# Chebyshev Polynomials in Distributed Consensus Applications 

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#### Abstract

In this paper we analyze the use of Chebyshev polynomials in distributed consensus applications. We study the properties of these polynomials to propose a distributed algorithm that reaches the consensus in a fast way. The algorithm is expressed in the form of a linear iteration and, at each step, the agents only require to transmit their current state to their neighbors. The difference with respect to previous approaches is that the update rule used by the network is based on the second order difference equation that describes the Chebyshev polynomials of first kind. As a consequence, we show that our algorithm achieves the consensus using far less iterations than other approaches. We characterize the main properties of the algorithm for both, fixed and switching communication topologies. The main contribution of the paper is the study of the properties of the Chebyshev polynomials in distributed consensus applications, proposing an algorithm that increases the convergence rate with respect to existing approaches. Theoretical results, as well as experiments with synthetic data, show the benefits using our algorithm.


Index Terms - Chebyshev polynomials, distributed consensus, convergence rate.

## I. Introduction

Chebyshev polynomials [1] are a powerful mathematical tool that has proven to be very helpful in many different fields of science. To name a few, they are used in the modeling of complex chemical reaction systems [2], the simulation satellite orbits around the Earth. [3], the numerical solution of diffusion-reactions equations with severely stiff reaction terms [4] or the recognition

[^0]of patterns in images using Support Vector Machine classification [5]. In this paper we study the use of these polynomials in the field of distributed consensus applications.

In sensor networks and multi-agent systems, the consensus problem consists of making the whole group of agents to reach a common estimation about a specific measurement. Within the control community many different distributed solutions have been proposed in the past years [6][12]. It is well known that the number of messages required to achieve the consensus depends on the network connectivity. Interesting analysis of convergence have been done in [13], [14], where consensus methods have been shown to behave in a similar manner as heat differential equations and electrical resistive networks respectively. Other interesting approaches analyze the convergence with stochastic link failures [15], switching random networks [16] and asynchronous consensus [17]. When the size of the network is large, communications between different pairs of agents become more difficult due to distance and power constraints. Under these circumstances the number of iterations required to reach the consensus is also large. For that reason a lot of research has been devoted to mitigate this problem, providing a variety of solutions that reduce the time to achieve the consensus.

Some works present continuous-time solutions to achieve consensus in finite time using non linear methods [18]-[20]. The use of numerical integrators affects the number of iterations in these approaches because they depend on the number of steps taken by the method. The approach in [21] proposes a link scheduling that reaches the consensus in a finite number of steps. However, in wireless networks, communications of direct neighbors depend on the distance that separates them and therefore, there might be situations in which this method cannot be used because not all the links are feasible. Other approaches speed up convergence by sending additional information in the messages. Following this idea a multi-hop protocol is presented in [22] and second order neighbors are considered in [23]. Unfortunately, the amount of additional information in both cases depends on the topology. This implies that there might be situations in which large messages must be sent.

The design of the adjacency matrix has been the focus of several works. For instance, the work in [24] provides the optimal weights for the matrix, as well as good approximations that do not require any global knowledge about the network topology. Different algorithms to solve the optimization problem of finding the best matrix are proposed in [25]. Another optimization method is proposed in [26], in this case considering a shift-registers method with a fixed gain.

These approaches indeed improve the convergence speed, nevertheless, they can still be combined with additional techniques in order to accelerate even more the consensus.

The distributed evaluation of polynomials, as well as the use of previous information in the algorithm, have turned out to be easy ways to speed up the consensus, also keeping the good properties found in standard methods. The minimal polynomial of the adjacency matrix is used in [27] and [28]. Once this polynomial is known, the network can achieve the consensus in a finite number of communication rounds. Unfortunately, when the topology of the network is time-varying this algorithm does not work and for large networks the computation of the polynomial can be inefficient. The approach in [29] uses a polynomial of fixed degree with coefficients computed assuming the network is known. A consensus predictor is considered in [30]. Different second order recurrences with fixed gains are used in [31], [32]. Finally, the distributed evaluation of Chebyshev polynomials for consensus has been proposed in [33], [34]. Although the convergence of some of these algorithms under switching topologies has been demonstrated in practice, to the authors' knowledge there is still a gap in the theoretical analysis of the behavior of polynomial evaluation in this case.

In this paper we try to fill this gap, extending the results presented in [33] about Chebyshev polynomials and their use in consensus applications. In [33] we introduced the algorithm, based on a second order difference equation, and we studied its convergence to consensus for stochastic symmetric matrices in fixed graphs. In this paper we extend the convergence result, considering non-symmetric matrices that can have complex eigenvalues. We also provide a complete study of the parameters that make the algorithm achieve the optimal convergence rate and we give bounds on the selection of these parameters to achieve a faster convergence than using the powers of the weighted adjacency matrix. Regarding the case of switching communication topologies, we are able to theoretically show that there always exist parameters that make the proposed algorithm converge to the consensus. Experiments with synthetic data show the benefits of using our algorithm compared to other methods.

The structure of the paper is the following: In section II we introduce some background about the Chebyshev polynomials and distributed consensus. In section III we present the new distributed consensus algorithm using Chebyshev polynomials. In sections IV and V we study the properties of the algorithm with fixed and switching communication topologies respectively. In section VI we analyze the behavior of the algorithm in a simulated setup. Finally in section

VII the conclusions of the work are presented. In order to simplify the reading of the manuscript we have moved to an appendix some of the proofs of the theoretical results in sections III and IV. We have left in the text only the proofs that contain convenient information to follow the analysis.

## II. Background on Chebyshev Polynomials and Distributed Consensus

In this paper we consider Chebyshev polynomials of the first kind [1]. We denote the Chebyshev polynomial of degree $n$ by $T_{n}(x)$. These polynomials satisfy

$$
\begin{equation*}
T_{n}(x)=\cos (n \arccos x), \text { for all } x \in[-1,1], \tag{1}
\end{equation*}
$$

and $\left|T_{n}(x)\right|>1$ when $|x|>1$, for all $n \in \mathbb{N}$. A more general way to define these polynomials in the real domain is using a second order recurrence,

$$
\begin{align*}
& T_{0}(x)=1, T_{1}(x)=x  \tag{2}\\
& T_{n}(x)=2 x T_{n-1}(x)-T_{n-2}(x), \quad n \geq 2
\end{align*}
$$

By the theory of difference equations [35], the direct expression of (2) is determined by the roots $\tau_{1}$ and $\tau_{2}$ of the characteristic equation,

$$
\begin{equation*}
T_{n}(x)=\frac{1}{2}\left(\tau_{1}(x)^{n}+\tau_{2}(x)^{n}\right), \tag{3}
\end{equation*}
$$

where $\tau_{1}(x)=x-\sqrt{x^{2}-1}$ and $\tau_{2}(x)=x+\sqrt{x^{2}-1}=1 / \tau_{1}(x)$. In the paper we take

$$
\tau(x)= \begin{cases}x-\sqrt{x^{2}-1}, & \text { if } x \geq 0  \tag{4}\\ x+\sqrt{x^{2}-1}, & \text { if } x<0\end{cases}
$$

so that $|\tau(x)|<1$ and $|\tau(x)|^{-1}>1$ for all $|x|>1$, and therefore,

$$
\begin{equation*}
T_{n}(x)=\frac{1}{2}\left(\tau(x)^{n}+\tau(x)^{-n}\right)=\frac{1}{2} \tau(x)^{-n}\left(1+\tau(x)^{2 n}\right) . \tag{5}
\end{equation*}
$$

It is clear that if $|x|>1$, then $T_{n}(x)$ goes to infinity as $n$ grows. If $|x|<1$, then $\tau(x)$ is a complex number with $|\tau(x)|=1$ and $\left|T_{n}(x)\right| \leq 1, \forall n$, as stated in eq. (1).

For the analysis in the paper, it is also convenient to describe the behavior of Chebyshev polynomials evaluated in complex numbers. For any $z \in \mathbb{C}$, Chebyshev polynomials, $T_{n}(z)$, on the complex plane can also be expressed by (5) where $\tau(z)$ is defined now by

$$
\tau(z)= \begin{cases}z-\sqrt{z^{2}-1}, & \text { if }\left|z-\sqrt{z^{2}-1}\right|<1  \tag{6}\\ z+\sqrt{z^{2}-1}, & \text { otherwise }\end{cases}
$$

and again $|\tau(z)| \leq 1$ and $|\tau(z)|^{-1} \geq 1$ for all $z$. However, note that Chebyshev polynomials evaluated in a complex number, $T_{n}(z)$, go always to infinity as $n$ grows.

Consider now a set of $N$ agents, $\mathcal{V}=\{1, \ldots, N\}$, with limited communication capabilities. A distributed algorithm achieves consensus if, starting with initial conditions $x_{i}(0) \in \mathbb{R}$, and using only local interactions between agents, $x_{i}(n)=x_{j}(n), \forall i, j \in \mathcal{V}$, as $n \rightarrow \infty$. The interactions between the agents are modeled using an undirected graph $\mathcal{G}=\{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ describes the communications between pairs of agents. In this way, agents $i$ and $j$ can communicate if and only if $(i, j) \in \mathcal{E}$. The neighbors of one agent $i \in \mathcal{V}$ are the subset of agents that can directly communicate with it; i.e., $\mathcal{N}_{i}=\{j \in \mathcal{V} \mid(i, j) \in \mathcal{E}\}$. Initially, let us assume that the communication graph is fixed and connected.

The discrete time distributed consensus algorithm based on the weighted adjacency matrix associated to the communication graph [6] is

$$
\begin{equation*}
x_{i}(n)=a_{i i} x_{i}(n-1)+\sum_{j \in \mathcal{N}_{i}} a_{i j} x_{j}(n-1), \tag{7}
\end{equation*}
$$

with $x_{i}(0)=x_{i}$. The algorithm can also be expressed in vectorial form as

$$
\begin{equation*}
\mathbf{x}(n)=\mathbf{A} \mathbf{x}(n-1), \tag{8}
\end{equation*}
$$

where $\mathbf{x}(n)=\left(x_{1}(n), \ldots, x_{N}(n)\right)^{T}$ and $\mathbf{A}=\left[a_{i j}\right] \in \mathbb{R}^{N \times N}$, is the weighted matrix.
Assumption 2.1 (Stochastic Weights): A is row stochastic and compatible with the underlying graph, $\mathcal{G}$, i.e., it is such that $a_{i i} \neq 0, a_{i j} \neq 0$ only if $(i, j) \in \mathcal{E}$ and $\mathbf{A 1}=\mathbf{1}$.

Since the communication graph is connected, by Assumption 2.1, A has one eigenvalue $\lambda_{1}=1$ with associated right eigenvector 1 and algebraic multiplicity equal to one. The rest of the eigenvalues, real or complex, satisfy $\left|\lambda_{i}\right|<1, i=2, \ldots, N$. Without loss of generality, let us suppose that all the eigenvalues are simple. We denote by $\lambda_{2}$ the second largest and $\lambda_{N}$ the smallest real eigenvalues and we assume that $\max \left\{\left|\lambda_{2}\right|,\left|\lambda_{N}\right|\right\}>\left|\lambda_{i}\right|, i=3, \ldots, N-1$.

