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Abstract—The number of available algorithms for the so-called Basis Pursuit Denoising problem (or the related LASSO-problem) is large and keeps growing. Similarly, the number of experiments to evaluate and compare these algorithms on different instances is growing.

In this note, we present a method to produce instances with exact solutions which is based on a simple observation which is related to the so called *source condition* from sparse regularization.

EDICS: DSP-RECO, DSP-ALGO

I. INTRODUCTION

"Lately, there has been a lot of fuss about sparse approximation." is the beginning of the paper [30] from 2006 and this note could have started with the same sentence. Three different minimization problems have gained much attention. We follow [31] and denote them as follows: For a matrix $A \in \mathbf{R}^{k \times n}$ and $b \in \mathbf{R}^k$ and positive numbers σ , λ and τ we define the Basis Pursuit Denoising ([7]) with constraint by

$$\min_{x} \|x\|_{1} \quad \text{subject to} \quad \|Ax - b\|_{2} \le \sigma, \tag{BP}_{\sigma}$$

the Basis Pursuit Denoising with penalty ([7]) by

$$\min_{x} \frac{1}{2} \|Ax - b\|_{2}^{2} + \lambda \|x\|_{1}, \qquad (QP_{\lambda})$$

and the LASSO (least absolute shrinkage and selection operator [29])

$$\min \|Ax - b\|_2 \quad \text{subject to} \quad \|x\|_1 \le \tau. \tag{LS}_{\tau}$$

All three problems are related: if we denote with $x_{\text{QP}}(\lambda)$ a solution of (QP_{λ}) , this also solves (BP_{σ}) for $\sigma = ||Ax_{\text{QP}}(\lambda) - b||_2$ and (LS_{τ}) for $\tau = ||x_{\text{QP}}(\lambda)||_1$ (see e.g. [25], [31]). However, this relation is implicit and relies in general on the knowledge of the solutions. Hence, it is not totally true that these problems are equivalent.

One may argue, that (BP_{σ}) is harder than the other problems since its objective is nonsmooth and shall be minimized over a complicated convex set (e.g. projecting on this set is difficult). Moreover, one may argue, that (QP_{λ}) is harder than (LS_{τ}) since the latter has a smooth objective (to be minimized over a somehow simple convex set) while the first has a nonsmooth objective. Computational experience with with these problems lead to the same conclusion.

Recently, minimization problems similar to Basis Pursuit Denoising have appeared in several contexts, e.g. group sparsity (or joint sparsity) [13], [26], [32] for sparse recovery, nuclear norm minimization for low-rank matrix recovery [28] to name just two.

A. Notation

With $||x||_p$ we denote the *p*-norm of a vector $x \in \mathbf{R}^n$, A^T is the transpose of a matrix A, the range of a matrix A is denoted with $\operatorname{rg} A$ and with $\operatorname{Sign}(x)$ we denote the multivalued sign, i.e.

$$y \in \operatorname{Sign}(x) \iff y_i \begin{cases} = 1 & \text{if } x_i > 0 \\ = -1 & \text{if } x_i < 0 \\ \in [-1, 1] & \text{if } x_i = 0 \end{cases}$$

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II. CONSTRUCTION OF INSTANCES WITH KNOWN SOLUTION

In this section we illustrate how instances (i.e. tuples (A, b, λ)) can be generated, such that the solution x^* of (QP_{λ}) is known up to machine precision. This is achieved by prescribing the solution x^* (and the matrix A and the value λ) and computing a corresponding right hand side b.

The basis is the following simple observation which has a one-line proof:

Lemma 1: Let $A \in \mathbf{R}^{k \times n}$, $\lambda > 0$ and $x^* \in \mathbf{R}^n$ and let $w \in \operatorname{rg} A^T$ fulfill $w \in \operatorname{Sign}(x^*)$. Then it holds: If y is a solution to $A^T y = w$ and b is defined by $b = \lambda y + Ax^*$, then x^* is a solution of $(\operatorname{QP}_{\lambda})$.

