On the Degrees of Freedom Achievable Through Interference Alignment in a MIMO Interference Channel*

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Abstract

Consider a K-user flat fading MIMO interference channel where the k-th transmitter (or receiver) is equipped with M_k (respectively N_k) antennas. If an exponential (in K) number of generic channel extensions are used either across time or frequency, Cadambe and Jafar [1] showed that the total achievable degrees of freedom (DoF) can be maximized via interference alignment, resulting in a total DoF that grows linearly with K even if M_k and N_k are bounded. In this work we consider the case where no channel extension is allowed, and establish a general condition that must be satisfied by any degrees of freedom tuple $(d_1, d_2, ..., d_K)$ achievable through linear interference alignment. For a symmetric system with $M_k = M$, $N_k = N$, $d_k = d$ for all k, this condition implies that the total achievable DoF cannot grow linearly with K, and is in fact no more than K(M+N)/(K+1). We also show that this bound is tight when the number of antennas at each transceiver is divisible by d, the number of data streams per user.

I. INTRODUCTION

Consider a multiuser communication system in which a number of transmitters must share common resources such as frequency, time, or space in order to send information to their respective receivers. The mathematical model for this communication scenario is the well-known *interference channel*, which consists of multiple transmitters simultaneously sending messages to their intended receivers while causing interference to each other.

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A central issue in the study of interfering multiuser systems is how to mitigate multiuser interference. In practice, there are several commonly used methods for dealing with interference. First, we can treat the interference as noise and just focus on extracting the desired signals. This approach is widely used in practice because of its simplicity and ease of implementation, but is known to be non-capacity achieving in general. An alternative technique is channel orthogonalization whereby transmitted signals are chosen to be nonoverlapping either in time, frequency or space, leading to Time Division Multiple Access, Frequency Division Multiple Access, or Space Division Multiple Access respectively. While channel orthogonalization effectively eliminates multiuser interference, it can lead to inefficient use of communication resources and is also generally non-capacity achieving. Another interference management technique is to decode and remove interference. Specifically, when interference is strong relative to desired signals, a user can decode the interference first, then subtract it from the received signal, and finally decode its own message. Unfortunately, none of the aforementioned interference management techniques can achieve the maximum system throughput in general.

Theoretically, what is the optimal transmit/receive strategy in a MIMO interference channel? The answer is related to the characterization of the capacity region of an interference channel, i.e., determining the set of rate tuples that can be achieved by the users simultaneously. In spite of intensive research on this subject over the past three decades, the capacity region of interference channels is still unknown (even for small number of users). The lack of progress to characterize the capacity region of the MIMO interference channel has motivated researchers to derive various approximations of the capacity region. For example, the maximum total degrees of freedom (DoF) corresponds to the first order approximation of sum-rate capacity in the high SNR regime. Specifically, in a *K*-user interference channel, we define the degrees of freedom region as the following [1]:

$$\mathcal{D} = \left\{ (d_1, d_2, \dots, d_K) \in \mathbb{R}_+^K \mid \forall (w_1, w_2, \dots, w_K) \in \mathbb{R}_+^K, \right.$$

$$\left. \sum_{k=1}^K w_k d_k \le \limsup_{\text{SNR} \to \infty} \left[\sup_{\mathbf{R} \in \mathcal{C}} \frac{1}{\log \text{SNR}} \sum_{k=1}^K w_k R_k \right] \right\}, \tag{1}$$

where C is the capacity region and R_k is the rate of user k. We can further define the total DoF in the system as the following:

$$\eta = \max_{(d_1, d_2, \dots, d_K) \in \mathcal{D}} d_1 + d_2 + \dots + d_K.$$

Intuitively, the total DoF is the number of independent data streams that we can communicate interference-

free in the channel.

It is well known that for a point-to-point MIMO channel with M antennas at the transmitter and N antennas at the receiver, the total DoF is $\eta = \min\{M, N\}$. Different approaches such as SVD precoder or V-BLAST can be used to achieve this DoF bound. For a 2-user MIMO fading interference channel with user k equipped with M_k transmit antennas and N_k receive antennas (k = 1, 2), Jafar and Fakhereddin [9] proved that the maximum total DoF is

$$\eta = \min \{M_1 + M_2, N_1 + N_2, \max\{M_1, N_2\}, \max\{M_2, N_1\}\}.$$

This result shows that for the case of $M_1=M_2=N_1=N_2$, the total DoF in the system is the same as the single user case. In other words, we do not gain more DoF by increasing the number of users from one to two. Interestingly, if generic channel extensions (drawn from a continuous probability distribution) are allowed either across time or frequency, Cadambe and Jafar [1] showed that the total DoF is $\eta=KM/2$ for a K-user MIMO interference channel, where M is the number of transmit/receive antennas per user. This result implies that each user can effectively utilize half of the total system resources in an interference-free manner by aligning the interference at all receivers¹. The principal assumption enabling this surprising result is that the channel extensions are exponentially long in K and are generic (e.g., drawn from a continuous probability distribution). If channel extensions are restricted to have a polynomial length or are not generic, the total DoF for a MIMO interference channel is still largely unknown even for the Single-Input-Single-Output (SISO) interference channel. For the 3-user special case, reference [7] provided a characterization of the total achievable DoF as a function of the diversity. In the absence of channel extensions, the computational complexity of numerically designing an interference alignment scheme has been shown to be NP-hard [12] in the number of users.

The main theoretical investigation pertaining to the current work is [2] by Yetis *et al.* who studied the maximum achievable DoF for a MIMO interference channel without channel extension. In general, linear interference alignment can be described by a set of bilinear equations which correspond to the zero-forcing conditions at each receiver. For a K-user system, there are a total of K(K-1) such coupled quadratic matrix equations whose unknowns are the transmit/receive beamforming matrices to be designed. Moreover, the achievability of a given tuple of DoF corresponds to these quadratic equations having a solution (in the form of beamforming matrices) whose individual matrix ranks are given by the

¹The idea of interference alignment was introduced in [3]–[5] and the terminology "interference alignment" was first used in [6].

DoFs. One can easily count the number of "independent unknowns" and the number of scalar equations in this quadratic system defining interference alignment. It is then tempting to conjecture, as was done in [2], that the interference alignment is feasible if and only if the number of equations is no more than the number of unknowns in each subsystem of the quadratic equations. When the latter is true, the authors of [2] called the corresponding system *proper*. However, except for some special cases involving a small number of users and antennas, the investigation of [2] was largely inconclusive.

In this paper, we settle the conjecture of [2] completely in one direction, and partially in the other. In particular, we consider the case where no channel extension is allowed, and use results from the field theory to establish a general condition that must be satisfied by any DoF tuple achievable through linear interference alignment. This condition shows that the improperness property (in the sense of [2]) indeed implies the infeasibility of interference alignment. For the symmetric system with $M_k = M$ and $N_k = N$ for all k, this condition implies that the total achievable DoF cannot grow linearly with the number of users, and is in fact no more than M+N-1. This is in sharp contrast to the case with independent channel extensions for which the total DoF can grow linearly with the number of users. For the converse direction, we show that if all users have the same DoF d and the number of antennas M_k , N_k are divisible by d for each k, then the properness of the quadratic system implies the feasibility of interference alignment for generic choice of channel coefficients (e.g., drawn from a continuous probability distribution). If in addition, $M_k = M$ and $N_k = N$ for all k and M, N are divisible by d, then our results imply that interference alignment is achievable if and only if $(M+N) \ge d(K+1)$. In the simulation section, we use these established DoF bounds to numerically benchmark the performance of several existing algorithms for interference alignment and sum-rate maximization.

II. SYSTEM MODEL

Consider a MIMO interference network consisting of K transmitter - receiver pairs, with transmitter k sending d_k independent data streams to receiver k. Let \mathbf{H}_{kj} be an $M_j \times N_k$ matrix that represents the channel gain matrix from transmitter j to receiver k where M_j and N_k denote the number of antennas at transmitter j and receiver k, respectively. The received signal at receiver k is given by

$$\mathbf{y}_k = \sum_{j=1}^K \mathbf{H}_{kj} \mathbf{x}_j + \mathbf{n}_k$$

where \mathbf{x}_j is an $M_j \times 1$ random vector that represents the transmitted signal of user j and $\mathbf{n}_k \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$ is a zero mean additive white Gaussian noise.

