nature.

For the case of a flat fading interference channel with full information[1] , it was shown [1] that Full Spread (FS) is a NE point, and a sufficient condition for its uniqueness was derived. It was further observed that in many cases the FS NE point leads to inefficient solutions. This happens when mutual FDM is better for both users than mutual FS but the system operates in a mutual FS since the users are subject to the prisoner's dilemma [11].

The full information assumption is not always practical because communicating channel gains between different users in a time varying channel within the channel coherence time may lead to large overhead.

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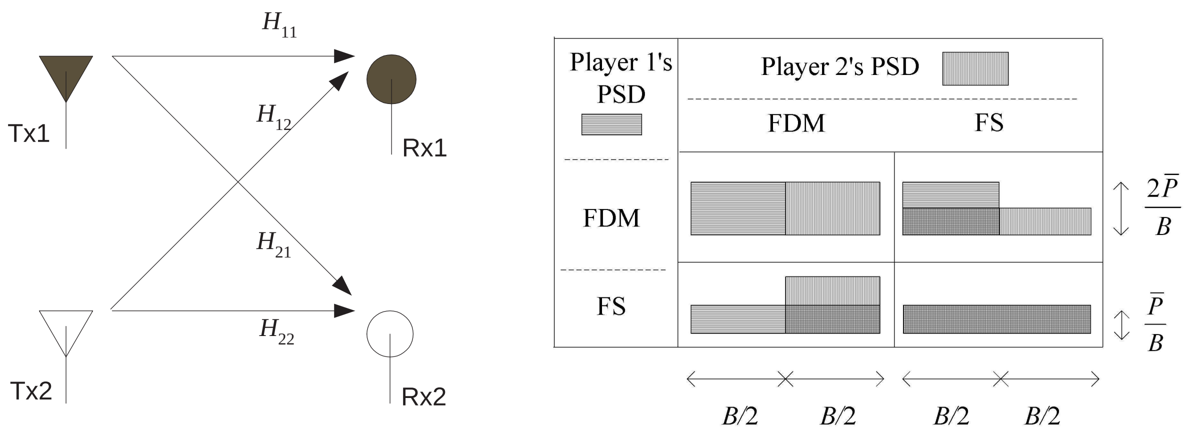
[1]By complete information, we mean that every user knows all the direct and cross channel gains of all users in the network.



(a) Wireless interference channel.

(b) Action space

Fig. 1. (a) A wireless interference scenario with incomplete information. Each player knows the square magnitudes of its direct and impinging channel gains and the statistics of its opponent's channel gains. For example player 1 knows $|H_{11}|^2$ and $|H_{12}|^2$ but knows only the statistics of $|H_{22}|^2$ and $|H_{21}|^2$. (b) the possible PSD configurations.

In this case, it is more appropriate to consider each channel coherence time as a one-stage game where players are only aware of their own channel gains and their opponent's channel statistics (which vary slowly compared to the channel gains and therefore can be communicated [3]). The interaction between the players may be repeated but with a different and independent channel realization each time and therefore is not a repeated game. This motivates the use of games with incomplete information, also known as Bayesian games [12, 13] which have been incorporated into wireless communications for problems such as power control [14–16] and spectrum management in the interference channel [2, 17]. In [15], a distributed uplink power control in a multiple access (MAC) fading channel was studied and shown to have a unique NE point. This result however does not apply to the interference channel which is radically different. In a MAC channel, user i's direct channel gain is equal to the gain of the interference he creates for the other users ($j \neq i$) while in the interference channel these parameters are independent. Thus, the interference channel is composed of a double number of parameters and therefore is more complicated.

In this paper we analyze a two-user interference channel with incomplete information in which each user knows the magnitudes of its direct channel and of the impinging channel gains and its noise power spectrum density (PSD) but is unaware of his opponent's direct and impinging channel gains but only knows it statistics (see Fig. 1(a)). Based on their measurements, users choose between pre-assigned FDM or FS (see Fig. **??**). This interaction may be repeated but a with different channel realization each time.

With the same incomplete information, it was shown [2] that in a symmetric[2] interference channel with a one-time interaction, FS is the only symmetric strategy profile[3] which is a NE point. This result however is limited to scenarios where all users statistically experience identical channel conditions (due to the symmetry assumption) and does not apply to interactions between weak and strong users.

A situation where both players use FS may lead to undesirable outcomes from a global network point of view and even for each user individually. Thus, it is desirable to derive non FS Nash equilibrium points which Pareto (which is component-wise larger) dominate the FS equilibrium point and lead to improved spectrum utilization. A first step toward this goal was made in [17] where it was shown that if users can coordinate in advance to use orthogonal FDM, there exists a non pure-FS NE point which Pareto dominates the pure-FS NE point. This result however is also limited to symmetric interference channels. This paper is aimed to fill this gap and derive NE points in the general case of arbitrary channel distributions. For example, scenarios of weak and strong users (where one experiences a high level of

[2]Symmetric in the sense that the all channel gains (i.e. $H_{i,q}$ for every i, q) are identically distributed.

[3]In symmetric strategy profile users are restricted to identical strategies.

interferences and the other experiences low interference), different fading effects and cases where one has a strong line of sight path and the other has no line of sight. The assumptions of arbitrary channel distribution together with the incomplete information that each user possesses about the other users in his vicinity are most suitable to the reality of selfish users operating independently in unlicensed frequency bands. This paper provides a closed form expression for non pure-FS ϵ-NE point that increases each user's throughput and therefore improves the spectrum utilization, and demonstrates that this performance gain can be substantial. The derived equilibrium point provides insights into the the behaviour of selfish and rational wireless users. Furthermore, it does not require a central authority that imposes compliance of the protocol. Thus, it provides guidelines for designing a protocol that users will choose voluntarily to follow.

The paper is orginazed as follows. In Section II we define the Bayesian Interference Game (BIG). This is a two user interference interaction with incomplete information where the channel's direct and crosstalks gains are arbitrarily distributed (but independent). In Section III we present the best response function which is a user's best action for his opponent's given strategy. We then provide a simple expression for the best response that depends only on the interference-to-signal ratio. In Section IV we show (Proposition 3) that non pure-FS NE points provide improved performance (with respect to pure-FS NE) to each user individually and therefore a better spectrum utilization. We then derive a closed form expression for non pure-FS ϵ NE points. Theorem 5 provides a sufficient condition for the existence of such points.

In section V we analyse the BIG in common wireless fading models (i.e. Rieghley and Rician and Nakagami) and learn the behaviour of selfish and rational wireless users in various wireless environments.

II. PROBLEM FORMULATION

A. Notation and Definitions

Consider a flat-fading interference channel with two players, where during a channel coherent time, player i's signal is given by (see Fig. 1(a))

$$W_i(n) = H_{ii}V_i(n) + H_{ij}V_j(n) + N_i(n) \tag{1}$$

where $i,j \in \{1,2\},\ i \neq j,\ V_i(n), V_j(n)$ are user i's and j's transmit signals respectively, $N_i(n)$ is a white Gaussian noise with variance σ_N^2 and $H_{iq}, i,q \in \{1,2\}$ are the channel fading coefficients which are random variables. Throughout this paper, the index j is never equal to i. Both players have a total power constraint $\bar{p}$. We denote user i's signal to noise ratio (SNR) and interference to noise ratio (INR) by $X_i = |H_{ii}|^2\bar{p}/\sigma_N^2$ and $Y_i = |H_{ij}|^2\bar{p}/\sigma_N^2$ respectively and denote $\overline{SNR}_i = \mathrm{E}\{X_i\}$, $\overline{INR}_i = \mathrm{E}\{Y_i\}$. We further denote the interference to signal ratio (ISR) by $Z_i = Y_i/X_i$. The realizations (sample points) of X_i, Y_i, Z_i are denoted by x_i, y_i, z_i respectively. When we want to stress that x_i, y_i, z_i are the observed values of the SNR, INR and ISR they will be denoted by SNR_i, INR_i and ISR_i respectively.

Assumption 1: The channel gains $|H_{iq}|^2,\ i,q \in \{1,2\}$ are continuous random variables with finite non zero moments and the probability density functions (PDF) $f_{|H_{iq}|^2}(h),\ i,q \in \{1,2\}$ are finite for every $h > 0$.

B. The Bayesian Interference Game (BIG)

In the BIG, user i's channel state information (CSI) at the transmitter side are the realized values of X_i and Y_i. It does not observe Y_j and X_j but only knows their distributions. The channel is divided into two equal sub-bands and player 1's and 2's actions are given by

$$\begin{aligned}\mathbf{p}_1(\theta_1) = \bar{p}[\theta_1,\ 1-\theta_1]^T \\ \mathbf{p}_2(\theta_2) = \bar{p}[1-\theta_2,\ \theta_2]^T\end{aligned} \tag{2}$$

respectively (see Fig. **??**), where $\theta_i \in \Theta_i = \{1, 1/2\}$ and $\bar{p}$ is the total power constraint. The actions $\theta = 1$ and $\theta = 1/2$ correspond to FDM and FS, respectively. This formalism implies that players coordinate

TABLE I
USER i'S PAYOFF $u_i(\theta_i, \theta_j, SNR_i, INR_i)$

	player j chooses FDM ($\theta_j = 1$)	player j chooses FS $\theta_j = 1/2$
player i chooses FDM ($\theta_i = 1$)	$\frac{1}{2}\log_2(1 + SNR_i)$	$\frac{1}{2}\log_2\left(1 + \frac{SNR_i}{1+INR_i/2}\right)$
player i chooses FS ($\theta_i = 1/2$)	$\frac{1}{2}\log_2\left(1 + \frac{SNR_i}{2}\right) + \frac{1}{2}\log_2\left(1 + \frac{SNR_i/2}{1+INR_i}\right)$	$\log_2\left(1 + \frac{SNR_i/2}{1+INR_i/2}\right)$

in advance to use disjoint subbands in the case of FDM. This coordination can be carried out by using Carrier Sense Multiple Access (CSMA) techniques (see e.g. [10]) where each player randomly chooses a subband and performs a random power backoff in case of collision. This is done at the first interaction when users exchange information (channel statistics).

We assume that during a single coherence period, players manage their spectrum only once, based on their knowledge. Therefore, if the interaction is repeated it will be with different and independent channel realizations. This represents a case where the channel vary fast or a case where simplicity requirements enable a single spectrum shaping every coherence period. Player i's utility function $u_i(\theta_i, \theta_j, SNR_i, INR_i)$ is given in Table I. We are now ready to define the Bayesian interference game.