Any initial conditions $\mathbf{x}(0)$ can be expressed as a sum of eigenvectors of $\mathbf{A}$,

$$
\mathbf{x}(0)=\mathbf{v}_{1}+\ldots+\mathbf{v}_{N}
$$

where $\mathbf{v}_{i}$ is a right eigenvector associated to the eigenvalue $\lambda_{i}$. Specifically, $\mathbf{v}_{1}$ will be of the form $\left(\mathbf{w}_{1}^{T} \mathbf{x}(0) / \mathbf{w}_{1}^{T} \mathbf{1}\right) \mathbf{1}$, with $\mathbf{w}_{1}$ a left eigenvector of $\mathbf{A}$ associated to $\lambda_{1}$. It is clear that

$$
\mathbf{x}(n)=\mathbf{A}^{n} \mathbf{x}(0)=\mathbf{v}_{1}+\lambda_{2}^{n} \mathbf{v}_{2}+\ldots+\lambda_{N}^{n} \mathbf{v}_{N},
$$

and since $\left|\lambda_{i}\right|<1, i \neq 1$, the consensus is asymptotically reached by all the agents in the network, i.e., $\lim _{n \rightarrow \infty} \mathbf{x}(n)=\mathbf{v}_{1}=\left(\mathbf{w}_{1}^{T} \mathbf{x}(0) / \mathbf{w}_{1}^{T} \mathbf{1}\right) \mathbf{1}$. The asymptotic convergence implies that the exact consensus value will not be achieved in a finite number of iterations. In practice, the consensus is said to be achieved when $\left|x_{i}(n)-x_{j}(n)\right|<$ tol for all $i$ and $j$, and a prefixed error tolerance tol. The convergence speed of (8) depends on $\max \left(\left|\lambda_{2}\right|,\left|\lambda_{N}\right|\right)$. When the size of the network is large or the number of links is small this value is usually close to one, which means that the algorithm requires many iterations before obtaining a good approximation of the final solution.

When the communication topology changes with the time, $\mathcal{G}(n)=\{\mathcal{V}, \mathcal{E}(n)\}$, eq. (8) becomes $\mathbf{x}(n)=\mathbf{A}(n) \mathbf{x}(n-1)$, where the different weight matrices are defined according to their respective underlying communication graphs. If the different weight matrices satisfy Assumption 2.1, and the sequence of matrices is not degenerated, the algorithm is still proved to achieve consensus. We refer the reader to [6] for further information about this case.

## III. Consensus algorithm using Chebyshev polynomials

The distributed evaluation of polynomials provides an easy way to speed up the consensus, keeping the good properties found in standard methods. The main idea consists in designing a distributed linear iteration such that the execution of a fixed number of $n$ steps is equivalent to the evaluation of some polynomial, $P_{n}(x)$, in the fixed matrix $\mathbf{A}$ [27], [29]. The polynomial must satisfy that $P_{n}(1)=1$ and $\left|P_{n}(x)\right|<1$ if $|x|<1$. In this way, successive evaluations of the polynomial in $\mathbf{A}$ will lead to the consensus. The choice of the polynomial determine the convergence speed of the algorithm, given by $\max _{\lambda_{i}}\left|P_{n}\left(\lambda_{i}\right)\right|$, with $\lambda_{i}$ the eigenvalues of $\mathbf{A}$.

Two reasons motivate the choice of Chebyshev polynomials for the consensus problem:

- By using the recurrent definition (2), instead of considering a polynomial of fixed degree we can evaluate Chebyshev polynomials of higher and higher degree as successive iterations of the algorithm are executed.
- Chebyshev polynomials have the mini-max property [1]. This property says that, among all the monic polynomials of degree $n$, the polynomial $2^{1-n} T_{n}(x)$ is the one that minimizes the uniform norm on the interval $[-1,1]$. This property is indeed quite convenient for our purposes. If the matrix $\mathbf{A}$ is unknown, using the Chebyshev polynomials we are minimizing $\max _{\lambda \in[-1,1]} P_{n}(\lambda)$, therefore, getting high chances to obtain a good convergence rate.

However, the monic version of the Chebyshev polynomials does not satisfy $2^{1-n} T_{n}(1)=1$. In order to keep this property we perform a linear transformation of $T_{n}(x)$, using two real coefficients $\lambda_{m}, \lambda_{M}$, with $1>\lambda_{M}>\lambda_{m}>-1$, bringing the interval $\left[\lambda_{m}, \lambda_{M}\right]$ to $[-1,1]$. In this way, we define the polynomial

$$
\begin{equation*}
P_{n}(x)=\frac{T_{n}(c x-d)}{T_{n}(c-d)}, \text { with } c=\frac{2}{\lambda_{M}-\lambda_{m}}, \quad d=\frac{\lambda_{M}+\lambda_{m}}{\lambda_{M}-\lambda_{m}}, \tag{9}
\end{equation*}
$$

which, for all $n$, has the following properties:

- if $x \in\left[\lambda_{m}, \lambda_{M}\right]$, then $c x-d \in[-1,1]$
- $P_{n}(1)=1$ and $P_{n}\left(\lambda_{M}+\lambda_{m}-1\right)=(-1)^{n}$
- $\left|P_{n}(x)\right|<1$ for all $x \in\left(\lambda_{M}+\lambda_{m}-1,1\right)$ and $\left|P_{n}(x)\right| \geq 1$ otherwise.

The polynomial defined in (9) satisfies the recurrence

$$
\begin{equation*}
P_{n}(x)=2 \frac{T_{n-1}(c-d)}{T_{n}(c-d)}(c x-d) P_{n-1}(x)-\frac{T_{n-2}(c-d)}{T_{n}(c-d)} P_{n-2}(x) \tag{10}
\end{equation*}
$$

and the consensus rule $\mathbf{x}(n)=P_{n}(\mathbf{A}) \mathbf{x}(0)$ is defined by

$$
\begin{align*}
\mathbf{x}(1) & =P_{1}(\mathbf{A}) \mathbf{x}(0)=\frac{1}{T_{1}(c-d)}(c \mathbf{A}-d \mathbf{I}) \mathbf{x}(0), \\
\mathbf{x}(n) & =P_{n}(\mathbf{A}) \mathbf{x}(0)=\left(2 \frac{T_{n-1}(c-d)}{T_{n}(c-d)}(c \mathbf{A}-d \mathbf{I}) P_{n-1}(\mathbf{A})-\frac{T_{n-2}(c-d)}{T_{n}(c-d)} P_{n-2}(\mathbf{A})\right) \mathbf{x}(0)  \tag{11}\\
& =2 \frac{T_{n-1}(c-d)}{T_{n}(c-d)}(c \mathbf{A}-d \mathbf{I}) \mathbf{x}(n-1)-\frac{T_{n-2}(c-d)}{T_{n}(c-d)} \mathbf{x}(n-2), n \geq 2,
\end{align*}
$$

with I the identity matrix of dimension $N$. Notice that this consensus rule is well designed to be executed in a distributed fashion.

When the topology of the network changes, the recurrent evaluation of Chebyshev polynomials (11) can still be used. The time-varying version of the algorithm is equivalent to (11) replacing the constant weight matrix $\mathbf{A}$ by the weight matrix at each step $\mathbf{A}(n)$. Although this is no longer equivalent to the distributed evaluation of a Chebyshev polynomial, a theoretical analysis about its convergence properties is still possible. Algorithm 1 shows a possible implementation of the algorithm. In the rest of the paper we analyze, both in theory and practice, the main properties of this algorithm for fixed and switching communication topologies.

```
Algorithm 1 Consensus algorithm using Chebyshev polynomials - agent \(i\)
Require: \(x_{i}(0)\), MaxIt \(\in \mathbb{N}, \lambda_{m}, \lambda_{M}\),
    - Initialization
    \(c=2 /\left(\lambda_{M}-\lambda_{m}\right) ; \quad d=\left(\lambda_{M}+\lambda_{m}\right) /\left(\lambda_{M}-\lambda_{m}\right) ;\)
    3: \(T(0)=1 ; \quad T(1)=c-d\);
    4: - First Communication Round
\[
x_{i}(1)=\frac{1}{T(1)}\left(c \sum_{j \in \mathcal{N}_{i}(n)} a_{i j} x_{j}(0)+\left(c a_{i i}-d\right) x_{i}(0)\right) ;
\]
for \(n=2, \ldots\), MaxIt do
\[
T(n)=2(c-d) T(n-1)-T(n-2) ;
\]
- Communication Between Neighbors
\[
x_{i}(n)=2 \frac{T(n-1)}{T(n)}\left(c \sum_{j \in \mathcal{N}_{i}(n)} a_{i j} x_{j}(n-1)+\left(c a_{i i}-d\right) x_{i}(n-1)\right)-\frac{T(n-2)}{T(n)} x_{i}(n-2) ;
\]
end for
```


## IV. Analysis with a Fixed Communication Topology

In this section we analyze the main properties of the proposed algorithm when the network topology is fixed. In particular we first study the convergence conditions of the algorithm. Next, we find the parameters that maximize the convergence speed. Finally, we give bounds on the selection of these parameters to satisfy that our algorithm achieves the consensus faster than (8).

Theorem 4.1 (Convergence of the algorithm): Let $\mathbf{A}$ be diagonalizable, fulfilling Assumption 2.1, and parameters $\lambda_{m}$ and $\lambda_{M}$ such that $1>\lambda_{M}>\lambda_{m}>-1$. If the minimum real eigenvalue of $\mathbf{A}$ satisfies $\lambda_{N}>\lambda_{m}+\lambda_{M}-1$ and the complex eigenvalues, $\lambda_{z}$, of $\mathbf{A}$ satisfy $\left|\tau\left(c \lambda_{z}-d\right)\right|>$ $\tau(c-d)$, then the recurrence in eq. (11) converges to the consensus state, $\lim _{n \rightarrow \infty} \mathbf{x}(n)=$ $\mathbf{w}_{1}^{T} \mathbf{x}(0) \mathbf{1} / \mathbf{w}_{1}^{T} \mathbf{1}$. Besides, the convergence rate is given by

$$
\begin{equation*}
\max _{\lambda_{i} \neq 1} \frac{\left|T_{n}\left(c \lambda_{i}-d\right)\right|}{T_{n}(c-d)} \tag{12}
\end{equation*}
$$

Proof. See the Appendix.
Note that the conditions in Theorem 4.1 are easy to fulfill without the necessity of knowing the eigenvalues of the matrix $\mathbf{A}$. For the real eigenvalues, any symmetric selection of the parameters, i.e., $-\lambda_{m}=\lambda_{M}, 0<\lambda_{M}<1$, satisfies the condition in Theorem 4.1. The condition on the complex eigenvalues has some geometrical meaning [1]. Imposing that $\left|\tau\left(c \lambda_{z}-d\right)\right|>\tau(c-d)$ is equivalent to require that $\lambda_{z}$ is inside an ellipse in the complex plane centered at $(d / c, 0)$, or matrix $\mathbf{A}$ has only real eigenvalues.


Fig. 1. Ellipse where all the eigenvalues must be contained in order to achieve the consensus. In this particular example we have chosen $\lambda_{M}=0.9$ and $\lambda_{m}=-0.5$. Note that when the imaginary part of the eigenvalues is zero convergence is achieved if $\lambda_{M}+\lambda_{m}-1>\lambda>1$ as stated in Theorem4.1
equivalently $\left(\left(\lambda_{M}+\lambda_{m}\right) / 2,0\right)$, and with semi-axis $e_{1}=(c-d) / c$ and $e_{2}=\left(\sqrt{(c-d)^{2}-1}\right) / c$ (see Fig 1). In practice, any parameters that ensure convergence for the real eigenvalues also ensure convergence for the complex ones. We have observed that if $\mathbf{A}$ is defined using well known distributed methods [24], the complex eigenvalues, when there are any of them, have always a very small modulus. For that reason, in the rest of the section we will assume that the

Next, we are interested in knowing the optimal selection of $\lambda_{m}$ and $\lambda_{M}$ to maximize the convergence speed. From Theorem 4.1 we know that the convergence rate is given by the factor

$$
\begin{equation*}
\max _{\lambda_{i} \neq 1} \frac{\left|T_{n}\left(c \lambda_{i}-d\right)\right|}{T_{n}(c-d)}=\max \left\{\frac{\left|T_{n}\left(c \lambda_{N}-d\right)\right|}{T_{n}(c-d)}, \frac{\left|T_{n}\left(c \lambda_{2}-d\right)\right|}{T_{n}(c-d)}\right\} . \tag{13}
\end{equation*}
$$

If the conditions in Theorem 4.1 are satisfied, for any $\lambda$, a simple calculation using eq. (5) leads to

$$
\begin{equation*}
\frac{\left|T_{n}(c \lambda-d)\right|}{T_{n}(c-d)}=\left(\frac{\tau(c-d)}{|\tau(c \lambda-d)|}\right)^{n} \frac{1+\tau(c \lambda-d)^{2 n}}{1+\tau(c-d)^{2 n}} \tag{14}
\end{equation*}
$$

