# Composite Cyclotomic Fourier Transforms with Reduced Complexities 

Xuebin Wu, Meghanad Wagh, Ning Chen, Ying Wang, and Zhiyuan Yan


#### Abstract

Discrete Fourier transforms (DFTs) over finite fields have widespread applications in digital communication and storage systems. Hence, reducing the computational complexities of DFTs is of great significance. Recently proposed cyclotomic fast Fourier transforms (CFFTs) are promising due to their low multiplicative complexities. Unfortunately, there are two issues with CFFTs: (1) they rely on efficient short cyclic convolution algorithms, which has not been sufficiently investigated in the literature, and (2) they have very high additive complexities when directly implemented. To address both issues, we make three main contributions in this paper. First, for any odd prime $p$, we reformulate a $p$-point cyclic convolution as a product of a $(p-1) \times(p-1)$ Toeplitz matrix vector products (TMVP), which can be obtained from well-known TMVP of very small sizes, leading to efficient bilinear algorithms for $p$-point cyclic convolutions. Second, to address the high additive complexities of CFFTs, we propose composite cyclotomic Fourier transforms (CCFTs). In comparison to previously proposed fast Fourier transforms, our CCFTs achieve lower overall complexities for moderate to long lengths, and the improvement significantly increases as the length grows. Third, our efficient algorithms for p-point cyclic convolution and CCFTs allow us to obtain longer DFTs over larger fields, e.g., 2047-point DFT over $\operatorname{GF}\left(2^{11}\right)$ and 4095-point DFT over $\operatorname{GF}\left(2^{12}\right)$, which are first efficient DFTs of such lengths to the best of our knowledge. Finally, our CCFTs are also advantageous for hardware implementations due to their regular and modular structure.


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## Index Terms

Discrete Fourier transforms, finite fields, cyclotomic fast Fourier transforms, prime-factor algorithm, Cooley-Turkey algorithm

## I. Introduction

Discrete Fourier transforms (DFTs) over finite fields [1] have widespread applications in error correction coding, which in turn is used in all digital communication and storage systems. For instance, both syndrome computation and Chien search in the syndrome based decoder of Reed-Solomon codes [2], [3], a family of widely used error control codes, can be formulated as polynomial evaluations and hence can be implemented efficiently using DFTs over finite fields. Implementing an $N$-point DFT directly requires $O\left(N^{2}\right)$ multiplications and $O\left(N^{2}\right)$ additions, and becomes costly when $N$ is large. Hence, reducing the computational complexities of DFTs is of great significance. Recently, efficient long DFTs have become particularly important as increasingly longer error control codes are chosen for digital communication and storage systems. For example, Reed-Solomon codes over $\operatorname{GF}\left(2^{12}\right)$ and with block length of several thousands are considered for hard drive [4] and tape storage [5] as well as optical communication systems [6] to achieve better error performance; the syndrome based decoder of such codes requires DFTs of lengths up to 4095 over $\mathrm{GF}\left(2^{12}\right)$. In addition to complexity, regular and modular structure of DFTs is desirable for efficient hardware implementations.

In the literature, fast Fourier transforms (FFTs) based on the prime-factor algorithm [7] and the CooleyTurkey algorithm [8] have been proposed for DFTs over the complex field. When FFTs based on the prime-factor algorithm are adapted to DFTs over finite fields [9], they still have high multiplicative complexities. In contrast, recently proposed cyclotomic FFTs (CFFTs) are promising since they have significantly lower multiplicative complexities [10], [11]. However, CFFTs have two issues. First, they rely on efficient algorithms for short cyclic convolutions, which do not always exist. For instance, CFFTs over $\mathrm{GF}\left(2^{11}\right)$ would require efficient algorithms for 11-point cyclic convolutions. Previous works (see, for example, [10]-[12]) have not investigated CFFTs over $\operatorname{GF}\left(2^{11}\right)$ partially due to the lack of efficient 11-point cyclic convolutions in the literature. Second, CFFTs have very high additive complexities when directly implemented, which can be reduced by techniques such as the common subexpression elimination (CSE) (see, for example, [12]-[15]). In particular, the CSE algorithm in [12] is effective for reducing the additive complexities of CFFTs over $\operatorname{GF}\left(2^{l}\right)$ for $l \leq 10$. However, although the CSE algorithm has a polynomial complexity [12, Sec. III-F], its time and memory requirements limit its effectiveness for
long DFTs. Due to these two issues, $\operatorname{CFFTs}$ over $\operatorname{GF}\left(2^{11}\right)$ and $\operatorname{GF}\left(2^{12}\right)$ have not been investigated in the literature.

In this paper, we address both aforementioned issues. The main contributions of our paper are as follows.

- For an odd prime $p$, we reformulate a $p$-point cyclic convolution over characteristic-2 finite fields as a product of a $(p-1) \times(p-1)$ Toeplitz matrix and a vector. Since $p-1$ is composite, this product can be readily obtained by multi-dimensional technology from well-known Toeplitz matrix vector products (TMVP) of very small sizes [16]-[20]. In comparison to other ad hoc techniques based on TMVP, our reformulation achieves lower multiplicative complexity, especially for small to moderate $p$. Hence, our reformulation leads to efficient bilinear algorithms for $p$-point cyclic convolution over characteristic-2 finite fields. Our reformulation can be readily extended to the real and complex fields as well as more general finite fields. Furthermore, by multi-dimensional technology, we can also obtain efficient algorithms for $p^{n}$-point cyclic convolutions. These algorithms are also key to long CFFTs.
- Due to the high additive complexities of CCFTs, we propose composite cyclotomic Fourier transforms (CCFTs), which are generalization of CFFTs. When the length $N$ of the DFT is factored, that is, $N=N_{1} \times N_{2}$, our CCFTs use $N_{1-}$ and $N_{2}$-point CFFTs as sub-DFTs via the prime-factor and Cooley-Turkey algorithms. Thus, CFFTs are simply a special case of our CCFTs, corresponding to the trivial factorization, i.e., $N=1 \times N$. This generalization reduces overall complexities in three ways. First, this divide-and-conquer strategy itself leads to lower complexities. Second, the moderate lengths of the sub-DFTs enable us to apply complexity-reducing techniques such as the CSE algorithm in [12] more effectively. Third, when the length $N$ admits different factorizations, the one with the lowest complexity is selected. In the end, while an $N$-point CCFT may have a higher multiplicative complexity than an $N$-point CFFT, the former achieves a lower overall complexity for long DFTs because of its significantly lower additive complexity. Moreover, when $N$ is composite, an $N$-point CCFT has a regular and modular structure, which is suitable for efficient hardware implementations. Our CCFTs provide a systematic approach to designing long DFTs with low complexity.
- Our efficient algorithms for $p$-point cyclic convolution and CCFTs allow us to obtain longer CFFTs over larger fields. For example, we propose CFFTs over $\operatorname{GF}\left(2^{11}\right)$, which are unavailable in the literature heretofore partially due to the lack of efficient 11-point cyclic convolution algorithms. Our

2047-point DFTs over $\operatorname{GF}\left(2^{11}\right)$ and 4095-point DFTs over $\operatorname{GF}\left(2^{12}\right)$ are also first efficient DFTs of such lengths to the best of our knowledge, and they are promising for emerging communication systems.

Our work in this paper extends and improves previous works [10], [12] on CFFTs over finite fields of characteristic two in several ways. First, previously proposed CFFTs focus on ( $2^{l}-1$ )-point CFFTs over $\operatorname{GF}\left(2^{l}\right)$ for $l \leq 10$. In contrast, our CCFTs allow us to derive long DFTs with low complexity over larger fields. Our approach can be applied to any finite field, but we present CCFTs over $\operatorname{GF}\left(2^{11}\right)$ and $\operatorname{GF}\left(2^{12}\right)$ due to their significance in applications. Furthermore, our work investigates $N$-point CFFTs over GF( $2^{l}$ ) for $N \mid 2^{l}-1$. Second, our CCFTs achieve lower overall complexities than all previously proposed FFTs for moderate to long lengths, and the improvement significantly increases as the length grows.

The rest of the paper is organized as follows. Sec. $\square$ briefly reviews the necessary background of this paper, such as the CFFT, the prime-factor algorithm, the Cooley-Turkey algorithm, and the CSE algorithm. We propose an efficient bilinear algorithm for $p$-point cyclic convolutions over $\mathrm{GF}\left(2^{l}\right)$ in Sec . [III We then use an 11-point cyclic convolution algorithm to construct 2047-point CFFT over $\operatorname{GF}\left(2^{11}\right)$ in Sec. V We also propose our CCFTs and compare their complexities with previously proposed FFTs in Sec. V The advantages of our CCFTs in hardware implementations are discussed in Sec. VI Concluding remarks are provided in Sec. VII.

## II. Background

## A. Cyclotomic Fast Fourier Transforms

In this paper, we consider DFTs over finite fields of characteristic two. Let $\alpha \in \operatorname{GF}\left(2^{l}\right)$ be an element with order $N$, which implies that $N \mid 2^{l}-1$ (otherwise $\alpha$ does not exist). Given an $N$-dimensional column vector $\mathbf{f}=\left(f_{0}, f_{1}, \cdots, f_{N-1}\right)^{T}$ over $\mathrm{GF}\left(2^{l}\right)$, the DFT of $\mathbf{f}$ is given by $\mathbf{F}=\left(F_{0}, F_{1}, \cdots, F_{N-1}\right)^{T}$, where

$$
\begin{equation*}
F_{j}=\sum_{i=0}^{N-1} f_{i} \alpha^{i j} . \tag{1}
\end{equation*}
$$

If we define $f(x)=\sum_{i=0}^{N-1} f_{i} x^{i}$, we have $F_{j}=f\left(\alpha^{j}\right)$. Directly computing the DFT requires $O\left(N^{2}\right)$ multiplications and $O\left(N^{2}\right)$ additions, and is impractical for large $N$ s. Cyclotomic FFTs (CFFTs) [10], [11] can reduce the multiplicative complexities greatly.