Definition 1: The Bayesian interference game (BIG) is defined by the following:

1) Set of players $\{1, 2\}$.
2) Action sets $\Theta_i = \{1, 1/2\}$, $i = 1, 2$. Let $\theta_i \in \Theta_i$ be the action chosen by player i, then according to (2), $\theta_i = 1$ corresponds to FDM and $\theta_i = 1/2$ corresponds to FS.
3) A set of positive and independent random variables $X_1,\ Y_1,\ X_2,\ Y_2$ whose distributions are common knowledge. Each player i observes the realized values of X_i, Y_i but does not observe X_j, Y_j.
4) A utility function $u_i(\theta_i, \theta_j, x_i, y_i)$ given in Table I.
5) A set of pure strategies $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ where every $S_i \in \mathcal{S}_i$ is a function that maps values of x_i, y_i to an action in Θ_i, i.e. $S_i : \mathcal{X}_i \times \mathcal{Y}_i \longrightarrow \Theta_i$, where $\mathcal{X}_i = \mathrm{Range}(X_i)$ and $\mathcal{Y}_i = \mathrm{Range}(Y_i)$.

Player i's objective is to maximize his conditional expected payoff given his private information x_i, y_i, i.e.:

$$\pi_i(S_i, S_j, x_i, y_i) \triangleq \mathrm{E}\left\{u_i(S_i, S_j, X_i, Y_i) | X_i = x_i, Y_i = y_i\right\},\ \forall x_i, y_i \in \mathcal{X}_i \times \mathcal{Y}_i \tag{3}$$

Definition 2: a NE point of the BIG is a strategy profile $\mathbf{S} = (S_i, S_j)$ such that for every strategy profile $\tilde{\mathbf{S}} = (\tilde{S}_i, \tilde{S}_j)$ and every $i \in \{1, 2\}$

$$\pi_i(S_i, S_j, x_i, y_i) \geq \pi(\tilde{S}_i, S_j, x_i, y_i)\ \forall\ \ x_i, y_i \in \mathcal{X}_i \times \mathcal{Y}_i \tag{4}$$

Since the action space is binary, a strategy $S_i(x_i, y_i)$ in the BIG is equivalent to a decision region $D_i \subseteq \mathcal{X}_i \times \mathcal{Y}_i$ such that $S_i(x_i, y_i) = 1$ (i.e. FDM) if $x_i, y_i \in D_i$ and $S_i(x_i, y_i) = 0.5$ if $x_i, y_i \in D_i^c$.

Two comments are in order:

- Only pure strategies are considered in the BIG; that is, player i's action is completely determined by his observed signal x_i, y_i. We do not consider mixed strategies which map values of the observed signal x_i, y_i to a probability distribution on Θ_i i.e., player i chooses randomly between FDM and FS with probability $a_i(x_i, y_i)$ and $1 - a_i(x_i, y_i)$ respectively. A well known theorem in game theory is the Purification Theorem [13, Theorem 6.2]. It asserts that under some regularity conditions (among others that each player's utility function $u_i(\theta_i, \theta_j, x_i, y_i)$ should not be a function of x_j and y_j), every mixed strategy has a pure strategy equivalent. Thus, all NE points can be reached using pure strategies. The conditions of the Purification Theorem are satisfied in the BIG.
- In the case where player j chooses FDM, FS is not the best action for player i. His payoff can be increased by performing waterfilling which will result in a higher rate. Therefore, it makes sense to modify the FS action with the waterfilling action as considered in [1, 6, 11] for interactions with complete information. There are, however, two important caveats. The first is that the waterfilling

solution in the interference channel must be carried out iteratively, where at every iteration players measure their interference and shape their spectrum accordingly. The process needs to be repeated within the channel coherence time until convergence[4]. This may lead to large overhead in time varying channels and therefore is impractical. Moreover, the iterative waterfilling procedure does not necessarily converge [8]. The second caveat is the analysis of the resulting game in the framework of incomplete information. The result is a game with incomplete information where in addition to not knowing their opponent's utility, players do not know their own utility function since it depends on their opponent's CSI. The analysis of such games is more complex and presents a greater challenge. For example, the Purification Theorem is not satisfied if players use iterative water filling.

III. Best response and approximate best response

An important notion in game theory is the best response function. The best response function of player i maps each of player j's strategies to an action for which player i's payoff is maximized. This function is used to derive NE points and is also important for understanding the players' preferences and the nature of the game.

In this section we present an expression for the best response function of the BIG. This expression, however, is too complex for deriving a closed form expression for NE points of the BIG. Worse, it does not provide insights into the game. For these reasons we obtain a simple approximation for the best response function which provides greater insights into the game and will enable us to obtain a closed form expression for near NE points of the BIG.

A. Best Response Function

We now derive player *i*'s best response to S_j - player j's strategy. Note that $u_i(1,1,x_i,y_i) > u_i(1,1/2,x_i,y_i)$ since the $\log$ is a monotonic function, and furthermore, due to Jenssen's inequality, $u_i(1/2,1/2,x_i,y_i) > u_i(1,1/2,x_i,y_i)$. Thus, the following situations are possible:

- A_i is the case in which $u_i(1,1,x_i,y_i) \geq u_i(1/2,1,x_i,y_i)$ which is equivalent to $INR_i > SNR_i/2$
- B_i is the case in which $u_i(1/2,1,x_i,y_i) \geq u_i(1,1,x_i,y_i)$ which is equivalent to $INR_i \leq SNR_i/2$

Recall that player i is not aware of the state of his opponent (A_j or B_j) but only of his probabilities.

If player i experiences situation B_i (which is $ISR_i \leq 1/2$), then FS is his best response. This is because FS is a strongly dominating action; that is, it produces a higher payoff to player i given any action of his opponent. It remains to find player i's best response for situation A_i; i.e. the case where $ISR_i > 1/2$, that is, strong interference. Let $P(S_j = 1)$ (the probability that player j chooses FDM), then player i's payoff is given by

$$\pi_i(\mathrm{FDM_i}, S_j, x_i, y_i) = P(S_j = 1)u_i(1,1,x_i,y_i) + (1 - P(S_j = 1))u_i(1,1/2,x_i,y_i) \tag{5}$$

$$\pi_i(\mathrm{FS_i}, S_j, x_i, y_i) = P(S_j = 1)u_i(1/2,1,x_i,y_i) + (1 - P(S_j = 1))u_i(1/2,1/2,x_i,y_i) \tag{6}$$

Observe that player i's payoff depends on his opponent's strategy and channel distribution only via $P(S_j = 1)$; hence the payoff will be denoted by $\pi_i(S_i, a_j, x_i, y_i)$ where

$$a_j = P(S_j = 1) \tag{7}$$

It follows that, player i's best response is invariant to strategies with equal probability for choosing FDM[5] and is dependent on S_j only via a_j.

Definition 3: Let S_j be player j's strategy with $a_j = P(S_j = 1)$. Player i's best response to S_j is defined by:

$$\check{S}_i(x_i, y_i, a_j) \triangleq \begin{cases} \theta_i = 1, & \text{if } \; e(x_i, y_i, a_j) > 0 \text{ and } y_i/x_i > 1/2 \\ \theta_i = 1/2, & \text{otherwise} \end{cases} \tag{8}$$

[4] See [6–8, 18–21] for further reference to the convergence of the iterative waterfilling procedure.

[5] From player i's point of view, $\mathcal{S}_j$ can be divided to into equivalent classes $\mathcal{S}_{a_j} = \{S_j : P(S_j = 1) = a_j\}$ such that $\mathcal{S}_j = \bigcup_{0 \leq a_j \leq 1} \mathcal{S}_{a_j}$.

where

$$\begin{aligned} e(x,y,a) &= \pi_i(\mathrm{FDM_i},a_j,x_i,y_i) - \pi_i(\mathrm{FS_i},a_j,x_i,y_i) = \tfrac{1}{2}a\log(1+x_i) - \tfrac{a}{2}\log\left(1+\tfrac{x_i}{2}\right) \\ &-\tfrac{1}{2}\log\left(1+\tfrac{x_i/2}{1+y_i}\right) - (1-a)\log\left(1+\tfrac{x_i}{1+y_i}\right) + \tfrac{1}{2}(1-a)\log\left(1+\tfrac{2x_i}{1+y_i}\right) \end{aligned} \tag{9}$$

Note that finding a NE point is equivalent to calculating $\hat{a}_1$ and $\hat{a}_2$ which solves the equations

$$\begin{aligned} a_1 = P\left(\check{S}_1(X_1,Y_1,a_2)=1\right) \\ a_2 = P\left(\check{S}_2(X_2,Y_2,a_1)=1\right) \end{aligned} \tag{10}$$

and that $a_1 = 0,\ a_2 = 0$ (pure-FS by both users) is a NE point regardless of the channel distribution since FS is the best response of each player if his opponent uses FS. In this case each player's payoff is $u_i(1/2,1/2,x_i,y_i)$. The pure-FS NE point may be very poor for both users as will demonstrated below.

B. Approximate Best Response

In order to analyze the best response function it will be simplified by an approximate best response. This approximate best response plays in important role in deriving equilibrium points and understanding each player's preferences. The following proposition is needed before presenting the approximate best response.

Proposition 1: Let

$$\begin{aligned} r(a,q) = \tfrac{\log_2(q)}{2} - \log_2(1+q) + \tfrac{1}{2}\log_2(2+q) \\ - a\left(1+\log_2(1+q) - \tfrac{1}{2}\log_2(2+q) - \tfrac{1}{2}\log_2(1+2q)\right) \end{aligned} \tag{11}$$

then, for every $0 < a \leq 1$ the following equation

$$r(a,q(a)) = 0 \tag{12}$$

has a unique solution $q(a) > 1/2$ and therefore it defines an implicit function $q : (0,1] \longrightarrow (0.5,\infty]$. Furthermore, $q(a)$ is continuous and monotonically decreasing.

Proof: see Appendix A.

Definition 4 (approximate best response): Let S_j be player j's strategy with $a_j = P(S_j = 1)$. Player i's approximate best response to S_j is defined by:

$$\tilde{S}_i(x_i,y_i,a_j) = \begin{cases} \theta_i = 1, & \text{if } ISR_i = y_i/x_i > q(a_j) \\ \theta_i = 1/2, & \text{otherwise} \end{cases} \tag{13}$$

i.e. the approximate best response compares the ISR to a threshold $q(a_j)$[6].