It is clear that when $n \rightarrow \infty$, the second fraction in the right side of goes to 1 . Therefore, the convergence rate is determined by

$$
\begin{equation*}
\max \left\{\frac{\tau(c-d)}{\left|\tau\left(c \lambda_{N}-d\right)\right|}, \frac{\tau(c-d)}{\left|\tau\left(c \lambda_{2}-d\right)\right|}\right\} \tag{15}
\end{equation*}
$$

If $\left[\lambda_{N}, \lambda_{2}\right] \subseteq\left[\lambda_{m}, \lambda_{M}\right]$, then $\max _{\lambda_{i}}\left|T_{n}\left(c \lambda_{i}-d\right)\right| \leq 1$ and therefore we can define the convergence factor as

$$
\nu(c, d)=\left\{\begin{array}{lr}
\tau(c-d), & \text { if }\left[\lambda_{N}, \lambda_{2}\right] \subseteq\left[\lambda_{m}, \lambda_{M}\right]  \tag{16}\\
\max \left\{\frac{\tau(c-d)}{\left|\tau\left(c \lambda_{N}-d\right)\right|}, \frac{\tau(c-d)}{\left|\tau\left(c \lambda_{2}-d\right)\right|}\right\}, & \text { otherwise. }
\end{array}\right.
$$

The optimum values of $\lambda_{m}$ and $\lambda_{M}$ will be those that lead to the minimum value of $\nu(c, d)$. In [33] it was proved that among the values of the parameters satisfying $\left[\lambda_{N}, \lambda_{2}\right] \subseteq\left[\lambda_{n}, \lambda_{M}\right]$, the ones that yield the minimum convergence factor are precisely $\lambda_{m}=\lambda_{N}$ and $\lambda_{M}=\lambda_{2}$. Let us see that they are also the optimum parameters in the case $\left[\lambda_{N}, \lambda_{2}\right] \nsubseteq\left[\lambda_{n}, \lambda_{M}\right]$.

Theorem 4.2 (Optimal parameters): The convergence rate $\nu(c, d)$ attains its minimum value for the parameters $c, d$ such that $\lambda_{M}=\lambda_{2}$ and $\lambda_{m}=\lambda_{N}$

Proof. See the Appendix.
This implies that in order to achieve the maximum convergence speed, some knowledge about the network is required. However, even if the network topology is unknown, it is important to study when the algorithm converges in a faster way than (8). Since the symmetric assignation of the parameters, $\lambda_{M}=-\lambda_{m}$, always ensures convergence, in the last result of this section we provide bounds for this particular case that also converge faster than (8).

Theorem 4.3 (Faster convergence than $\boldsymbol{A}^{n}$ ): For any matrix A satisfying Assumption 2.1, let $\lambda=\max \left(\left|\lambda_{2}\right|,\left|\lambda_{N}\right|\right)$ be the convergence rate in (8). For any

$$
\begin{equation*}
0<\lambda_{M}<\frac{2 \lambda}{\lambda^{2}+1}, \text { and } \lambda_{m}=-\lambda_{M}, \tag{17}
\end{equation*}
$$

$P_{n}(\lambda)$ goes to zero faster than $\lambda^{n}$ when $n$ goes to infinity. Therefore the algorithm in eq. (11) converges to the consensus faster than the one in eq. (8).

Proof. See [33].
Remark 4.4: The above result shows that there always exist parameters that make the proposed algorithm faster than (8). Therefore, if the algorithm is executed using the optimal parameters, it will also converge to the average faster than (8).

Finally, a graphical comparison of $x^{n}, T_{n}(x)$ and $P_{n}(x)$ is depicted in Fig. 2 for $n=4$, in the interval $[-1,1]$. Note that $T_{n}(x)$ cannot be used in the consensus process because at some points it would not reduce the error. On the other hand, as we have shown along the section, $P_{n}(x)$ satisfies the conditions required to achieve consensus. Also notice that $P_{n}(x)$ has closer


Fig. 2. Plot of the polynomials $x^{n}, T_{n}(x)$ and $P_{n}(x)$. In the figure $n=4, \lambda_{m}=-0.95$ and $\lambda_{M}=0.95$.
values to zero than $x^{n}$ in points close to -1 and 1 , which supports the theory that the error associated to eigenvalues in that regions will be reduced faster.

## V. Analysis with a Switching Communication Topology

We are interested now in the study of the recursive evaluation of (11) when the topology of the network, and therefore the matrix $\mathbf{A}$, changes at different iterations. Given initial conditions $\mathbf{x}(0)$, the distributed recurrence now looks:

$$
\begin{align*}
& \mathbf{x}(1)=\frac{1}{T_{1}(c-d)}(c \mathbf{A}(1)-d \mathbf{I}) \mathbf{x}(0) \\
& \mathbf{x}(n)=2 \frac{T_{n-1}(c-d)}{T_{n}(c-d)}(c \mathbf{A}(n)-d \mathbf{I}) \mathbf{x}(n-1)-\frac{T_{n-2}(c-d)}{T_{n}(c-d)} \mathbf{x}(n-2), n \geq 2 . \tag{18}
\end{align*}
$$

Note that this recurrence is suitable for switching weight matrices. However, the evaluation of the recurrence is no longer equivalent to $P_{n}(\mathbf{A}) \mathbf{x}(0)$, for some matrix $\mathbf{A}$. This means that we are not exactly evaluating the transformed Chebyshev polynomials in the eigenvalues of some matrix anymore. Nevertheless, a theoretical analysis is still possible.

For this analysis, the matrices $\mathbf{A}(n)$ now require the following assumption.
Assumption 5.1 (Non-Degenerate Stochastic Weights): The matrices A(n) are row stochastic, symmetric, non-degenerate and compatible with the underlying graphs, $\mathcal{G}(n)$, for all $n$, i.e., they are such that $\mathbf{A}(n) \mathbf{1}=\mathbf{1}, a_{i i}(n)>\epsilon$ and $a_{i j}(n) \in\{0\} \cup[\epsilon, 1)$ with $0<\epsilon<1$ some fixed constant.

Recalling the analysis done in the previous section, the evaluation of $P_{n}(\mathbf{A})$ was separated into the evaluation of its eigenvalues and eigenvectors, $P_{n}\left(\lambda_{i}\right) \mathbf{v}_{i}=T_{n}\left(c \lambda_{i}-d\right) / T_{n}(c-d) \mathbf{v}_{i}$. In the
switching case we must take into account that both $\lambda_{i}$ and $\mathbf{v}_{i}$ change at each iteration. Moreover, since the eigenvectors of different matrices are related we must also consider these relations. For the moment, as a first simplification of the problem, let us forget about the changes in $\mathbf{v}_{i}$ and the parameters $c$ and $d$ and let us study the scalar evaluation of the Chebyshev recurrence (2) with different $\lambda_{i}$ at each iteration. That is,

$$
\begin{equation*}
T_{0}(\Lambda)=1, T_{1}(\Lambda)=\lambda(1), T_{n}(\Lambda)=2 \lambda(n) T_{n-1}(\Lambda)-T_{n-2}(\Lambda), \tag{19}
\end{equation*}
$$

where $\Lambda=\{\lambda(n)\}, n \in \mathbb{N}$ is a succession of real numbers. Specifically, we are interested in the behavior of $\left|T_{n}(\Lambda)\right|$.

Proposition 5.2: Suppose there exists values $\lambda_{\text {min }}$ and $\lambda_{\text {max }}$ such that $\lambda(n) \in\left[\lambda_{\min }, \lambda_{\text {max }}\right], \forall n \in$ $\mathbb{N}, \lambda_{\text {min }}<0<\lambda_{\text {max }}$ and $\left|\lambda_{\text {min }}\right| \leq \lambda_{\text {max }}$. Then

$$
\begin{equation*}
\left|T_{n}(\Lambda)\right| \leq\left|T_{n}\left(\Lambda^{*}\right)\right| \tag{20}
\end{equation*}
$$

where $\Lambda^{*}=\left\{\lambda^{*}(n)\right\}$ is a succession defined by

$$
\lambda^{*}(n)= \begin{cases}\lambda_{\max } & \text { if } n \text { odd }  \tag{21}\\ \lambda_{\min } & \text { if } n \text { even }\end{cases}
$$

Proof. For abbreviation, in the proof we will denote the sign of $T_{n}(\Lambda)$ by $s\left(T_{n}\right)$.
Let us note that, if $s\left(T_{n-1}\right)=s\left(T_{n-2}\right)$, by choosing $\lambda(n)<0$, then

$$
\begin{equation*}
\left|T_{n}(\Lambda)\right|=\left|2 \lambda(n) T_{n-1}(\Lambda)-T_{n-2}(\Lambda)\right|=\left|2 \lambda(n) T_{n-1}(\Lambda)\right|+\left|T_{n-2}(\Lambda)\right|, \tag{22}
\end{equation*}
$$

independently of $n$. The choice of $\lambda(n)>0$ when $s\left(T_{n-1}\right)=s\left(T_{n-2}\right)$ implies that

$$
\begin{equation*}
\left|T_{n}(\Lambda)\right|=\left|2 \lambda(n) T_{n-1}(\Lambda)-T_{n-2}(\Lambda)\right|<\left|2 \lambda(n) T_{n-1}(\Lambda)\right|+\left|T_{n-2}(\Lambda)\right| . \tag{23}
\end{equation*}
$$

Taking these two facts into account we can see that

$$
\begin{equation*}
s\left(T_{n-1}\right)=s\left(T_{n-2}\right) \Rightarrow \arg \max _{\lambda(n)}\left|T_{n}(\Lambda)\right|=\lambda_{\min } \tag{24}
\end{equation*}
$$

Besides, in this situation, choosing $\lambda(n)<0$ yields $s\left(T_{n}\right) \neq s\left(T_{n-1}\right)$.
Now, if $s\left(T_{n-1}\right) \neq s\left(T_{n-2}\right)$ and $\lambda(n)>0$, then eq. (22) is again true. On the other hand, choosing $\lambda(n)<0$ in this situation implies (23). Thus,

$$
\begin{equation*}
s\left(T_{n-1}\right) \neq s\left(T_{n-2}\right) \Rightarrow \arg \max _{\lambda(n)}\left|T_{n}(\Lambda)\right|=\lambda_{\max } \tag{25}
\end{equation*}
$$

Also, if $s\left(T_{n-1}\right) \neq s\left(T_{n-2}\right)$ and $\lambda(n)>0$, then $s\left(T_{n}\right)=s\left(T_{n-1}\right)$.

Finally, noting that inequality (20) holds for $n=0$ and 1 , and $s\left(T_{0}\left(\Lambda^{*}\right)\right)=s\left(T_{1}\left(\Lambda^{*}\right)\right)$, then using (24) and (25) the succession (21) is obtained and the result is proved.

Corollary 5.3: If $\left|\lambda_{\min }\right|>\lambda_{\max }$ then the bound in eq. (20) is true taking $\Lambda^{*}=\left\{\lambda^{*}(n)\right\}$ with

$$
\lambda^{*}(n)= \begin{cases}\lambda_{\max } & \text { if } n \text { even }  \tag{26}\\ \lambda_{\min } & \text { if } n \text { odd }\end{cases}
$$

The previous proposition reveals that the Chebyshev recurrence evaluated in a succession of different real numbers does not keep the behavior shown when it is evaluated with a constant value. The next Lemma provides a bound for the direct expression of this behavior.