We first partition the integer set $\{0,1, \cdots, N-1\}$ into $m$ cyclotomic cosets modulo $N$ with respect to GF(2) [3]: $C_{s_{0}}, C_{s_{1}}, \cdots, C_{s_{m-1}}$, where $C_{s_{k}}=\left\{2^{0} s_{k}, 2^{1} s_{k}, \cdots, 2^{m_{k}-1} s_{k}\right\}(\bmod N)$ and $s_{k}=2^{m_{k}} s_{k}$ $(\bmod N)$. A polynomial $L(x)=\sum_{i} l_{i} x^{2^{i}}$, where $l_{i} \in \mathrm{GF}\left(2^{l}\right)$, is called a linearized polynomial over
$\mathrm{GF}\left(2^{l}\right)$, since it has a linear property $L(x+y)=L(x)+L(y)$ for $x, y \in \mathrm{GF}\left(2^{l}\right)$. With the help of cyclotomic cosets, $f(x)$ can be decomposed as a sum of linearized polynomials

$$
f(x)=\sum_{k=0}^{m-1} L_{k}\left(x^{s_{k}}\right), \quad L_{k}(x)=\sum_{j=0}^{m_{k}-1} f_{s_{k} 2^{j} \bmod N} x^{2^{j}}
$$

Therefore $F_{j}=\sum_{k=0}^{m-1} L_{k}\left(\alpha^{j s_{k}}\right)$, and each $\alpha^{j s_{k}}$ lies in the subfield $\mathrm{GF}\left(2^{m_{k}}\right) \subseteq \mathrm{GF}\left(2^{l}\right)$.
Using a normal basis $\left\{\gamma_{k}^{2^{0}}, \gamma_{k}^{2^{1}}, \cdots, \gamma_{k}^{2^{m_{k}-1}}\right\}$ in $\operatorname{GF}\left(2^{m_{k}}\right), \alpha^{j s_{k}}$ can be expressed by $\sum_{i=0}^{m_{k}-1} a_{i, j, k} \gamma_{k}^{2^{i}}$, where $a_{i, j, k} \in\{0,1\}$. By the linear property of $L_{i}(x)$ 's, $F_{j}=\sum_{k=0}^{m-1} \sum_{i=0}^{m_{k}-1} a_{i, j, k} L_{k}\left(\gamma_{k}^{2^{i}}\right)$. Written in the matrix form, the DFT of $\mathbf{f}$ is given by $\mathbf{F}=\mathbf{A L I f}$, where $\mathbf{A}$ is an $N \times N$ binary matrix constructed from the binary coefficients $a_{i, j, k}, \boldsymbol{\Pi}$ is an $N \times N$ permutation matrix, $\mathbf{L}=\operatorname{diag}\left(Ł_{0}, Ł_{1}, \cdots, Ł_{m-1}\right)$ is a block diagonal matrix, and $\mathbf{L}_{k}$ 's are $m_{k} \times m_{k}$ square matrices. The permutation matrix $\boldsymbol{\Pi}$ reorders the vector $\mathbf{f}$ into $\mathbf{f}^{\prime}=\left(\mathbf{f}^{\prime T}, \mathbf{f}^{\prime}{ }_{1}^{T}, \cdots, \mathbf{f}^{\prime T}{ }_{m-1}\right)^{T}$, and $\mathbf{f}^{\prime}{ }_{k}=\left(f_{s_{k} 2^{0} \bmod N}, f_{s_{k} 2^{1} \bmod N}, \cdots, f_{s_{k} 2^{m_{k}-1} \bmod N}\right)^{T}$.

Though the idea of cyclotomic decomposition dates back to [21], the normal basis representation is a key step [10]. Since $\gamma_{k}^{2^{m_{k}}}=\gamma_{k}$, the $k$-th block $\mathbf{L}_{k}$ of $\mathbf{L}$ is actually a circulant matrix, which is given by

$$
\mathrm{Ł}_{k}=\left[\begin{array}{cccc}
\gamma_{k}^{2^{0}} & \gamma_{k}^{2^{1}} & \cdots & \gamma_{k}^{2^{m_{k}-1}} \\
\gamma_{k}^{2^{1}} & \gamma_{k}^{2^{2}} & \cdots & \gamma_{k}^{2^{0}} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{k}^{2^{m_{k}-1}} & \gamma_{k}^{2^{0}} & \cdots & \gamma_{k}^{2^{m_{k}-2}}
\end{array}\right]
$$

Hence the multiplication between $Ł_{k}$ and $\mathbf{f}_{k}^{\prime}$ can be formulated as an $m_{k}$-point cyclic convolution between $\mathbf{b}_{k}=\left(\gamma_{k}^{2^{0}}, \gamma_{k}^{2^{m_{k}}-1}, \gamma_{k}^{2^{m_{k}}-2} \cdots, \gamma_{k}^{2^{1}}\right)^{T}$ and $\mathbf{f}_{k}^{\prime}$. Since $m_{k}$ is usually small, we can use efficient bilinear form algorithms [1] for short cyclic convolutions to compute $\mathbf{L}_{k} \mathbf{f}^{\prime}{ }_{k}$. Those bilinear form algorithms have the following form,

$$
\mathfrak{Ł}_{k} \mathbf{f}_{k}^{\prime}=\mathbf{b}_{k} \otimes \mathbf{f}_{k}^{\prime}=\mathbf{Q}_{k}\left(\mathbf{R}_{k} \mathbf{b}_{k} \cdot \mathbf{P}_{k} \mathbf{f}_{k}^{\prime}\right)=\mathbf{Q}_{k}\left(\mathbf{c}_{k} \cdot \mathbf{P}_{k} \mathbf{f}_{k}^{\prime}\right)
$$

where $\mathbf{P}_{k}, \mathbf{Q}_{k}$, and $\mathbf{R}_{k}$ are all binary matrices, $\mathbf{c}_{k}=\mathbf{R}_{k} \mathbf{b}_{k}$ is a precomputed constant vector, and . denotes an component-wise multiplication between two vectors. Combining all the matrices, we get

$$
\begin{equation*}
\mathbf{F}=\mathbf{A Q}\left(\mathbf{c} \cdot \mathbf{P f}^{\prime}\right) \tag{2}
\end{equation*}
$$

where $\mathbf{Q}=\operatorname{diag}\left(\mathbf{Q}_{0}, \mathbf{Q}_{1}, \cdots, \mathbf{Q}_{m-1}\right), \mathbf{P}=\operatorname{diag}\left(\mathbf{P}_{0}, \mathbf{P}_{1}, \cdots, \mathbf{P}_{m-1}\right)$, and $\mathbf{c}=\left(\mathbf{c}_{0}^{T}, \mathbf{c}_{1}^{T}, \cdots, \mathbf{c}_{m-1}^{T}\right)^{T}$.
The multiplications required by (2) are due to the component-wise multiplication between $\mathbf{c}$ and $\mathbf{P} \mathbf{f}^{\prime}$, and the additions required by (2) are for multiplications between binary matrices and vectors. Direct implementation of CFFT in (2) requires much fewer multiplications than the direct implementation of DFT, at the expense of a very high additive complexity.

## B. Common Subexpression Elimination

Given an $N \times M$ binary matrix $\mathbf{M}$ and an $M$-dimensional vector x over a field $\mathbb{F}$. The matrix vector multiplication $\mathbf{M x}$ can be done by additions over $\mathbb{F}$ only, the number of which is denoted by $\mathcal{C}(\mathbf{M})$ since the complexity is determined by $\mathbf{M}$, when x is arbitrary. The problem of determining the minimal number of additions, denoted by $\mathcal{C}_{\text {opt }}(\mathbf{M})$, has been shown to be NP-complete [22]. Instead, different common subexpression elimination algorithms (see, e.g., [13]-[15]) have been proposed to reduce $\mathcal{C}(\mathbf{M})$. The CSE algorithm proposed in [12] takes advantage of the differential savings and recursive savings, and can greatly reduce the number of additions in calculating $\mathbf{M x}$, although the reduced additive complexity, denoted by $\mathcal{C}_{\mathrm{CSE}}(\mathbf{M})$, is not guaranteed to be the minimum. Like other CSE algorithms, the CSE algorithm in [12] is randomized, and the reduction results of different runs are not necessarily the same. Therefore in practice, a better result can be obtained by first running the CSE algorithm many times and then selecting the smallest number of additions. The CSE algorithm in [12] greatly reduces the additive and overall complexities of CFFTs with lengths up to 1023, but it is much more difficult to reduce the additive complexities of longer CFFTs. This is because though the CSE algorithm in [12] has a polynomial complexity (it is shown that its complexity is $O\left(N^{4}+N^{3} M^{3}\right)$ ), the runtime and memory requirements become prohibitive when $M$ and $N$ are very large, which occurs for long CFFTs.

## C. Prime-Factor and Cooley-Turkey Algorithms

Both the prime-factor algorithm and Cooley-Turkey algorithm first decompose an $N$-point DFT into shorter sub-DFTs, and then construct the $N$-point DFT from the sub-DFTs [1]. The prime-factor algorithm requires that the length $N$ has at least two co-prime factors, i.e., there exist two co-prime numbers $N_{1}$ and $N_{2}$ such that $N=N_{1} N_{2}$. For an integer $i \in\{0,1, \cdots, N-1\}$, there is a unique integer pair $\left(i_{1}, i_{2}\right)$ such that $0 \leq i_{1} \leq N_{1}-1,0 \leq i_{2} \leq N_{2}-1$, and $i=i_{1} N_{2}+i_{2} N_{1}(\bmod N)$, since $N_{1}$ and $N_{2}$ are co-prime. For any integer $j \in\{0,1, \cdots, N-1\}$, let $j_{1}=j\left(\bmod N_{1}\right), \quad j_{2}=j\left(\bmod N_{2}\right)$, where $0 \leq j_{1} \leq N_{1}-1$ and $0 \leq j_{2} \leq N_{2}-1$. By Chinese remainder theorem, $\left(j_{1}, j_{2}\right)$ uniquely determines $j$, and $j$ can be represented by $j=j_{1} N_{2}^{-1} N_{2}+j_{2} N_{1}^{-1} N_{1}(\bmod N)$, where $N_{2}^{-1} N_{2}=1$ $\left(\bmod N_{1}\right)$ and $N_{1}^{-1} N_{1}=1\left(\bmod N_{2}\right)$. Substituting the above representation of $i$ and $j$ in (1), we get $\alpha^{i j}=\left(\alpha^{N_{2}}\right)^{i_{1} j_{1}}\left(\alpha^{N_{1}}\right)^{i_{2} j_{2}}$, where $\alpha^{N_{2}}$ and $\alpha^{N_{1}}$ are the $N_{1}$-th root and $N_{2}$-th root of 1 , respectively.