The intuition behind the approximation is now described. First consider the case of $SNR_i >> 1$ (recall that x_i, y_i are used interchangeably with SNR_i, INR_i respectively). In this case

$$e(SNR_i, INR_i, a) \approx \hat{e}(SNR_i, INR_i, a) \tag{14}$$

where

$$\begin{aligned} \hat{e}(SNR_i,INR_i,a) \triangleq \tfrac{1}{2}a_j\log(SNR_i) + \tfrac{1}{2}(1-a_j)\log\left(1+\tfrac{2SNR_i}{INR_i}\right) \\ -(1-a_j)\log\left(1+\tfrac{SNR_i}{INR_i}\right) - \tfrac{a_j}{2}\log\left(\tfrac{SNR_i}{2}\right) - \tfrac{1}{2}\log\left(1+\tfrac{SNR_i}{2INR_i}\right) \end{aligned} \tag{15}$$

Thus, $\check{S}_i(x_i,y_i,a_j)$ can be approximated by replacing $e(SNR_i,INR_i,a)$ with $\hat{e}(SNR_i,INR_i,a)$. Furthermore, note that

$$\hat{e}(SNR_i, qSNR_i, a) = r(a_j, q) \tag{16}$$

[6]For $a = 0$, we define $q(0) = \lim_{a\to 0} q(a) = \infty$. Under this definition, player i's best response to the case where his opponent always chooses FS is to choose FS.

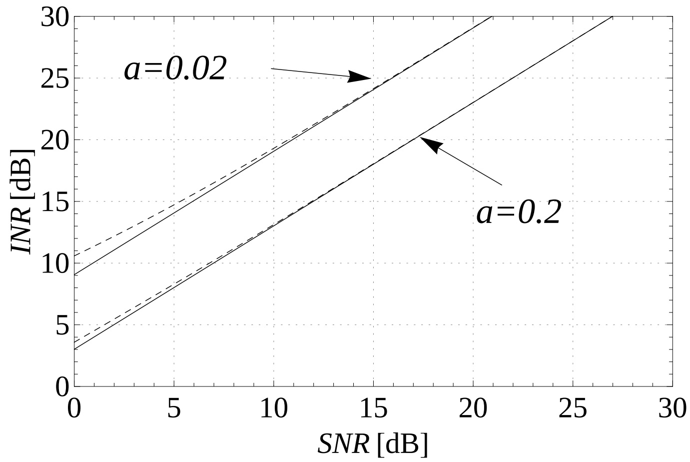


Fig. 2. Numerical evaluation of the best response function regions given in (8) for different values of a_j (a in the plot). For a given a, points above the corresponding line belong to the FDM region.

and recall that the equation $r(a, q) = 0$ (see (12)) defines the function $q(a)$. Therefore, $q(a)$ represents an ISR level for which FDM and FS yield approximately equal payoffs. Thus, if $SNR_i >> 1$, $\check{S}_i(x_i, y_i, a_j)$ can be approximated by a simple strategy which only compares the ISR to a threshold and choose action accordingly, i.e. it chooses FDM if

$$INR_i/SNR_i = ISR_i > q(a_j) \tag{17}$$

and chooses FS otherwise.

It remains to approximate (8) for the case where $SNR_i >> 1$ is not satisfied. If $INR_i >> 1$ and $ISR_i > 1/2$ it can be shown that (8) chooses FDM for every $0 < a_j \leq 1$ and if $a_j = 0$, it chooses FS. Thus, (17) is the best response in this case as well since ISR_i is greater than $q(a_j)$ (which is finite for every $0 < a_j \leq 1$ and is infinite for $a_j = 0$[7]). In the case of $ISR_i \leq 1/2$, the best response in (8) (which always chooses FS because it is a strictly dominant strategy for player i) and the approximate best response (17) coincide. This is because $q(a_j) \geq 1/2$ for every $0 \leq a_j \leq 1$. For the case where $INR_i > 1/2SNR_i$ but INR_i and SNR_i are in the same magnitude as 1, the best response in (8) cannot be simplified. However, numerical evaluation indicates that (8) is well approximated by (17) as is depicted in Figure 2.

We now present this idea formally. To establish the relation between the approximate and the ordinary best responses, define:

$$\check{D}_i^{a_j} = \{(x, y) : e(x, y, a) > 0, \;\; \text{and } y > 0.5x\} \tag{18}$$
$$\tilde{D}_i^{a_j} = \{(x, y) : y > q(a)x\} = \{(x, y) : \hat{e}(x, y, a) > 0, \;\; \text{and } y > 0.5x\} \tag{19}$$

where (19) is obtained by substituting $y = qx$ in (15) and then invoking Proposition 1. The following lemma describes precisely the sense in which $\tilde{S}_i(x_i, y_i, a_j)$ is approximately the best response. It shows that in the high transmit power regime, the best response converges in probability to the approximate best response. Thus, each player is "approximately" indifferent to whether his opponent uses the approximate best response or the true best response.

Lemma 2: Assume the channel gains $|H_{iq}|^2, \; i, q \in \{1, 2\}$ are continuous random variables, then for every $\epsilon > 0$, there exist some $\overline{SNR}_0$ such that for every $\overline{SNR}_i > \overline{SNR}_0, \; i = 1, 2$ (or equivalently, for every $\bar{p} > \bar{p}_0$)

$$P\left(\check{S}_i\left(X_i, Y_i, a_j\right) \neq \tilde{S}_i\left(X_i, Y_i, a_j\right)\right) < \epsilon \tag{20}$$

[7]under the convention that $\infty > \infty$ is fuels.

furthermore, if $|H_{iq}|^2,\ i,q \in \{1,2\}$ satisfy the regularity conditions in Assumption 1, ϵ decreases like

$$\epsilon \leq O\left(\frac{\sigma_N^2}{\bar{p}^2} + \sum_{q=1}^{2} F_{|H_{ii}|^2}\left(\frac{\sigma_N^2}{\bar{p}^{1-\nu}}\right) F_{|H_{iq}|^2}\left(\frac{\sigma_N^2}{\bar{p}^{1-\nu}}\right)\right) \tag{21}$$

for every $0 < \nu < 1$.

Proof: see Appendix B.

IV. NE AND ϵ-NE POINTS OF THE BIG

A trivial NE point in the BIG is the pure-FS strategy profile . We would like to derive additional NE points which are non-FS. These points are of interest because (if they exist) they Pareto dominate pure-FS NE points as shown in the following proposition.

Proposition 3: Let S_1, S_2 be a non pure-FS NE point (i.e. $P(S_1 = 1), P(S_2 = 1) \neq 0$), then it Pareto dominates the pure-FS NE point, i.e. $\pi_i(S_i, S_j, x_i, y_i) \geq u_i(1/2, 1/2, x_i, y_i)$ for all x_i, y_i and i.

Proof: See Appendix C.

In the sequel, it is shown that if users are allowed to coordinate in advance to use disjoint subbands in the case of FDM (as implied in (2)), FDM is possible from a game theoretic point of view and also increases the total system throughput as well as the individual throughput.

A. Derivation of non pure-FS NE points

Proposition 3 shows that non pure-FS NE points are attractive. However, deriving such points analytically is not always possible. For a symmetric game where all channel magnitudes are identically distributed, NE points were derived in [17] where it was shown that in addition to the pure-FS NE point, there exists a non pure-FS NE given by the following strategy profile:

$$S_i(x_i, y_i) = \begin{cases} \theta_i = 1, & \text{if } y_i > x_i \\ \theta_i = 1/2, & \text{otherwise} \end{cases} \tag{22}$$

However, in the general case of arbitrary distributions, NE points cannot be computed analytically. This makes them impossible to implement and analyze. We therefore address to near NE points.

Definition 5: For $\epsilon \geq 0$, an ϵ-near NE point is a strategy profile $(\hat{S}_1, \hat{S}_2)$ such that

$$\pi_i\left(\hat{S}_i, \hat{S}_j, x_i, y_i\right) \geq \sup_{S_i \in \mathcal{S}_i} \pi_i\left(S_i, \hat{S}_j, x_i, y_i\right) - \epsilon,\ \forall x_i, y_i \tag{23}$$

It is straightforward to show that for sufficiently small ϵ, ϵ-near NE points also Pareto dominate the pure FS NE point (this follows from the continuity of the expected payoff with respect to a).

The main idea behind ϵ-near NE points is that if one of the players deviates from it, he can gain no more than ϵ additional payoff. From a practical point of view, for sufficiently small ϵ, ϵ-near NE points are as stable as ordinary NE points.

We are now ready to introduce the main theorem which provides an analytic expression for such points:

Theorem 4: Assume the channel gains $|H_{iq}|^2,\ i,q \in \{1,2\}$ are continuous random variables, then for every $\epsilon > 0$, there exists some $\overline{SNR}_0$ such that for every $\overline{SNR}_i > \overline{SNR}_0,\ i = 1,2$ (or equivalently, for every $\bar{p} > \bar{p}_0$) the following strategy profile is an ϵ-near NE point:

$$\hat{S}_1 = \check{S}_1(x_1, y_1, \hat{a}_2) \tag{24}$$
$$\hat{S}_2 = \check{S}_2(x_2, y_2, \hat{a}_1) \tag{25}$$

where $\check{S}_i$ is the best response given in (8), and $(\hat{a}_1, \hat{a}_2)$ is a solution to the following equation system

$$a_1 = 1 - F_{Z_1}\left(q(a_2)\right) \tag{26}$$
$$a_2 = 1 - F_{Z_2}\left(q(a_1)\right) \tag{27}$$

where $F_{Z_i}(z)$ is the distribution function of the ISR. Furthermore, if the channel gains $|H_{iq}|^2,\ i,q \in \{1,2\}$ satisfy the regularity conditions in Assumption 1 ϵ decreases like

$$\epsilon \leq O\left(\frac{\sigma_N^2}{\bar{p}^2} + \sum_{q=1}^{2} F_{|H_{ii}|^2}\left(\frac{\sigma_N^2}{\bar{p}^{1-\nu}}\right) F_{|H_{iq}|^2}\left(\frac{\sigma_N^2}{\bar{p}^{1-\nu}}\right)\right) \tag{28}$$

for every $0 < \nu < 1$.

Proof: see Appendix D.

Theorem 4 provides a procedure to calculate ϵ-near NE points in the high averaged received SNR regime. First, $\hat{a}_1$ and $\hat{a}_2$ are obtained by solving equations (26) and (27), then ϵ-near NE points are given by (24) and (25). Each $\hat{a}_i$ is associated with a unique threshold $\widehat{ISR}_i = q(\hat{a}_i)$ where above it FDM is approximately the best strategy and below it, FS is the approximately the best strategy.

