Beamforming Design for Multiuser Two-Way Relaying: A Unified Approach via Max-Min SINR

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Abstract

In this paper, we develop a unified framework for beamforming designs in non-regenerative multiuser two-way relaying (TWR). The core of our framework is the solution to the max-min signal-to-interference-plus-noise-ratio (SINR) problem for multiuser TWR. We solve this problem using a Dinkelbach-type algorithm with near-optimal performance and superlinear convergence. We show that, using the max-min SINR solution as a corner stone, the beamforming designs under various important criteria, such as weighted sum-rate maximization, weighted sum mean-square-error (MSE) minimization, and average bit-error-rate (BER) or symbol-error-rate (SER) minimization, etc, can be reformulated into a monotonic program. A polyblock outer approximation algorithm is then used to find the desired solutions with guaranteed convergence and optimal performance (provided that the core max-min SINR solver is optimal). Furthermore, the proposed unified approach can provide important insights for tackling the optimal beamforming designs in other emerging network models and settings. For instances, we extend the proposed framework to address the beamforming design in collaborative TWR and multi-pair MIMO TWR. Extensive numerical results are presented to demonstrate the merits of the proposed beamforming solutions.

Keywords: Two-way relaying, beamforming, fractional program, semi-definite program, monotonic optimization.

Submission date: March 24, 2013Revision Date: July 2, 2018Associate Editor: Prof. Eduard Jorswieck

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I. INTRODUCTION

Relay communications have been long studied to enhance the capacity and expand the coverage of wireless networks. For conventional communications between two users via a single relay, four transmission phases in time or frequency are typically required: two used for user-to-relay, and the other two for relay-to-user. To improve spectral efficiency, a two-way relaying (TWR) method, referred to as physical-layer network coding (PNC) [1], was proposed to accomplish bidirectional data exchange in two phases. This PNC technique is remarkable for its potential to double the system throughput.

PNC for two-way relay channels has gained a growing interest in recent years [1]–[4]. Various relaying strategies have been proposed to exploit the benefit of PNC, including but not limited to, decode-and-forward [1], compress-and-forward [2], amplify-and-forward (AF) [3], and compute-and-forward [4]. Particularly, it was shown in [5] that PNC with nested lattice coding can achieve the capacity of the single-input single-output Gaussian two-way relay channel within $\frac{1}{2}$ bit. Later, the authors in [6], [7] showed that lattice-coding techniques can be efficiently incorporated into multiple-input multiple-output (MIMO) TWR, where the users and the relay are equipped with multiple antennas. It was revealed therein that near-capacity performance can be achieved in MIMO two-way relay channels.

More recently, multiuser two-way relaying, in which multiple users exchange data via a single relay in a pairwise or non-pairwise manner, has been intensively studied in the literature [8]–[15]. In these approaches, analogue network coding (ANC) is employed, i.e., simple AF operations are implemented at the relay and self interference is canceled at the user ends; multiple antennas are deployed at the relay to provide extra degrees of freedom, which enables a potential boost of the system throughput. However, to fully exploit this potential requires a proper design of the beamforming (or called *precoding*) matrix at the relay, which is in general a difficult problem. To date, only approximate algorithms have been proposed based on specific design criteria, such as zero-forcing [10], power minimization [11], max-min signal-to-interference-plus-noise ratio (SINR) [14], and maximum sum-rate [13], [15].

In this paper, we develop a unified framework to solve the beamforming optimization problems for

multiuser TWR. We use the classic max-min SINR problem as the core of our framework. Our major contribution is to show that the max-min SINR solution can be used as a corner stone to pursue the optimal beamforming designs based on arbitrary utility functions that are monotonic in the user SINRs. Our framework works for various optimization criteria, such as power minimization, weighted sum-rate maximization, average symbol-error-rate (SER) or bit-error rate (BER) minimization, etc. Relying on solving a series of max-min SINR problems, a polyblock outer approximation algorithm is developed to find the desired solutions with guaranteed convergence and global optimality (provided that the core max-min SINR solver yields the optimal solution).

The optimality and efficiency of our proposed framework depends on the choice of the max-min SINR solver. In our approach, the max-min SINR problem, treated as a max-min fractional program, is solved using a Dinkelbach-type algorithm [18]. This algorithm is optimal for the two-user case and can provide near-optimal performance for the general case of multiple pairs of users. It is worth mentioning that the max-min SINR problem can be alternatively solved using the bisection search method in [14] with linear (i.e., geometrically fast) convergence. In contrast, the proposed Dinkelbach-type algorithm has a quotient-(Q-)superlinear convergence speed [18], and hence in general exhibit faster convergence (and thus reduced computation) than the bisection search method.