Throughout this paper, we focus on *linear* transmit and receive strategies that can maximize system throughput. In this case, transmitter k uses a beamforming matrix V_k in order to send a signal vector s_k to its intended receiver k. On the other side, receiver k estimates the transmitted data vector s_k by using a linear beamforming matrix U_k , i.e.,

$$\mathbf{x}_k = \mathbf{V}_k \; \mathbf{s}_k, \qquad \hat{\mathbf{s}}_k = \mathbf{U}_k^H \mathbf{y}_k$$

where the power of the data vector $\mathbf{s}_k \in \mathbb{R}^{d_k \times 1}$ is normalized such that $E[\mathbf{s}_k \mathbf{s}_k^H] = \mathbf{I}$, and $\hat{\mathbf{s}}_k$ is the estimate of \mathbf{s}_k at the k-th receiver. The matrices $\mathbf{V}_k \in \mathbb{C}^{M_k \times d_k}$ and $\mathbf{U}_k \in \mathbb{C}^{N_k \times d_k}$ are the beamforming matrices at the k-th transmitter and receiver respectively. Without channel extension, the linear interference alignment conditions can be described by the following zero-forcing conditions [2], [12]

$$\mathbf{U}_k^H \mathbf{H}_{kj} \mathbf{V}_j = \mathbf{0}, \quad \forall \ j \neq k, \tag{2}$$

$$\operatorname{rank}\left(\mathbf{U}_{k}^{H}\mathbf{H}_{kk}\mathbf{V}_{k}\right) = d_{k}, \quad \forall \ k. \tag{3}$$

The first equation guarantees that all the interfering signals at receiver k lie in the subspace orthogonal to \mathbf{U}_k , while the second one assures that the signal subspace $\mathbf{H}_{kk}\mathbf{V}_k$ has dimension d_k and is linearly independent of the interference subspace. Intuitively, as the number of users K increases, the number of constraints on the beamformers $\{\mathbf{U}_k, \mathbf{V}_k\}$ increases quadratically in K, while the number of design variables in $\{\mathbf{U}_k, \mathbf{V}_k\}$ only increases linearly. This suggests the above interference alignment can not have a solution unless K or d_k is small.

The interference alignment conditions (2) and (3) imply that each transmitter k can use a linear transmit/receive strategy to communicate d_k interference-free independent data streams to receiver k (per channel use). In this case, it can be checked that d_k represents the DoF achieved by the k-th transmitter/receiver pair in the information theoretic sense of (1). In other words, the vector $(d_1, d_2, ..., d_K)$ in (2) and (3) represents the tuple of DoF achieved by linear interference alignment. Intuitively, the larger the values of $d_1, d_2, ..., d_K$, the more difficult it is to satisfy the interference alignment conditions (2) and (3).

III. BOUNDING THE TOTAL DOF ACHIEVABLE VIA LINEAR INTERFERENCE ALIGNMENT

Our goal is to study the solvability of the interference alignment problem (2)-(3) and derive a general condition that must be satisfied by any DoF tuple $(d_1, d_2, ..., d_K)$ achievable through linear interference alignment for generic choice of channel matrices. We will also provide some conditions under which this upper bound is achievable.

Let us denote the polynomial equations in (3) by the index set

$$\mathcal{J} \triangleq \{(k,j) \mid 1 \le k \ne j \le K\}.$$

The following theorem provides an upper bound on the total achievable DoF when no channel extension is allowed.

Theorem 1 Consider a K-user flat fading MIMO interference channel where the channel matrices $\{\mathbf{H}_{ij}\}_{i,j=1}^{K}$ are generic (e.g., drawn from a continuous probability distribution). Assume no channel extension is allowed. Then any tuple of degrees of freedom $(d_1, d_2, ..., d_K)$ that is achievable through linear interference alignment (2) and (3) must satisfy the following inequalities

$$\min\{M_k, N_k\} > d_k, \quad \forall k, \tag{4}$$

$$\max\{M_k, N_j\} \ge d_k + d_j, \quad \forall \ k, j, k \ne j, \tag{5}$$

$$\sum_{k:(k,j)\in\mathcal{I}} (M_k - d_k) d_k + \sum_{j:(k,j)\in\mathcal{I}} (N_j - d_j) d_j \ge \sum_{(k,j)\in\mathcal{I}} d_k d_j, \quad \forall \, \mathcal{I} \subseteq \mathcal{J}.$$
 (6)

Condition (6) in Theorem 1 can be used to bound the total DoF achievable in a MIMO interference channel. The following corollary is immediate.

Corollary 1 Assume the setting of Theorem 1. Then the following upper bounds hold true.

(a) In the case of $d_k = d$ for all k, interference alignment is impossible unless

$$d \le \frac{1}{K(K+1)} \sum_{k=1}^{K} (M_k + N_k).$$

(b) In the case of $M_k + N_k = M + N$, interference alignment requires

$$\left(\sum_{k=1}^{K} d_k\right)^2 + \sum_{k=1}^{K} d_k^2 \le (M+N) \sum_{k=1}^{K} d_k$$

which further implies

$$\sum_{k=1}^{K} d_k < (M+N).$$

Part (b) of Corollary 1 shows that the total achievable DoF in a MIMO interference channel is bounded by a constant M + N - 1, regardless of how many users are present in the system. While this bound is an improvement over the single user case which has a maximum DoF of min $\{M, N\}$, it is significantly weaker than the maximum achievable total DoF for a diagonal frequency selective (or time varying)

interference channel. The latter grows linearly with the number of users in the system [1].

The rest of this section is devoted to the proof of Theorem 1 and its converse. Since we will use several concepts and results from the field theory [11] and algebraic geometry [14], [16], we first provide a brief review of the necessary algebraic background.

A. Algebraic Preliminaries

Let \mathcal{K}, \mathcal{F} be two fields such that $\mathcal{K} \subseteq \mathcal{F}$. In this case, we say \mathcal{F} is an extension of \mathcal{K} , denoted by \mathcal{F}/\mathcal{K} . Let us use $\mathcal{K}[z_1, z_2, \ldots, z_n]$ to denote the ring of polynomials with coefficients drawn from \mathcal{K} . We say $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{F}$ are algebraically dependent over \mathcal{K} if there exists a nonzero polynomial $f(z_1, z_2, \ldots, z_n) \in \mathcal{K}[z_1, z_2, \ldots, z_n]$ such that

$$f(\alpha_1, \alpha_2, \dots, \alpha_n) = 0. (7)$$

Otherwise, we say that they are algebraically independent over \mathcal{K} . The largest cardinality of an algebraically independent set is called the transcendence degree of \mathcal{F} over \mathcal{K} . An element $\alpha \in \mathcal{F}$ is said to be algebraic over \mathcal{K} if there exists a nonzero polynomial $f \in \mathcal{K}[z]$ such that $f(\alpha) = 0$; else, we say α is transcendental over \mathcal{K} .

Example 1. Let $\mathcal{K} = \mathbb{C}$ be the field of complex numbers and $\mathcal{F} = \mathbb{C}(x_1, x_2)$ be the field of rational functions in variables x_1, x_2 . Then, the polynomials

$$g_1 = x_1^2 x_2, \quad g_2 = x_2^2, \quad g_3 = x_1 x_2$$

are algebraically dependent over \mathbb{C} because $f(g_1, g_2, g_3) = 0$ identically for all (x_1, x_2) , where $f(z_1, z_2, z_3) = z_1^2 z_2 - z_3^4$.

Example 2. The two complex numbers $a = \sqrt{\pi}$, $b = 3\pi + 2$ are algebraically dependent over the field of rational numbers because by defining $f(z_1, z_2) = 3z_1^2 - z_2 + 2$, we have f(a, b) = 0.