Although Theorem 4 is proven rigourously in Section D, we now explain it intuitively. The idea behind the proof is to approximate player i's best response $\check{S}_i(x_i, y_i, \hat{a}_j)$ by the simple approximate best response $\tilde{S}_i(x_i, y_i, \hat{a}_j)$ that satisfies

$$P\left(\tilde{S}_i(X_i, Y_i, \hat{a}_j) = 1\right) \approx P\left(\check{S}_i(X_i, Y_i, \hat{a}_j) = 1\right) \tag{29}$$

Note that the LHS of (29) can be expressed in closed form. This way, the equations in (10) are approximated by (26)-(27). This enables us to obtain $\hat{a}_1,\ \hat{a}_2$ analytically with the corresponding ϵ-near NE point given in (24-25).

B. *Existence of ϵ-near NE Points*

Now that a procedure to derive ϵ-near NE points has been established, we investigate the existence properties of such points. The following theorem presents a sufficient condition for the existence of a ϵ-near NE point.

Theorem 5: Assume that $Z_i,\ i = 1, 2$ are continuous random variables such that $P(Z_i < 0.5) < 1$ and denote the corresponding densities by $f_{Z_i}(z)$. A sufficient condition for the existence of a solution to equations (26), (27) is that

$$\lim_{b\to\infty} f_{Z_i}(b) b^2 \log(b) = \infty \tag{30}$$

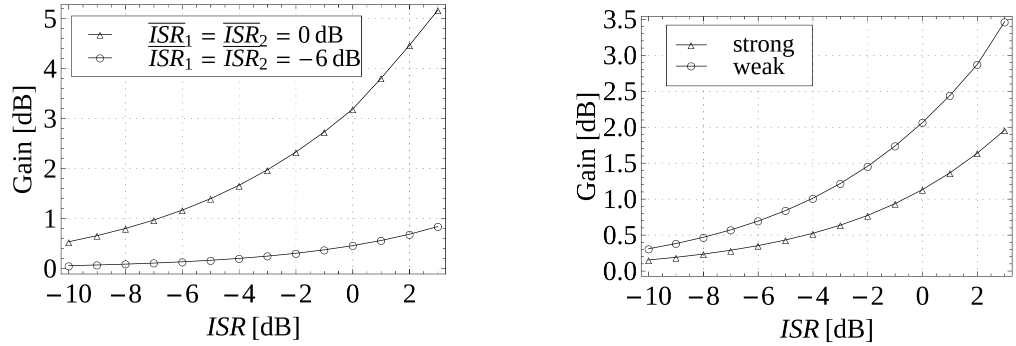
for every $i \in \{1, 2\}$.

Proof: see Section E.

Theorem 5 asserts that if the ISR's PDF is tail heavy (as given exactly in (30)), non pure-FS strategies are possible and beneficial to both users. This condition is satisfied in important channel models including Rayleigh, Rician and Nakagami fading (as demonstrated in Section V).

V. The BIG in Common Channel Models

In this section we study the BIG in practical channel models such as Rayleigh, Nakagami and Rician. We will study the effect of different fading intensities on the players' preferences, the existence and uniqueness properties of NE points and the performance gain.



(a) symmetric scenarios

(b) weak strong scenario

Fig. 3. The difference (dB) between the conditional expected payoffs of non pure-FS and pure-FS NE points as a function of ISR. The channel distributions are Rayleigh. (a) Two symmetric game scenarios: weak-weak (0 dB) and strong-strong (-6 dB). Each curve represents the gain in the corresponding scenario. (b) Weak-strong scenario, the weak $\overline{ISR} = -6$ dB whereas the strong $\overline{ISR} = 0$ dB.

A. *Nakagami channel*

The Nakagami distribution [see e.g. 10, Sec. 3.2.2] is parameterized by averaged received magnitude and fading parameter m, i.e. X's PDF is given by

$$f_X(x) = \left(\frac{m}{\overline{SNR}}\right)^m \frac{x^{m-1}}{\Gamma(m)} \exp\left(\frac{-mx}{\overline{SNR}}\right) \tag{31}$$

where $\overline{SNR}$ is the averaged level of the SNR.

We now study the existence of non pure-FS NE points using Theorem 5. Denote the averages and the fading parameters of X and Y by $\overline{SNR}$, m_1 and $\overline{INR}$, m_2 respectively. Using the formula for transformation of random variables [see e.g. 22], the PDF of $Z = X/Y$ is given by

$$f_Z(z) = \frac{\overline{ISR}^{m_1} m_1^{m_1} m_2^{m_2} \Gamma(m_1 + m_2)}{\Gamma(m_1)\Gamma(m_2)} \frac{z^{m_2-1}}{\left(m_2 z + \overline{ISR} m_1\right)^{m_1+m_2}} \tag{32}$$

where $\overline{ISR} = \overline{INR}/\overline{SNR}$. Thus, by applying Theorem 5, a sufficient condition for the existence of a non pure-FS NE point is that the fading coefficient of the direct channel of both users must satisfy

$$m_1 \leq 1 \tag{33}$$

In particular, this condition is satisfied in Rayleigh fading channels.

Figure 3 shows the benefit of non pure-FS over pure-FS NE points for different values of $\overline{ISR}$ in Rayleigh fading channel (i.e. $m = 1$ for all paths). Figure 3(a) depicts a symmetric weak-weak scenario and a symmetric strong-strong scenario. In both cases the conditional expected payoff is higher for both players and increases with the ISR. However, in the weak-weak scenario, the gain is significant. Figure 3(b) depicts a weak-strong scenario ($\overline{ISR} = -7$ corresponds to the strong). In this case, it is clear that the weak player gains more than the strong one, but non pure-FS is better for both of the players.

In order to obtain insights into the BIG in Nakigimi channels, we address to numerical evaluation of the approximate best response function (13) for different values of distribution parameters. To study the effect of m_1, the fading parameter in the direct channel, Figure 4 depicts the threshold ISR of the approximate best response of player i as a function of m_1. This is evaluated for different values of $\overline{ISR}$. It is shown that the threshold ISR is a decreasing function of m_1. This is violated only if interference is very strong ($\overline{ISR}_i = 10$ dB) whereas the threshold ISR is hardly affected by the values of m_1. From this we deduce that a low fading effect (smaller probabilities of deep fade) in the direct channel (i.e. high values of m_1) encourages the use of FS (since the threshold ISR increases).

In Figure 5, we study the effect of m_2, the fading parameter of the interfering channel. In this case we see that the effect of m_2 on the threshold ISR of the approximate best response depends on other factors

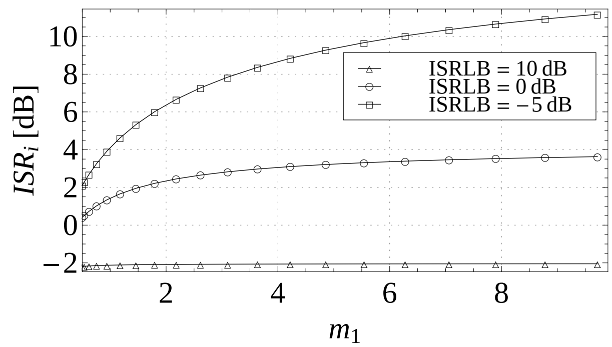
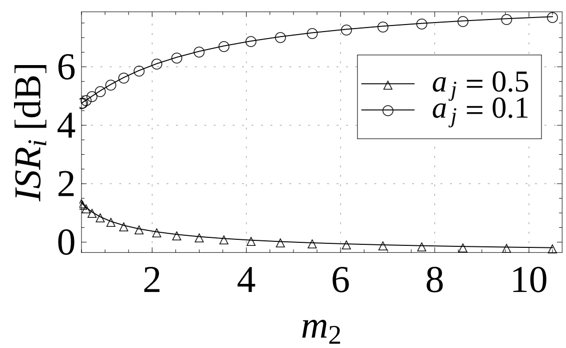
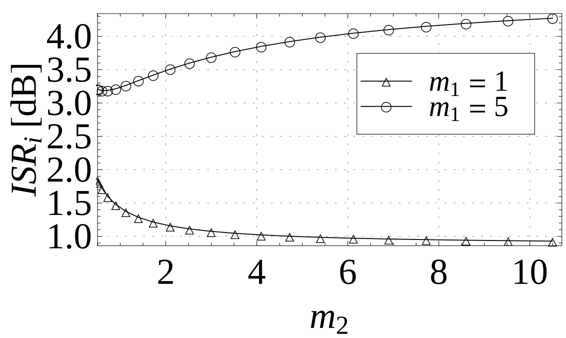


Fig. 4. Numerical evaluation of the threshold ISR of the approximate best response functions for Nakagami fading as a function of m_1 - the fading coefficient in the direct channel. $\overline{ISR} = \overline{INR}/\overline{SNR}$ stands for the ratio between the averaged received INR and SNR. The value of m_1 is fixed and equal to 1 and the opponent's probability of choosing FDM is $a_j = 0.2$.



(a)

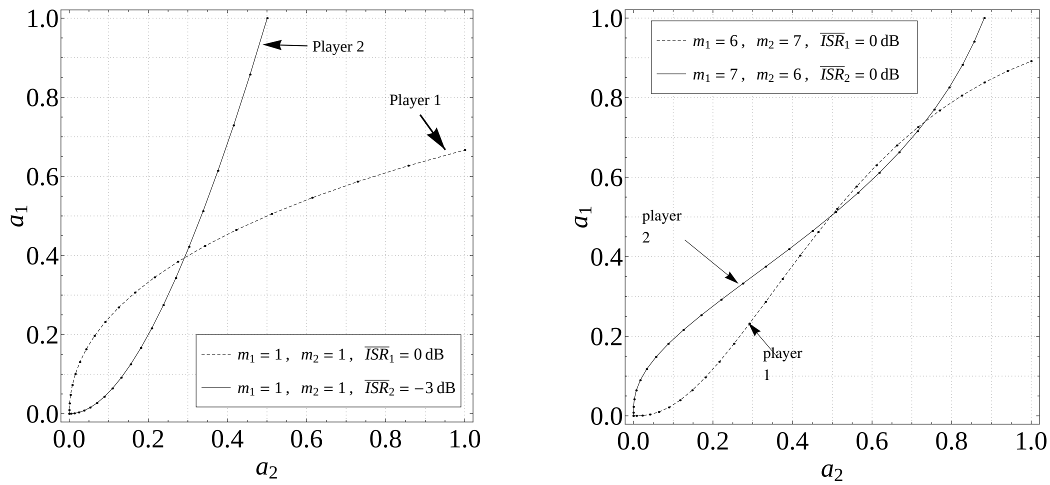


(b)

Fig. 5. The threshold ISR (above which player i chooses FDM) as determined by the approximate best response in a Nakagami fading channel. The horizontal axes represent the fading coefficient of the interference channel m_2. Figure 5(a) depicts the threshold ISR for both low and high level of a_j (the opponent's probability of choosing FDM) with fixed value of the fading parameter in the direct channel $(m_1 = 1)$. Figure 5(b) depicts the threshold ISR for two levels of m_1 with $a_j = 0.1$.

such a m_1 and a_j. For low levels of a_j, it can be seen in Figure 5(a) that the threshold ISR is a increasing function of m_2 while it is a decreasing function for higher values on a_j. In other words, if your (assuming that you are player i) opponent favours (does not favour) FDM, you should consider FDM (FS) more strongly as the interference to your receiver becomes more dominant by the line of sight path than by the reflected paths. Figure 5(b) shows the same for the parameter m_1; i.e. if a player experiences high probability of fading in the direct channel, he should consider FDM (FS) more strongly if the interference to his receiver becomes more dominant by the line of sight than by the multipath.