Lemma 5.4: Let us suppose that the conditions of Proposition 5.2 are true. Then

$$
\begin{equation*}
\left|T_{n}\left(\Lambda^{*}\right)\right| \leq \kappa_{1}\left(\lambda_{\max }\right)^{n}, \text { where } \kappa_{1}\left(\lambda_{\max }\right)=\lambda_{\max }+\sqrt{\lambda_{\max }^{2}+1} \tag{27}
\end{equation*}
$$

Proof. Let us define the recurrence

$$
\begin{equation*}
T_{0}^{*}(\lambda)=1, T_{1}^{*}(\lambda)=\lambda, T_{n}^{*}(\lambda)=2 \lambda T_{n-1}^{*}(\lambda)+T_{n-2}^{*}(\lambda), \tag{28}
\end{equation*}
$$

which satisfies that

$$
\begin{equation*}
\left|T_{n}\left(\Lambda^{*}\right)\right| \leq T_{n}^{*}\left(\lambda_{\max }\right) \tag{29}
\end{equation*}
$$

According to recurrence (28), the succession $\left\{T_{n}^{*}\left(\lambda_{\max }\right), n=0,1, \ldots\right\}$ satisfies the homogeneous difference equation $T_{n}^{*}\left(\lambda_{\max }\right)-2 \lambda_{\max } T_{n-1}^{*}\left(\lambda_{\max }\right)-T_{n-2}^{*}\left(\lambda_{\max }\right)=0$. By the theory of difference equations [35], the solution to this equation is determined by the roots $\kappa_{1}$ and $\kappa_{2}$ of the characteristic polynomial. In this case

$$
\begin{equation*}
\kappa_{1}\left(\lambda_{\max }\right)=\lambda_{\max }+\sqrt{\lambda_{\max }^{2}+1}>1, \text { and } \kappa_{2}\left(\lambda_{\max }\right)=\lambda_{\max }-\sqrt{\lambda_{\max }^{2}+1}=-1 / \kappa_{1}\left(\lambda_{\max }\right) . \tag{30}
\end{equation*}
$$

Since $\kappa_{1}\left(\lambda_{\max }\right) \neq \kappa_{2}\left(\lambda_{\max }\right)$, the direct expression of $T_{n}^{*}\left(\lambda_{\max }\right)$ is

$$
\begin{equation*}
T_{n}^{*}\left(\lambda_{\max }\right)=A \kappa_{1}\left(\lambda_{\max }\right)^{n}+B \kappa_{2}\left(\lambda_{\max }\right)^{n} \tag{31}
\end{equation*}
$$

where $A$ and $B$ depend on the initial conditions $T_{0}^{*}\left(\lambda_{\max }\right)$ and $T_{1}^{*}\left(\lambda_{\max }\right)$. In our case $A=B=$ $1 / 2$ and

$$
\begin{equation*}
\left|T_{n}\left(\Lambda^{*}\right)\right| \leq T_{n}^{*}\left(\lambda_{\max }\right)=\frac{1}{2}\left(\kappa_{1}\left(\lambda_{\max }\right)^{n}+\left(-1 / \kappa_{1}\left(\lambda_{\max }\right)\right)^{n}\right) \leq \kappa_{1}\left(\lambda_{\max }\right)^{n} \tag{32}
\end{equation*}
$$

This direct expression (27) will be helpful in the development of the convergence analysis dealing with changing matrices and the parameters $c$ and $d$. We provide now the main result, showing the convergence of the algorithm for the switching case.

Theorem 5.5: Allow the communication graph, $\mathcal{G}(n)$, to arbitrarily change in such a way that it is connected for all $n$, with the weight matrices, $\mathbf{A}(n)$, designed according to Assumption 5.1 . Let us denote $\lambda_{i}(n), i=1, \ldots, N$, the eigenvalues of $\mathbf{A}(n)$ and

$$
\begin{equation*}
\lambda_{\max }=\max _{n} \max _{i=2, \ldots, N} \lambda_{i}(n), \text { and } \lambda_{\min }=\min _{n} \min _{i=2, \ldots, N} \lambda_{i}(n) . \tag{33}
\end{equation*}
$$

Given fixed parameters $c$ and $d$, a sufficient condition to guarantee convergence to consensus of iteration (18) is

$$
\begin{equation*}
\kappa_{1}\left(\max \left\{\left|c \lambda_{\max }-d\right|,\left|c \lambda_{\min }-d\right|\right\}\right) \tau(c-d)<1 . \tag{34}
\end{equation*}
$$

Proof. See the Appendix.
The next corollaries give more specific values of $\lambda_{M}$ and $\lambda_{m}$, and therefore on $c$ and $d$, that satisfy the condition in the theorem to achieve convergence.

Corollary 5.6: Assume $\left|c \lambda_{\max }-d\right|>\left|c \lambda_{\min }-d\right|$ and a symmetric assignation, $-\lambda_{m}=\lambda_{M}=$ $\lambda$, of the parameters. Then if

$$
\begin{equation*}
\lambda^{2}<\left(1-\lambda_{\max }^{2}\right), \tag{35}
\end{equation*}
$$

the algorithm converges.
Proof. Recall that with this assignation $c=1 / \lambda$ and $d=0$. Substituting $\kappa_{1}$ and $\tau$ by their values in eq. (34) and doing some simplifications eq. (35) is obtained.

If we prefer to assign non-symmetric values to the parameters, the following corollary provides a possible assignation that satisfies Theorem 5.5 .

Corollary 5.7: Assume now that the values of $\lambda_{\max }$ and $\lambda_{\min }$, or some bounds, are known. If $\lambda_{M}$ and $\lambda_{m}$ satisfy that

$$
\begin{equation*}
\lambda_{M}+\lambda_{m}=\lambda_{\max }+\lambda_{\min }, \tag{36}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda_{M}-\lambda_{m}<\sqrt{4\left(1-\lambda_{\max }\right)\left(1-\lambda_{\min }\right)}, \tag{37}
\end{equation*}
$$

then the algorithm achieves the consensus.

Proof. If we know the values of $\lambda_{\max }$ and $\lambda_{\min }$, the choice of $\lambda_{m}$ and $\lambda_{M}$ can be done in such a way that

$$
\begin{equation*}
\left|c \lambda_{\min }-d\right|=\left|c \lambda_{\max }-d\right| . \tag{38}
\end{equation*}
$$

With this assignation we are minimizing the value of $\max \left\{\left|c \lambda_{\max }-d\right|,\left|c \lambda_{\text {min }}-d\right|\right\}$ and therefore, the convergence condition is easier to fulfill. Clearing (38) yields (36). With this first condition, doing some, rather tedious, calculations in eq. (34) the second condition (37) is obtained.

We discuss now in detail the meaning of the theorem and its implications.
Remark 5.8: Note that the theorem provides just a sufficient condition to ensure convergence. This means that although the given bounds seem very restrictive, in practice, even if we choose large values of $\lambda_{M}$ and $\lambda_{m}$, there will be convergence. Moreover, an important consequence of corollaries 5.6 and 5.7 is that, independently on the changes of the network topology, there are always parameters such that the method converges to the consensus.

Remark 5.9: It is also interesting to note the different behavior of the algorithm when the topology changes with respect to the fixed case. In the latter case, in general it is better to select the parameters $\lambda_{M}$ and $\lambda_{m}$ with large modulus to ensure that all the eigenvalues of the weight matrix are included in the interval $\left[\lambda_{m}, \lambda_{M}\right]$. However, in the switching case, it is necessary to choose them small so that $c-d$ is large enough to guarantee convergence. This happens because the more variation on the eigenvalues of the weight matrices, the larger $\kappa_{1}\left(\max \left\{\left|c \lambda_{\max }-d\right|,\left|c \lambda_{\min }-d\right|\right\}\right)$ is. Therefore, the larger $N$, the smaller (in modulus) $\lambda_{M}$ and $\lambda_{m}$ should be chosen.

Remark 5.10: The analysis followed to proof convergence of our algorithm is also interesting because it can be applied to more general consensus algorithms based on recurrences of order greater than one. Given a recurrence similar to (18), if a scalar difference equation is found such that its solution bounds the original one in the worst case, a convergence result using the behavior of this recurrence can be obtained. To the authors' knowledge, this is the first theoretical result proving convergence of a distributed algorithm based on polynomials under switching communication topologies.

Finally, we provide a discussion about the assumptions we have made to proof convergence.

- Symmetric weight matrices: If the weight matrices are not symmetric, then we cannot ensure that the norm of the matrices used to change the base of eigenvectors is equal to 1 . In such
a case the convergence condition in Theorem 5.5 would be $K \kappa_{1}\left(\max \left\{\left|c \lambda_{\max }-d\right|, \mid c \lambda_{\min }-\right.\right.$ $d \mid\}) \tau(c-d)<1$, with $K \geq 1$ some positive constant. It is also important to remark that, in this situation, the left eigenvector associated to $\lambda_{1}(n)$ is not constant anymore for different matrices. This makes the theoretical analysis of the behavior more tedious because at each iteration it is affected by these eigenvectors, which do not tend to zero with $n$. However, convergence can still be achieved.
- Connectivity of the graphs: The assumption about the connectivity of each graph is more restrictive than in other approaches, e.g., [9], where only joint connectivity is imposed. In our analysis, if one graph is disconnected, then $\lambda_{\max }=1$ and the sufficient condition (34) is never satisfied. This, of course, is caused because we are considering the worst case scenario, so that we can model the behavior of the Chebyshev recurrence as the $n$th power of some quantity. However, in practice, even if some graphs are disconnected, the errors associated to the eigenvectors associated to the eigenvalue 1 are also canceled. We show this in simulations in section VI.


## VI. Simulations

In this section we analyze our algorithm in a simulated environment. Monte Carlo experiments have been designed to study the convergence of the method and the influence of the parameters $\lambda_{m}$ and $\lambda_{M}$ in the algorithm.

## A. Evaluation with a fixed communication topology

In a first step we study the algorithm when the topology of the network is fixed. We analyze the convergence speed for different weight matrices, comparing it with other approaches, and the influence of the parameters $\lambda_{M}$ and $\lambda_{m}$ in the performance of the algorithm.

In the experiments we have considered 100 random networks of 100 nodes. For each network the nodes have been randomly positioned in a square of $200 \times 200$ meters. Two nodes communicate if they are at a distance lower than 20 meters. The networks are also forced to be connected so that the algorithms converge. After that, 100 different random initial values have been generated in the interval $(0,1)^{N}$, giving a total of 10000 trials to test the algorithm.

1) Convergence speed of the algorithm: We evaluate how our algorithm behaves compared to other methods using different weighted adjacency matrices. For each communication network we have computed 4 different weighted adjacency matrices. The first one, $\mathbf{A}_{l d}$, uses the "local degree weights", the second one, $\mathbf{A}_{b c}$, uses the "best constant factor" and the third one, $\mathbf{A}_{o s}$, computes an approximation of the "optimal symmetric weights". For more information about these matrices we refer the reader to [24]. These three matrices are symmetric, for that reason we have included in the experiment a fourth non-symmetric matrix, $\mathbf{A}_{n s}$, computed by $a_{i j}=1 /\left(\mathcal{N}_{i}+1\right)$ if $j \in \mathcal{N}_{i} \cup i$ and $a_{i j}=0$ otherwise.

We have compared our method with the powers of the matrices using (8), the Newton's interpolation polynomial of degree 2 proposed in [29], $N_{2}(x)=(x-\alpha)^{2} /(1-\alpha)^{2}$, and the second order recurrence with fixed weights proposed in [32], $F_{n}(x)=\beta x F_{n-1}(x)+(1-\beta) F_{n-2}(x)$. We have used the values $\alpha=\left(\lambda_{2}+\lambda_{N}\right) / 2$ and $\beta=2 /\left(1+\sqrt{1-\lambda_{2}^{2}}\right)$, which give the best convergence rate for the two algorithms. For the Chebyshev polynomials we have also assigned the optimal parameters $\lambda_{M}=\lambda_{2}$ and $\lambda_{m}=\lambda_{N}$. We have measured the average number of iterations required to obtain an error, $e=\left\|\mathbf{x}(n)-\left(\mathbf{w}_{1}^{T} \mathbf{x}(0) / \mathbf{w}_{1}^{T} \mathbf{1}\right) \mathbf{1}\right\|_{\infty}$, smaller than a given tolerance.