Proof: Simply check

$$-A^{T}(Ax^{*} - b) = -A^{T}(Ax^{*} - \lambda y - Ax^{*})$$
$$= \lambda A^{T}y = \lambda w \in \lambda \operatorname{Sign}(x^{*}).$$

Hence x^* fulfills the necessary and sufficient condition for optimality.

Remark 2: The existence of the vector w is exactly the source condition used in sparse regularization of ill-posed problems. There one shows that a vector x^{\dagger} for which such a vector w exists can be reconstructed from noisy measurements b^{δ} with $||Ax^{\dagger} - b^{\delta}||_2 \leq \delta$ by solving (QP_{λ}) with b^{δ} instead of b and $\lambda \approx \delta$ and that one achieves a linear convergence rate, i.e. for the solution x_{λ}^{δ} one gets $||x_{\lambda}^{\delta} - x^{\dagger}||_1 = \mathcal{O}(\delta)$, see [16], [17], [23].

The following corollary reformulates the above lemma in a way which is more suitable for an algorithmic reformulation.

Corollary 3: Let $\{1, \ldots, n\}$ be partitioned into sets \mathcal{I} , \mathcal{A}_+ and \mathcal{A}_- and let $x^* \in \mathbf{R}^n$ be any vector such that

$$\begin{aligned} x_i^* &> 0, \quad i \in \mathcal{A}_+ \\ x_i^* &< 0, \quad i \in \mathcal{A}_- \\ x_i^* &= 0, \quad i \in \mathcal{T} \end{aligned} \tag{1}$$

and let $\lambda > 0$. Furthermore assume that $y \in \mathbf{R}^k$ fulfills

$$(A^{T}y)_{i} = 1, \quad i \in \mathcal{A}_{+}$$

$$(A^{T}y)_{i} = -1, \quad i \in \mathcal{A}_{-}$$

$$|A^{T}y|_{i} \leq 1, \quad i \in \mathcal{I}$$
(2)

and define $b = \lambda y + Ax^*$. Then x^* is a solution of (QP_{λ}) . According to this corollary we can construct an instance (A, b, λ)

with known solution x^* as follows:

- Specify A ∈ R^{m×n} and a sign-pattern (given by the partition A₊, A₋, I).
- 2) Construct a vector $y \in \mathbf{R}^m$ which fulfills (2).
- Choose any λ > 0 and any x^{*} ∈ Rⁿ which complies with the sign-pattern, i.e. (1) holds.
- 4) Define $b = \lambda y + Ax$.

The vector y can be constructed by several methods which are outline in Appendix A. These methods have been implemented in the Matlab package L1TestPack in the function construct_bpdn_rhs¹. One should note that a vector y as in Corollary 3 need not to exist.

¹The package is available at http://www.tu-braunschweig.de/iaa/personal/ lorenz/l1testpack. Indeed, for a fixed matrix A not every sign-pattern of x^* can occur as a minimizer of any (QP_{λ}) .

Remark 4: For injective A everything is much simpler: Since A^T is surjective, we can just choose some $w \in \text{Sign}(x^*)$, solve $A^T y = w$ and set $b = \lambda y + Ax^*$.

We discuss advantages and disadvantages of our approach:

Advantages:

- The algorithm is independent of the value of λ while the performance of solvers for (QP_λ) usually deteriorates for smaller λ, see, e.g. [11], [12] and Section III-A.
- The algorithm is independent of the dynamic range of the optimal value x^* , however, several experiments have recorded that the performance of solvers for (QP_{λ}) depends greatly on the dynamic range, see, e.g. [4] and Section III-C.
- For square matrices A with full rank, one immediately get a desired vector y by solving $A^T y = w$ for some vector $w \in \text{Sign}(x^*)$. While this setting is unusual, e.g., in compressed sensing, one encounters such situations in regularization with sparsity constraints, see [6], [9], [10], [18], [22], [27].