In Figures 6,7 we study the existence properties of ϵ-near NE points in different channel configurations. Figure 6(a) shows a Rayleigh fading channel with two users and illustrates the ϵ-near NE point. Figure



(a) Rayleigh fading

(b) Multiple equilibrium points

Fig. 6. Numerical evaluation of the ϵ-near NE points for Nakagami fading in different scenarios. The dashed (solid) lines represent player 1's (2's) best response for given values of m_1 and m_2. Each intersection between dashed and solid lines is a ϵ-near NE point. A user is considered "strong" ("weak") if its $\overline{ISR}$ his 10 dB (0 dB).

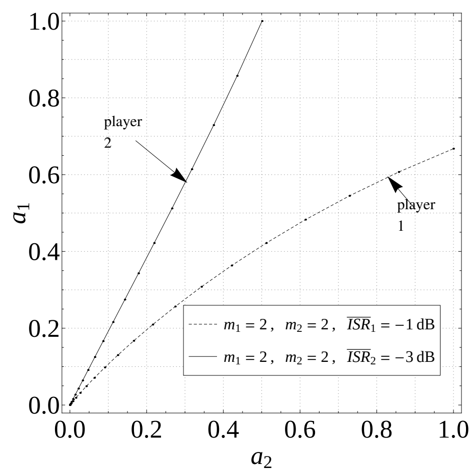


Fig. 7. A scenario where the conditions of Theorem 5 are not satisfied

6(b) shows that ϵ-near NE points are not necessarily unique. In Figure 7 we show a scenario the conditions of Theorem 5 are not satisfied.

VI. Conclusions

In this paper we applied Bayesian games to analyze a two user wireless interference channel with incomplete information. Each player knows its own direct and interfering channel magnitudes but only knows the statistics of its opponent's channel.

The main result of this paper is the derivation of a non pure-FS ϵ-NE point in the BIG with minimal coordination between users. This is a much better alternative than the pure-FS NE point which may be very inefficient. The non pure-FS point offers better spectrum utilization efficiency than the pure-FS Nash equilibrium. This is true for each user individually and in terms of a global network. Through numerical examples, we demonstrated that this performance gain can be substantial. We further provided a sufficient condition for the existence of non pure-FS ϵ-NE and which is satisfied in particular in a Rayleigh fading channel. We also demonstrated numerically that such points exist in many other scenarios.

In addition to the derivation of the non pure-FS NE points, in Section III we presented the best response and the approximate best response function that converges in probability to the best response as the transmitted-power to noise ratio increases. The approximated best response funcntion simply compares the measured interference-to-noise ratio to a threshold that depends on the opponents's probability of choosing FDM and on channel distribution. These results were later used in Section V to analyse selfish and rational behaviour of wireless users as a function of the channel parameters. It was shown that:

- Strong fading (high probabilities for deep fade) in the direct channel encourages wireless selfish users to use FDM.
- Strong fading in the interfering channel encourages selfish wireless users with strong fading in the direct channel to use FDM, while it has the opposite effect on users with weak fading in the direct channel.
- Strong fading in the interfering channel encourages selfish wireless users to use FDM if the opponent chooses FDM with high probabilities, while it has the opposite effect if the opponent chooses FDM frequently.

APPENDIX

A. Proof of Proposition 1

Observe that $r(a,b)$ is a continuous, differentiable and strictly increasing function of b for every a. It can be shown that $r(a,1/2)<0$ and that $\lim_{b\longrightarrow\infty} r(a,b)>0$ for all $a>0$. Thus, $r(a,q(a))=0$ defines an implicit differentiable function $q(a)$ that satisfies $q(a)>1/2$ for every $0<a\leq 1$.

We show that $q(a)$ is a strictly monotonic decreasing function of a. This can be established by observing the derivative of $q(a)$

$$q'(a)=\left(-\frac{q(a)(1+q(a))(2+q(a))(1+2\,q(a))}{2+4q(a)-a\,q(a)+aq^2(a)}\right)(2+2\log(1+q(a))-\log(2+q(a))-\log(1+2\,q(a))) \quad (34)$$

Since $q(a)>1/2$ the derivative is negative. $\square$

B. Proof of Lemma 2

Since $\check{S}_i$ and $\tilde{S}_i$ are binaries in their range, it is sufficient to show that

$$|P((X_i,Y_i)\in\check{D}_i^{a_j})-P((X_i,Y_i)\in\tilde{D}_i^{a_j})|\leq\epsilon,\ \ \forall\,\bar{p}>\bar{p}_0 \quad (35)$$

Henceforth, the indices i,j are omitted, a will denote a_j and $\check{D}^a,\ \tilde{D}^a$ will denote $\check{D}_i^{a_j},\ \tilde{D}_i^{a_j}$.

Let $\bar{p}_n,\ \alpha_n$ be sequences satisfying $\lim_{n\to\infty}\bar{p}_n,\alpha_n=\infty$ such that $\alpha_n=o\left(\bar{p}_n\right)$,[8] denote $X^n=\bar{p}_n|H_{ii}|^2/\sigma_N^2,\ Y^n=\bar{p}_n|H_{ij}|^2/\sigma_N^2,\ P_n(A)=P\left((X^n,Y^n)\in A\right)$. Further denote $\mathcal{A}_n=\{X^n>\alpha_n\}$ and $\mathcal{B}_n=\{Y^n>\alpha_n/2\}$.

Define

$$\Psi_n^i=\left(\check{D}^a\Delta\tilde{D}^a\right)\bigcap G_i \quad (36)$$

(see Figure 8 for illustration) where $G_1=\mathcal{A}_n\bigcap\mathcal{B}_n$, $G_2=\mathcal{A}_n^c\bigcap\mathcal{B}_n$, $G_3=\mathcal{A}_n\bigcap\mathcal{B}_n^c$ and $G_4=\mathcal{A}_n^c\bigcap\mathcal{B}_n^c$. This partition satisfies

$$P_n(\check{D}^a\Delta\tilde{D}^a)=\sum_{q=1}^{4}P_n\left(\Psi_n^q\right) \quad (37)$$

[8]For deterministic sequences α_n,β_n with $\lim_{n\to\infty}\alpha_n/\beta_n=M$ we say that $\alpha_n=O(\beta_n)$ if M is finite and non zero and $\alpha_n=o(\beta_n)$ if $M=0$.

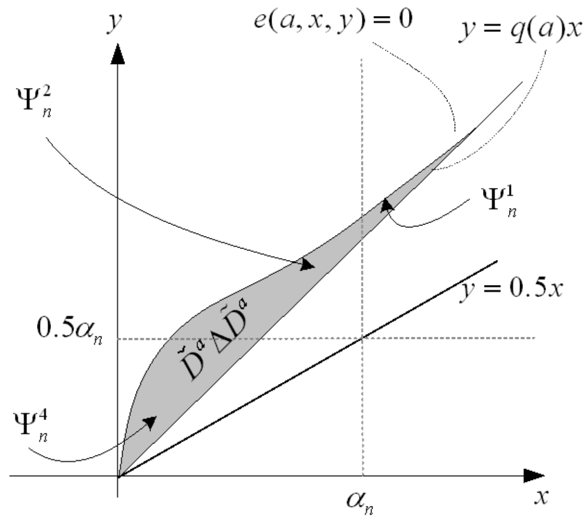


Fig. 8. Graphic illustration of the partition in the proof of Lemma 2.

and

$$P_n\left(\Psi_n^3\right)=0 \tag{38}$$

$$P_n\left(\Psi_n^2\right)\leq F_{|H_{ii}|^2}\left(\sigma_N^2\frac{\alpha_n}{\bar{p}_n}\right) \tag{39}$$

$$P_n\left(\Psi_n^4\right)\leq F_{|H_{ii}|^2}\left(\sigma_N^2\frac{\alpha_n}{\bar{p}_n}\right)F_{|H_{ij}|^2}\left(\sigma_N^2\frac{\alpha_n}{2\bar{p}_n}\right) \tag{40}$$

where (38) is true because both strategies are identical if $y\leq 0.5x$. Therefore, to show the first part of the Lemma (Equation (20)) , it is sufficient to show that $P_n\left(\Psi_n^1\right)=o(1)$. This follows from the fact that for every $a>0$,

$$\lim_{n\to\infty} e(\alpha_n,q(a)\alpha_n,a)-\hat{e}(\alpha_n,q\alpha_n,a)=0 \tag{41}$$

Thus

$$\limsup_n \Psi_n^1=\phi \tag{42}$$

and from the continuity from above of measures [see e.g. 23, Theorem 1.8] it follows that

$$\lim_{n\to\infty} P_n\left(\Psi_n\right)=0 \tag{43}$$

which establishes the first part of the Lemma.