Therefore, (1) becomes

$$
\begin{equation*}
F_{j}=\underbrace{\sum_{i_{1}=0}^{N_{1}-1}(\overbrace{\sum_{i_{2}=0}^{N_{2}-1} f_{i_{1} N_{2}+i_{2} N_{1}} \alpha^{N_{1} i_{2} j_{2}}}^{N_{2} \text {-point DFT }}) \alpha^{N_{2} i_{1} j_{1}}}_{N_{1}-\text { point DFT }} . \tag{3}
\end{equation*}
$$

In this way, the $N$-point DFT is obtained by using $N_{1-}$ and $N_{2}$-point sub-DFTs. The $N$-point DFT result is derived by first carrying out $N_{1} N_{2}$-point DFTs and $N_{2} N_{1}$-point DFTs, and then combining the results according to the representation of $j$. The prime-factor algorithm can also be applied to $N_{1}$ - and $N_{2}$-point DFTs if they have co-prime factors.

The Cooley-Turkey algorithm has a different decomposition strategy from the prime-factor algorithm. Let $N=N_{1} N_{2}$, where $N_{1}$ and $N_{2}$ do not have to be co-prime. Let $i=i_{1}+i_{2} N_{1}$, where $0 \leq i_{1} \leq N_{1}-1$ and $0 \leq i_{2} \leq N_{2}-1$, and $j=j_{1} N_{2}+j_{2}$, where $0 \leq j_{1} \leq N_{1}-1$ and $0 \leq j_{2} \leq N_{2}-1$. Then (1) becomes

$$
\begin{equation*}
F_{j}=\underbrace{\sum_{i_{1}=0}^{N_{1}-1}(\overbrace{\sum_{i_{2}=0}^{N_{2}-1} f_{i_{1}+i_{2} N_{1}} \alpha^{N_{1} i_{2} j_{2}}}^{N_{2} \text {-point DFT }}) \alpha^{i_{1} j_{2}} \alpha^{N_{2} i_{1} j_{1}}}_{N_{1}-\text { point DFT }} . \tag{4}
\end{equation*}
$$

In this way, the Cooley-Turkey algorithm also decomposes the $N$-point DFT into $N_{1-}$ and $N_{2}$-point DFTs. However, compared with (3), (4) has an extra term $\alpha^{i_{1} j_{2}}$, which is called twiddle factor and incurs additional multiplicative complexity. The Cooley-Turkey algorithm can be used for arbitrary non-prime length $N$, including the prime powers to which case the prime-factor algorithm cannot be applied. The Cooley-Turkey algorithm is very suitable if $N$ has a lot of small factors, for example, $2^{n}$-point DFT by the Cooley-Turkey algorithm requires $O\left(n \cdot 2^{n}\right)$ multiplications.

## III. $p$-point Cyclic Convolutions over $\operatorname{GF}\left(2^{m}\right)$

Efficient short cyclic convolution algorithms play an essential role in the multiplicative complexity reduction of CFFTs. Note the lengths of cyclic convolutions involved in CFFTs are the same as the sizes of the conjugate classes. Since the sizes of all possible conjugate classes in $\operatorname{GF}\left(2^{m}\right)$ are divisors of $m$, efficient algorithms for only short cyclic convolutions are needed, since they determine the multiplicative complexities of CFFTs.

Despite their significance, there is no general algorithms for efficient cyclic convolutions of arbitrary length over finite fields. Of course, efficient ad hoc algorithms for 2- to 9-point cyclic convolution can be found in the literature (4- and 8-point can be found in [23]-[25], and their details are included in

Appendix B due to their limited access, and the rest can be found in [1] and [2]). Furthermore, cyclic convolutions with composite length can be constructed with multi-dimensional technology described in [1]. For instance, 10-point cyclic convolution algorithms can be constructed based on 2- and 5-point algorithms, while 12-point cyclic convolution algorithm is constructed based on 3- and 4-point algorithms. However, an efficient algorithm for cyclic convolutions of larger prime length (for example, 11- or 13-point) is not available in the open literature. We can implement these cyclic convolutions via the convolution theorem. Although the DFTs and IDFT can be implemented by the Winograd algorithm [26] or the Rader algorithm [27], this approach remains inefficient, especially for small to moderate lengths. In [28], strategies to derive cyclic convolution algorithms directly over any finite field $\mathrm{GF}\left(q^{m}\right)$ were developed. Unfortunately, these methods are applicable only to lengths $q^{m}-1$ or their factors.

Herein for an odd prime $p$, we reformulate a $p$-point cyclic convolution as a product of a $(p-1) \times$ ( $p-1$ ) Toeplitz matrix and a vector. Since $p-1$ is composite, this product can be readily obtained by multi-dimensional technology from well-known TMVP of very small sizes, leading to efficient bilinear algorithms for $p$-point cyclic convolutions. Since these cyclic convolutions will be used for CFFTs over $\operatorname{GF}\left(2^{l}\right)$, we focus on cyclic convolutions over $\operatorname{GF}\left(2^{l}\right)$. However, our reformulation can be readily extended to the real and complex fields as well as more general finite fields. Furthermore, by multi-dimensional technology, we can also obtain efficient algorithms for $p^{n}$-point cyclic convolutions. These algorithms are also key to long CFFTs.

For a $p$-dimensional vector $\mathbf{x}=\left(x_{0}, x_{1}, \cdots, x_{p-1}\right)^{T}$ over some field $\operatorname{GF}\left(2^{l}\right)$, where $p$ is any odd prime integer, we consider its corresponding polynomial $X(w)=\sum_{i=0}^{p-1} x_{i} w^{i}$. Assuming that the $p$-point cyclic convolution of two vectors $\mathbf{x}$ and $\mathbf{y}$ is $\mathbf{z}$, all of which are $p$-dimensional vectors over $\operatorname{GF}\left(2^{l}\right)$, their corresponding polynomials are related by [1]

$$
\begin{equation*}
Z(w)=X(w) Y(w) \quad\left(\bmod w^{p}+1\right) \tag{5}
\end{equation*}
$$

Note that $w^{p}+1=(w+1)\left(w^{p-1}+w^{p-2}+\cdots+1\right)$, and $w+1$ and $w^{p-1}+w^{p-2}+\cdots+1$ are co-prime in $\operatorname{GF}\left(2^{l}\right)$. Hence by Chinese remainder theorem, $Z(w)$ can be uniquely determined by $Z_{0}$ and $Z^{\prime}(w)=\sum_{i=0}^{p-2} Z_{i}^{\prime} w^{i}$, where

$$
\begin{align*}
Z_{0} & =Z(w) \quad(\bmod w+1)  \tag{6}\\
Z^{\prime}(w) & =Z(w) \quad\left(\bmod w^{p-1}+w^{p-2}+\cdots+1\right)
\end{align*}
$$

It is easy to see that $Z_{0}=\sum_{i=0}^{p-1} z_{i}, Z_{i}^{\prime}=z_{i}+z_{p-1}$, and the vector $\mathbf{Z}^{\dagger}=\left(Z_{0}, Z_{0}^{\prime}, Z_{1}^{\prime}, \cdots, Z_{p-2}^{\prime}\right)^{T}$ can
be derived by multiplying the vector $\mathbf{z}$ with an $p \times p$ matrix $\mathbf{B}$ with structure

$$
\mathbf{B}=\left[\begin{array}{cccc}
1 & 1 & \ldots & 1 \\
& & & 1 \\
& \mathbf{I}_{p-1} & & \vdots \\
& & & 1
\end{array}\right]
$$

where $\mathbf{I}_{p-1}$ is a $(p-1) \times(p-1)$ identity matrix. That is, $\mathbf{Z}^{\dagger}=\mathbf{B z}$.
To compute the $p$-point cyclic convolution of $\mathbf{x}$ and $\mathbf{y}$, we first compute $\mathbf{X}^{\dagger}=\mathbf{B x}$ and $\mathbf{Y}^{\dagger}=\mathbf{B y}$, then compute $\mathbf{Z}^{\dagger}$ from $\mathbf{X}^{\dagger}$ and $\mathbf{Y}^{\dagger}$, and finally, $\mathbf{z}=\mathbf{B}^{-1} \mathbf{Z}^{\dagger}$. With the same partitioning scheme aforementioned and equations (5) and (6), it is easy to see that $Z_{0}=X_{0} Y_{0}$, and

$$
\begin{equation*}
Z^{\prime}(w)=X^{\prime}(w) Y^{\prime}(w) \quad\left(\bmod w^{p-1}+w^{p-2}+\cdots+1\right) \tag{7}
\end{equation*}
$$

and hence we can compute $\mathbf{Z}^{\dagger}=\left(Z_{0}, \mathbf{Z}^{T}\right)^{T}$.
From (7), the polynomial product can be computed as

$$
\begin{align*}
X^{\prime}(w) Y^{\prime}(w)= & \sum_{k=0}^{p-2} \sum_{j=0}^{p-2}\left(Y_{k-j}^{\prime}+Y_{k-j+p}^{\prime}+Y_{p-1-j}^{\prime}\right) X_{j}^{\prime} w^{k} \\
& \left(\bmod w^{p-1}+w^{p-2}+\cdots+1\right) \tag{8}
\end{align*}
$$

and hence the vector $\mathbf{Z}^{\prime}$ can be computed through a matrix product $\mathbf{Z}^{\prime}=\mathbf{M} \mathbf{X}^{\prime}$, where the elements of matrix $\mathbf{M}$ are

$$
\begin{equation*}
M_{k, j}=Y_{k-j}^{\prime}+Y_{k-j+p}^{\prime}+Y_{p-1-j}^{\prime} \tag{9}
\end{equation*}
$$

Note that in (8) and (9), $Y_{i}^{\prime}$ are considered as zero outside its valid range, i.e., $Y_{i}^{\prime}=0$ if $i<0$ or $i>p-2$.

We can check that $\mathbf{B}$ is an invertible matrix, and $\mathbf{B}^{-1}$ is given by

$$
\mathbf{B}^{-1}=\left[\begin{array}{cc}
1 & \mathbf{A}_{1} \\
\mathbf{A}_{2} & \mathbf{A}_{3}
\end{array}\right]
$$

where the length- $(p-1)$ row vector $\mathbf{A}_{1}=(0,1,1, \cdots, 1)$, the length- $(p-1)$ column vector $\mathbf{A}_{2}=$ $(1,1, \cdots, 1)^{T}$, and $(p-1) \times(p-1)$ matrix $\mathbf{A}_{3}$ has 0 on the first upper diagonal and 1 everywhere else.