Disadvantages

- The construction of b from x^* leads to a specific noise model, namely, the noise is given by λy . Hence, there is no control about the noise distribution². This limits the use of instances constructed in this way to the comparison of solvers for basis pursuit denoising. For other sparse reconstruction methods like matching pursuit algorithms they seem to be useless.
- The algorithm produces one particular element w ∈ Sign(x*) and it is not clear if this has any additional properties. Usually, several w ∈ Sign(x*) ∩ rg A^T exist and probably the proposed method favors a particular form of w.

III. ILLUSTRATIVE INSTANCES

Numerous papers contain comparisons of different solvers for the three problems (BP_{σ}) , (QP_{λ}) and (LS_{τ}) , see e.g. [3], [4], [11], [18], [20], [25], [31], [34]. Hence, we not aim at yet another comparison of solvers but try to illustrate, how different features of the measurement matrix and the solution influence the difficulty of the problem.

From the zoo of available solvers we have chosen four. The choice was not uniformly at random but to represent four different classes: fpc [20] as a simple tuning of the basic iterative thresholding algorithm, FISTA [3] as a representative of the "optimal algorithms" in the sense of worst case complexity, GPSR [11] as a highly tuned basic gradient method and YALL1 [33] as a member of the class of alternating directions methods³. All these solvers proceed iteratively and use (basically) one application of A and one of A^T for each iteration. Hence, the runtime of these algorithms is mainly related to the number of iterations. We did not include higher order solvers like fss [21] or ssn [18] and also did not use any variant of homotopy approaches [24].

For algorithms we overrode the implemented stopping criteria by the criterion that the relative error in the reconstruction

$$R_n = \frac{\|x_n - x^*\|}{\|x^*\|}$$

falls below a given threshold.

²However, one observes that the noise level $||Ax^* - b|| = \lambda ||y||$ is proportional to λ which, again, motivates that one should choose λ proportional to the noise level.

³Sources: fpc version 2.0 http://www.caam.rice.edu/~optimization/L1/fpc/, GPSR version 6.0 http://www.lx.it.pt/~mtf/GPSR/, YALL1 version 1.0 http: //yall1.blogs.rice.edu/ and an own implementation of FISTA.



Fig. 2. Results for from Section III-B on the influence of the sparsity level s.

A. Influence of the parameter λ

Here we consider a standard example from compressed sensing, namely a sensing matrix A which consists of random rows of a DCT matrix. The setup is as follows:

Dimensions:

$$n = 1000$$
 variables,

• k = 200 measurements

Matrix A:

Random rows of a DCT matrix

Solution x^* :

s = 20 non-zero entries, magnitude normally distributed with mean zero and variance one.

 $10^{-1}, 10^{-2}, 10^{-4}$

Results:

 λ :

In general, all solver slow down for smaller values of λ . However, some solvers depend greatly on the size of λ , see Figure 1.

B. Influence of the sparsity level

While the construction of a test instance is independent of the parameter λ , it gets harder for less sparsity. The behavior of the solvers with respect to the sparsity level is illustrated by this example: Dimensions:

•
$$n = 2000$$
 variables,

• k = 200 measurements

Matrix A:

Bernoulli ensemble, i.e. random ± 1

Solution x^* :

s = 4,80 non-zero entries, respectively; magnitude normally distributed with mean zero and variance one.

 λ : 10⁻

Results:

Most solvers take longer for less sparsity; however, surprisingly, YALL1 is even faster for lower sparsity, see Figure 2.

C. Influence of the dynamic range of the entries in x^*

As claimed in the introduction, the dynamic range

$$\Theta(x^*) = \frac{\max\{|x^*| : x^* \neq 0\}}{\min\{|x^*| : x^* \neq 0\}}$$

also influences the performance.



Fig. 1. Results for from Section III-A on the influence of λ .