For the second part of the Lemma, we will show that

$$P_n(\check{D}^a\Delta\tilde{D}^a)\leq O\left(\frac{\sigma_N^2}{\bar{p}_n}+\left(F_{|H_{ii}|^2}\left(\sigma_N^2\frac{\alpha_n^2}{\bar{p}_n}\right)\right)^2+\prod_{q=1}^{2}\left(F_{|H_{iq}|^2}\left(\frac{\sigma_N^2}{\bar{p}_n^{1-\nu}}\right)\right)\right) \tag{44}$$

This requires an additional analysis of $P_n(\Psi_n^1)$ and $P_n(\Psi_n^2)$. For the term $P_n\left(\Psi_n^1\right)$, we first assume that that $Y^n>q(a)X^n$. In this case

$$P_n\left(\Psi_n^1\middle|\, Z>q(a),\mathcal{A}_n\right)=P\left(\,e(X^n,Y^n,a)<0\,|\mathcal{A}_n,Y^n>q(a)X^n\right)=P_n^1+P_n^2 \tag{45}$$

where

$$P_n^1=P\left(e(X^n,Y^n,a)<0,q(a)<Z<q(a)+\tfrac{1}{\gamma_n}\middle|\,\mathcal{A}_n,Y^n>q(a)X^n\right) \tag{46}$$

$$P_n^2 = P\left(e(X^n, Y^n, a) < 0, q(a) + \tfrac{1}{\gamma_n} \le Z \,|\, \mathcal{A}_n, Y^n > q(a)X^n\right) \tag{47}$$

and $\gamma_n = o(\alpha_n)$. Before evaluating P_n^1 and P_n^2 the function $e(x, y, a)$ will be simplified by substituting $y = zx$ (which is possible because $x, y > 0$)

$$\begin{aligned} e(x, z\,x, a) =& \tfrac{1}{2}a\log(x+1) - a\left(\tfrac{1}{2}\log\left(\tfrac{x}{2(xz+1)}+1\right) + \tfrac{1}{2}\log\left(\tfrac{x}{2}+1\right)\right) \\ & -(1-a)\log\left(\tfrac{x}{xz+2}+1\right) + \tfrac{1}{2}(1-a)\log\left(\tfrac{2x}{xz+2}+1\right) \end{aligned} \tag{48}$$

which is a bounded function of z for every x. Furthermore, it is easy to verify that the function $e(\frac{1}{w}, \frac{z}{w}, a)$ is infinitely differentiable with bounded derivatives at $w = 0$. Thus, since

$$\begin{aligned} & t(z, a) \triangleq \lim_{w\to 0} e(\tfrac{1}{w}, \tfrac{z}{w}, a) = \tfrac{1}{2}\left(a\left(1 - \log\left(\tfrac{1}{2z}+1\right)\right)\right. \\ & \left. +(1-a)\log\left(1+\tfrac{2}{z}\right) + 2(a-1)\log\left(\tfrac{1}{z}+1\right)\right) \end{aligned} \tag{49}$$

is bounded, continuous and differentiable on $z > 0.5$ for every a it is possible to expand $e(x, z\,x, a)$ with respect to $1/x$ and obtain

$$e(x, z\,x, a) = r(a, z) + Q(a, z)\frac{1}{x} + O\left(\frac{1}{x^2}\right) \tag{50}$$

where $r(a, z)$ is defined in (11), the residual absolute value can be bounded by M/x^2 where M is finite and

$$Q(a, z) = \frac{-2az^4 - 7az^3 - 6az^2 - 7az - 2a + 8z + 4}{2z(z+1)(z+2)(2z+1)} \tag{51}$$

is bounded for every z, a; furthermore, since $r(a, z)$ is a continuous and increasing function of z for every $a > 0$ (as shown in Proposition 1) that satisfies $r(a, q(a)) = 0$, it follows that

$$r(a, z) = R(a)(z - q(a)) + O\left((z - q(a))^2\right) \tag{52}$$

where the residual absolute value can be bounded by $M/(z - q(a))^2$ where M is finite for every $a > 0$ and

$$R(a) = \frac{(aq(a)^2 - aq(a) + 4q(a) + 2)}{2q(a)(q(a)+1)(q(a)+2)(2q(a)+1)} \tag{53}$$

is bounded and positive for every $0 \le a \le 1$ because $q(a) > 1/2$.

In what follows it is shown that for sufficiently large n, $P_n^2 = 0$. To see this, observe that for every $z \ge (a) + \frac{1}{\gamma_n}$

$$\begin{aligned} e(x, z\,x, a) &\ge r(a, z) - \tfrac{M_1}{x} - \tfrac{M_2}{x^2} \ge r(a, z) - \tfrac{M_1}{\alpha_n} - \tfrac{M_2}{\alpha_n^2} \\ &\ge \tfrac{R(a)}{\gamma_n} + O\left(\tfrac{1}{\gamma_n^2}\right) - \tfrac{M_1}{\alpha_n} - \tfrac{M_2}{\alpha_n^2} \end{aligned} \tag{54}$$

where M_1, M_2 are positive and finite for every z and a. Therefore,

$$\gamma_n e(x, zx, a) \ge R(a) + O\left(\frac{1}{\gamma_n}\right) - \frac{M_1\gamma_n}{\alpha_n} - \frac{M_3\gamma_n}{\alpha_n^2} \tag{55}$$

and becomes the $R(a) > 0, \forall a > 0$ and because M_1 and M_2 are bounded, it follows that $P_n^2 = 0$ for sufficiently large n.

It remains to show that P_n^1 decreases like $1/\bar{p}$. By substituting the series expansions of $r(a, z)$ into (50) it follows that for every $z \in (q(a), q(a) + 1/\gamma_n),\ x > \alpha_n$

$$\begin{aligned} & e(x, y, a) = R(a)(z - q) + \tfrac{Q(a,z)}{x} + O\left(\tfrac{1}{x^2}\right) + O(z-q)^2 \\ & \le R(a)\left(z - q(a)\right) - \tfrac{M_1}{x} - \tfrac{M_2}{x^2} - M_3(z-q)^2 \le \eta_n\left(z - q(a)\right) - \tfrac{\xi_n}{x} \end{aligned} \tag{56}$$

where $\eta_n = R(a) - M_3/\gamma_n$ and $\xi_n = M_1 + M_2/\alpha_n$.

Thus,

$$\begin{aligned} P_n^1 &\leq \frac{P(0<Y^n-q(a)X^n<\min(\xi_n/\eta_n, X^n/\gamma_n), \mathcal{A}_n)}{P(Y^n>X^n q(a), \mathcal{A}_n)} = \frac{\int\limits_{\alpha_n}^{\infty} \int\limits_{xq(a)}^{q(a)x+\xi_n/\eta_n} f_{Y^n}(y) f_{X^n}(x) dy dx}{\int\limits_{\alpha_n}^{\infty} \int\limits_{xq(a)}^{\infty} f_{Y^n}(y) f_{X^n}(x) dy dx} \\ &= \frac{\int\limits_{\alpha_n}^{\infty} (F_{Y^n}(q(a)x+\xi_n/\eta_n) - F_{Y^n}(q(a)x)) f_{X^n}(x) dx}{\int\limits_{\alpha_n}^{\infty} (1-F_{Y^n}(q(a)x)) f_X(x) dx} = \frac{\mu_n}{\lambda_n} \end{aligned} \tag{57}$$

Note that $\liminf_n \lambda_n >$0, because

$$\begin{aligned} \lambda_n &= \int\limits_{\alpha_n}^{\infty} \left(1 - F_{|H_{ij}|}^2 \left(\frac{\sigma_v^2 q(a) x}{\bar{p}_n}\right)\right) f_{X^n}(x) dx \geq \int\limits_{\alpha_n}^{\bar{p}_n} \left(1 - F_{|H_{ij}|^2} \left(\frac{\sigma_v^2 q(a) x}{\bar{p}_n}\right)\right) f_{X^n}(x) dx \\ &\geq \int\limits_{\alpha_n}^{\bar{p}_n} \left(1 - F_{|H_{ij}|^2} \left(\sigma_v^2 q(a)\right)\right) f_{X^n}(x) dx = \left(1 - F_{|H_{ij}|^2} \left(\sigma_v^2 q(a)\right)\right) \times \Big(F_{|H_{ij}|^2} \left(\sigma_v^2 q(a)\right) \\ &\quad - F_{|H_{ij}|^2} \left(\frac{\sigma_v^2 q(a) \alpha_n}{\bar{p}_n}\right)\Big) \xrightarrow[n\to\infty]{} \left(1 - F_{|H_{ij}|^2} \left(\sigma_v^2 q(a)\right)\right) F_{|H_{ij}|^2} \left(\sigma_v^2 q(a)\right) > 0, \end{aligned} \tag{58}$$

therefore, the first term of P_n^1 decreases like μ_n

$$\begin{aligned} \mu_n &\leq \int\limits_0^{\infty} \left(F_{|H_{ij}|^2} \left(\frac{\sigma_v^2 (q(a)x+\xi_n/\eta_n)}{\bar{p}_n}\right) - F_{|H_{ij}|^2} \left(\frac{\sigma_v^2 q(a) x}{\bar{p}_n}\right)\right) \frac{\sigma_v^2}{\bar{p}_n} f_{|H_{ii}|^2} \left(\frac{\sigma_v^2 x}{\bar{p}_n}\right) dx \\ &= \frac{\sigma_v^2 \xi_n}{\eta_n \bar{p}_n} \int\limits_0^{\infty} \frac{F_{|H_{ij}|^2}\left(q(a)v + \sigma_v^2 \xi_n/(\eta_n \bar{p}_n)\right) - F_{|H_{ij}|^2}(q(a)v)}{\sigma_v^2 \xi_n/(\eta_n \bar{p}_n)} f_{|H_{ii}|^2}(v)\, dv \end{aligned} \tag{59}$$

Recall that by hypothesis $f_{|H_{iq}|^2}(v)$ is bounded for every $v > 0$. Thus by the LaGrange mean value theorem, for every $v \geq \delta$

$$\frac{F_{|H_{ij}|^2}\left(q(a)v + \sigma_v^2 \xi_n/\eta_n \bar{p}_n\right) - F_{|H_{ij}|^2}(q(a)v)}{\sigma_v^2 \xi_n/(\eta_n \bar{p}_n)} \leq \sup_{\theta \in [0,1]} \left(f_{|H_{ij}|^2} \left(q(a)v + \frac{\theta \xi_n \sigma_v^2}{\eta_n \bar{p}_n}\right)\right) \leq M \tag{60}$$

by invoking the dominant convergence theorem [see e.g. 23, Theorem 2.24] on the integral in (59)

$$\begin{aligned} &\lim_{\delta\to 0} \lim_{n\to\infty} \int\limits_{\delta}^{\infty} \frac{F_{|H_{ij}|^2}\left(q(a)v + \sigma_v^2 \xi_n/(\eta_n \bar{p}_n)\right) - F_{|H_{ij}|^2}(q(a)v)}{\sigma_v^2 \xi_n/(\eta_n \bar{p}_n)} f_{|H_{ii}|^2}(v)\, dv \\ &= \lim_{\delta\to 0} \int\limits_{\delta}^{\infty} f_{|H_{ij}|^2}(q(a)v) f_{|H_{ii}|^2}(v)\, dv = \int\limits_0^{\infty} f_{|H_{ij}|^2}(q(a)v) f_{|H_{ii}|^2}(v)\, dv \end{aligned} \tag{61}$$

where (61) is true because $f_{|H_{iq}|^2}(v)\,,\ i, q \in \{1, 2\}$ are probability densities. Furthermore, it is positive and finite for every a. From this it follows that

$$P_n^1 \leq O\left(\frac{\sigma_v^2}{\bar{p}_n}\right) \tag{62}$$

and therefore

$$P_n\left(\Psi_n^1/\, Z > q(a), \mathcal{A}_n\right) \leq O\left(\frac{1}{\bar{p}_n}\right) \tag{63}$$