Now consider the product of $\mathbf{B}^{-1}$ and a length $p$ column vector $\mathbf{U}$ :

$$
\mathbf{B}^{-1}\left[\begin{array}{l}
U_{0} \\
\mathbf{U}^{\prime}
\end{array}\right]=\left[\begin{array}{cc}
1 & \mathbf{A}_{1} \\
\mathbf{A}_{2} & \mathbf{A}_{3}
\end{array}\right]\left[\begin{array}{l}
U_{0} \\
\mathbf{U}^{\prime}
\end{array}\right]=\left[\begin{array}{c}
V_{0} \\
\mathbf{V}^{\prime}
\end{array}\right]
$$

where $U_{0}, \mathbf{U}^{\prime}$, and $V_{0}, \mathbf{V}^{\prime}$ are appropriate partitions of the vector $\mathbf{U}$ and the multiplication result vector $\mathbf{V}$, respectively. Values of $V_{0}$ and $\mathbf{V}^{\prime}$ can be computed as $V_{0}=U_{0}+\mathbf{A}_{1} \mathbf{U}^{\prime}$ and $\mathbf{V}^{\prime}=\mathbf{A}_{2} U_{0}+\mathbf{A}_{3} \mathbf{U}^{\prime}$.

Note that $\mathbf{A}_{1}$ and $\mathbf{A}_{3}$ are related as $\mathbf{A}_{1}=(1,1, \ldots, 1) \mathbf{A}_{3}$. This implies that the sum of the components of $\mathbf{A}_{3} \mathbf{U}^{\prime}$ gives $\mathbf{A}_{1} \mathbf{U}^{\prime}$. Furthermore, $\mathbf{A}_{2}$ contains only 1's. Thus the computation of $V_{0}$ and $\mathbf{V}^{\prime}$ reduces to

$$
\begin{align*}
V_{0} & =U_{0}+\sum\left(\mathbf{A}_{3} \mathbf{U}^{\prime}\right)  \tag{10}\\
\mathbf{V}^{\prime} & =\left[U_{0}, U_{0}, \ldots, U_{0}\right]^{\mathrm{T}}+\mathbf{A}_{3} \mathbf{U}^{\prime} .
\end{align*}
$$

Eq. (10) shows that multiplying a vector with $\mathbf{B}^{-1}$ needs only an evaluation of $\mathbf{A}_{3} \mathbf{U}^{\prime}$.
The cyclic convolution result $\mathbf{z}$ is obtained by first multiplying $\mathbf{A}_{3}$ and $\mathbf{Z}^{\prime}$. Thus one need to compute $\mathbf{R X} \mathbf{X}^{\prime}$ where the $(p-1) \times(p-1)$ matrix $\mathbf{R}=\mathbf{A}_{3} \mathbf{M}$. We now show by direct computation that $\mathbf{R}$ is a Toeplitz matrix. From the structure of $\mathbf{A}_{3}$, we have

$$
\begin{equation*}
R_{i, j}=M_{i+1, j}+\sum_{k=0}^{p-2} M_{k, j} . \tag{11}
\end{equation*}
$$

¿From (9), using appropriate ranges for the three terms we get

$$
\begin{equation*}
\sum_{k=0}^{p-2} M_{k, j}=Y_{p-1-j}^{\prime}+\sum_{s=0}^{p-2} Y_{s}^{\prime} . \tag{12}
\end{equation*}
$$

Finally, combining (9), (11) and (12) gives

$$
\begin{equation*}
R_{i, j}=Y_{i-j+1}^{\prime}+Y_{i-j+p+1}^{\prime}+\sum_{s=0}^{p-2} Y_{s}^{\prime} . \tag{13}
\end{equation*}
$$

Since $R_{i, j}$ is a function of only $i-j, \mathbf{R}$ is a Toeplitz matrix. Recall that $Y_{i}^{\prime}$ is assumed zero if its index is outside the valid range from 0 to $p-2$. Thus in (13), at most one of the first two terms is valid for any combination of $i$ and $j$.

Fig. 1 illustrates our algorithm for $p$-point cyclic convolutions, which relies on the implementation of $\mathbf{R X} \mathbf{X}^{\prime}$. Direct implementation of $\mathbf{R} \mathbf{X}^{\prime}$ requires $(p-1)^{2}$ multiplications, but we can reduce it since $\mathbf{R}$ is a Toeplitz matrix. For any odd prime $p>3, p-1$ is composite and $\mathbf{R X} \mathbf{X}^{\prime}$ can be obtained by using multidimensional technology from TMVP of smaller sizes [16]-[20]. For example, CFFTs over $\operatorname{GF}\left(2^{11}\right)$, $\mathrm{GF}\left(2^{13}\right), \operatorname{GF}\left(2^{17}\right)$, and $\mathrm{GF}\left(2^{19}\right)$, involve 11-, 13 -, 17 -, and 19-point cyclic convolutions, respectively. Using our reformulations, these cyclic convolutions can be obtained from a TMVP of $2 \times 2,3 \times 3$, and $5 \times 5$, which are provided in Appendix A Hence our reformulation leads to efficient cyclic convolution algorithms for odd prime $p$ for $p \leq 19$, which are sufficient for all CFFTs over characteristic-2 fields as large as $\mathrm{GF}\left(2^{19}\right)$.

This reformulation is also applicable to a prime greater than 19 , where $p-1$ may have a prime factor $p^{\prime}$ greater than five. In this case, one can use two ad hoc techniques to proceed. First, one can break a $p^{\prime} \times p^{\prime}$


Fig. 1. $p$-point cyclic convolution.
matrix into blocks, and treat them separately. Second, one can extend the $p^{\prime} \times p^{\prime}$ matrix to a larger matrix so that it remains a Toeplitz matrix and its size becomes composite again. The complexities of cyclic convolution algorithms obtained through this reformulation are much smaller than direct implementation. For example, we can first extend the $(p-1) \times(p-1)$ Toeplitz matrix to a $2^{\left\lceil\log _{2}(p-1)\right\rceil} \times 2^{\left\lceil\log _{2}(p-1)\right\rceil}$ matrix, and it requires fewer than $3^{\left[\log _{2}(p-1)\right\rceil}$ multiplications if we use the two-way split method described in [20].

We note that a $p$-point cyclic convolution can be formulated as a $p \times p$ circulant matrix vector product. Since a circulant matrix is a special case of Toeplitz matrix, one can of course apply the two ad hoc techniques described above to this $p \times p$ Toeplitz matrix directly. However, since our reformulation turns a $p$-point cyclic convolution into a $(p-1) \times(p-1)$ TMVP, which directly benefit from multi-dimensional technologies, at the expense of only one extra multiplication, we believe our reformulation will lead to lower multiplicative complexity. We cannot prove this analytically, but will illustrate this point below
with an example.
We also remark that our reformulation leads to bilinear algorithms for cyclic convolutions, which can be implemented efficiently since the pre- and post-addition matrices are all binary.

## A. Example: 11-point Convolution Algorithm over $G F\left(2^{m}\right)$

To illustrate the advantages of our reformulation above, we derive our efficient 11-point cyclic convolution algorithm over $\operatorname{GF}\left(2^{11}\right)$ and compare its multiplicative complexity with some other approaches. By using well-known $2 \times 2$ and $5 \times 5$ TMVP, we obtain an 11-point cyclic convolution algorithm $\mathbf{z}=\mathbf{Q}^{(11)}\left(\mathbf{R}^{(11)} \mathbf{y} \cdot \mathbf{P}^{(11)} \mathbf{x}\right)$, where the matrices $\mathbf{Q}^{(11)}, \mathbf{P}^{(11)}$, and $\mathbf{R}^{(11)}$ are given in Appendix B Since the $10 \times 10$ TMVP requires 42 multiplications, our 11-point cyclic convolution requires 43 multiplications.

Let us compare this multiplicative complexity with the two ad hoc techniques. First, we can partition the $11 \times 11$ circulant matrix into a $10 \times 10$ Toeplitz matrix, a $10 \times 1$ column vector, a $1 \times 10$ row vector, and a single element, and then apply the multi-dimensional technology to the $10 \times 10$ TMVP. In addition to the $10 \times 10$ TMVP, this approach requires 21 extra multiplications, as opposed to one in our approach. Second, we can extend the $11 \times 11$ circulant matrix to a $12 \times 12$ Toeplitz matrix, and then apply the multi-dimensional technology to this matrix. A $12 \times 12$ TMVP requires $54=3 \times 3 \times 6$ multiplications. Taking into account that we pad a zero to the $11 \times 1$ vector and that the last element of the TMVP is not needed, two multiplications can be saved, and we need 52 multiplications in total (note that this total multiplicative complexity is the same regardless of the order of decomposition of 12). We can also extend the $11 \times 11$ circulant matrix to a $15 \times 15$ Toeplitz matrix or a $16 \times 16$ one, which require 66 and 60 multiplications, respectively. Our reformulation is more efficient than these ad hoc techniques in terms of the multiplicative complexity. This is because our reformulation turns a $p$-point cyclic convolution into a $(p-1) \times(p-1)$ TMVP, which directly benefit from multi-dimensional technologies, at the expense of only one extra multiplication.

We also compare our result with the implementation via convolution theorem, i.e., first multiply the DFTs of the two vector component-wisely, and then compute the inverse DFT of the resulting vector. If we use the Rader's algorithm to implement the DFT and inverse DFT, it needs 101 multiplications in total. Hence this approach is less efficient than ours.

By using the CSE algorithm in [12], our 11-point cyclic convolution algorithm requires 43 multiplications and 164 additions. When we use this algorithm in $\operatorname{CFFTs}$ over $\operatorname{GF}\left(2^{11}\right)$, one of the two inputs is known in advance. Our algorithm requires 42 multiplications since one of the multiplication has an operand of one, and 120 additions because the additions involving the known input can be pre-computed.

## IV. Long Cyclotomic Fourier Transforms

## A. 2047-point CFFT over $G F\left(2^{11}\right)$

The efficient algorithm for 11-point cyclic convolution we designed in III-A is the key to the CFFTs over $\mathrm{GF}\left(2^{11}\right)$. Direct implementation of 2047-point CFFT with this cyclic convolution algorithm requires 7812 multiplications and 2130248 additions. The prohibitively high additive complexity is dominated by the multiplication between the $2047 \times 2047$ matrix A and a 2047 -dimensional vector, which requires 2095280 additions. Unfortunately, if we use the CSE algorithm in [12] to reduce its additive complexity, the time complexity of the CSE algorithm itself is too high (it needs months to finish).