## TABLE I

NUMBER OF ITERATIONS FOR DIFFERENT ALGORITHMS AND TOLERANCES

| Method $\backslash$ Tolerance | $10^{-2}$ | $10^{-3}$ | $10^{-4}$ | $10^{-5}$ | Method $\backslash$ Tolerance | $10^{-2}$ | $10^{-3}$ | $10^{-4}$ | $10^{-5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{A}_{l d}^{n}$ | 396.1 | 899.0 | 1422.9 | 1902.9 | $N_{2}\left(\mathbf{A}_{l d}\right)$ | 381.4 | 748.9 | 1120.8 | 1474.5 |
| $\mathbf{A}_{b c}^{n}$ | 470.5 | 892.4 | 1307.4 | 1691.5 | $N_{2}\left(\mathbf{A}_{b c}\right)$ | 475.7 | 897.0 | 1109.9 | 1493.7 |
| $\mathbf{A}_{o s}^{n}$ | 390.8 | 735.1 | 1092.0 | 1446.0 | $N_{2}\left(\mathbf{A}_{o s}\right)$ | 426.8 | 792.4 | 964.3 | 1225.2 |
| $\mathbf{A}_{n s}^{n}$ | 308.9 | 698.4 | 1116.7 | 1521.2 | $N_{2}\left(\mathbf{A}_{n s}\right)$ | 302.6 | 604.1 | 911.5 | 1216.4 |
| $F_{n}\left(\mathbf{A}_{l d}\right)$ | 45.7 | 71.9 | 98.0 | 124.2 | $P_{n}\left(\mathbf{A}_{l d}\right)$ | 41.8 | 62.2 | 82.6 | 103.0 |
| $F_{n}\left(\mathbf{A}_{b c}\right)$ | 45.2 | 67.4 | 91.2 | 114.6 | $P_{n}\left(\mathbf{A}_{b c}\right)$ | 44.6 | 66.4 | 88.1 | 109.9 |
| $F_{n}\left(\mathbf{A}_{o s}\right)$ | 42.2 | 62.9 | 83.3 | 103.6 | $P_{n}\left(\mathbf{A}_{o s}\right)$ | 42.1 | 62.6 | 83.0 | 103.4 |
| $F_{n}\left(\mathbf{A}_{n s}\right)$ | 40.8 | 63.9 | 86.8 | 109.8 | $P_{n}\left(\mathbf{A}_{n s}\right)$ | 38.6 | 57.1 | 75.6 | 94.1 |

Table I shows the results of the experiment. For any matrix our algorithm is the one that reaches the consensus first. It is remarkable the speed up compared to the powers and the Newton method. Moreover, considering that the initial error is upper bounded by 1 , note that
our algorithm is able to reduce the error by five orders of magnitude $\left(10^{-5}\right)$ in around $N=100$ iterations (103.0, 109.9, 103.4 and 94.1 iterations in the table), which is the size of the network.

An interesting detail is that our algorithm converges faster using the "local degree weights", $\mathbf{A}_{l d}(103.0)$, and the "non-symmetric weights", $\mathbf{A}_{n s}(94.1)$, than using the other two matrices (109.9 and 103.4), even though the second largest eigenvalue of the other two matrices is smaller. This behavior happens because the eigenvalues of $\mathbf{A}_{b c}$ and $\mathbf{A}_{o s}$ are symmetrically placed with respect to zero whereas for $\mathbf{A}_{l d}$ and $\mathbf{A}_{n s}\left|\lambda_{N}\right|<\lambda_{2}$ (an example can be found in [24]). As a consequence, $c-d$ is larger and the algorithm converges faster. This is indeed very convenient because the "local degree weights" and the "non-symmetric weights" can be easily computed in a distributed way without global information, whereas the other two require the knowledge of the whole topology.

Regarding the non-symmetric weights, we have observed that $\lambda_{2}$ is, in general, small compared to the second eigenvalue of the symmetric matrices. Since the eigenvalues of $\mathbf{A}_{n s}$ also satisfy that $\left|\lambda_{N}\right|<\lambda_{2}$, the convergence for this matrix is the fastest. Also note that these matrices are the easiest to compute. On the other hand, when using symmetric weight matrices the convergence value is known to be the average of the initial conditions whereas when using non-symmetric weights the convergence value depends on the matrix.
2) Dependence on the parameters $\lambda_{M}$ and $\lambda_{m}$ : So far we have evaluated the convergence speed of our algorithm only considering the optimal parameters, which implies the knowledge of the eigenvalues of the weight matrix. However, in most situations the nodes will have no knowledge about these eigenvalues. We analyze now the convergence rates of our algorithm when it is run using sub-optimal parameters. In this case, for simplicity we have only considered $\mathbf{A}_{l d}$ in the experiment.

The results are in Table [II. The table shows the average number of iterations required to have an error lower than $10^{-3}$. The number of iterations is in all the cases larger than in Table $\square$ ( 62.2 iterations) but anyway, the results are in most cases also good. The only problem appears when $\lambda_{M}+\lambda_{m}-1>\lambda_{N}$ because the algorithm diverges (cells with $\infty$ in the table). Nevertheless, the number of iterations is almost always smaller than using the powers of $\mathbf{A}_{l d}$ and the Newton polynomial (899.0 and 748.9 iterations in Table $\rrbracket$ respectively). The results compared to $F_{n}$ evaluated with the optimal parameter ( 71.9 it . in Table $\rrbracket$ ) seem to be poor. However, the optimal $\beta$ requires the knowledge of $\lambda_{2}$ which, right now, we are assuming it is unknown. For

TABLE II
NUMBER OF ITERATIONS USING SUB-OPTIMAL PARAMETERS AND TOLERANCE $10^{-3}$

| $\lambda_{m} \backslash \lambda_{M}$ | 0.2 | 0.5 | 0.8 | 0.9 | 0.95 | 0.999 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| -0.2 | 713.8 | 563.7 | 355.2 | $\infty$ | $\infty$ | $\infty$ |
| -0.5 | 798.1 | 630.4 | 397.2 | 279.0 | 194.5 | 75.9 |
| -0.8 | 874.3 | 690.6 | 435.2 | 305.6 | 213.1 | 83.1 |
| -0.9 | 898.3 | 709.5 | 447.0 | 314.0 | 219.0 | 85.4 |
| -0.95 | 910.0 | 718.8 | 453.0 | 318.1 | 221.8 | 86.5 |
| -0.999 | 919.4 | 726.0 | 457.6 | 321.3 | 224.0 | 87.4 |
| $F_{n}$ | 757.5 | 672.4 | 463.9 | 320.9 | 227.1 | 93.0 |

that reason, in the last row of Table [II we have included the results using $F_{n}$ evaluated with $\beta=2 /\left(1+\sqrt{1-\lambda_{M}^{2}}\right)$, i.e., with the same estimation of $\lambda_{2}$ used for the Chebyshev polynomials. In this case we observe again that both methods present a similar performance when using the same parameters. The degree of freedom given by $\lambda_{m}$ is what differs in the algorithms. By adjusting this parameter we can reduce the number of iterations in our algorithm.

Another advantage of using our algorithm with the weight matrix $\mathbf{A}_{l d}$, besides the computation using local information, is that usually its smallest eigenvalue, $\lambda_{N}$, is a negative value close to zero (in our simulations it has never valued less than -0.5 ). The second largest eigenvalue depends on how many nodes has the network and the number of links, but in general this eigenvalue is close to one. Therefore by choosing $\lambda_{m}=-0.5$ and $\lambda_{M} \simeq 1$ there is a great chance to obtain a good convergence rate and almost no risk of divergence, see for example the cell in the second row and sixth column of Table $\Pi$ (153.7). A safer choice of parameters is $\lambda_{m}=-\lambda_{M}$, which we know that has good convergence rates. In this case it is also convenient to choose $\lambda_{M} \simeq 1$ to ensure that all the eigenvalues are contained in $\left[\lambda_{m}, \lambda_{M}\right]$.

## B. Evaluation with a switching communication topology

Let us see how the algorithm behaves when the topology of the network changes at different iterations. We start by showing the convergence in an illustrative example where the conditions of Theorem 5.5 are satisfied. After that we run again Monte Carlo experiments to analyze the algorithm in more realistic situations, where the conditions of Theorem 5.5 do not always hold.

1) Illustrative Example: The communication network considered, composed by 20 nodes, is depicted in Fig. 3 (top left), which is connected. In order to satisfy the conditions of Theorem 5.5 at each iteration we have randomly added some links to the network. In this way all the topologies remain connected and the parameters $\lambda_{\max }$ and $\lambda_{\min }$ correspond to the second maximum and the smallest eigenvalues of the initial weight matrix. Using the local degree weights, which return a symmetric matrix, these parameters are $\lambda_{\max }=0.9477$ and $\lambda_{\min }=-0.1922$. Figure 3 top middle and top right depict the evolution of $x(n)$ using (18) with the parameters of Corollary 5.6. $\lambda_{M}=-\lambda_{m}=0.3190$, and Corollary 5.7, $\lambda_{M}=0.6274, \lambda_{m}=0.1282$, respectively. The evolution of $\mathbf{x}(n)$ using (8) is shown in Fig. 3 bottom left. It is interesting to note the similarity of this graphic with the Chebyshev recurrence using the symmetric parameters given by Corollary 5.6 (top middle). Finally, to remark that the condition of Theorem 5.5 is a sufficient condition in Fig. 3 bottom middle and bottom right we show that the algorithm also converges to the consensus choosing parameters with larger modulus. In the example we have chosen the parameters using the criteria analyzed for the fixed topology situation. Moreover, we can see in the graphics that the consensus is achieved in both cases in less iterations (the lines overlap earlier in the graphics). Finally, note that the symmetry in all the weight matrices implies that, in all the cases, the value of the consensus is the average of the initial conditions.
2) Analysis of convergence depending on the evolution of network and the parameters of the algorithm: We have generated again 100 random networks of 100 nodes like in the fixed topology case. To model the changes in the communication topology we have considered three different scenarios in the experiment. The first one assumes a fixed initial communication topology and, at each iteration the links can fail with constant probability equal to 0.05 (Link Failures). This is a usual way to model networks with unreliable or noisy communications. In the second scenario we consider a set of mobile agents that randomly move in the environment. In this way, at each iteration the communication topology evolves with the proximity graph defined by the new positions of the agents (Evolution with Motion). The last scenario assumes a new random network at each iteration (Random Network). Although in reality this situation will be uncommon, it is interesting to analyze it in order to study the properties of our algorithm. In the three scenarios we have used the local degree weights to define the weight matrix at each iteration. We have not worried about the network connectivity, letting the experiment to possibly have several iterations with disconnected networks. We have set a maximum of 3000 iterations



$\Pi_{n} \mathbf{A}(n) \mathbf{x}(0)$
$\lambda_{M}=-\lambda_{m}=0.3190$

$\lambda_{M}=0.8, \lambda_{m}=-0.8$
$\lambda_{M}=0.6274, \lambda_{m}=0.1282$


$$
\lambda_{M}=0.9, \lambda_{m}=-0.5
$$

Fig. 3. Illustrative example of the convergence speed of the algorithm with a switching communication topology. The initial network is shown at the top left graphic. The evolution using (8) is shown at the bottom left and four different executions of (18) with the same changes in the topology and different parameters are depicted in the rest of the graphics. Notice that even when the conditions of Theorem 5.5 are not satisfied (bottom middle and bottom right graphics), the algorithm still achieves the consensus.

Table III shows the number of iterations required by iteration (8) to achieve a precision of $10^{-3}$. We can see that when the network has link failures or evolves with the motion of the nodes the number of iterations required by the algorithm is slightly greater than when the topology of the network remains fixed (1087.2 and 1032.4 compared to 899.0 in Table $\mathbb{\square}$ ). On the other hand, when the network randomly changes at each step, in a few iterations (9.4) the consensus is achieved, which makes sense because in this situation the information is spread in a fast way.

TABLE III

$$
\text { NUMBER OF ITERATIONS WITH TOLERANCE } 10^{-3}
$$

| Link Failures | Evolution with Motion | Random Networks |
| :---: | :---: | :---: |
| 1087.2 | 1032.4 | 9.4 |

The number of iterations required to achieve the same accuracy (tolerance of $10^{-3}$ ) using (18) with different parameters is shown in Tables IV, V and VI for the Link Failures, Evolution with Motion and Random Networks scenarios respectively.