Dimensions:

- n = 3000 variables,
- k = 1000 measurements

Matrix A:

Union of three orthonormal basis: the identity matrix, the DCT matrix and an orthonormalized random matrix Solution x^* :

s = 50 non-zero entries, with a dynamic range of approximately 9, 701 and 55.000, respectively. 10^{-1}

 λ : 1 Results:

Some solvers dramatically slow down for larger dynamic range, see Figure 3

D. Influence of the coherence of A

To illustrate that also a large coherence can cause solvers to slow down, we have chosen the following setup: We considered square matrices $A \in \mathbf{R}^{n \times n}$ which are zero expect on the diagonal and a certain number K of lower off-diagonals, scaled to have ||A|| = 1:

$$A_{K} = c \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 1 & & \ddots & \ddots & & \vdots \\ 0 & \ddots & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & \underbrace{1 & \cdots & 1}_{K \text{ columns}} \end{bmatrix}.$$

We also considered the extreme case K = n, also known as the Heaviside matrix. Denoting the columns of A_K by a_j , we calculate the coherence of the matrix A_K as

$$\mu = \max_{i \neq j} \frac{\langle a_i | a_j \rangle}{\|a_i\| \|a_j\|} = \sqrt{\frac{K-1}{K}}.$$

Dimensions:

• n = 300 variables,

• k = 300 measurements

Matrix A:

Increasingly coherent matrices with K = 5, 40, 100, 300Solution x^* :

s = 30 non-zero entries, Bernoulli, i.e. randomly selected +1 and -1.

 $\lambda: 10^{-1}$

Results:

This problem, while with an square and invertible matrix, is known the be notoriously hard. Especially for large K all solvers deteriorate, see Figure 4.

APPENDIX A ALGORITHMS

Instead of $y \in \mathbf{R}^m$ we construct a vector $w \in \mathbf{R}^n$ such that

$$w \in \operatorname{rg} A^{I} \cap \operatorname{Sign}(x^{*})$$

which can be reformulated as

$$w_i = 1, \quad i \in \mathcal{A}_+$$

 $w_i = -1, \quad i \in \mathcal{A}_-$
 $w_i \leq 1, \quad i \in \mathcal{I}$

and $w \in \operatorname{rg} A^T$. Then y can be found by solving $A^T y = w$.

A. Solution by projection onto convex sets

The condition $w \in \operatorname{rg} A^T \cap \operatorname{Sign}(x^*)$ can be seen as a convex feasibility problem [1] since both the sets $\operatorname{rg} A^T$ and $\operatorname{Sign}(x^*)$ are convex. Moreover, the projection onto each set is computationally feasible: The projection onto the range of A^T can be calculated explicitly, e.g. with the help of QR factorization. If $A^T = QR$ with orthonormal Q and upper triangular R, the projection $P_{\operatorname{rg} A^T}$ is given by $P_{rgA^T} = Q(:, 1:k)Q(:, 1:k)^T$. Projecting onto the convex set $\operatorname{Sign}(x^*)$ is even simpler: Set the fixed components to ± 1 respectively and clip the others by $x \mapsto \max(\min(x, 1)x, -1)$. We done the projection onto $\operatorname{Sign}(x^*)$ by $P_{\operatorname{Sign}(x^*)}$.

Now we find w by alternatingly project an initial guess onto both sets, a strategy knows as *projection onto convex sets* (POCS) [8], [19]. This is given as pseudo code in Algorithm 1.

B. Solution by quadratic programming

We sketch another approach by quadratic programming: We call $\mathcal{A} = \mathcal{A}_+ \cup \mathcal{A}_-$ the *active set* and \mathcal{I} the *inactive set* and define $s \in \mathbf{R}^{\mathcal{A}}$ by

$$s_i = 1, \quad i \in \mathcal{A}_+ \tag{3}$$

$$s_i = -1, \quad i \in \mathcal{A}_-.$$

Furthermore we denote with $P_{\mathcal{A}} : \mathbf{R}^n \to \mathbf{R}^{\mathcal{A}}$ the projection which deletes the "inactive" components and with $P_{\mathcal{I}} : \mathbf{R}^n \to \mathbf{R}^{\mathcal{I}}$ the projection which deletes in "active" components and the respective



Fig. 3. Results for from Section III-C on the influence of the dynamic range.