Due to the high time complexity of the CSE algorithm in [12], we have tried a simplified CSE algorithm with limited success. In the original CSE algorithm in [12], only one of the patterns with the greatest recursive savings is selected and removed in one round of iteration. Instead of selecting only one pattern, our simplified CSE algorithm has a reduced time complexity as it removes multiple patterns at one time. The reduced time complexity of the simplified CSE algorithm allows us to reduce the additive complexity for the 2047 -point CFFT to 529720 additions, about one fourth of that for the direct implementation. Despite this improvement, the effectiveness of this simplified CSE algorithm is rather limited.

## B. Difficulty with Long CFFTs

Consider an $N$-point CFFT over $\operatorname{GF}\left(2^{l}\right)$. Suppose $C_{s_{0}}, C_{s_{1}}, \cdots, C_{s_{m-1}}$ are $m$ cyclotomic cosets modulo $N$ over $\operatorname{GF}(2)$, and $\left|C_{s_{k}}\right|=m_{k}$. Suppose an $m_{k}$-point cyclic convolution can be done with $\mathcal{M}\left(m_{k}\right)$ multiplications, and hence implementing the $N$-point DFT with the CFFT directly requires $\sum_{k=0}^{m-1} \mathcal{M}\left(m_{k}\right)$ multiplications and $\mathcal{C}(\mathbf{A Q})+\mathcal{C}(\mathbf{P})$ additions, where $\mathcal{C}(\cdot)$ denotes the number of additions we need to evaluate the product of a binary matrix and a vector. The multiplicative complexity can be further reduced because we can pre-compute the vector $\mathbf{c}$ in (2) and some of its elements may be unitary. Then the CSE algorithm can be applied to the matrices $\mathbf{A Q}$ and $\mathbf{P}$ to reduce $\mathcal{C}(\mathbf{A Q})$ and $\mathcal{C}(\mathbf{P})$ to $\mathcal{C}_{\mathrm{CSE}}(\mathbf{A Q})$ and $\mathcal{C}_{\mathrm{CSE}}(\mathbf{P})$, respectively. Since $\mathbf{P}=\operatorname{diag}\left(\mathbf{P}_{0}, \mathbf{P}_{1}, \cdots, \mathbf{P}_{m-1}\right)$ is a block diagonal matrix, we have $\mathcal{C}_{\mathrm{CSE}}(\mathbf{P})=\sum_{i=0}^{m-1} \mathcal{C}_{\mathrm{CSE}}\left(\mathbf{P}_{i}\right)$. Therefore, we can reduce the additive complexity of each $\mathbf{P}_{i}$ to get a better result of $\mathcal{C}(\mathbf{P})$. Since the size of $\mathbf{P}_{i}$ is much smaller than that of $\mathbf{P}$, it allows us to run the CSE algorithm many times to achieve a smaller additive complexity. However, the matrix AQ is not a block diagonal matrix, and therefore we have to apply the CSE algorithm directly to AQ. When the size of AQ is large, the CSE algorithm in [12] requires a lot of time and memory and hence it is impractical for extremely long DFTs. As mentioned above, it would take months for the CSE algorithm in [12] to reduce
the additive complexity of 2047-point CFFT over $\operatorname{GF}\left(2^{11}\right)$, let alone 4095 -point CFFTs over $\mathrm{GF}\left(2^{12}\right)$. The prohibitively high time complexity of the CSE algorithm in [12] and the limited effectiveness of the simplified CSE algorithm motivate our composite cyclotomic Fourier transforms.

## V. Composite Cyclotomic Fourier Transforms

## A. Composite Cyclotomic Fourier Transforms

Instead of simplifying the CSE algorithm or designing other low complexity optimization algorithms, we propose composite cyclotomic Fourier transforms by first decomposing a long DFT into shorter subDFTs, via the prime-factor or Cooley-Turkey algorithms, and then implementing the sub-DFTs by CFFTs. Note that both the decompositions require only that $\alpha$ is a primitive $N$-th root of 1 , hence they can be extended to finite fields easily. When $N$ is prime, our CCFTs reduce to CFFTs. When $N$ is composite, we first decompose the DFT into shorter sub-DFTs, and then combine the sub-DFT results according to (3) or (4). The shorter sub-DFTs are implemented by CFFTs to reduce their multiplicative complexities, and then we use the CSE algorithm in [12] to reduce their additive complexities. Finally, when $N$ has multiple factors, the factorization can be carried out recursively.

Suppose the length of the DFT is composite, i.e., $N=N_{1} N_{2}$. Either the prime-factor or the CooleyTurkey algorithms can be used to decompose the $N$-point DFT into sub-DFTs when $N_{1}$ and $N_{2}$ are co-prime. When $N_{1}$ and $N_{2}$ are not co-prime, only the Cooley-Turkey algorithm can be used. It is easy to show that if $N_{1}$ and $N_{2}$ are co-prime, the prime-factor and Cooley-Turkey algorithms lead to the same additive complexity for CCFTs, but the Cooley-Turkey algorithm results in a higher multiplicative complexity due to the extra multiplications of twiddle factors. Hence the prime-factor algorithm is better than the Cooley-Turkey algorithm in this case, and the Cooley-Turkey algorithm is used only if the prime-factor algorithm cannot be applied.

We denote the multiplicative and additive complexity of an $N$-point DFT by $\mathcal{K}^{\text {mult }}(N)$ and $\mathcal{K}^{\text {add }}(N)$, respectively, and the algorithm used to implement this DFT is specified in the subscription of $\mathcal{K}$. Suppose $N=\prod_{i=1}^{s} N_{i}$, and the total number of non-unitary twiddle factors required by the Cooley-Turkey algorithm decompositions is denoted by $T$, then the complexity of this decomposition is given by

$$
\begin{align*}
\mathcal{K}_{\mathrm{CCFT}}^{\mathrm{add}}(N) & =\sum_{i=1}^{s} \frac{N}{N_{i}} \mathcal{K}_{\mathrm{CFFT}}^{\mathrm{add}}\left(N_{i}\right),  \tag{14}\\
\mathcal{K}_{\mathrm{CCFT}}^{\mathrm{mult}}(N) & =\sum_{i=1}^{s} \frac{N}{N_{i}} \mathcal{K}_{\mathrm{CFFT}}^{\mathrm{mult}}\left(N_{i}\right)+T . \tag{15}
\end{align*}
$$

For $N \mid 2^{l}-1$ for $4 \leq l \leq 12$, there is at most one pair of $N_{i}$ 's that are not co-prime in the decomposition of $N$, say $N_{1}$ and $N_{2}$, without loss of generality. In this case, $T=\frac{N}{N_{1} N_{2}}\left(N_{1}-1\right)\left(N_{2}-1\right)$. If all the elements in the decomposition of $N$ are co-prime to each other, then $T=0$.

The decomposition allows our CCFTs to achieve low complexities for several reasons. First, this divide-and-conquer strategy is used in many fast Fourier transforms. If we assume CFFTs have quadratic additive complexities with their length $N$ when directly implemented (this assumption is at least supported by the additive complexities of the CFFTs without CSE in Table IV], the CCFT decomposition reduces the additive complexity from $O\left(N^{2}\right)$ to $O\left(N \sum_{i=1}^{s} N_{i}\right)$. Second, the lengths of the sub-DFTs are much shorter, which enables us to apply several powerful but complicated techniques to reduce the complexities of the sub-DFTs. For example, it takes much less time and memory to apply the CSE algorithm in [12] to the sub-DFTs, and thus we can run it multiple times to get a better reduction result. Third, when the length of the DFT admits different factorizations (for example, $2^{6}-1=63=3 \times 21=9 \times 7$ ), we choose the decomposition(s) with the lowest complexity.

## B. Complexity Reduction

We reduce the additive complexities of our CCFTs in three steps. First, we reduce the complexities of short cyclic convolutions. Second, we use these short cyclic convolutions to construct CFFTs of moderate lengths. Third, we use CFFTs of moderate lengths as sub-DFTs to construct our CCFTs.

Efficient short cyclic convolution algorithms are the keys to the multiplicative complexity reduction of CFFTs and our CCFTs, and hence our first step is to reduce the computational complexities of small size cyclic convolutions. Suppose an $L$-point cyclic convolution $\mathbf{b}^{(L)} \otimes \mathbf{a}^{(L)}$ is calculated with the bilinear form $\mathbf{Q}^{(L)}\left(\mathbf{R}^{(L)} \mathbf{b}^{(L)} \cdot \mathbf{P}^{(L)} \mathbf{a}^{(L)}\right)$. Since $\mathbf{b}^{(L)}$ is the normal basis in our CCFTs, $\mathbf{R}^{(L)} \mathbf{b}^{(L)}$ can be precomputed to reduce multiplicative complexity. We apply the CSE algorithm in [12] to reduce the additive complexities in the multiplication with binary matrices $\mathbf{Q}^{(L)}$ and $\mathbf{P}^{(L)}$. The complexity reduction results $\mathcal{C}_{\mathrm{CSE}}\left(\mathbf{Q}^{(L)}\right)$, $\mathcal{C}_{\mathrm{CSE}}\left(\mathbf{P}^{(L)}\right)$, the total additive complexity $\mathcal{C}_{\mathrm{CSE}}\left(\mathbf{Q}^{(L)}\right)+\mathcal{C}_{\mathrm{CSE}}\left(\mathbf{P}^{(L)}\right)$, and the multiplicative complexities are listed in Table II

The second step is to reduce the additive complexity of CFFTs with moderate lengths, which will be used to build our CCFTs. Their moderate lengths allow us to use multiple techniques to reduce their additive complexities.

- First, for any CFFT, we run the CSE algorithm in [12] multiple times and then choose the best results.