TABLE IV
Number of iterations for Link Failures

| $\lambda_{m} \backslash \lambda_{M}$ | 0.25 | 0.5 | 0.75 | 0.9 | 0.95 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| -0.25 | $\geq 3000$ | $\geq 3000$ | 1298.1 | 383.3 | 267.9 |
| -0.5 | $\geq 3000$ | $\geq 3000$ | 1328.6 | 418.9 | 293.5 |
| -0.75 | $\geq 3000$ | $\geq 3000$ | 1356.6 | 452.3 | 316.8 |
| -0.9 | $\geq 3000$ | $\geq 3000$ | 1321.0 | 470.9 | 330.0 |
| -0.95 | $\geq 3000$ | $\geq 3000$ | 1326.4 | 476.9 | 334.5 |

TABLE V
Number of iterations for Evolution with Motion

| $\lambda_{m} \backslash \lambda_{M}$ | 0.25 | 0.5 | 0.75 | 0.9 | 0.95 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| -0.25 | $\geq 3000$ | 1738.0 | 600.1 | 457.2 | 260.9 |
| -0.5 | $\geq 3000$ | 1765.2 | 665.6 | 461.9 | 306.5 |
| -0.75 | 1726.5 | 1793.5 | 703.6 | 506.3 | 309.8 |
| -0.9 | 1740.0 | 1813.0 | 708.5 | 564.9 | 311.0 |
| -0.95 | 1744.5 | 1818.0 | 710.4 | 564.9 | 311.5 |

TABLE VI
Number of iterations for Random Networks

| $\lambda_{m} \backslash \lambda_{M}$ | 0.25 | 0.5 | 0.75 | 0.9 | 0.95 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| -0.25 | 8.1 | 8.3 | 11.8 | 25.4 | $\infty$ |
| -0.5 | 8.3 | 8.9 | 11.6 | 22.3 | 42.1 |
| -0.75 | 8.7 | 9.6 | 11.8 | 21.7 | 37.8 |
| -0.9 | 8.9 | 10.0 | 12.0 | 21.7 | 36.8 |
| -0.95 | 9.0 | 10.1 | 12.0 | 21.7 | 36.5 |

With these results we can extract some interesting remarks. First of all, for the parameters tested in the experiment, the algorithm is convergent in almost all the cases. Only in the Random Networks the algorithm diverges when $\lambda_{M}=0.95$ and $\lambda_{m}=-0.25$ (Table VI first row and sixth column). The cells with " $\geq 3000$ " iterations point that for these parameters the algorithm converges but in a slow way. A second interesting detail is that, similarly to the fixed topology case, we can always find parameters that make our algorithm achieve the consensus faster than using (8) (results of Table III). However, it is surprising which parameters achieve this goal in the different scenarios. For the Link Failures and the Evolution with Motion, the best parameters are exactly the parameters that make the algorithm diverge for the Random Networks scenario, i.e., $\lambda_{M}=0.95$ and $\lambda_{m}=-0.25$ with 267.9 and 260.9 iterations respectively. On the other hand, the best parameters for the Random Networks are those who give the slowest convergence rate for the other two scenarios, i.e., $\lambda_{M}=0.25$ and $\lambda_{m}=-0.25$ with 8.1 iterations in Table VI versus more than 3000 in Tables IV and V. The explanation for this phenomenon appears in the variability of the eigenvectors of the weight matrices. When the topology changes arbitrarily at each iteration, there is a great variability in the eigenvectors of the weight matrices, which turns out in a great variability of $\mathbf{x}(n)$. This situation is closer to the worst case we have shown in section IV to proof the convergence of the algorithm. Therefore, a good convergence rate requires a large value of $c-d$, achieved when $\lambda_{M}$ and $\lambda_{m}$ have small modulus. When the topology changes smoothly, as in the Link Failures and the Motion Evolution, the eigenvectors almost do not change and the algorithm behaves similarly to the fixed case. For that reason, the parameters that achieve the best convergence rate are the same as in the fixed case. However, we must be careful because for larger values of $\lambda_{M}$ the algorithm may diverge.

A final detail is that, in all the cases, the convergence seems to be more affected by $\lambda_{M}$ than $\lambda_{m}$. This is explained by the use of the local degree weights. As we have mentioned earlier, these matrices do not have symmetric eigenvalues with respect to zero. In these matrices $\lambda_{\max }$ dominates the convergence rate, so the convergence is more sensible to the parameter $\lambda_{M}$.

In conclusion, when the topology of the network changes, the parameters should be chosen taking into account the nature of these changes. For small changes similar parameters to the fixed case should be assigned whereas if the network is expected to change a lot we should pick small parameters for the algorithm to guarantee convergence.

## VII. Conclusions

In this paper we have analyzed the properties of Chebyshev polynomials to design a fast distributed consensus algorithm. We have shown that the proposed algorithm significantly reduces the number of communication rounds required by the network to achieve the consensus. We have provided a theoretical analysis of the properties of the algorithm in both fixed and switching communication topologies. We have also evaluated our method with an extensive set of simulations. Both theoretical and empirical analysis show the goodness of our proposal.

## VIII. Acknowledgments

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## References

[1] J.C. Mason and D. C. Handscomb. Chebyshev Polynomials. Chapman and Hall, 2002.
[2] C. Naik, H.H. Cartensen, and A. M. Dean. Reaction rate representation using chebyshev polynomials. In Western States Section 2002 Spring Meeting of the Combustion Institute, pages 5714-5719, March 2002.
[3] D. L. Richardson, D. Schmidt, and J. Mitchell. Improved chebyshev methods for the numerical integration of first-order differential equations. In Spaceflight mechanics 1998; Proceedings of the AAS/AIAA Space Flight Mechanics Meeting, pages 1533-1544, February 1998.
[4] J.G. Verwer and B.P. Sommeijer. An implicit-explicit runge-kutta-chebyshev scheme for diffusion-reaction equations. SIAM Journal on Scientific Computing, 25(5):1824-1835, May 2004.
[5] S. Ozera, C. H. Chenb, and H. A. Cirpanc. A set of new chebyshev kernel functions for support vector machine pattern classification. Pattern Recognition, 44(7):1435-1447, July 2011.
[6] F. Bullo, J. Cortés, and S. Martínez. Distributed Control of Robotic Networks. Applied Mathematics Series. Princeton University Press, 2009. Electronically available at http://coordinationbook.info.
[7] W. Ren and R. W. Beard. Distributed Consensus in Multi-vehicle Cooperative Control. Communications and Control Engineering. Springer-Verlag, London, 2008.
[8] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. IEEE Transactions on Automatic Control, 49(9):1520-1533, September 2004.
[9] A. Jadbabaie, J. Lin, and A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. IEEE Transactions on Automatic Control, 48(6):988-1001, June 2003.
[10] M. Zhu and S. Martínez. Discrete-time dynamic average consensus. Automatica, 46(2):322-329, February 2010.
[11] M. Mehyar, D. Spanos, J. Pongsajapan, S. H. Low, and R. M. Murray. Asynchronous distributed averaging on communication networks. IEEE/ACM Transactions on Networking, 15(3):512-520, June 2007.
[12] L. Xiao, S. Boyd, and S. Lall. A scheme for robust distributed sensor fusion based on average consensus. In International Conference on Information Processing in Sensor Networks, pages 63-70, Los Angeles, April 2005.
[13] G. Ferrari-Trecate, A. Buffa, and M. Gati. Analysis of coordination in multi-agent systems through partial difference equations. IEEE Transactions on Automatic Control, 51(6):1058-1063, June 2006.
[14] E. Lovisari, F. Garin, and S. Zampieri. A resistance-based approach to performance analysis of the consensus algorithm. In IEEE Int. Conference on Decision and Control, pages 5714-5719, December 2010.
[15] S. Patterson, B. Bamieh, and A. E. Abbadi. Convergence rates of distributed average consensus with stochastic link failures. IEEE Transactions on Automatic Control, 55(4):880-892, April 2010.
[16] J. Zhou and Q. Wang. Convergence speed in distributed consensus over dynamically switching random networks. Automatica, 45(6):1455-1461, June 2009.
[17] M. Zhu and S. Martínez. On the convergence time of asynchronous distributed quantized averaging algorithms. IEEE Transactions on Automatic Control, 56(2):386-390, February 2011.
[18] F. Jiang and L. Wang. Finite-time information consensus for multi-agent systems with fixed and switching topologies. Physica D, 238(16):1550-1560, August 2009.
[19] J. Cortés. Finite-time convergent gradient flows with applications to network consensus. Automatica, 42(11):1993-2000, November 2006.
[20] L. Wang and F. Xiao. Finite-time consensus problems for networks of dynamic agents. IEEE Transactions on Automatic Control, 55(4):950-955, April 2010.
[21] C.K. Ko and X. Gao. On matrix factorization and finite-time average-consensus. In IEEE Int. Conference on Decision and Control, pages 5798-5803, December 2009.
[22] Z. Jin and R.M. Murray. Multi-hop relay protocols for fast consensus seeking. In IEEE Int. Conference on Decision and Control, pages 1001-1006, December 2006.
[23] D. Yuan, S. Xu, H. Zhaoa, and Y. Chub. Accelerating distributed average consensus by exploring the information of second-order neighbors. Physica Letters A, 37(4):24382445, May 2010.
[24] L. Xiao and S. Boyd. Fast linear iterations for distributed averaging. Systems and Control Letters, 53:65-78, September 2004.
[25] Y. Kima, D.W. Gub, and I. Postlethwaite. Spectral radius minimization for optimal average consensus and output feedback stabilization. Automatica, 45(6):1379-1386, June 2009.
[26] B. Johansson and M. Johansson. Faster linear iterations for distributed averaging. In $17^{\text {th }}$ IFAC World Congress, July 2008.
[27] S. Sundaram and C. N. Hadjicostis. Finite-time distributed consensus in graphs with time-invariant topologies. In American Control Conference, pages 711-716, New York, July 2007.
[28] Y. Yuan, G. Stan, L. Shi, and J. Gonçalves. Decentralised final value theorem for discrete-time lti systems with application to minimal-time distributed consensus. In IEEE Int. Conference on Decision and Control, pages 2664-2669, December 2009.
[29] E. Kokiopoulou and P. Frossard. Polynomial filtering for fast convergence in distributed consensus. IEEE Transactions on Signal Processing, 57(1):342354, January 2009.
[30] T.C. Aysal, B. Oreshkin, and M.J. Coates. Accelerated distributed average consensus via localized node state prediction. IEEE Transactions on Signal Processing, 57(4):1563-1576, April 2009.
[31] B. Oreshkin, M. J Coates, , and M. Rabbat. Optimization and analysis of distributed averaging with short node memory. IEEE Transactions on Signal Processing, 58(5):2850-2865, May 2010.
[32] B. Ghosh S. Muthukrishnan and M. H. Schultz. First- and second-order diffusive methods for rapid, coarse, distributed load balancing. Theory of Computing Systems, 31(4):331-354, December 1998.
[33] E. Montijano, J. I. Montijano, and C. Sagues. Fast Distributed Consensus with Chebyshev Polynomials. In American Control Conference, 2011. (to appear).
[34] R. L. G. Cavalgante, A. Rogers, and N. R. Jennings. Consensus Acceleration in Multiagent Systems with the Chebyshev Semi-iterative Method. In $10^{\text {th }}$ Int. Conf. on Autonomous Agents and Multiagent Systems, 2011. (to appear).
[35] R. P. Agarwal. Difference equations and inequalities, Theory, Methods and Applications. Dekker, 1992.