Notice that the definition of algebraic independence is in many ways similar to the standard notion of linear independence from linear algebra. In fact, if the function f in (7) is required to be linear, then algebraic independence reduces to the usual concept of linear independence. Similar to linear algebra, we can define a basis for the field \mathcal{F} using the notion of algebraic independence. In particular, given any algebraically independent set S over the field \mathcal{K} , let $\mathcal{K}(S)$ denote the field of rational functions in

S with coefficients taken from the field K. For any field extension F/K, it is always possible to find a set S in F, algebraically independent over K, such that F is an algebraic extension of K(S). Such a set S is called a transcendence basis of F over K. All transcendence bases have the same cardinality, equal to the transcendence degree of the extension F/K. If every element in F is algebraic over K, then we say F/K is an algebraic extension. In this case, the transcendence degree of F over K is zero.

Example 3. The two polynomials g_1 and g_2 in Example 1 are algebraically independent over \mathbb{C} . Together, they constitute a transcendental basis for $\mathbb{C}(x_1, x_2)$ over \mathbb{C} .

The following table shows similar concepts between linear algebra and transcendental field extension (see [11], [16] for more details).

Linear algebra	Transcendental field extension
linear independence	algebraic independence
$A\subseteq \operatorname{span}(B)$	A algebraically dependent on B
linear basis	transcendence basis
dimension	transcendence degree

In linear algebra, it is well known that any (n+1) vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n+1}$ in an n-dimensional vector space must be linearly dependent. In other words, there exists a nonzero linear function $f(z_1, z_2, ..., z_{n+1})$ such that $f(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n+1}) = 0$. A similar result holds for algebraic independence. For example, any (n+1) polynomials $g_1, g_2, ..., g_{n+1}$ defined on n variables $(x_1, x_2, ..., x_n)$ must be algebraically dependent. Consequently, there exists a nonzero polynomial $f(z_1, z_2, ..., z_{n+1})$ such that

$$f(g_1, g_2, ..., g_{n+1}) = 0, \quad \forall (x_1, x_2, ..., x_n).$$

Example 1 is an instance of this property with n = 2. The following example states this property, to be used in the proof of Theorem 1, in a more formal setting.

Example 4. Let $\mathbb{C}(z_1, z_2, \dots, z_n)$ denote the field of rational functions in n variables with coefficients in \mathbb{C} . The set $\{z_1, z_2, \dots, z_n\}$ is a maximal algebraically independent set in $\mathbb{C}(z_1, z_2, \dots, z_n)$. Hence the transcendence degree of the field extension $\mathbb{C}(z_1, z_2, \dots, z_n)/\mathbb{C}$ is n. Furthermore, for any m polynomials

$$g_1(z_1, z_2, \ldots, z_n), g_2(z_1, z_2, \ldots, z_n), \ldots, g_m(z_1, z_2, \ldots, z_n),$$

where m > n, there exists a nonzero polynomial $f(\cdot)$ such that $f(g_1, g_2, \dots, g_m) = 0, \ \forall \ z_1, z_2, \dots, z_n$.

Next we describe a useful local expansion of a multivariate polynomial function. Recall that for any univariate polynomial f and any $\bar{x} \in \mathbb{C}$, there holds

$$f(x) = f(\bar{x}) + (x - \bar{x})g(x)$$
, for all $x \in \mathbb{C}$,

where g is some polynomial dependent on \bar{x} and the coefficients of f only. Similarly, for a n-variate polynomial f defined on the variables $\mathbf{x} = (x_1, x_2, ..., x_n)$ and any $\bar{\mathbf{x}} \in \mathbb{C}^n$, we have

$$f(\mathbf{x}) = f(\bar{\mathbf{x}}) + \sum_{i=1}^{n} (x_i - \bar{x}_i) g_i(\mathbf{x}) = f(\bar{\mathbf{x}}) + (\mathbf{x} - \bar{\mathbf{x}})^T \mathbf{g}(\mathbf{x}), \ \forall \ \mathbf{x} \in \mathbb{C}^n,$$

where each g_i is some polynomial dependent on $\bar{\mathbf{x}}$ and the coefficients of f only. If we replace the scalar variable x_i by a matrix variable \mathbf{X}_i , then we can write

$$f(\mathbf{X}) = f(\bar{\mathbf{X}}) + \sum_{i=1}^{n} \operatorname{Tr}\left((\mathbf{X}_{i} - \bar{\mathbf{X}}_{i})\mathbf{G}_{i}(\mathbf{X})\right), \ \forall \ \mathbf{X},$$
(8)

where each G_i is a matrix whose entries are polynomials dependent on the entries of \bar{X} and the coefficients of f only. The local expansion (8) will be used in the proof of Theorem 1.

To prove the converse of Theorem 1, we will use the concepts of Zariski topology and a Zariski constructible set. We briefly review these concepts next (see [14] for more details). Consider \mathbb{C}^n , the n-dimensional vector space over the field of complex numbers \mathbb{C} . [One can replace \mathbb{C} by any algebraically closed field.] The Zariski topology for \mathbb{C}^n is defined by specifying its closed sets, and these are taken simply to be all the algebraic sets in \mathbb{C}^n . That is, the closed sets under Zariski topology are those of the form

$$S = {\mathbf{x} \in \mathbb{C}^n \mid f_i(\mathbf{x}) = 0, \ i = 1, 2, ..., m}$$

where $\{f_i\}_{i=1}^m$ is any set if polynomials with coefficients taken from \mathbb{C} . For example, the entire space \mathbb{C}^n is Zariski closed (Take m=1 and f_1 to be the zero function, i.e., $f_1(x)=0, \ \forall \ x$). All other Zariski closed sets have zero measure. A nonempty Zariski open set (the complement of a Zariski closed set) always has dimension n. If a property holds over a Zariski open set, we say the property holds generically.

In topology, a set is locally closed if it is the intersection of an open set with a closed set. A *constructible* set is defined as a finite union of locally closed sets. Thus, a Zariski constructible set is simply a finite

collection of sets, each defined by the feasible set of finitely many polynomial equations and polynomial inequalities. Clearly, if a Zariski constructible set has dimension n, then it must contain a Zariski open subset.

Let $\phi_1, \phi_2, \dots, \phi_n$ be polynomials in x_1, x_2, \dots, x_n with coefficients from \mathbb{C} . They define a map $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ as follows: $\Phi(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \dots, \phi_n(\mathbf{x})) \in \mathbb{C}^n$. Chevalley's Theorem says that the image of this map is a constructible set (see [16] for more details).

Example 5. Let $\Phi: \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $\Phi(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}))$ where $\phi_1(x) = x_1$ and $\phi_2(x) = x_1x_2$. Let \mathcal{L} be the line $\{\mathbf{x} \in \mathbb{C}^2 : x_1 = 0\}$. The image of Φ is the union of two locally closed sets, $\mathbb{C}^2 \setminus \mathcal{L}$ (which is in fact open) and the point (0,0) (which is indeed closed).

Let the image of Φ be the union of locally closed subsets $\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_p$ where $\mathcal{W}_i = \mathcal{U}_i \cap \mathcal{V}_i$ and \mathcal{V}_i is closed and \mathcal{U}_i is open. Assume the Jacobian of $\phi_1, \phi_2, \dots, \phi_n$ is nonsingular at some point $\mathbf{x} \in \mathbb{C}^n$. The Implicit Function Theorem says that the image of Φ contains a small open disc around $\Phi(\mathbf{x})$, hence the measure of the image is nonzero. This implies that for some i, $\mathcal{V}_i = \mathbb{C}^n$ and $\mathcal{W}_i = \mathcal{U}_i$, i.e., the image of the map $\Phi(\cdot)$ contains a Zariski open set. Thus, if a certain property is shown to hold over the image of a polynomial map $\Phi: \mathbb{C}^n \mapsto \mathbb{C}^n$ whose Jacobian is nonsingular at some point, then this property must hold generically. We will use this approach to establish the generic feasibility of interference alignment for certain MIMO interference channels (Theorem 2).