TABLE I
COMPLEXITIES OF SHORT CYCLIC CONVOLUTIONS OVER GF $\left(2^{l}\right)$.

| $L$ | mult. | additive complexities |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  |  | $\mathcal{C}_{\mathrm{CSE}}\left(\mathbf{Q}^{(L)}\right)$ | $\mathcal{C}_{\mathrm{CSE}}\left(\mathbf{P}^{(L)}\right)$ | total |
| 2 | 1 | 2 | 1 | 3 |
| 3 | 3 | 5 | 4 | 9 |
| 4 | 5 | 9 | 4 | 13 |
| 5 | 9 | 16 | 10 | 26 |
| 6 | 10 | 21 | 11 | 32 |
| 7 | 12 | 24 | 23 | 47 |
| 8 | 19 | 35 | 16 | 51 |
| 9 | 18 | 40 | 31 | 71 |
| 10 | 28 | 52 | 31 | 83 |
| 11 | 42 | 76 | 44 | 120 |
| 12 | 32 | 53 | 34 | 87 |

- Second, for each CFFT in (2), we may reduce $\mathcal{C}(\mathbf{A Q})$ together as a whole, or reduce $\mathcal{C}(\mathbf{A})$ and $\mathcal{C}(\mathbf{Q})$ separately. Since $(\mathbf{A Q}) \mathbf{v}=\mathbf{A}(\mathbf{Q v}), \mathcal{C}_{\text {opt }}(\mathbf{A Q}) \leq \mathcal{C}_{\text {opt }}(\mathbf{A})+\mathcal{C}_{\text {opt }}(\mathbf{Q})$. However, this property may not hold for the CSE algorithm because the CSE algorithm may not find the optimal solutions. Furthermore, we may benefit from reducing $\mathcal{C}(\mathbf{A})$ and $\mathcal{C}(\mathbf{Q})$ separately for the following reasons. First, $\mathbf{Q}$ has a block diagonal structure, which is similar as $\mathbf{P}$, therefore we can find a better reduction result for $\mathcal{C}(\mathbf{Q})$. Second, AQ has much more columns than $\mathbf{A}$, and hence the CSE algorithm requires less memory and time to reduce $\mathbf{A}$ than to reduce AQ.
- Third, there is flexibility in terms of normal bases used to construct the matrix A in (2), and this flexibility can be used to further reduce the additive complexity of any CFFT. For each cyclotomic coset, a normal basis is needed. A normal basis is not unique in finite fields, and any normal basis can be used in the construction of the matrix $\mathbf{A}$, leading to the same multiplicative complexity. But different normal bases result in different $\mathbf{A}$ and hence different additive complexities due to A. There are several options regarding the normal basis. One can simply choose a fixed normal basis for all cyclotomic cosets of the same size as in [12]. A more ideal option is to enumerate all possible normal bases and their corresponding $\mathbf{A}$ and to select the smallest additive complexity. However, when the underlying field is large, the number of possible normal basis is very large, and hence it becomes infeasible to enumerate all possible constructions. Thus, in this paper we use a
compromise of these two options: for each cyclotomic coset we choose a normal basis at random and the combination of random normal bases leads to $\mathbf{A}$; we minimize the complexity over as many combinations as complexity permits. We refer to this as a random normal basis option.

We emphasize that all three techniques require multiple runs of the CSE algorithm. Since the time and memory requirements of the CSE algorithm grows with the length of DFT, the moderate lengths of the sub-DFTs is the key enabler of these techniques.

For any $k \leq 320$ so that $k \mid 2^{l}-1(4 \leq l \leq 12)$, the multiplicative and additive complexities of the $k$-point CFFT are shown in Table $\Pi$ Table $\Pi$ shows four different schemes to reduce the additive complexity for CFFTs. Schemes A and B both use the fixed normal basis option in the construction of the matrix A, while schemes C and D are based on the random normal basis option. Schemes A and C reduce $\mathcal{C}(\mathbf{A})$ and $\mathcal{C}(\mathbf{Q})$ separately, while schemes B and D reduces $\mathcal{C}(\mathbf{A Q})$ as a whole. For smaller CFFTs, we typically minimize the complexity over hundreds of combinations of normal bases, and fewer combinations for longer CFFTs. In Table II the smallest additive complexities are in boldface font. We observe that the random normal basis option offers further additive complexity reduction in most of the cases. However, since the fixed normal basis is not necessarily one of the combinations, in some cases the fixed normal basis option outperforms the random normal basis option. Also, sometimes applying the CSE to AQ together as a whole leads to lower complexity, and in some cases it is better to apply the CSE to $\mathbf{A}$ and $\mathbf{Q}$ separately.

In the third step, we use the CFFTs with moderate lengths in Table $\Pi$ as sub-DFTs to construct our CCFTs. With (14) and (15), the computational complexities of our CCFTs over GF $\left(2^{l}\right)(4 \leq l \leq 12)$ with non-prime lengths can be calculated. The results are summarized in Table IIII, where the factorizations in parentheses are not co-prime and the Cooley-Turkey algorithm is used in these cases. We have tried all the decompositions with lengths smaller than 320, and the decompositions with the smallest overall complexities are listed in Table III. Note that for each sub-DFT, the scheme with the smallest additive complexity listed in Table $\Pi$ is used in the CCFT implementation to reduce the total additive complexity. We also note that all DFT lengths in Table IIII are composite. The prime lengths are omitted because when $N$ is prime, an $N$-point CCFT reduces to an $N$-point CFFT, which can be found in Table $\Pi$

Since some lengths of the DFTs have more than one decomposition, it is possible that one decomposition scheme has a smaller additive complexity but a larger multiplicative complexity than another one. Therefore, we need a metric to compare the overall complexities between different decompositions. In this paper, we follow our previous work [12] and assume that the complexity of a multiplication over $\mathrm{GF}\left(2^{l}\right)$ is $2 l-1$ times of that of an addition over the same field, and the total complexity of a DFT

TABLE II
THE COMPLEXITIES OF THE CFFTS WHOSE LENGTHS ARE LESS THAN 320 AND ARE FACTORS OF $2^{l}-1$ FOR $1 \leq l \leq 12$.

| $N$ | $l$ | mult. | additive complexities |  |  |  |
| :--- | :--- | :---: | :---: | :---: | :---: | :---: |
|  |  |  | A | B | C | D |
| 3 | 2 | 1 | $\mathbf{6}$ | $\mathbf{6}$ | $\mathbf{6}$ | $\mathbf{6}$ |
| 5 | 4 | 5 | 20 | $\mathbf{1 6}$ | 20 | $\mathbf{1 6}$ |
| 7 | 3 | 6 | 31 | $\mathbf{2 4}$ | 31 | $\mathbf{2 4}$ |
| 9 | 6 | 11 | 51 | $\mathbf{4 8}$ | 51 | $\mathbf{4 8}$ |
| 11 | 10 | 28 | 109 | 102 | 102 | $\mathbf{8 4}$ |
| 13 | 12 | 32 | 125 | 100 | 110 | $\mathbf{9 1}$ |
| 15 | 4 | 16 | 87 | $\mathbf{7 4}$ | 87 | $\mathbf{7 4}$ |
| 17 | 8 | 38 | 153 | 163 | $\mathbf{1 5 1}$ | 153 |
| 21 | 6 | 27 | 167 | 179 | $\mathbf{1 4 7}$ | 153 |
| 23 | 11 | 84 | 335 | 407 | $\mathbf{3 2 3}$ | 357 |
| 31 | 5 | 54 | 354 | $\mathbf{2 9 9}$ | 335 | 350 |
| 33 | 10 | 85 | 413 | 440 | $\mathbf{4 0 4}$ | 434 |
| 35 | 12 | 75 | 406 | 303 | 358 | $\mathbf{2 9 9}$ |
| 39 | 12 | 97 | 502 | 425 | 472 | $\mathbf{3 9 1}$ |
| 45 | 12 | 90 | 481 | 415 | 498 | $\mathbf{4 1 4}$ |
| 51 | 8 | 115 | $\mathbf{6 4 1}$ | 755 | 676 | 739 |
| 63 | 6 | 97 | 798 | $\mathbf{7 5 9}$ | 806 | 1031 |
| 65 | 12 | 165 | 1092 | $\mathbf{9 0 1}$ | 1114 | 915 |
| 73 | 9 | 144 | 1498 | 1567 | $\mathbf{1 4 4 7}$ | 1526 |
| 85 | 8 | 195 | 1601 | 1816 | $\mathbf{1 5 8 9}$ | 1810 |
| 89 | 11 | 336 | $\mathbf{2 0 8 5}$ | 4326 | 2247 | 3973 |
| 91 | 12 | 230 | 1668 | 1431 | 1596 | $\mathbf{1 4 2 1}$ |
| 93 | 10 | 223 | 1772 | 1939 | $\mathbf{1 7 3 6}$ | 1788 |
| 105 | 12 | 234 | 1762 | 1481 | 1776 | $\mathbf{1 3 3 3}$ |
| 117 | 12 | 299 | 2304 | 2028 | 2366 | $\mathbf{1 9 4 7}$ |
| 195 | 12 | 496 | 4900 | 4230 | 4942 | $\mathbf{4 1 6 6}$ |
| 273 | 12 | 699 | 8064 | $\mathbf{7 2 1 7}$ | 8082 | 7223 |
| 315 | 12 | 752 | 8965 | $\mathbf{8 0 3 2}$ | 9899 | 8099 |
|  |  |  |  |  |  |  |

is a weighted sum of the additive and multiplicative complexities, i.e., total $=(2 l-1) \times$ mult + add. This assumption is based on both the software and hardware implementation considerations [12]. Table IIII lists the decompositions with the smallest overall complexities.

Tables IIII provide complexities of all $N$-point DFTs over GF( $2^{l}$ ) when $N \mid 2^{l}-1$ and $4 \leq l \leq 12$.

TABLE III
The smallest complexity of our $N$-point CCFTS over GF $\left(2^{l}\right)$ For composite $N$ and $N \mid 2^{l}-1$ for $4 \leq l \leq 12$ (WE ASSUME THE SUB-DFTS ARE SHORTER THAN 320).

| $l$ | Length | Decomposition | mult. | add. | total |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | 15 | $1 \times 15$ | 16 | 74 | 186 |
| 6 | 9 | $(3 \times 3)$ | 10 | 36 | 146 |
|  | 21 | $3 \times 7$ | 25 | 114 | 389 |
|  | 63 | $(3 \times 3) \times 7$ | 124 | 468 | 1832 |
| 8 | 51 | $1 \times 51$ | 115 | 641 | 2366 |
|  | 85 | $1 \times 85$ | 195 | 1590 | 4515 |
|  | 255 | $3 \times 85$ | 670 | 5277 | 15327 |
| 9 | 511 | $7 \times 73$ | 1446 | 11881 | 36463 |
| 10 | 33 | $1 \times 33$ | 85 | 404 | 2019 |
|  | 93 | $3 \times 31$ | 193 | 1083 | 4750 |
|  | 341 | $1 \times 341$ | 922 | 15184 | 32702 |
|  | 1023 | $33 \times 31$ | 4417 | 22391 | 106314 |
| 11 | 2047 | $23 \times 89$ | 15204 | 76702 | 395986 |
| 12 | 35 | $5 \times 7$ | 65 | 232 | 1727 |
|  | 39 | $1 \times 39$ | 97 | 391 | 2622 |
|  | 45 | $(3 \times 15)$ | 91 | 312 | 2405 |
|  | 65 | $1 \times 65$ | 165 | 902 | 4697 |
|  | 91 | $1 \times 93$ | 230 | 1421 | 6711 |
|  | 105 | $7 \times 15$ | 202 | 878 | 5524 |
|  | 117 | $1 \times 117$ | 299 | 1947 | 8824 |
|  | 195 | $3 \times 65$ | 560 | 3093 | 15973 |
|  | 273 | $3 \times 91$ | 781 | 4809 | 22772 |
|  | 315 | $5 \times 63$ | 800 | 4803 | 23203 |
|  | 455 | $7 \times 65$ | 1545 | 7867 | 43402 |
|  | 585 | $5 \times 117$ | 2080 | 11607 | 59447 |
|  | 819 | $7 \times 117$ | 2795 | 16437 | 80722 |
|  | 1365 | $7 \times 195$ | 4642 | 33842 | 140608 |
|  | 4095 | $65 \times 63$ | 16700 | 106098 | 490198 |