We now assume that that $Y^n \leq q(a) X^n$. In this case

$$P_n\left(\Psi_n^1 \middle|\, Z \leq q(a), \mathcal{A}_n\right) = P\left(\, e(X^n, Y^n, a) > 0 \,|\mathcal{A}_n, Y^n \leq q(a) X^n\right) = P_n^1 + P_n^2 \tag{64}$$

where

$$P_n^1 = P\left(e(X^n, Y^n, a) \geq 0, q(a) - \tfrac{1}{\gamma_n} < Z < q(a) \middle|\, \mathcal{A}_n, Y^n \leq q(a) X^n\right) \tag{65}$$

$$P_n^2 = P\left(e(X^n, Y^n, a) \geq 0, Z \leq q(a) - \tfrac{1}{\gamma_n} \middle| \mathcal{A}_n, Y^n \leq q(a) X^n \right) \tag{66}$$

In what follows it is shown that for sufficiently large n, $P_n^2 = 0$. To see this, observe that for every $z \leq q(a) - \frac{1}{\gamma_n}$

$$e(x, z\,x, a) \leq r(a,z) + \frac{M_1}{x} + \frac{M_2}{x^2} \leq r(a,z) + \frac{M_1}{\alpha_n} + \frac{M_2}{\alpha_n^2} \tag{67}$$

$$\leq -\frac{R(a)}{\gamma_n} + O\left(\frac{1}{\gamma_n^2}\right) + \frac{M_1}{\alpha_n} + \frac{M_2}{\alpha_n^2} \tag{68}$$

where M_1, M_2 are positive and finite for every z and a. Therefore,

$$\gamma_n e(x, zx, a) \leq -R(a) + O\left(\frac{1}{\gamma_n}\right) + \frac{M_1 \gamma_n}{\alpha_n} + \frac{M_3 \gamma_n}{\alpha_n^2} \tag{69}$$

and become the $R(a) > 0, \forall a > 0$ and because M_1 and M_2 are bounded, it follows that $P_n^2 = 0$ for sufficiently large n.

It remains to show that P_n^1 decreases like $1/\bar{p}$. By substituting the series expansions of $r(a,z)$ into (50) it follows that for every $z \in (q(a) - 1/\gamma_n, q(a)),\ x > \alpha_n$

$$e(x,y,a) \leq R(a)\left(z - q(a)\right) + \tfrac{M_1}{x} + \tfrac{M_2}{x^2} + M_3 (z-q)^2 \leq \eta_n \left(z - q(a)\right) + \tfrac{\xi_n}{x} \tag{70}$$

where $\eta_n = R(a) + M_3/\gamma_n$ and $\xi_n = M_1 + M_2/\alpha_n$. Thus,

$$\begin{aligned} P_n^1 &\leq \tfrac{P(-\min(\xi_n/\eta_n, X^n/\gamma_n) < Y^n - q(a) X^n < 0, \mathcal{A}_n)}{P(Y^n \leq X^n q(a), \mathcal{A}_n)} \\ &= \frac{\int_{\alpha_n}^{\infty} (F_{Y^n}(q(a)x) - F_{Y^n}(q(a)x - \xi_n/\eta_n)) f_{X^n}(x) dx}{\int_{\alpha_n}^{\infty} F_{Y^n}(q(a)x) f_X(x) dx} = \tfrac{\mu_n}{\lambda_n} \end{aligned} \tag{71}$$

Note that $\liminf_n \lambda_n >0$, to see this

$$\begin{aligned} \lambda_n &= \int_{\alpha_n}^{\infty} F_{|H_{ij}|}^2 \left(\tfrac{\sigma_v^2 q(a) x}{\bar{p}_n}\right) f_{X^n}(x) dx \geq \int_{\bar{p}_n}^{\infty} F_{|H_{ij}|^2} \left(\tfrac{\sigma_v^2 q(a) x}{\bar{p}_n}\right) f_{X^n}(x) dx \\ &\geq \int_{\bar{p}_n}^{\infty} F_{|H_{ij}|^2} \left(\sigma_v^2 q(a)\right) f_{X^n}(x) dx = F_{|H_{ij}|^2} \left(\sigma_v^2 q(a)\right) \times \left(1 - F_{|H_{ij}|^2} \left(\sigma_v^2 q(a)\right)\right) \end{aligned} \tag{72}$$

therefore, the first term of P_n^1 decreases like μ_n

$$\begin{aligned} \mu_n &\leq \int_0^{\infty} \left(F_{|H_{ij}|^2} \left(\tfrac{\sigma_v^2 (q(a)x)}{\bar{p}_n}\right) - F_{|H_{ij}|^2} \left(\tfrac{\sigma_v^2 (q(a)x - \xi_n/\eta_n)}{\bar{p}_n}\right) \right) \tfrac{\sigma_v^2}{\bar{p}_n} f_{|H_{ii}|^2} \left(\tfrac{\sigma_v^2 x}{\bar{p}_n}\right) dx \\ &= \tfrac{\sigma_v^2 \xi_n}{\eta_n \bar{p}_n} \int_0^{\infty} \tfrac{F_{|H_{ij}|^2}(q(a)v) - F_{|H_{ij}|^2}\left(q(a)v - \sigma_v^2 \xi_n/(\eta_n \bar{p}_n)\right)}{\sigma_v^2 \xi_n/(\eta_n \bar{p}_n)} f_{|H_{ii}|^2}(v)\, dv \leq O\left(\tfrac{\sigma_v^2}{\bar{p}_n}\right) \end{aligned} \tag{73}$$

which leads to

$$P_n \left(\Psi_n^1 / \, Z \leq q(a), \mathcal{A}_n \right) \leq O\left(\frac{1}{\bar{p}_n}\right) \tag{74}$$

and therefore

$$P_n(\Psi_n^1) \leq O\left(1/\bar{p}_n\right) \tag{75}$$

It remains to evaluate $P_n\left(\Psi_n^2\right)$. Note that

$$P_n\left(\Psi_n^2\right) = P\left(\mathcal{B}_n, \mathcal{A}_n^c\right) P_n\left(\Psi_n^2 \middle| \mathcal{B}_n, \mathcal{A}_n^c\right) \tag{76}$$

and

$$P\left(\mathcal{B}_n, \mathcal{A}_n^c\right) \leq O\left(F_{|H_{ii}|^2}\left(\frac{\sigma_v^2 \alpha_n}{\bar{p}_n}\right)\right) \tag{77}$$

it follows that

$$P_n\left(\Psi_n^2\right) \leq P_n\left(\Psi_n^2\middle| \mathcal{C}_n\right) O\left(F_{|H_{ii}|^2}\left(\frac{\sigma_v^2 \alpha_n}{\bar{p}_n}\right)\right) \tag{78}$$

where $\mathcal{C}_n = \mathcal{A}_n^c \bigcap \mathcal{B}_n$. Furthermore

$$\begin{aligned} P_n\left(\Psi_n^2\middle| \mathcal{C}_n\right) &= P_n\left(\Psi_n^2\middle| \mathcal{C}_n, Z > \alpha_n/2\right) P\left(Z > \alpha_n/2\middle| \mathcal{C}_n\right) + P_n\left(\Psi_n^2\middle| \mathcal{C}_n, Z \leq \alpha_n/2\right) P\left(Z \leq \alpha_n/2\middle| \mathcal{C}_n\right) \\ &\leq P_n\left(\Psi_n^2\middle| \mathcal{C}_n, Z > \alpha_n/2\right) P\left(Z > \alpha_n/2\middle| \mathcal{C}_n\right) + F_{|H_{ij}|^2}\left(\frac{\sigma_v^2 \alpha_n^2}{\bar{p}_n}\right) \end{aligned} \tag{79}$$

where the last inequality is due to

$$\begin{aligned} P\left(Z \leq \alpha_n/2\middle| \mathcal{C}_n\right) &= \frac{P(\alpha_n/2 < Y^n < X^n \alpha_n/2, X^n \leq \alpha_n)}{P(Y^n > \alpha_n/2, X^n \leq \alpha_n)} \\ &\leq \frac{P\left(\alpha_n/2 < Y^n < \alpha^2{}_n/2, X^n \leq \alpha_n\right)}{P(Y^n > \alpha_n/2, X^n \leq \alpha_n)} = P\left(\alpha_n/2 < Y^n < \alpha^2{}_n/2\right) \leq F_{|H_{ij}|^2}\left(\frac{\sigma_v^2 \alpha_n^2}{\bar{p}_n}\right) \end{aligned} \tag{80}$$

It remains to calculate the term

$$P\left(\Psi_n^2\middle| \mathcal{C}_n, Z > \alpha_n/2\right) = P\left(e\left(X^n, Y^n, a\right) < 0\middle| \mathcal{C}_n, Y^n/X^n > \alpha_n/2\right) \tag{81}$$

To evaluate (81), consider the function $e\left(x, y, a\right) - T(x)$ where $T(x) = \frac{a}{2}\log\left(1 + 2x/(x+2)\right)$. Similar to the derivation of (50) we obtain

$$e\left(\frac{y}{z}, y, a\right) - T\left(\frac{y}{z}\right) = r\left(a, z\right) + \frac{az^2 - 5az - 2a + 8z + 4}{2(z+1)(z+2)(2z+1)}\frac{1}{y} + O\left(\frac{1}{y^2}\right) \tag{82}$$

and because $r\left(a, z\right)$ is an increasing and positive function of z for $z > q(a)$ and for every a and because $T\left(y/z\right) \geq 0$ for every $y, z \geq 0$, the RHS of (81) is equal to zero for sufficiently large n. Thus, by combining (80) and (78), it follows that

$$P\left((X^n, Y^n) \in \Psi_n^2\right) \leq O\left(\left(F_{|H_{ii}|^2}\left(\frac{\alpha_n^2 \sigma_N^2}{\bar{p}}\right)\right)^2\right) \tag{83}$$

and by combining it with (38), (40) and (75) we obtain the desired result. □

C. *Proof of Proposition 3*

Player i's conditional expected payoff is

$$\pi_i(S_i, S_j, x_i, y_i) = \max\left\{a_j u_i(1, 1, x_i, y_i) + (1 - a_j)u_i(1, 1/2, x_i, y_i)\right. \tag{84}$$

$$\left., a_j u_i(1/2, 1, x_i, y_i) + (1 - a_j)u_i(1/2, 1/2, x_i, y_i)\right\} \tag{85}$$

where $a_j = P(S_j = 1)$. Thus, it is sufficient to show that

$$a_j u_i(1/2, 1, x_i, y_i) + (1 - a_j)u_i(1/2, 1/2, x_i, y_i) > u_i(1/2, 1/2, x_i, y_i),\ \forall x_i, y_i \in \mathcal{X}_i \times \mathcal{Y}_i \tag{86}$$

This is equivalent to

$$\begin{aligned} &\frac{y_i^3 x_i^2}{2} + 2y_i^3 x_i + \frac{y_i^2 x_i^3}{2} + 4y_i^2 x_i^2 + 8y_i^2 x_i + \frac{x_i^4 y_i}{8} \\ &+2x_i^3 y_i + 9x_i^2 y_i + 12y_i x_i + \frac{x_i^4}{4} + 2x_i^3 + 6x_i^2 + 6x_i > 0 \end{aligned} \tag{87}$$

which is always true. □

D. Proof of Theorem 4

We begin with the following definition:

Definition 6: An approximate NE point is the strategy profile $(\tilde{S}_1(x_1, y_1, \hat{a}_2), \tilde{S}_2(x_2, y_2, \hat{a}_1))$ where $\hat{a}_1$ and $\hat{a}_2$ are a solution to equations (26) and (27).