B. Proof of Theorem 1

We now use the transcendental field extension theory to establish Theorem 1.

Proof: The inequality (4) is obvious due to (3). To prove (5), assume $M_j \leq N_k$. Since \mathbf{H}_{kj} is generic, $\operatorname{rank}(\mathbf{H}_{kj}\mathbf{V}_j) = d_j$. Furthermore, due to (3), the beamformer \mathbf{U}_k must be full rank and hence $d_k + d_j$ must be no more than the total dimension N_k . Similar argument shows that $d_k + d_j \leq M_j$ when $M_j \geq N_k$. Thus, $d_k + d_j \leq \max\{M_j, N_k\}$.

For simplicity of notations, we prove (6) for the case $\mathcal{I} = \mathcal{J}$. When $\mathcal{I} \subset \mathcal{J}$, the proof is the same except that we need to focus on a subset of equations/variables. Now, we prove (6) for the case of $\mathcal{I} = \mathcal{J}$

by contradiction. Assume the contrary that

$$\sum_{k=1}^{K} (M_k - d_k) d_k + \sum_{j=1}^{K} (N_j - d_j) d_j < \sum_{k,j=1, k \neq j}^{K} d_k d_j,$$
(9)

and the interference alignment conditions in (2) and (3) are satisfied. The interference alignment condition (3) implies that U_k and V_k must have full column rank. By applying appropriate linear transformations to the rows of U_k and V_k , we can write

$$\mathbf{U}_{k} = \mathbf{P}_{k}^{u} \begin{bmatrix} \mathbf{I} \\ \bar{\mathbf{U}}_{k} \end{bmatrix} \mathbf{Q}_{k}^{u}, \quad \mathbf{V}_{k} = \mathbf{P}_{k}^{v} \begin{bmatrix} \mathbf{I} \\ \bar{\mathbf{V}}_{k} \end{bmatrix} \mathbf{Q}_{k}^{v}, \quad \forall k,$$
 (10)

where $\bar{\mathbf{U}}_k$ and $\bar{\mathbf{V}}_k$ are some matrices of size $(N_k - d_k) \times d_k$ and $(M_k - d_k) \times d_k$ respectively. The matrices \mathbf{P}_k^u and \mathbf{P}_k^v are square permutation matrices of size $N_k \times N_k$ and $M_k \times M_k$ respectively, while \mathbf{Q}_k^u , \mathbf{Q}_k^v are some invertible matrices of size $d_k \times d_k$. Define $\bar{\mathbf{H}}_{ij} = \mathbf{P}_i^{u-1} \mathbf{H}_{ij} \mathbf{P}_j^{v-1}$ to be the permuted version of \mathbf{H}_{kj} . We can partition the matrix $\bar{\mathbf{H}}_{kj}$ as

$$ar{\mathbf{H}}_{kj} = \left[egin{array}{ccc} ar{\mathbf{H}}_{kj}^{(1)} & ar{\mathbf{H}}_{kj}^{(2)} \ ar{\mathbf{H}}_{kj}^{(3)} & ar{\mathbf{H}}_{kj}^{(4)} \end{array}
ight]$$

where $\bar{\mathbf{H}}_{kj}^{(1)}$ is of size $d_k \times d_j$. Since the channel matrices $\{\mathbf{H}_{kj}\}_{k\neq j}$ are drawn from a continuous probability distribution, the transformed channel matrices $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k\neq j}$ remain generic. Rewriting the linear interference alignment condition (2) in terms of $\bar{\mathbf{U}}_k$ and $\bar{\mathbf{V}}_k$, we obtain

$$\begin{bmatrix} \mathbf{I} & \bar{\mathbf{U}}_k^H \end{bmatrix} \begin{bmatrix} \bar{\mathbf{H}}_{kj}^{(1)} & \bar{\mathbf{H}}_{kj}^{(2)} \\ \bar{\mathbf{H}}_{kj}^{(3)} & \bar{\mathbf{H}}_{kj}^{(4)} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \bar{\mathbf{V}}_j \end{bmatrix} = \mathbf{0}$$
(11)

or equivalently

$$\bar{\mathbf{H}}_{kj}^{(1)} + \bar{\mathbf{U}}_{k}^{H} \bar{\mathbf{H}}_{kj}^{(3)} + \bar{\mathbf{H}}_{kj}^{(2)} \bar{\mathbf{V}}_{j} + \bar{\mathbf{U}}_{k}^{H} \mathbf{H}_{kj}^{(4)} \bar{\mathbf{V}}_{j} = \mathbf{0}, \quad \forall \ j \neq k.$$
 (12)

The above system of quadratic equations, first derived in [2], is equivalent to the interference alignment condition (2). The number of scalar equations in (12) is

$$\sum_{j,k=1,j\neq k}^{K} d_k d_j,$$

while the total number of scalar variables (i.e., the scalar entries of the unknown matrices $\{\bar{\mathbf{U}}_k\}$'s and

$$\{\bar{\mathbf{V}}_k\}$$
's) is

$$\sum_{k=1}^{K} (M_k - d_k) d_k + \sum_{k=1}^{K} (N_k - d_k) d_k = \sum_{k=1}^{K} (M_k + N_k - 2d_k) d_k.$$

So if

$$\sum_{k=1}^{K} (M_k + N_k - 2d_k) d_k < \sum_{j,k=1, j \neq k}^{K} d_k d_j, \tag{13}$$

then we would have more constraints than unknowns in the interference alignment condition (12), which we will argue cannot hold.

Let us consider the field \mathcal{F} defined over the field of complex numbers \mathbb{C} , consisting of all rational functions in the entries of the matrices $\{\bar{\mathbf{U}}_k\}_{k=1}^K$ and $\{\bar{\mathbf{V}}_k\}_{k=1}^K$. Note that the entries of the matrices $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_k\}_{k=1}^K$ form a transcendence basis for \mathcal{F} over \mathbb{C} . Thus, the transcendence degree of \mathcal{F} is $\sum_{k=1}^K (M_k + N_k - 2d_k) d_k$, which is equal to the number of entries in the matrices $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_k\}_{k=1}^K$.

Now, let us consider the matrices $\mathbf{H}_{kj}^{(2)}, \mathbf{H}_{kj}^{(3)}, \mathbf{H}_{kj}^{(4)}$ for all $k, j, k \neq j$ and define the matrix \mathbf{F}_{kj} :

$$\mathbf{F}_{kj}(\bar{\mathbf{U}}, \bar{\mathbf{V}}) \triangleq -\left(\bar{\mathbf{U}}_k^H \bar{\mathbf{H}}_{kj}^{(3)} + \bar{\mathbf{H}}_{kj}^{(2)} \bar{\mathbf{V}}_j + \bar{\mathbf{U}}_k^H \bar{\mathbf{H}}_{kj}^{(4)} \bar{\mathbf{V}}_j\right),\tag{14}$$

for all k, j with $k \neq j$. Note that \mathbf{F}_{kj} is a $d_k \times d_j$ matrix, with each entry being a quadratic polynomial function of the entries in the matrices $\bar{\mathbf{U}}_k$ and $\bar{\mathbf{V}}_k$. As a result, the entries of \mathbf{F}_{kj} belong to the field \mathcal{F} . Moreover, if (13) holds, then the number of quadratic polynomials given in the matrices $\{\mathbf{F}_{kj}\}_{k\neq j}$ is strictly larger than the transcendence degree of \mathcal{F} over \mathbb{C} . Hence, as we discussed in the algebraic preliminaries (Section III-A; see also [11, Chapter 8]), these quadratic polynomials in \mathcal{F} must be algebraically dependent. This implies that there exists a nonzero polynomial p which vanishes at the quadratic polynomials corresponding to the entries of the matrices $\{\mathbf{F}_{kj}\}_{k\neq j}$, i.e.,