Note that the decomposition corresponding to $1 \times N$ is merely the $N$-point $\operatorname{CFFT}$ over $\operatorname{GF}\left(2^{l}\right)$. We have used the simplified CSE algorithm described in Sec. IV-A to reduce the complexity of the 2047-point CFFTs over GF $\left(2^{11}\right)$, and applied the CSE algorithm in [12] to the other CFFTs. Thus, we have expanded the results of [[12], where only the $\left(2^{l}-1\right)$-point CFFTs over $\mathrm{GF}\left(2^{l}\right)$ were given. We also observe that
for some short lengths (see, for example, $N=15,33$, or 65 ), the $N$-point CFFTs lead to the lowest complexity for the $N$-point CCFTs. For the DFTs with lengths larger than 320, i.e., 511-point CFFTs over $\operatorname{GF}\left(2^{9}\right)$, 341-point CFFTs over $\operatorname{GF}\left(2^{10}\right)$, and 455-, 585 -, 819 -, and 1365 -point CFFTs over $\operatorname{GF}\left(2^{12}\right)$, the time complexity of the CSE algorithm in [12] is still considerable. Thus, we cannot minimize their complexities using schemes $\mathrm{A}, \mathrm{B}, \mathrm{C}$, and D , and hence they are not listed in Table II

Although the twiddle factors in the Cooley-Turkey algorithm decomposition incur extra multiplicative complexity, Tables III show that the Cooley-Turkey algorithm decomposition reduces the total complexity of our CCFTs in some cases (the decompositions in parentheses). For example, while 9-point CFFT requires 11 multiplications and 48 additions, $3 \times 3$ CCFT based on the Cooley-Turkey algorithm decomposition requires 10 multiplications and 36 additions. Despite the twiddle factors, the CCFT based on the Cooley-Turkey algorithm decomposition have lower multiplicative and additive complexities, because the Cooley-Turkey algorithm decomposition allows us to take advantage of the low complexity of the 3-point DFT.

## C. Complexity Comparison and Analysis

TABLE IV
Comparison of the complexities our $N$-point CCFTs with FFTs available in the literature.

| $N$ | Field | Wang and Zhu [29] |  |  | Trung et al. [9] |  |  | CFFT |  |  |  |  | CCFT |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | mult. | add. | total | mult. | add. | total | mult. | w/o CSE |  | w/ CSE [12] |  | mult. | add. | total |
|  |  |  |  |  |  |  |  |  | add. | total | add. | total |  |  |  |
| 15 | $\mathrm{GF}\left(2^{4}\right)$ | 41 | 97 | 384 | - | - | - | 16 | 201 | 313 | 74 | 186 | 20 | 78 | 218 |
| 63 | $\operatorname{GF}\left(2^{6}\right)$ | 801 | 801 | 9612 | - | - | - | 97 | 2527 | 3594 | 759 | 1826 | 124 | 468 | 1832 |
| 255 | $\mathrm{GF}\left(2^{8}\right)$ | 1665 | 5377 | 30352 | 1135 | 3887 | 20902 | 586 | 34783 | 43573 | 6736 | 15526 | 670 | 5277 | 15327 |
| 511 | $\mathrm{GF}\left(2^{9}\right)$ | 13313 | 13313 | 239634 | 6516 | 17506 | 128278 | 1014 | 141710 | 158948 | 23130 | 40368 | 1446 | 11881 | 36463 |
| 1023 | $\mathrm{GF}\left(2^{10}\right)$ | 32257 | 32257 | 645140 | 5915 | 30547 | 142932 | 2827 | 536093 | 589806 | 75360 | 129073 | 4417 | 22391 | 106314 |
| 2047 | $\mathrm{GF}\left(2^{11}\right)$ | 78601 | 78601 | 1689622 | - | - | - | 7812 | 2130248 | 2294300 | - | - | 15204 | 76702 | 395986 |
| 4095 | $\operatorname{GF}\left(2^{12}\right)$ | 180225 | 180225 | 4325400 | - | - | - | 10832 | 8434414 | 8683550 | - | - | 16700 | 106098 | 490198 |

We compare the complexities of our CCFTs with those of previously proposed FFTs in the literature in Table IV] For each length, the lowest total complexity is in boldface font. In Table IV, our CCFTs achieve the lowest complexities for $N \geq 255$. Although the algorithm in [29] is proved asymptotically fast, the complexities of our CCFTs are only a fraction of those in [29], and the advantage grows as the length increases. Although the FFTs in [9] are also based on the prime-factor algorithm, our CCFTs achieve lower complexities for two reasons. Since our CCFTs use CFFTs as the sub-DFTs,
the multiplicative complexities of our CCFTs are greatly reduced compared with the FFTs in [9]. For example, the multiplicative complexity of our 511-point CCFT is only one fourth of the prime-factor algorithm in [9]. Furthermore, using the powerful CSE algorithm in [12], the additive complexities of our CCFTs are also greatly reduced. Compared with the CFFTs, our CCFTs have a somewhat higher multiplicative complexities, but this is more than made up by reduced additive complexities of our CCFTs. The additive complexities of our CCFTs are only a small fraction of those of CFFTs when directly implemented. Compared with the CFFTs with reduced additive complexities in [12], our CCFTs still have much smaller additive complexities due to their decomposition structure for $N \geq 63$. For example, the additive complexities of our CCFT is only about half of that of the CFFT for $N=511$, and one third for $N=1023$. Due to the significant reduction of the additive complexities, the total complexities of our CCFTs with $N \geq 255$ are lower than those of CFFTs. In comparison to CFFTs, the improvement by our CCFTs also grows as the length increases.

For the DFTs whose lengths are prime, such as 31-point DFT over $\operatorname{GF}\left(2^{5}\right)$, 127 -point DFT over $\operatorname{GF}\left(2^{7}\right)$, and 8191-point DFT over $\operatorname{GF}\left(2^{13}\right)$, our CCFTs reduce to the CFFTs, and they have the same computational complexities.

## VI. Regular and Modular Structure of Our CCFTs

We have shown that our CCFTs lead to lower complexities for moderate to long lengths. Regardless of the length, our CCFTs also have advantages in hardware implementations due to their regular and modular structure.


Fig. 2. The structure of the CFFTs.

The CFFT algorithm has a bilinear form, and therefore its circuitry can be divided into three parts as shown in Fig. 2. The input vector $\mathbf{f}$ first goes through an pre-addition network, which reorders $\mathbf{f}$ into $\mathbf{f}^{\prime}$ and then computes $\mathbf{P f}^{\prime}$. Then the resulting vector is sent to a multiplicative network, in which the component-wise product of $\mathbf{c}$ and $\mathbf{P f}^{\prime}$ is computed. The DFT result $\mathbf{F}$ is finally computed in the
post-addition network which corresponds to the linear transform AQ. While the structure of the CFFT looks simple, the two additive networks are very complex for long DFTs. Although we can reduce the additive complexity by the CSE algorithm, the resulted additive networks still require a large number of additions. Furthermore, the additions due to $\mathbf{A}$ or $\mathbf{A Q}$ (the second additive network in Fig. (2) lack regularity, and hence it is hard to use architectural techniques such as folding and pipelining to achieve smaller area or high throughput.

In contrast, our CCFTs have regular and modular structure since they are decomposed into shorter sub-DFTs. The sub-DFTs can be implemented much easier than the long ones, and they can be reused in the CCFT architecture. Fig. 3 shows the regular and modular structure of a $3 \times 5$ CCFT. Instead of designing the 15 -point CFFT directly, we only need to design a 3 -point CFFT module and a 5 -point CFFT module, and compute the 15 -point CCFT by reusing these modules according to the structure shown in Fig. 3 It is much easier to apply architectural techniques such as folding and pipelining to this regular and modular structure, leading to efficient hardware implementations.


Fig. 3. The regular and modular structure of our 15 -point CCFT based on a $3 \times 5$ decomposition.

## VII. Conclusion

For any odd prime integer $p$, we reformulate $p$-point cyclic convolution as a $(p-1) \times(p-1)$ Toeplitz matrix vector product, leading to efficient cyclic convolution algorithms. Based on this reformulation, we have obtained efficient 11 -point cyclic convolution algorithm and derived the CFFTs over $\operatorname{GF}\left(2^{11}\right)$. We have shown that our composite cyclotomic Fourier transform algorithm leads to lower complexities through decomposing long DFTs into shorter ones using the prime-factor or Cooley-Turkey algorithms.

Our CCFTs over $\operatorname{GF}\left(2^{l}\right)(4 \leq l \leq 12)$, have lower complexities than previously known FFTs over finite fields. They also have a regular and modular structure, which is desirable in hardware implementations.

## Appendix A

Short Toeplitz Matrix Vector Product over GF(2l)
An $n \times n$ TMVP over $\operatorname{GF}\left(2^{l}\right)$ as

$$
\left[\begin{array}{c}
u_{0} \\
u_{1} \\
\vdots \\
u_{n-1}
\end{array}\right]=\left[\begin{array}{cccc}
r_{n-1} & r_{n} & \cdots & r_{2 n-2} \\
r_{n-2} & r_{n-1} & \cdots & r_{2 n-3} \\
\vdots & \vdots & \ddots & \vdots \\
r_{0} & r_{1} & \cdots & r_{n-1}
\end{array}\right]\left[\begin{array}{c}
v_{0} \\
v_{1} \\
\vdots \\
v_{n-1}
\end{array}\right] .
$$

can be computed with bilinear algorithm $\mathbf{E}^{(n)}\left(\mathbf{G}^{(n)} \mathbf{r} \cdot \mathbf{H}^{(n)} \mathbf{v}\right)$, where $\mathbf{r}=\left(r_{0}, r_{1}, \cdots, r_{2 n-2}\right)^{T}, \mathbf{v}=$ $\left(v_{0}, v_{1}, \cdots, v_{n-1}\right)^{T}$, and $\mathbf{E}^{(n)}, \mathbf{G}^{(n)}$ and $\mathbf{H}^{(n)}$ are all binary matrices.