It remains to show that if there exists an approximate NE point, then there exists a ϵ-near NE point given by (24) and (25). Let

$$\tilde{a}_j = P\left(\check{S}_j\left(X_i, Y_i, \hat{a}_i\right) = 1\right) \tag{88}$$

$$\tilde{a}_i = P\left(\check{S}_i\left(X_i, Y_i, \tilde{a}_j\right) = 1\right) \tag{89}$$

In words, $\tilde{a}_j$ is the probability that player j chooses FDM if he is not deviating from the ϵ-near NE point and $\tilde{a}_i$ is the probability that player i chooses FDM if he "cheats" and uses his best response to player j's true probability for choosing FDM $\tilde{a}_i$ rather than the probability $\hat{a}_j$.

To show that $\left(\check{S}_i\left(x_i, y_i, \hat{a}_j\right), \check{S}_j\left(x_j, y_j, \hat{a}_i\right)\right)$ satisfies (23), one needs to show that for every $x_i, y_i \in \mathcal{X}_i \times \mathcal{Y}_i$ and for sufficiently large $\bar{p}$

$$\Delta\pi_i(x_i, y_i) = |\pi_i\left(\check{S}_i\left(x_i, y_i, \tilde{a}_j\right), \check{S}_j\left(x_j, y_j, \hat{a}_i\right)\right) - \pi_i\left(\check{S}_i\left(x_i, y_i, \hat{a}_j\right), \check{S}_j\left(x_j, y_j, \hat{a}_i\right)\right)| < \epsilon \tag{90}$$

Note that $\Delta\pi_i(x_i, y_i) \neq 0$ if and only if $(x_i, y_i) \in \check{D}^{\hat{a}_j} \Delta \check{D}_i^{\tilde{a}_j}$ (since player j's true probability for choosing FDM is identical in both cases and is equal to $\tilde{a}_j$), thus

$$\Delta\pi_i(x_i, y_i, \hat{a}_j, \tilde{a}_j) = |e(x_i, y_i, \tilde{a}_i)| I_{\check{D}^{\hat{a}_j} \Delta \check{D}_i^{\tilde{a}_j}}(x_i, y_i) \tag{91}$$

where $I_A(x, y)$ denotes the indicator function, i.e. it is equal to 1 if $(x, y) \in A$ and zero otherwise. Since $(x_i, y_i) \in \check{D}^{\hat{a}_j} \Delta \check{D}_i^{\tilde{a}_j}$ is equivalent to $e(x_i, y_i, \hat{a}_j) > 0$ and $e(x_i, y_i, \tilde{a}_j) \leq 0$ or vice versa, and because $e(x, y, a)$ is a continuous function of a, for every $\hat{a}_j, \tilde{a}_j$ there exists some a^* in the interval between $\hat{a}_j$ and $\tilde{a}_j$ such that $e(x_i, y_i, a^*) = 0$. By Lemma 2, we know that $\tilde{a}_j \xrightarrow[\bar{p} \to \infty]{} \hat{a}_j$, thus $e(x_i, y_i, \tilde{a}_j) \xrightarrow[\bar{p} \to \infty]{} 0$, furthermore because $e(x, y, a)$ is bounded for every x, y and is a linear function of *a* it follows that

$$|e(x_i, y_i, \tilde{a}_i)| I_{\check{D}^{\hat{a}_j} \Delta \check{D}_i^{\tilde{a}_j}}(x_i, y_i) = O\left(\tilde{a}_i - \hat{a}_j\right) \tag{92}$$

□

E. Proof of Theorem 5

Denote $w_i(a_j) = 1 - F_{Z_i}(q(a_j))$ for $i \neq j$. Thus

$$w_i'(a_j) = -f_{Z_i}(q(a_j))q'(a_j) \tag{93}$$

Before analyzing (93) recall that $\lim q(a)_{a \to 0} = \infty$, furthermore, it can be verified that

$$\lim_{a \to 0} \frac{q'(a)}{q^2(a)\log(q(a))} = M \tag{94}$$

(this follows immediately from (34). Thus, if (30) is satisfied

$$\lim_{a_j \to 0} w'(a_j) = \infty \tag{95}$$

Consider the curves (26) and (27) in a two-dimensional cartesian system where a_1 and a_2 are given by the horizontal and the vertical coordinates respectively. Both curves are continuous and differentiable. Furthermore, the point $(0, 0)$ is a common point of the two curves and the points $(1 - F_{Z_1}(0.5), 1)$, $(1, 1 - F_{Z_2}(0.5))$ lie on curves (26) and (27) respectively. Since the slop of curve (26) tends to zero as $a_1 \to 0$ and the slope of curve (27) tends to infinity as $a_1 \to 0$, the two curves must intersect at least once. □

References

[1] R. Etkin, A. Parekh, and D. Tse, "Spectrum sharing for unlicensed bands," *IEEE J. Sel. Areas Comm.*, vol. 25, pp. 517–527, April 2007.
[2] S. Adlakha, R. Johari, and A. Goldsmith, "Competition in wireless systems via Bayesian interference games." arXiv:0709.0516v1, Aug. 2007.
[3] A. Leshem and E. Zehavi, "Cooperative game theory and the Gaussian interference channel," *IEEE J. Sel. Areas Comm.*, vol. 26, pp. 1078–1088, Sep 2008.
[4] S. Hayashi and Z.-Q. Luo, "Dynamic spectrum management: When is FDMA sum-rate optimal," in *IEEE Int. Conf. Acoust. Speech, Signal Process. (ICASSP)*, vol. 3, pp. 609–612, IEEE, April 2007.
[5] Z. Ji and K. Liu, "Cognitive radios for dynamic spectrum access-dynamic spectrum sharing: A game theoretical overview," *IEEE Comm. Mag.*, vol. 45, no. 5, pp. 88–94, 2007.
[6] W. Yu, G. Ginis, and J. Cioffi, "Distributed multiuser power control for digital subscriber lines," *IEEE J. Sel. Areas Comm.*, vol. 20, pp. 1105–1115, june 2002.
[7] G. Scutari, D. Palomar, and S. Barbarossa, "Asynchronous iterative water-filling for Gaussain frequency-selective interference channles," *IEEE Trans. Inf. Theory*, vol. 54, pp. 2868–2878, July 2008.
[8] A. Leshem and E. Zehavi, "Game theory and the frequency selective interference channel," *IEEE Sig. Process. Mag.*, vol. 26, pp. 28–40, Sep. 2009.
[9] E. A. Jorswieck, E. G. Larsson, M. Luise, and H. V. Poor, eds., *Game theory in signal processing & communication*, vol. 26. IEEE Sig. Process. Mag., Sep. 2009. special issue.
[10] A. Goldsmith, *Wireless communications*. Cambridge University Press, 2005.
[11] A. Laufer, A. Leshem, and H. Messer, "Game theoretic aspects of distributed spectral coordination with application to DSL networks." arXiv:cs/0602014, 2005.
[12] J. Harsanyi, "Games with incomplete information played by Bayesian players," *Management Science*, no. 14, pp. 159–182,320–334,486–502, 1967–68.
[13] D. Fudenberg and J. Tirole, *Game Theory*. MIT Press, 1991.
[14] H. T, "A minimax game of power control in a wireless network under incomplete information," Technical Report 99-43, DIMACS, August 1999.
[15] G. He, M. Debbah, and E. Altman, "A Bayesian game-theoretic approach for distributed resource allocation in fading multiple access channels." submitted to EURASIP J. on Wireless Commun. and Net., available at: http://www.supelec.fr/d2ri/flexibleradio/pub/Journals2010/J1018.pdf, 2009.
[16] S. Jean and B. Jabbari, "Bayesian game-theoretic modeling of transmit power determination in a self-organizing CDMA wireless network," in *Proc. IEEE VTC Fall 2004*, vol. 5, pp. 3496–3500, Sep. 2004.
[17] Y. Noam, A. Leshem, and H. Messer, "Competitive spectrum sharing in symmetric fading channel with incomplete information." to appear, ICASSP 2010.
[18] S. T. Chung, S. J. Kim, J. Lee, and J. Cioffi, "A game-theoretic approach to power allocation in frequency-selective gaussian interference channels," in *IEEE Int. Symposium Inf. Theory (ISIT)*, pp. 316–316, 2003.
[19] Z. Luo and J. Pang, "Analysis of iterative waterfilling algorithm for multiuser power control in digital sibcriber lines," *EURASIP J. Applied Signal Process.*, vol. 2006, pp. 1–10, 2006.
[20] K. W. Shum, K.-K. Leung, and C. W. Sung, "Convergence of iterative waterfilling algorithm for Gaussian interference channels," *IEEE J. Sel. Areas Comm.*, vol. 25, pp. 1091–1099, August 2007.
[21] J.-S. Pang, G. Scutari, F. Facchinei, and C. Wang, "Distributed power allocation with rate constraints in Gaussian parallel interference channels," *IEEE Trans. Inf. Theory*, vol. 54, pp. 3471–3489, Aug. 2008.
[22] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*. New York: McGraw-Hill, third ed., 1991.
[23] G. Folland, *Real Analysis: Modern Techniques and their Applications*. New York, NY: John Wiley

& sons, 1984.