$$p\left(\mathbf{F}_{12}(\bar{\mathbf{U}},\bar{\mathbf{V}}),\mathbf{F}_{13}(\bar{\mathbf{U}},\bar{\mathbf{V}}),\ldots,\mathbf{F}_{K(K-1)}(\bar{\mathbf{U}},\bar{\mathbf{V}})\right)=0,$$

for all $\{\bar{\mathbf{U}}_k,\ \bar{\mathbf{V}}_k\}_{k=1}^K$. Notice that the polynomial p is independent of the channel matrices $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k\neq j}^K$, even though it does depend on the matrices $\{\bar{\mathbf{H}}_{kj}^{(2)}, \bar{\mathbf{H}}_{kj}^{(3)}, \bar{\mathbf{H}}_{kj}^{(4)}\}_{k\neq j}^K$. When viewed as a polynomial of the matrix variable $\mathbf{X} := (\bar{\mathbf{H}}_{12}^{(1)}, \bar{\mathbf{H}}_{13}^{(1)}, \dots, \bar{\mathbf{H}}_{K(K-1)}^{(1)}), p(\cdot)$ can be expanded locally at $\bar{\mathbf{X}} := (\mathbf{F}_{12}(\bar{\mathbf{U}}, \bar{\mathbf{V}}), \mathbf{F}_{13}(\bar{\mathbf{U}}, \bar{\mathbf{V}}), \dots, \mathbf{F}_{K(K-1)}^K)$

using (8):

$$p\left(\bar{\mathbf{H}}_{12}^{(1)}, \bar{\mathbf{H}}_{13}^{(1)}, \dots, \bar{\mathbf{H}}_{K(K-1)}^{(1)}\right)$$

$$= p\left(\mathbf{F}_{12}(\bar{\mathbf{U}}, \bar{\mathbf{V}}), \mathbf{F}_{13}(\bar{\mathbf{U}}, \bar{\mathbf{V}}), \dots, \mathbf{F}_{K(K-1)}(\bar{\mathbf{U}}, \bar{\mathbf{V}})\right)$$

$$+ \sum_{k \neq j} \operatorname{Tr}\left((\bar{\mathbf{H}}_{kj}^{(1)} - \mathbf{F}_{kj}(\bar{\mathbf{U}}, \bar{\mathbf{V}}))\mathbf{Q}_{kj}(\bar{\mathbf{U}}, \bar{\mathbf{V}})\right),$$

for all $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_k\}_{k=1}^K$, where \mathbf{Q}_{kj} is some polynomial matrix of size $d_j \times d_k$. Combining the above two identities yields

$$p\left(\bar{\mathbf{H}}_{12}^{(1)}, \bar{\mathbf{H}}_{13}^{(1)}, \dots, \bar{\mathbf{H}}_{K(K-1)}^{(1)}\right)$$

$$= \sum_{k \neq j} \operatorname{Tr}\left(\left(\bar{\mathbf{H}}_{kj}^{(1)} - \mathbf{F}_{kj}(\bar{\mathbf{U}}, \bar{\mathbf{V}})\right) \mathbf{Q}_{kj}(\bar{\mathbf{U}}, \bar{\mathbf{V}})\right). \tag{15}$$

Notice that this equality holds for all choices of $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_k\}_{k=1}^K$. If the interference alignment condition (12) holds, then we have

$$\bar{\mathbf{H}}_{kj}^{(1)} - \mathbf{F}_{kj}(\bar{\mathbf{U}}, \bar{\mathbf{V}}) = 0$$
, for all k, j with $k \neq j$,

for some special choices of the matrices $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_k\}_{k=1}^K$. Substituting this condition into the right hand side of (15), we obtain

$$p\left(\bar{\mathbf{H}}_{12}^{(1)}, \bar{\mathbf{H}}_{13}^{(1)}, \dots, \bar{\mathbf{H}}_{K(K-1)}^{(1)}\right) = 0.$$
(16)

Notice that the polynomial p is independent of the channel matrices $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k\neq j}$. Under our channel model, the channel matrices $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k\neq j}$ are drawn from a continuous probability distribution. It follows that the condition (16) cannot hold unless p is identically zero, which contradicts the requirement $p \neq 0$.

Theorem 1 settles the conjecture of [2] in one direction, namely, the improperness of polynomial system (2) and (3) implies the infeasibility of interference alignment. From the proof of Theorem 1, it can be seen that the upper bound (6) holds for any choice of fixed channel matrices $\{\bar{\mathbf{H}}_{kj}^{(2)}, \bar{\mathbf{H}}_{kj}^{(3)}, \bar{\mathbf{H}}_{kj}^{(4)}\}_{k\neq j}$ as long as the channel matrices $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k\neq j}$ are generic.

Also, we remark that the proof technique for Theorem 1 can be used to bound the DoF for a single antenna parallel interference channel (e.g., the OFDM channel). In particular, consider a single input single output interference channel with M channel extensions, i.e., the channel matrices are diagonal and of the size $M \times M$. Assuming each user transmits one data stream ($d_k = 1$ for all k), we can check

that the properness of the interference alignment condition (2)-(3) is equivalent to $K+1 \le 2M$ (see [2, Theorem 1]). Using a completely identical proof, we can show that the properness condition $K+1 \le 2M$ is a necessary condition for the feasibility of interference alignment. This implies that for the single beam case the total DoF per channel extension is upper bounded by 2, regardless of the number of channel extensions. This DoF bound has also been proposed recently in [17].

C. The Converse Direction

In the remainder of this section, we consider the converse of Theorem 1. In particular, we show that the upper bound in Theorem 1 is tight for a special case where all users have the same DoF d and number of antennas is divisible by d. In this case, we have K(K-1) matrix equations in (12), each giving rise to d^2 scalar equations. For any subset of these matrix equations indexed by \mathcal{I} , with $\mathcal{I} \subseteq \mathcal{J}$, the number of corresponding scalar equations is equal to $d^2|\mathcal{I}|$, whereas the number of scalar variables involved in the equations indexed by \mathcal{I} is

$$\left(\sum_{k:(k,j)\in\mathcal{I}} (M_k - d) + \sum_{j:(k,j)\in\mathcal{I}} (N_j - d)\right) d.$$

The next result shows that the bound in Theorem 1 is tight if the polynomial system (12) defining interference alignment is proper, i.e., for each $\mathcal{I}\subseteq\mathcal{J}$, the number of variables involved in each set of equations indexed by \mathcal{I} is no less than $d^2|\mathcal{I}|$, the number of scalar equations. The proof of this result uses the Implicit Function Theorem which involves checking the Jacobian matrix of the polynomial map (14) is nonsingular at some channel realization $\{\bar{\mathbf{H}}_{kj}\}_{k\neq j}$. Notice that the feasibility of interference alignment condition (12) at a given channel realization $\{\bar{\mathbf{H}}_{kj}\}_{k\neq j}$ is equivalent to $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k\neq j}$ being contained in the image of the polynomial map (14) which is defined by $\{\bar{\mathbf{H}}_{kj}^{(2)}, \bar{\mathbf{H}}_{kj}^{(3)}, \bar{\mathbf{H}}_{kj}^{(4)}\}_{k\neq j}$. Fix a generic choice of $\{\bar{\mathbf{H}}_{kj}^{(2)}, \bar{\mathbf{H}}_{kj}^{(3)}, \bar{\mathbf{H}}_{kj}^{(4)}\}_{k\neq j}$ for which the Jacobian of the polynomial map (14) is nonsingular. The Implicit Function Theorem allows us to establish the existence of a locally invertible map from the space of channel submatrices $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k\neq j}\}_{k\neq j}$ to the space of beamforming matrices, and that the image of this polynomial map (14) is locally full-dimensional. Therefore, for all channel submatrices near the given channel realization $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k\neq j}$, the interference alignment condition (12) can be satisfied by some beamforming matrices. By Chevalley's Theorem from algebraic geometry [14] (see also the discussion at the end of Section III-A), the "local full-dimensionality" of the image of (14) implies that this image, which is a constructible set, must contain a nonempty Zariski open set. As a result, the whole image

of polynomial map (14) contains all generically generated channel sub-matrices $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k\neq j}$. Since the choice of channel submatrices $\{\bar{\mathbf{H}}_{kj}^{(2)}, \bar{\mathbf{H}}_{kj}^{(3)}, \bar{\mathbf{H}}_{kj}^{(4)}\}_{k\neq j}$ is also generic, this then establishes the feasibility of interference alignment for all generically generated channel matrices $\{\bar{\mathbf{H}}_{kj}\}_{k\neq j}$.