## APPENDIX

## A. Proof of Theorem 4.1

We introduce two auxiliary results to proof the convergence.
Lemma 1.1: Given $x_{1}>1$, for any $x_{2}$ such that $\left|x_{2}\right|<x_{1}$ it holds that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{T_{n}\left(x_{2}\right)}{T_{n}\left(x_{1}\right)}=0 \tag{39}
\end{equation*}
$$

Proof. For $\left|x_{2}\right| \leq 1,\left|T_{n}\left(x_{2}\right)\right| \leq 1, \forall n$, and since $T_{n}\left(x_{1}\right) \rightarrow \infty$ with $n$, eq. (39) is true. Now, if $1<\left|x_{2}\right|<x_{1}$, then using (5) we have

$$
\begin{equation*}
\frac{T_{n}\left(x_{2}\right)}{T_{n}\left(x_{1}\right)}=\frac{\tau\left(x_{1}\right)^{n}}{\tau\left(x_{2}\right)^{n}} \frac{1+\tau\left(x_{2}\right)^{2 n}}{1+\tau\left(x_{1}\right)^{2 n}} . \tag{40}
\end{equation*}
$$

But in this case $1>\left|\tau\left(x_{2}\right)\right|>\tau\left(x_{1}\right)>0$ and the result holds immediately.
Lemma 1.2: Given $x>1$, for any complex number $z$, such that $|\tau(z)|=\min \left\{\left|z+\sqrt{z^{2}-1}\right|, \mid z-\right.$ $\left.\sqrt{z^{2}-1} \mid\right\}>\tau(x)$, then $\lim _{n \rightarrow \infty} T_{n}(z) / T_{n}(x)=0$.

Proof. It is a straightforward consequence of (40).
Proof of Theorem 4.1. Let $\mathbf{Q}=\mathbf{A}-\mathbf{1} \mathbf{w}_{1}^{T} / \mathbf{w}_{1}^{T} \mathbf{1}$, whose eigenvalues are 0 , with $\mathbf{v}_{1}$ its corresponding right eigenvector, and $\lambda_{2}, \ldots, \lambda_{N}$ with the same eigenvectors as $\mathbf{A}$. Since $\mathbf{v}_{1}=\mathbf{w}_{1}^{T} \mathbf{x}(0) \mathbf{1} / \mathbf{w}_{1}^{T} \mathbf{1}$, then $\mathbf{1} \mathbf{w}_{1}^{T}\left(\mathbf{x}(0)-\mathbf{v}_{1}\right)=0$. Taking this into account it is easy to see that

$$
\begin{equation*}
\mathbf{A}^{n}\left(\mathbf{x}(0)-\mathbf{v}_{1}\right)=\mathbf{Q}^{n}\left(\mathbf{x}(0)-\mathbf{v}_{1}\right), \quad \forall n \in \mathbb{N} \tag{41}
\end{equation*}
$$

and therefore $P_{n}(\mathbf{A})\left(\mathbf{x}(0)-\mathbf{v}_{1}\right)=P_{n}(\mathbf{Q})\left(\mathbf{x}(0)-\mathbf{v}_{1}\right)$.
Also $\mathbf{A} \mathbf{v}_{1}=\mathbf{v}_{1}$ and $P_{n}(1)=1$, then $P_{n}(\mathbf{A}) \mathbf{v}_{1}=\mathbf{v}_{1}$ and

$$
\begin{equation*}
\left\|\mathbf{x}(n)-\mathbf{v}_{1}\right\|_{2}=\left\|P_{n}(\mathbf{A})\left(\mathbf{x}(0)-\mathbf{v}_{1}\right)\right\|_{2}=\left\|P_{n}(\mathbf{Q})\left(\mathbf{x}(0)-\mathbf{v}_{1}\right)\right\|_{2} \leq\left\|P_{n}(\mathbf{Q})\right\|_{2}\left\|\mathbf{x}(0)-\mathbf{v}_{1}\right\|_{2} . \tag{42}
\end{equation*}
$$

In addition, since $\mathbf{A}$ is diagonalizable, so is $\mathbf{Q}$, which implies that $\mathbf{Q}$ can be decomposed, $\mathbf{Q}=\mathbf{P D P}{ }^{-1}$, with $\mathbf{D}=\operatorname{diag}\left(0, \lambda_{2}, \ldots, \lambda_{N}\right)$. Using algebra rules we get that $P_{n}(\mathbf{Q})=\mathbf{P} P_{n}(\mathbf{D}) \mathbf{P}^{-1}$
and then

$$
\begin{equation*}
\left\|P_{n}(\mathbf{Q})\right\|_{2} \leq\|\mathbf{P}\|_{2} \rho\left(P_{n}(\mathbf{Q})\right)\left\|\mathbf{P}^{-1}\right\|_{2}=K \max _{i \neq 1}\left|P_{n}\left(\lambda_{i}\right)\right|=K \max _{i \neq 1} \frac{\left|T_{n}\left(c \lambda_{i}-d\right)\right|}{T_{n}(c-d)}, \tag{43}
\end{equation*}
$$

with $K$ the condition number of $\mathbf{P}$.
For any $x \in\left(\lambda_{M}+\lambda_{m}-1,1\right)$ we have that $|c x-d|<c-d$, then for all the real eigenvalues of A but $\lambda_{1},\left|c \lambda_{i}-d\right|<c-d$. Noting that $c-d$ is strictly larger than 1 and $\tau(c-d)<\tau\left(c \lambda_{z}-d\right)$, for any complex eigenvalue $\lambda_{z}$, by Lemmas 1.1 and 1.2 , $p_{n}\left(\lambda_{i}\right) \rightarrow 0$ for all $i \neq 1$, which proves the convergence of the algorithm.

## B. Proof of Theorem 4.2

In order to proof Theorem 4.2 we will use the following auxiliary results.
Lemma 1.3: Let $\lambda_{m}, \lambda_{M}$ such that $\left[\lambda_{N}, \lambda_{2}\right] \nsubseteq\left[\lambda_{m}, \lambda_{M}\right]$ and $\left|c \lambda_{N}-d\right|<c \lambda_{2}-d$. Then, for fixed $c, \nu(c, d)$ is a decreasing function of $d$.

Proof. Let us see that $\partial \nu(c, d) / \partial d<0$.

$$
\nu(c, d)=\frac{\tau(c-d)}{\left|\tau\left(c \lambda_{2}-d\right)\right|}=\frac{\tau(c-d)}{\tau\left(c \lambda_{2}-d\right)}>0
$$

Then

$$
\frac{\partial \nu}{\partial d}=\frac{-\tau^{\prime}(c-d) \tau\left(c \lambda_{2}-d\right)+\tau(c-d) \tau^{\prime}\left(c \lambda_{2}-d\right)}{\tau\left(c \lambda_{2}-d\right)^{2}} .
$$

But since for $x>0, \tau^{\prime}(x)=-\tau(x) / \sqrt{x^{2}-1}$, then

$$
\frac{\partial \nu}{\partial d}=\frac{\tau(c-d)}{\tau\left(c \lambda_{2}-d\right)}\left[\frac{1}{\sqrt{(c-d)^{2}-1}}-\frac{1}{\sqrt{\left(c \lambda_{2}-d\right)^{2}-1}}\right]
$$

which is negative because $1<\left(c \lambda_{2}-d\right)^{2}<(c-d)^{2}$.
Lemma 1.4: Let $\lambda_{m}, \lambda_{M}$ such that $\left[\lambda_{N}, \lambda_{2}\right] \nsubseteq\left[\lambda_{m}, \lambda_{M}\right]$ and $\left|c \lambda_{N}-d\right|>\left|c \lambda_{2}-d\right|$ with $c \lambda_{N}-d<0$. Then, for fixed $c, \nu(c, d)$ is an increasing function of $d$.
Proof. Let us see that $\partial \nu(c, d) / \partial d>0$.

$$
\nu(c, d)=\frac{\tau(c-d)}{\left|\tau\left(c \lambda_{N}-d\right)\right|}=\frac{\tau(c-d)}{-\tau\left(c \lambda_{N}-d\right)}>0
$$

Then

$$
\frac{\partial \nu}{\partial d}=\frac{\tau^{\prime}(c-d) \tau\left(c \lambda_{N}-d\right)-\tau(c-d) \tau^{\prime}\left(c \lambda_{N}-d\right)}{\tau\left(c \lambda_{N}-d\right)^{2}}
$$

But since, for $x<0, \tau^{\prime}(x)=\tau(x) / \sqrt{x^{2}-1}$, then

$$
\frac{\partial \nu}{\partial d}=\frac{\tau(c-d)}{-\tau\left(c \lambda_{N}-d\right)}\left[\frac{1}{\sqrt{(c-d)^{2}-1}}+\frac{1}{\sqrt{\left(c \lambda_{2}-d\right)^{2}-1}}\right]
$$

which is positive.
Proposition 1.5: Let $\lambda_{m}, \lambda_{M}$ such that $\lambda_{M}-\lambda_{m}=2 / c$ is fixed and $\left[\lambda_{N}, \lambda_{2}\right] \nsubseteq\left[\lambda_{m}, \lambda_{M}\right]$. Then
i) If $\lambda_{2}-\lambda_{N}>\lambda_{M}-\lambda_{m}, \nu(c, d) \geq \nu\left(c, d^{*}\right)$, $d^{*}$ being the value such that $\lambda_{M}+\lambda_{m}=\lambda_{2}+\lambda_{N}$, that is, for a fixed $c, \nu(c, d)$ is minimum when $\lambda_{m}, \lambda_{M}$ are symmetrically placed with respect to $\lambda_{N}, \lambda_{2}$.
ii) If $\lambda_{2}-\lambda_{N} \leq \lambda_{M}-\lambda_{m}$ and $\lambda_{M}<\lambda_{2}$ then $\nu(c, d) \geq \nu\left(c, d^{*}\right), d^{*}$ being such that $\lambda_{M}=\lambda_{2}$, and in this case $\left[\lambda_{N}, \lambda_{2}\right] \subseteq\left[\lambda_{m}, \lambda_{M}\right]$
iii) If $\lambda_{2}-\lambda_{N} \leq \lambda_{M}-\lambda_{m}$ and $\lambda_{m}>\lambda_{N}$ then $\nu(c, d) \geq \nu\left(c, d^{*}\right), d^{*}$ being such that $\lambda_{m}=\lambda_{N}$, and in this case $\left[\lambda_{N}, \lambda_{2}\right] \subseteq\left[\lambda_{m}, \lambda_{M}\right]$

## Proof.

i) The result follows from Lemmas 1.3 and 1.4. If $\lambda_{2}>\lambda_{M}$, then $c \lambda_{2}-d>\left|c \lambda_{N}-d\right|$ and $\nu(c, d)$ is a decreasing function of $d=\left(\lambda_{M}+\lambda_{m}\right) c / 2$ which means that it decreases as $\lambda_{M}$ increases. The maximum value of $\lambda_{M}$ for which these conditions hold is $\lambda_{M}=$ $1 / c+\left(\lambda_{2}+\lambda_{N}\right) / 2$ for which $c \lambda_{2}-d=\left|c \lambda_{N}-d\right|$. If $\lambda_{N}<\lambda_{m}$, then $c \lambda_{2}-d<\left|c \lambda_{N}-d\right|$ and $\nu(c, d)$ is an increasing function of $d=$ $\left(\lambda_{M}+\lambda_{m}\right) c / 2$ which means that it increases when $\lambda_{M}$ increaseses. The minimum value of $\lambda_{M}$ for which these conditions hold is $\lambda_{M}=1 / c+\left(\lambda_{2}+\lambda_{N}\right) / 2$ for which $c \lambda_{2}-d=\left|c \lambda_{N}-d\right|$.
ii) In this case $c \lambda_{2}-d>\left|c \lambda_{N}-d\right|$, and $\nu(c, d)$ is a decreasing function of $d=\left(\lambda_{M}+\lambda_{m}\right) c / 2$ which means that it decreases when $\lambda_{M}$ increases. The maximum value of $\lambda_{M}$ for which these conditions hold is $\lambda_{M}=\lambda_{2}$.
iii) In this case $c \lambda_{2}-d<\left|c \lambda_{N}-d\right|$, and $\nu(c, d)$ is an increasing function of $d=\left(\lambda_{M}+\lambda_{m}\right) c / 2$ which means that it increases when $\lambda_{m}$ increases. The minimum value of $\lambda_{m}$ for which these conditions hold is $\lambda_{m}=\lambda_{N}$.