Fig. 4. Results for from Section III-D on the influence of the coherence.

Algorithm 1 Calculation of y by POCS

Require: Input $A \in \mathbb{R}^{m \times n}$, a partition \mathcal{A}_+ , \mathcal{A}_- and \mathcal{I} of $\{1, \ldots, n\}$ (coded as $\operatorname{Sign}(x^*)$), a tolerance $\epsilon > 0$ and an initial guess w_0 . 1: for $i = 0, 1, \ldots$ do 2: $v^n = P_{\operatorname{rg} A^T} w^n$ 3: $w^{n+1} = P_{\operatorname{Sign}(x^*)} v^n$ 4: if $\max(\|v^n - w^n\|, \|w^{n+1} - v^n\|) \le \epsilon$ then 5: break 6: end if 7: end for 8: Solve $A^T y = w$ 9: return y

adjoint $P_{\mathcal{A}}^T$ and $P_{\mathcal{I}}^T$ which fill up the vectors be zeros. With this notation, we aim at finding $w \in \operatorname{rg} A^T$ such that

$$P_{\mathcal{A}}w = s$$
, and $||P_{\mathcal{I}}w||_{\infty} \leq 1$.

To fulfill the condition $w \in \operatorname{rg} A^T$ we use the orthogonal projection on $\operatorname{rg} A^T$, denoted by $P_{\operatorname{rg} A^T}$ and require $P_{\operatorname{rg} A^T} w = w$. Since w is determined on the active set \mathcal{A} we rewrite is as

$$w = P_{\mathcal{A}}^T s + P_{\mathcal{I}}^T z \tag{4}$$

with a $z \in \mathbf{R}^{\mathcal{I}}$. Putting this together we have to find a vector $z \in \mathbf{R}^{\mathcal{I}}$ such that

$$(P_{\operatorname{rg} A^T} - \operatorname{Id})P_{\mathcal{I}}^T z = (\operatorname{Id} - P_{\operatorname{rg} A^T})P_{\mathcal{A}}^T s, \quad ||z||_{\infty} \le 1.$$

We the abbreviations

$$\bar{P} = (P_{\mathrm{rg}\,A^T} - \mathrm{Id})P_{\mathcal{I}}^T
\bar{v} = (\mathrm{Id} - P_{\mathrm{rg}\,A^T})P_{\mathcal{A}}^T s$$
(5)

we reformulate this as the optimization problem

$$\min_{\boldsymbol{\epsilon}\in\mathbf{R}^{\mathcal{I}}} \frac{1}{2} \left\| \bar{P}z - \bar{v} \right\|^2 \text{ s.t. } \left\| z \right\|_{\infty} \le 1.$$
(6)

This quadratic programming or constrained regression problem can be solved by various methods [5] including the simple gradient projection [15] or the conditional gradient method [2], [14]. Note that we require that the optimal value of (6) is indeed zero.

Algorithm 2 gives pseudo-code for calculating y.

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Algorithm 2 Calculation of y by quadratic programming

Require: Input $A \in \mathbf{R}^{m \times n}$ and a partition \mathcal{A}_+ , \mathcal{A}_- and \mathcal{I} of $\{1, \ldots, n\}$.

1: Set *s* according to (3).

- 2: Calculate the projection matrix $P_{\text{rg}AT}$ (e.g. by QR-factorization or singular value decomposition) and define \bar{P} and \bar{v} according to (5).
- 3: Calculate z as a solution of (6).
- 4: if $\bar{P}z = \bar{v}$ then
- 5: calculate w according to (4)
- 6: **else**
- 7: return "Error: No solution with this sign-pattern"

8: end if

- 9: Solve $A^T y = w$
- 10: **return** y
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