Theorem 2 Assume that all users have the same DoF $d_k = d$, where $1 \le d \le \min\{M_k, N_k\}$, $\forall k$. Furthermore, suppose that M_k and N_k are divisible by d for all k. Then interference alignment is achievable for generic channel coefficients if and only if for each subset \mathcal{I} of equations in (12), the number of variables involved in these equations is no less than the number of matrix equations times d^2 , or equivalently,

$$|\mathcal{I}|d \leq \sum_{k:(k,j)\in\mathcal{I}} (M_k - d) + \sum_{j:(k,j)\in\mathcal{I}} (N_j - d), \qquad \forall \, \mathcal{I} \text{ with } \mathcal{I} \subseteq \mathcal{J}.$$

$$(17)$$

Proof: First of all, the "only if" direction is a direct consequence of Theorem 1. We now focus on the "if" direction. Consider the polynomial map that we get by concatenating all maps in (14) for all $(k,j) \in \mathcal{J}$, i.e.,

$$\mathbf{F}_{12}(\bar{\mathbf{U}}, \bar{\mathbf{V}}) = -\left(\bar{\mathbf{U}}_{1}^{H} \bar{\mathbf{H}}_{12}^{(3)} + \bar{\mathbf{H}}_{12}^{(2)} \bar{\mathbf{V}}_{2} + \bar{\mathbf{U}}_{1}^{H} \bar{\mathbf{H}}_{12}^{(4)} \bar{\mathbf{V}}_{2}\right),$$

$$\mathbf{F}_{13}(\bar{\mathbf{U}}, \bar{\mathbf{V}}) = -\left(\bar{\mathbf{U}}_{1}^{H} \bar{\mathbf{H}}_{13}^{(3)} + \bar{\mathbf{H}}_{13}^{(2)} \bar{\mathbf{V}}_{3} + \bar{\mathbf{U}}_{1}^{H} \bar{\mathbf{H}}_{13}^{(4)} \bar{\mathbf{V}}_{3}\right),$$

$$\vdots$$
(18)

$$\mathbf{F}_{K(K-1)}(\bar{\mathbf{U}}, \bar{\mathbf{V}}) = -\left(\bar{\mathbf{U}}_K^H \bar{\mathbf{H}}_{K(K-1)}^{(3)} + \bar{\mathbf{H}}_{K(K-1)}^{(2)} \bar{\mathbf{V}}_{K-1} + \bar{\mathbf{U}}_K^H \bar{\mathbf{H}}_{K(K-1)}^{(4)} \bar{\mathbf{V}}_{K-1}\right),\,$$

which maps the variables $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_k\}_{k=1}^K$ to the $\{\mathbf{F}_{k,j}\}_{k\neq j}$ space. We will first show that for a specific set of channel matrices, the rank of the Jacobian of this polynomial map is $K(K-1)d^2$, equal to the number of equations. Hence, if we restrict the equations to a subset of variables of size $K(K-1)d^2$, the determinant of the Jacobian matrix of the polynomial map (18) does not vanish identically. This step will establish the existence of a locally invertible map from the space of beamforming matrices to the space of channel matrices. By Chevalley's Theorem (see [14, Chapter 2, 6.E.]), this image is a constructible subset under Zariski topology. This, plus the fact that the image is locally full-dimensional, implies that the interference alignment condition (12) is feasible for all generically chosen channel matrices. This then will show the "if" direction of Theorem 2.

To show the nonsingularity of the Jacobian matrix, we need to remove some redundant variables in $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_j\}_{k,j}$ (this occurs when there are more variables than equations), and then construct a specific set of channel matrices $\{\mathbf{H}_{kj}^{(2)}, \mathbf{H}_{kj}^{(3)}, \mathbf{H}_{kj}^{(4)}\}_{k\neq j}$ and a solution $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_j\}_{k,j}$ at which the Jacobian matrix of

(18) is nonsingular. Before providing a rigorous description for such a construction, we first consider a toy example with K = 3 users where $M_k = 3$, $N_k = 2$, $d_k = 1$, for k = 1, 2, 3. For this specific example, the assumption (17) is satisfied and the equations in (18) can be rewritten as

$$\begin{split} \mathbf{F}_{12}(\bar{\mathbf{U}},\bar{\mathbf{V}}) &= -\left(\bar{\mathbf{U}}_{1}^{H}\bar{\mathbf{H}}_{12}^{(3)} + \bar{\mathbf{H}}_{12}^{(2)}\bar{\mathbf{V}}_{2} + \bar{\mathbf{U}}_{1}^{H}\bar{\mathbf{H}}_{12}^{(4)}\bar{\mathbf{V}}_{2}\right), \\ \mathbf{F}_{13}(\bar{\mathbf{U}},\bar{\mathbf{V}}) &= -\left(\bar{\mathbf{U}}_{1}^{H}\bar{\mathbf{H}}_{13}^{(3)} + \bar{\mathbf{H}}_{13}^{(2)}\bar{\mathbf{V}}_{3} + \bar{\mathbf{U}}_{1}^{H}\bar{\mathbf{H}}_{13}^{(4)}\bar{\mathbf{V}}_{3}\right), \\ \mathbf{F}_{21}(\bar{\mathbf{U}},\bar{\mathbf{V}}) &= -\left(\bar{\mathbf{U}}_{2}^{H}\bar{\mathbf{H}}_{21}^{(3)} + \bar{\mathbf{H}}_{21}^{(2)}\bar{\mathbf{V}}_{1} + \bar{\mathbf{U}}_{2}^{H}\bar{\mathbf{H}}_{21}^{(4)}\bar{\mathbf{V}}_{1}\right), \\ \mathbf{F}_{23}(\bar{\mathbf{U}},\bar{\mathbf{V}}) &= -\left(\bar{\mathbf{U}}_{2}^{H}\bar{\mathbf{H}}_{23}^{(3)} + \bar{\mathbf{H}}_{23}^{(2)}\bar{\mathbf{V}}_{3} + \bar{\mathbf{U}}_{2}^{H}\bar{\mathbf{H}}_{23}^{(4)}\bar{\mathbf{V}}_{3}\right), \\ \mathbf{F}_{31}(\bar{\mathbf{U}},\bar{\mathbf{V}}) &= -\left(\bar{\mathbf{U}}_{3}^{H}\bar{\mathbf{H}}_{31}^{(3)} + \bar{\mathbf{H}}_{31}^{(2)}\bar{\mathbf{V}}_{1} + \bar{\mathbf{U}}_{3}^{H}\bar{\mathbf{H}}_{31}^{(4)}\bar{\mathbf{V}}_{1}\right), \\ \mathbf{F}_{32}(\bar{\mathbf{U}},\bar{\mathbf{V}}) &= -\left(\bar{\mathbf{U}}_{3}^{H}\bar{\mathbf{H}}_{32}^{(3)} + \bar{\mathbf{H}}_{32}^{(2)}\bar{\mathbf{V}}_{2} + \bar{\mathbf{U}}_{3}^{H}\bar{\mathbf{H}}_{32}^{(4)}\bar{\mathbf{V}}_{2}\right), \end{split}$$