And finally, we are able to proof the theorem.
Proof of Theorem 4.2. If $\left[\lambda_{2}, \lambda_{N}\right] \subseteq\left[\lambda_{m}, \lambda_{M}\right]$ the result was proved in [33]. Let us suppose then that $\left[\lambda_{2}, \lambda_{N}\right] \nsubseteq\left[\lambda_{m}, \lambda_{M}\right]$. If $\lambda_{2}-\lambda_{N} \leq \lambda_{M}-\lambda_{m}$, it has been shown in Proposition 1.1 that
$\nu(c, d)$ has smaller values for $c, d$ such that $\left[\lambda_{N}, \lambda_{2}\right] \subseteq\left[\lambda_{m}, \lambda_{M}\right]$, and in this case $\lambda_{2}=\lambda_{M}$ and $\lambda_{N}=\lambda_{m}$ yields to the minimum $\nu(c, d)$.

If $\lambda_{2}-\lambda_{N}>\lambda_{M}-\lambda_{m}$, we have seen in Proposition 1.1 that $\nu(c, d)$ is smaller for $c, d$ such that $\lambda_{m}, \lambda_{M}$ are symmetrically placed with respect to $\lambda_{N}, \lambda_{2}$, that is, $\lambda_{M}=\lambda_{2}-\alpha$ and $\lambda_{m}=\lambda_{N}+\alpha$, $\alpha \geq 0$. Let us see that $\nu(c, d)$ is minimum for $\alpha=0$. First, note that

$$
c=\frac{2}{\lambda_{M}-\lambda_{m}}=\frac{2}{\lambda_{2}-\lambda_{N}-2 \alpha}, \text { and } d=\frac{\lambda_{M}+\lambda_{m}}{\lambda_{M}-\lambda_{m}}=\frac{\lambda_{2}+\lambda_{N}}{\lambda_{2}-\lambda_{N}-2 \alpha} .
$$

Thus

$$
\nu(c, d)=\frac{\tau(c-d)}{\tau\left(c \lambda_{2}-d\right)}=\frac{\tau(c-d)}{-\tau\left(c \lambda_{N}-d\right)}
$$

and taking into account that

$$
\begin{gathered}
\frac{\mathrm{d}}{\mathrm{~d} \alpha}(c \lambda-d)=2 \frac{2 \lambda-\lambda_{2}-\lambda_{N}}{\left(\lambda_{2}-\lambda_{N}-2 \alpha\right)^{2}}=2 \frac{c \lambda-d}{\left(\lambda_{2}-\lambda_{N}-2 \alpha\right)}, \\
\frac{\mathrm{d} \nu(c, d)}{\mathrm{d} \alpha}=\frac{-2 \tau(c-d)}{\tau\left(c \lambda_{2}-d\right)\left(\lambda_{2}-\lambda_{N}-2 \alpha\right)}\left[\frac{c-d}{\sqrt{(c-d)^{2}-1}}-\frac{c \lambda_{2}-d}{\sqrt{\left(c \lambda_{2}-d\right)^{2}-1}}\right]>0 .
\end{gathered}
$$

Then $\nu(c, d)$ is increasing with $\alpha$ and the minimum value is obtained for $\alpha=0$.

## C. Proof of Theorem 5.5

First of all, let us state the notation we will follow along the proof. For any weight matrix $\mathbf{A}(n)$ we denote its eigenvectors by $\mathbf{v}_{i}(n), i=1, \ldots, N$. Let us denote $\mathbf{V}(n)=\left[\mathbf{v}_{1}(n), \ldots, \mathbf{v}_{N}(n)\right]$ the matrix with all the eigenvectors of $\mathbf{A}(n)$. Thus, $\mathbf{A}(n) \mathbf{V}(n)=\mathbf{V}(n) \mathbf{D}(n)$, with $\mathbf{D}(n)=$ $\operatorname{diag}\left(\lambda_{1}(n), \ldots, \lambda_{N}(n)\right)$. Since $\mathbf{A}(n)$ is symmetric, it is diagonalizable and we can choose the base of eigenvectors in such a way that $\mathbf{V}(n)$ is orthogonal. Therefore, $\mathbf{v}_{1}(n)^{T} \mathbf{v}_{i}(n)=0, \forall i=$ $2, \ldots, N$, and $\mathbf{v}_{1}(n)=\mathbf{1} / \sqrt{N}=\mathbf{v}_{1}$, for all $n$.

Let $\mathbf{Q}(n)=\mathbf{A}(n)-\frac{1}{N} \mathbf{1 1}^{T}$, whose eigenvalues are 0 , with $\mathbf{v}_{1}(n)=\mathbf{1} / \sqrt{N}$ its corresponding eigenvector, and $\lambda_{2}(n), \ldots, \lambda_{N}(n)$, with the same eigenvectors as $\mathbf{A}(n)$. Taking all of this into account it is easy to see that $\mathbf{1 1}^{T}\left(\mathbf{x}(0)-\left(\mathbf{1}^{T} \mathbf{x}(0)\right) \mathbf{v}_{1}\right)=0$, and

$$
\begin{equation*}
\mathbf{A}(n)\left(\mathbf{x}(n)-\left(\mathbf{1}^{T} \mathbf{x}(0)\right) \mathbf{v}_{1}\right)=\mathbf{Q}(n)\left(\mathbf{x}(n)-\left(\mathbf{1}^{T} \mathbf{x}(0)\right) \mathbf{v}_{1}\right) \tag{44}
\end{equation*}
$$

Given two consecutive matrices, $\mathbf{Q}(n)$ and $\mathbf{Q}(n-1)$, let $\mathbf{P}(n)$ be the matrix such that $\mathbf{V}(n-1)=$ $\mathbf{V}(n) \mathbf{P}(n)$, that is, the matrix that changes from the base of eigenvectors of $\mathbf{Q}(n-1)$ to the
base of eigenvectors of $\mathbf{Q}(n)$. In a similar way, $\mathbf{R}(n)$ will be such that $\mathbf{V}(n-2)=\mathbf{V}(n) \mathbf{R}(n)$. The orthogonality of $\mathbf{V}(n)$, implies that the matrices $\mathbf{P}(n)=\mathbf{V}(n)^{-1} \mathbf{V}(n-1)$ and $\mathbf{R}(n)=$ $\mathbf{V}(n)^{-1} \mathbf{V}(n-2)$ are also orthogonal, and $\|\mathbf{P}(n)\|_{2}=\|\mathbf{R}(n)\|_{2}=1$.

Recalling the Chebyshev recurrence (18), we define the error at iteration $n$ by $\mathbf{x}(n)-\left(\mathbf{1}^{T} \mathbf{x}(0)\right) \mathbf{v}_{1}$. The equivalence

$$
\begin{equation*}
\mathbf{v}_{1}=2 \frac{T_{n}(c-d)}{T_{n+1}(c-d)}(c \mathbf{A}(n)-d \mathbf{I}) \mathbf{v}_{1}-\frac{T_{n-1}(c-d)}{T_{n+1}(c-d)} \mathbf{v}_{1} \tag{45}
\end{equation*}
$$

allows us to express the error by $\mathbf{e}(n) / T_{n}(c-d)$, with $\mathbf{e}(0)=\mathbf{x}(0)-\left(\mathbf{1}^{T} \mathbf{x}(0)\right) \mathbf{v}_{1}, \mathbf{e}(1)=$ $(c \mathbf{Q}(1)-d \mathbf{I}) \mathbf{e}(0)$ and

$$
\begin{equation*}
\mathbf{e}(n)=2(c \mathbf{Q}(n)-d \mathbf{I}) \mathbf{e}(n-1)-\mathbf{e}(n-2) . \tag{46}
\end{equation*}
$$

Each vector $\mathbf{e}(n)$ can be expressed as a linear combination of the eigenvectors of $\mathbf{Q}(n)$,

$$
\begin{equation*}
\mathbf{e}(n)=\sum_{i=1}^{N} \alpha_{i}(n) \mathbf{v}_{i}(n)=\mathbf{V}(n) \boldsymbol{\alpha}(n) \tag{47}
\end{equation*}
$$

Replacing $\mathbf{e}(n)$ by (47) in (46),

$$
\begin{align*}
\mathbf{e}(n) & =2(c \mathbf{Q}(n)-d \mathbf{I}) \mathbf{V}(n-1) \boldsymbol{\alpha}(n-1)-\mathbf{V}(n-2) \boldsymbol{\alpha}(n-1) \\
& =2(c \mathbf{Q}(n)-d \mathbf{I}) \mathbf{V}(n) \mathbf{P}(n) \boldsymbol{\alpha}(n-1)-\mathbf{V}(n) \mathbf{R}(n) \boldsymbol{\alpha}(n-2)  \tag{48}\\
& =2 \mathbf{V}(n)(c \mathbf{D}(n)-d \mathbf{I}) \mathbf{P}(n) \boldsymbol{\alpha}(n-1)-\mathbf{V}(n) \mathbf{R}(n) \boldsymbol{\alpha}(n-2) \\
& =\mathbf{V}(n)[2(c \mathbf{D}(n)-d \mathbf{I}) \mathbf{P}(n) \boldsymbol{\alpha}(n-1)-\mathbf{R}(n) \boldsymbol{\alpha}(n-2)]=\mathbf{V}(n) \boldsymbol{\alpha}(n) .
\end{align*}
$$

Therefore, the vectors $\boldsymbol{\alpha}(n)$ satisfy the recurrence

$$
\begin{equation*}
\boldsymbol{\alpha}(n)=2(c \mathbf{D}(n)-d \mathbf{I}) \mathbf{P}(n) \boldsymbol{\alpha}(n-1)-\mathbf{R}(n) \boldsymbol{\alpha}(n-2), \tag{49}
\end{equation*}
$$

with $\boldsymbol{\alpha}(0)=\boldsymbol{\alpha}(1)$.
Taking spectral norms,

$$
\begin{align*}
\|\boldsymbol{\alpha}(n)\|_{2} & =\|2(c \mathbf{D}(n)-d \mathbf{I}) \mathbf{P}(n) \boldsymbol{\alpha}(n-1)-\mathbf{R}(n) \boldsymbol{\alpha}(n-2)\|_{2} \leq \\
& \leq 2\|(c \mathbf{D}(n)-d \mathbf{I})\|_{2}\|\mathbf{P}(n)\|_{2}\|\boldsymbol{\alpha}(n-1)\|_{2}+\|\mathbf{R}(n)\|_{2}\|\boldsymbol{\alpha}(n-2)\|_{2} \leq  \tag{50}\\
& \leq\left(2 \max _{i}\left|c \lambda_{i}(n)-d\right|\|\boldsymbol{\alpha}(n-1)\|_{2}+\|\boldsymbol{\alpha}(n-2)\|_{2}\right) .
\end{align*}
$$

By Lemma 5.4 we can bound the norm of $\|\boldsymbol{\alpha}(n)\|$ by

$$
\begin{equation*}
\|\boldsymbol{\alpha}(n)\| \leq \kappa_{1}\left(x_{\max }\right)^{n}\|\boldsymbol{\alpha}(0)\| \tag{51}
\end{equation*}
$$

where the parameter $x_{\max }$ in this case is

$$
\begin{align*}
x_{\max } & =\max _{n} \max _{i=2, \ldots, N}\left|c \lambda_{i}(n)-d\right|=\max _{n}\left\{\left|c \lambda_{2}(n)-d\right|,\left|c \lambda_{N}(n)-d\right|\right\}=  \tag{52}\\
& =\max \left\{\left|c \lambda_{\max }-d\right|,\left|c \lambda_{\min }-d\right|\right\}
\end{align*}
$$

Using (5)

$$
\begin{equation*}
\frac{\kappa_{1}\left(x_{\max }\right)^{n}}{T_{n}(c-d)}=\frac{\kappa_{1}\left(x_{\max }\right)^{n} \tau(c-d)^{n}}{1+\tau(c-d)^{2 n}} \tag{54}
\end{equation*}
$$

which goes to zero if $\kappa_{1}\left(x_{\max }\right) \tau(c-d)<1$. When this happens $\lim _{n \rightarrow \infty} \mathbf{x}(n)=\left(\mathbf{1}^{T} \mathbf{x}(0) / \mathbf{1}^{T} \mathbf{1}\right) \mathbf{1}$, and the consensus is achieved.


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