For $n=2$ (see, for example, [18], [20]),

$$
\mathbf{E}^{(2)}=\left[\begin{array}{ll}
1 & 0
\end{array} 1\right.
$$

For $n=3$ (see, for example, [20]),

$$
\left(\mathbf{E}^{(3)}\right)^{T}=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{array}\right], \mathbf{G}^{(3)}=\left[\begin{array}{llll}
0 & 0 & 1 & 1
\end{array}\right]
$$

For $n=5$,

$$
\begin{aligned}
& \mathbf{H}^{(5)}=\left[\begin{array}{l}
00001000010111 \\
00010001001011 \\
00100010101100 \\
01000100110001 \\
10000111000001
\end{array}\right] .
\end{aligned}
$$

## Appendix B

4 -, 8 -, and 11-point Cyclic Convolution Algorithms over GF ( $2^{l}$ )
For 4-point cyclic convolutions, [24]

$$
\mathbf{Q}^{(4)}=\left[\begin{array}{lllllllll}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}\right]
$$

$$
\mathbf{R}^{(4)}=\left[\begin{array}{llll}
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1
\end{array}\right], \mathbf{P}^{(4)}=\left[\begin{array}{llll}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{array}\right] .
$$

For 8-point cyclic convolution [25],

$$
\begin{aligned}
\mathbf{Q}^{(8)}=\left[\begin{array}{l}
11000011011000000000000110 \\
10100010110100000000000111 \\
11000011000011000000110000 \\
101000101000101000000111000 \\
110110110110000000110000000 \\
101101101101000000111000000 \\
110110110000110110000000000 \\
101101101000101111000000000
\end{array}\right], \\
\left(\mathbf{R}^{(8)}\right)^{T}=\left[\begin{array}{l}
111111111111111111111111111 \\
001001001001011011011011011 \\
000000000000111111111111111 \\
000010010000001011011011011 \\
000111111000000111111111111 \\
000001001010000011011011011 \\
000000000111000111111111111 \\
010010010011010011011011011
\end{array}\right],
\end{aligned}
$$

and

$$
\left(\mathbf{P}^{(8)}\right)^{T}=\left[\begin{array}{c}
101101000000101101000000000 \\
110110000000110111000000000 \\
101101000101000000101000000 \\
110110000110000000111000000 \\
101101101000101000000101000 \\
110110110000110000000111000 \\
101101101101000000000000101 \\
110110110110000000000000111
\end{array}\right] .
$$

For our new 11-point cyclic convolutions, $\mathbf{Q}^{(11)}$ is given by
$\left[\begin{array}{l}1000000000000001111100000000011111000000000 \\ 1000010000101110000100001011100000000000000 \\ 1000100010010110001000100101100000000000000 \\ 1001000101011000010001010110000000000000000 \\ 1010001001100010100010011000100000000000000 \\ 1100001110000011000011100000100000000000000 \\ 1000010000101110000000000000000001000010111 \\ 1000100010010110000000000000000010001001011 \\ 1001000101011000000000000000000100010101100 \\ 1010001001100010000000000000001000100110001 \\ 1100001110000010000000000000010000111000001\end{array}\right]$,
the transpose of $\mathbf{R}^{(11)}$ is given by
$\left[\begin{array}{l}1100000101101011111100001100001111000011000 \\ 1000001111111101111110101001111111101010011 \\ 1000010110011011111000110000011111001100000 \\ 1000110001011011110101000000011111010000010 \\ 1001111101011011101110000001011111100000100 \\ 1011110101011011011100000010011111000011000 \\ 1111110101011010111100001100011111101010011 \\ 1111110101011011111110101001111110001100000 \\ 1111100101011011111100110000011101010000000 \\ 1111000101011111111101000001011011100000010 \\ 1110000101010011111110000010010111000000100\end{array}\right]$,
and the transpose of $\mathbf{P}^{(11)}$ is given by
$\left[\begin{array}{c}1100001110000010000000000000010000111000001 \\ 1010001001100010000000000000001000100110001 \\ 1001000101011000000000000000000100010101100 \\ 1000100010010110000000000000000010001001011 \\ 1000010000101110000000000000000001000010111 \\ 1100001110000011000011100000100000000000000 \\ 1010001001100010100010011000100000000000000 \\ 1001000101011000010001010110000000000000000 \\ 1000100010010110001000100101100000000000000 \\ 1000010000101110000100001011100000000000000 \\ 1000000000000001111100000000011111000000000\end{array}\right]$.
[1] R. E. Blahut, Fast Algorithms for Digital Signal Processing. Addison-Wesley, 1985.
[2] ——, Theory and Practice of Error Control Codes. Addison-Wesley, 1984.
[3] S. B. Wicker, Error Control Systems for Digital Communications and Storage. Upper Saddle River, NJ: Prentice Hall, 1995.
[4] "Hard disk drive long data sector white paper," April 20 2007. [Online]. Available: http://www.idema.org/
[5] Y. Han, W. E. Ryan, and R. Wesel, "Dual-mode decoding of product codes with application to tape storage," in Proc. IEEE Global Telecommunications Conference, vol. 3, Nov. 28-Dec. 2, 2005, pp. 1255-1260.
[6] T. Buerner, R. Dohmen, A. Zottmann, M. Saeger, and A. J. van Wijngaarden, "On a high-speed Reed-Solomon codec architecture for $43 \mathrm{~Gb} / \mathrm{s}$ optical transmission systems," in Proc. 24th International Conference on Microelectronics, vol. 2, May 16-19, 2004, pp. 743-746.
[7] I. J. Good, "The interaction algorithm and practical Fourier analysis," Journal of the Royal Statistical Society. Series B (Methodological), vol. 20, no. 2, pp. 361-372, 1958. [Online]. Available: http://www.jstor.org/stable/2983896
[8] J. W. Cooley and J. W. Tukey, "An algorithm for the machine calculation of complex Fourier series," Mathematics of Computation, vol. 19, no. 90, pp. 297-301, 1965.
[9] T. K. Truong, P. D. Chen, L. J. Wang, Y. Chang, and I. S. Reed, "Fast, prime factor, discrete Fourier transform algorithms over $G F\left(2^{m}\right)$ for $8 \leq m \leq 10$," Inf. Sci., vol. 176, no. 1, pp. 1-26, 2006.
[10] P. V. Trifonov and S. V. Fedorenko, "A method for fast computation of the Fourier transform over a finite field," Probl. Inf. Transm., vol. 39, no. 3, pp. 231-238, 2003.
[11] S. V. Fedorenko, "A method for computation of the discrete fourier transform over a finite field," Probl. Inf. Transm., vol. 42, pp. 139-151, 2006.
[12] N. Chen and Z. Yan, "Cyclotomic FFTs with reduced additive complexities based on a novel common subexpression elimination algorithm," IEEE Trans. Signal Process., vol. 57, no. 3, pp. 1010-1020, Mar. 2009.
[13] P. Trifonov, "Matrix-vector multiplication via erasure decoding," in Proceedings of XI International Symposium on Problems of Redundancy in Information and Control Systems, Jul. 2007.
[14] A. V. Aho, J. E. Hopcroft, and J. D. Ullman, The Design and Analysis of Computer Algorithms. Reading, MA: AddisonWesley, 1974.
[15] O. Gustafsson and M. Olofsson, "Complexity reduction of constant matrix computations over the binary field," in WAIFI '07: Proceedings of the 1st international workshop on Arithmetic of Finite Fields. Berlin, Heidelberg: Springer-Verlag, 2007, pp. 103-115.
[16] J. C. Allwright, "Real factorisation of noncyclic-convolution operators with application to fast convolution," Electronics Letters, vol. 7, no. 24, pp. 718-719, 1971.
[17] D. A. Pitassi, "Fast convolution using the Walsh transforms," IEEE Trans. Electromagn. Compat., no. 3, pp. 130-133, 1971.
[18] R. Agarwal and C. Burrus, "Fast one-dimensional digital convolution by multidimensional techniques," IEEE Trans. Acoust., Speech, Signal Process., vol. 22, no. 1, pp. 1-10, 1974.
[19] S. Winograd, Arithmetic Complexity of Computations. SIAM, 1980.
[20] H. Fan and M. A. Hasan, "A new approach to subquadratic space complexity parallel multipliers for extended binary fields," IEEE Trans. Comput., vol. 56, no. 2, pp. 224-233, 2007.
[21] T. G. Zakharova, "Fourier transform evaluation in fields of characteristic 2," Probl. Peredachi Inf., vol. 28, pp. 62-77, 1992.
[22] M. R. Garey and D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness (Series of Books in the Mathematical Sciences). W. H. Freeman, January 1979.
[23] P. V. Trifonov, private communication.
[24] V. Afanasyev and I. Grushko, "FFT algorithms for the fields $\mathrm{GF}\left(2^{m}\right)$," in Pomekhoustoichivoe kodirovanie i nadezhnost' EVM (Error-Resistant Coding and Reliability of Computers), Nauka, Moscow, 1987, pp. 33-55, in Russian.
[25] N. Churkov and S. Fedorenko, "A method for construction of composite length cyclic convolutions over finite fields,"
in 32nd conference "Nedelya nauki Sankt-Peterburgskogo gosudarstvennogo polytechnicheskogo universiteta" ("Scientific week of Saint-Petersberg State Polytechnic University"), vol. 5, 2004, pp. 180-181.
[26] S. Winograd, "On computing the discrete Fourier transforms," Proc. National Academy of Sciences USA, vol. 73, no. 4, pp. 1005 - 1006, Apr. 1976.
[27] C. M. Rader, "Discrete fourier transforms when the number of data samples is prime," Proceedings of the IEEE, vol. 56, no. 6, pp. 1107 - 1108, Jun. 1968.
[28] M. Wagh and S. Morgera, "A new structured design method for convolutions over finite fields, part I," IEEE Trans. Inf. Theory, vol. 29, no. 4, pp. 583-595, Jul. 1983.
[29] Y. Wang and X. Zhu, "A fast algorithm for the Fourier transform over finite fields and its VLSI implementation," IEEE Journal on Selected Areas in Communications, vol. 6, no. 3, pp. 572-577, Apr. 1988.