where $\bar{\mathbf{V}}_k = [v_{k_1} \ v_{k_2}]^T \in \mathbb{C}^{2 \times 1}$, $\bar{\mathbf{U}}_k = [u_k] \in \mathbb{C}$, for k = 1, 2, 3, and $\bar{\mathbf{H}}_{kj}^{(2)} = [\bar{h}_{kj}^{(2),1} \ \bar{h}_{kj}^{(2),2}]^T \in \mathbb{C}^{2 \times 1}$, $\bar{\mathbf{H}}_{kj}^{(3)} = [\bar{h}_{kj}^{(3)}] \in \mathbb{C}$, for $k \neq j$. If we set $\bar{\mathbf{H}}_{kj}^{(4)} = 0$ for all channels, one can write the Jacobian of $[\mathbf{F}_{12} \ \mathbf{F}_{13} \ \mathbf{F}_{21} \ \mathbf{F}_{23} \ \mathbf{F}_{31} \ \mathbf{F}_{32}]$ with respect to the variables $[u_1 \ u_2 \ u_3 \ v_{1_1} \ v_{1_2} \ v_{2_1} \ v_{2_2} \ v_{3_1} \ v_{3_2}]$ as

$$\begin{bmatrix} -\bar{h}_{12}^{(3)} & -\bar{h}_{13}^{(3)} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\bar{h}_{21}^{(3)} & -\bar{h}_{23}^{(3)} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\bar{h}_{31}^{(3)} & -\bar{h}_{32}^{(3)} \\ 0 & 0 & -\bar{h}_{21}^{(2),1} & 0 & -\bar{h}_{31}^{(2),1} & 0 \\ 0 & 0 & -\bar{h}_{21}^{(2),2} & 0 & -\bar{h}_{31}^{(2),2} & 0 \\ -\bar{h}_{12}^{(2),1} & 0 & 0 & 0 & 0 & -\bar{h}_{32}^{(2),1} \\ -\bar{h}_{12}^{(2),2} & 0 & 0 & 0 & 0 & -\bar{h}_{32}^{(2),2} \\ 0 & -\bar{h}_{13}^{(2),1} & 0 & -\bar{h}_{23}^{(2),1} & 0 & 0 \\ 0 & -\bar{h}_{13}^{(2),2} & 0 & 0 & -\bar{h}_{23}^{(2),2} & 0 & 0 \end{bmatrix} .$$

One can easily observe that by removing the variables $\{v_{1_1},v_{2_1},v_{3_2}\}$ and setting

$$\bar{h}_{12}^{(3)} = \bar{h}_{23}^{(3)} = \bar{h}_{31}^{(3)} = \bar{h}_{13}^{(2),1} = \bar{h}_{21}^{(2),2} = \bar{h}_{32}^{(2),2} = 1,$$

$$\bar{h}_{13}^{(3)} = \bar{h}_{21}^{(3)} = \bar{h}_{32}^{(3)} = \bar{h}_{12}^{(2),2} = \bar{h}_{31}^{(2),2} = \bar{h}_{23}^{(2),1} = 0,$$

the Jacobian of the mapping (18) with respect to the remaining variables becomes

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which is clearly nonsingular since there exists exactly one nonzero element in each column/row.

Next we argue that the above construction procedure can be generalized to the case where M_k and N_k are divisible by d, provided that the assumption (17) is satisfied. The construction of these channel/beamforming matrices and the removal of redundant variables are outlined below. First, we set $\mathbf{H}_{kj}^{(4)} = \mathbf{0}$, for all $k \neq j$. Then we choose $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_j\}_{k,j}$ arbitrarily. It remains to specify $\{\bar{\mathbf{H}}_{kj}^{(3)}, \bar{\mathbf{H}}_{kj}^{(2)}\}_{k\neq j}$. We should do so to ensure that the corresponding Jacobian matrix of (18) at $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_j\}_{k,j}$ is nonsingular. Since M_k and N_k are divisible by d, we can partition our variables into blocks of size $d \times d$ and rewrite the mapping (18) as

$$\mathbf{F}_{kj}(\bar{\mathbf{U}}, \bar{\mathbf{V}}) = -\begin{bmatrix} \bar{\mathbf{U}}_{k_1}^H \, \bar{\mathbf{U}}_{k_2}^H \dots \bar{\mathbf{U}}_{k_{s_k}}^H \end{bmatrix} \begin{bmatrix} \bar{\mathbf{H}}_{kj}^{(3),1} \\ \bar{\mathbf{H}}_{kj}^{(3),2} \\ \vdots \\ \bar{\mathbf{H}}_{kj}^{(3),s_k} \end{bmatrix} - \begin{bmatrix} \bar{\mathbf{H}}_{kj}^{(2),1} \, \bar{\mathbf{H}}_{kj}^{(2),2} \dots \bar{\mathbf{H}}_{kj}^{(2),t_j} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{V}}_{j_1} \\ \bar{\mathbf{V}}_{j_2} \\ \vdots \\ \bar{\mathbf{V}}_{j_{t_j}} \end{bmatrix}, \quad \forall \, k \neq j,$$
(19)

where $s_k = \frac{M_k}{d} - 1$, $t_j = \frac{N_j}{d} - 1$, and $\bar{\mathbf{U}}_{k_i}$, $\bar{\mathbf{V}}_{j_\ell}$, $\bar{\mathbf{H}}_{kj}^{(2),i}$, $\bar{\mathbf{H}}_{kj}^{(3),\ell} \in \mathbb{C}^{d\times d}$. Consider a bipartite graph G where the vertices are partitioned into two sets \mathcal{X} and \mathcal{Y} . Each block of variables will correspond to a node in \mathcal{X} , while each matrix equation in (19) will correspond to a node in \mathcal{Y} . We draw an edge between a node $x \in \mathcal{X}$ and a node $y \in \mathcal{Y}$ if the block of variables corresponding to node x appears in the equation corresponding to node y. When viewed on the bipartite graph G, the assumption (17) simply says that for any given set of nodes $\mathcal{S} \subseteq \mathcal{Y}$, the cardinality of the neighbors of \mathcal{S} in \mathcal{X} is no smaller than the cardinality of \mathcal{S} . This condition is precisely what is required to ensure the existence of a complete matching in G covering all nodes in \mathcal{Y} (Hall's theorem, see [15, Theorem 3.1.11]). Now consider a fixed complete matching in G. Let $G \subseteq \mathcal{X}$ be the set of vertices that are not matched to a node in \mathcal{Y} . Then, we can set to zero all the blocks of the variables corresponding to the vertices in G, i.e., we can remove them from our equations. Now we choose the rest of the channel matrices so that the determinant of the

Jacobian with respect to the remaining variables is nonzero. To this end, we set $\bar{\mathbf{H}}_{kj}^{(3),p}=0$ if the node for $\bar{\mathbf{U}}_{k_p}$ is not matched to the node in \mathcal{Y} corresponding to the equation \mathbf{F}_{kj} . Similarly, we set $\bar{\mathbf{H}}_{kj}^{(2),q}=0$ if $\bar{\mathbf{V}}_{j_q}$ is not matched to \mathbf{F}_{kj} . Moreover, we set all the remaining channel sub-matrices to the $d\times d$ identity matrix. Since this construction is based on a complete matching, it is not hard to see that the Jacobian for the whole system is a block permutation matrix, with nonzero blocks equal to the negative $d\times d$ identity matrix. Hence the determinant of the Jacobian matrix is equal to the product of the determinant of all nonzero blocks (up to sign), which is clearly nonzero in our case. This completes the description of the procedure to remove potential redundant variables, as well as the procedure to construct all the channel matrices $\{\mathbf{H}_{kj}^{(2)},\mathbf{H}_{kj}^{(3)},\mathbf{H}_{kj}^{(4)}\}_{k\neq j}$ and the beamforming solution $\{\bar{\mathbf{U}}_k,\bar{\mathbf{V}}_j\}_{k,j}$. The Jacobian matrix of (18) is nonsingular at this constructed channel realization and beamforming solution. Figure 1 illustrates the construction of graph G and a complete matching (in solid lines) for the aforementioned toy example.

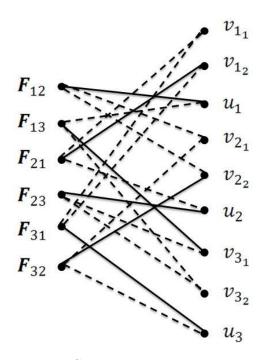


Fig. 1: The bipartite graph G and a complete matching for the toy example

To complete the proof, we fix a generic choice of $\{\bar{\mathbf{H}}_{kj}^{(2)}, \bar{\mathbf{H}}_{kj}^{(3)}, \bar{\mathbf{H}}_{kj}^{(4)}\}_{k\neq j}$ for which the Jacobian of (18) is nonsingular. Let n be the total number of remaining scalar variables in $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_j\}_{k,j}$ after removing the redundant variables. Notice that n is the same as the number of scalar equations, i.e., $n=d^2K(K-1)$. Let $R_1=\mathbb{C}[h_1,h_2,\ldots,h_n]$ and $R_2=\mathbb{C}[\bar{u}_1,\bar{u}_2,\ldots,\bar{u}_m,\bar{v}_1,\ldots,\bar{v}_{n-m}]$ be two polynomial rings where \bar{u}_i 's and \bar{v}_j 's are the entries of the matrices $\{\bar{\mathbf{U}}_k\}_{k=1}^K$ and $\{\bar{\mathbf{V}}_k\}_{k=1}^K$ (after removing the redundant variables),

and h_1, h_2, \ldots, h_n are the entries of the matrices $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k \neq j}$. Consider $\{f_i\}_{i=1}^n$ (the components of \mathbf{F}_{kj} 's in (18)) as the functions of \bar{u} 's and \bar{v} 's, i.e., f_i 's are polynomials in R_2 . These polynomials define a map a_{ϕ} which maps a point $c=(c_1,c_2,\ldots,c_n)$ to $(f_1(c),f_2(c),\ldots,f_n(c))$. According to the Chevalley Theorem (see [14, Chapter 2, 6.E.]), the image of this map is a Zariski constructible subset of $A_{\mathbb{C},1}^n$, where $A_{\mathbb{C},1}^n$ is the corresponding affine space of R_1 . Since the Jacobian of the set $\{f_1,f_2,\ldots,f_n\}$ with respect to the variables $\{\bar{u}_1,\bar{u}_2,\ldots,\bar{u}_m,\bar{v}_1,\ldots,\bar{v}_{n-m}\}$ is nonsingular generically for all channel realizations, it follows from the Implicit Function Theorem that the dimension of the image of a_{ϕ} is n. Note that the image of a_{ϕ} is a Zariski constructible subset of $A_{\mathbb{C},1}^n$ (see Chevalley Theorem [14], Ch. 2, 6.E.) and it has full dimension. Hence, the image contains a Zariski open subset of $A_{\mathbb{C},1}^n$ (see the discussion in section III-A). Let \mathcal{U} be that Zariski open subset of $A_{\mathbb{C},1}^n$ in the image. Since \mathcal{U} is in the image of the map a_{ϕ} , there exists a solution for interference alignment equations for any choice of $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k \neq j}$ in \mathcal{U} , which implies that interference alignment is feasible for generic choice of $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k \neq j}$. Since the choice of channel matrices $\{\bar{\mathbf{H}}_{kj}^{(2)}, \bar{\mathbf{H}}_{kj}^{(3)}, \bar{\mathbf{H}}_{kj}^{(4)}\}_{k \neq j}$ is also generic, this completes the proof of the "if" direction.

Notice that the condition (17) is equivalent to the properness of the polynomial system (12) defining interference alignment. For symmetric systems with $M_k = M$, $N_k = N$ for all k, this condition simplifies to $M+N \geq d(K+1)$ (see [2, Theorem 1]). Thus, each user can achieve d degrees of freedom as long as $M+N \geq d(K+1)$ and that d divides both M and N. In a concurrent work, the authors of [8] obtained a similar result under a different set of assumptions. More specifically, they considered the symmetric case with $M_k = N_k = M$, $d_k = d$ for all k, and proved that the feasibility of interference alignment in this case is equivalent to $2M \geq d(K+1)$. This result and Theorem 2 are complementary to each other. In particular, Theorem 2 is applicable to non-symmetric systems, but does require an extra condition about the divisibility of the number of antennas by the number of data streams. When K is odd and K+1 and K+1 and K+1 and K+1 and K+1 are the case where K+1 is even and K+1 and K+1 and K+1 are the covered by both Theorem 2 and the result in [8]. However, for the case where K+1 is even and K+1 and K+1 are the covered by both Theorem 2 is no longer applicable, whereas [8] shows that the interference alignment is achievable.

A few other remarks are in order.

1) Reference [2] also considered the case $d_k = 1$ and used the Bernshtein's theorem to numerically compute the number of solutions, and therefore prove the feasibility, for the resulting polynomial system (2)-(3) when the number of antennas are small. In contrast, Theorem 2 shows the feasibility of single beam interference alignment for all values of M_k , N_k as long as the system is proper.

- 2) As shown in Theorem 1, the condition (5) is necessary. For example, the system K=2, M=N=3, d=2 satisfies the inequality (6). However, the system of equations (2)-(3) is infeasible for generic choice of channel coefficients. This further shows that the *properness* property in [2] does not imply feasibility in general, a fact that was first pointed out in [2, example 17].
- 3) Theorem 2 does not contradict the NP-hardness result of [12]. Given a set of channel matrices, checking the feasibility of the interference alignment conditions (2)-(3) when $M_k \geq 3$ and $N_k \geq 3$, is NP-hard. It is true that, under the setting of Theorem 2, the interference alignment fails only for a measure zero set of channels. However, for systems not satisfying the conditions of Theorem 2, checking the feasibility of interference alignment can be hard. Moreover, the results of [12] imply that, even if a given tuple of DoF is known to be achievable via interference alignment, finding the actual linear transmit/receive beamformers to achieve it is still a NP-hard problem when the number of users is large.
- 4) The condition (17) implies the condition (5) if the number of antennas at each transceiver is divisible by d. In fact, by choosing $\mathcal{I} = \{(k, j)\}$, condition (17) implies that $d \leq M_k + N_j 2d$ and hence the condition (5) is satisfied.
- 5) Theorem 2 assumes that both M_k and N_k are divisible by d. This condition can be weakened for a symmetric system where $M_k = M$, $N_k = N$, $d_k = d$, for all k. In particular, assume that only M (not N) is divisible by d and $M, N \geq d$. If the properness condition $(K+1)d \leq M+N$ holds, then we can construct a reduced MIMO interference channel with N' = M d(K+1) receive antennas for each user, where M+N'=d(K+1) and M,N' are divisible by d. By Theorem 2, the interference alignment condition for the reduced interference channel is feasible and therefore, so is the interference alignment condition for the original channel since the latter has more antennas. This shows that if M is divisible by d and $M,N \geq d$, then the interference alignment system (2)–(3) is feasible for generic choice of channel coefficients if and only if $(K+1)d \leq M+N$. By symmetry, the same conclusion holds for the case where N is divisible by d.

IV. SIMULATION RESULTS

In this section, we use the theoretical DoF upper bounds to benchmark an existing algorithm for sumrate maximization. We generate MIMO interference channels using the standard Rayleigh fading model. The numerical experiments are averaged over 100 Monte Carlo runs.

We consider a MIMO interference channel where each transmitter/receiver is equipped with 3 antennas. For different number of users in the system, we maximize the sum-rate using the WMMSE algorithm [13]

at increasingly high SNRs. We estimate the slope of the sum-rate versus SNR and use it to approximate the achievable total DoF. We then compare it with the value of theoretical upper bound given by the conditions in Theorem 1. The maximum gap of the two curves is one, but it is not clear if the gap is due to the weakness of the WMMSE algorithm or the DoF upper bound.

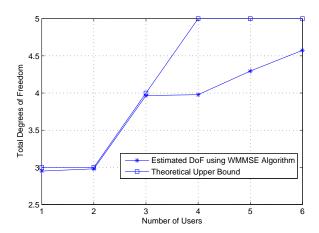


Fig. 2: Achievable DoF and theoretical upper bound

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