Furthermore, the proposed unified approach can provide important insights for tackling the optimal beamforming designs in other emerging network models and settings. For instances, we extend the proposed framework to cover the beamforming design in collaborative TWR and multi-pair MIMO TWR. Specifically, for collaborative TWR, we propose the beamforming design under an individual power constraint at each relay node, which is more practical than the settings in [16], [17] (where the relays share a total power budget). For multi-pair MIMO TWR, an iterative optimization algorithm is developed to jointly optimize the transmit and receive beamforming vectors of each user, together with the relay precoding matrix. Extensive numerical results are presented to demonstrate the merits of the proposed beamforming solutions.

The rest of this paper is organized as follows. Section II outlines the notations in use and the system model. Section III discusses the max-min SINR problem and its solution, as well as the relation between the power minimization design and the max-min SINR design. A unified approach for beamforming designs is presented in Section IV. Sections V and VI discuss generalizations of the proposed framework to collaborative beamforming for multi-pair multi-relay TWR, as well as to multi-pair MIMO TWR. The proposed schemes are tested and compared with existing alternatives in Section VII, followed by the conclusions in Section VIII.

II. PRELIMINARIES

A. Notation

The following notation is used throughout this paper. Boldface fonts denote vectors or matrices, the *i*th entry of a vector, say a, is denoted by a_i ; $\mathbb{R}^{K \times M}$ and $\mathbb{C}^{K \times M}$ denote the *K*-by-*M* dimensional real and complex space, respectively. $\mathbb{R}^K_+ := \{a \in \mathbb{R}^{K \times 1} \mid a \ge 0\}$. Note that the vector inequalities, such as $a \ge 0$, are defined element-wise. $\lceil x \rceil$ denotes the nearest integer greater than or equal to x; $(\cdot)^*$ denotes complex conjugate, $(\cdot)^T$ denotes transpose, and $(\cdot)^H$ conjugate transpose; \otimes represents the Kronecker product; \odot denotes the Schur-Hadamard (element-wise) product; tr(A) denotes trace operator for matrix A, vec(A) operator creates a column vector from a matrix A by stacking its column vectors below one another, $A^{1/2}$ denotes the square-root of a positive semi-definite matrix A, diag (A_1, \ldots, A_M) denotes a block-diagonal matrix with A_1, \ldots, A_M as the submatrices in the diagonal; $\|\cdot\|$ denotes the Euclidean norm for vectors, and $|\cdot|$ denotes norm of a complex scalar; 0 and 1 denote all-zero and all-one vectors; $A \succeq 0$ means that a square matrix A is positive semi-definite; a circularly symmetric complex Gaussian random vector x with mean \bar{x} and covariance matrix Σ is denoted as $x \sim C\mathcal{N}(\bar{x}, \Sigma)$, where \sim stands for "distributed as"; $\mathcal{A} \setminus \mathcal{B}$ denotes the set obtained by excluding all the elements of set \mathcal{B} from set \mathcal{A} .



Fig. 1. A multi-pair two-way relaying system.

B. System Model for Multi-Pair TWR

As shown in Fig. 1, we consider a two-way relay (bidirectional) communication between K pairs of users, where the relay is equipped with M antennas and each user has a single antenna [8], [9]. Without loss of generality, it is assumed that the (2k - 1)th and the (2k)th users communicate with each other, k = 1, ..., K, through two phases. The communication channels between the relay and users are assumed to be flat-fading over a common narrow band. Following the convention in [8], [9], [13], [15], we assume global channel state information (CSI), i.e., all the users and the relay have full CSI.

In the first phase of the two-way relaying communication, all users transmit to the relay simultaneously, and the received signal $\boldsymbol{y}_R(t) \in \mathbb{C}^{M \times 1}$ at the relay is

$$\boldsymbol{y}_{R}(t) = \sum_{i=1}^{2K} \boldsymbol{h}_{i} \sqrt{p_{i}} s_{i}(t) + \boldsymbol{n}_{R}(t), \qquad (1)$$

where h_i , p_i and $s_i(t)$ denote the channel coefficient vector from user i to the relay, transmit power of user i, and unit-power transmitted symbol from user i, respectively, and $n_R(t) \in \mathbb{C}^{M \times 1}$ denotes the noise vector. With a given covariance matrix Λ_R , it is assumed $n_R(t) \sim C\mathcal{N}(0, \Lambda_R)$.

Upon receiving $\boldsymbol{y}_R(t)$, the non-regenerative relay amplifies and forwards the signal $\boldsymbol{x}_R(t) = \boldsymbol{A}\boldsymbol{y}_R(t)$ to all users in the next phase, where $\boldsymbol{A} \in \mathbb{C}^{M \times M}$ is the relay beamforming matrix. The transmit power at the relay is

$$p_R(\boldsymbol{A}) = E \|\boldsymbol{x}_R(t)\|^2$$

= $E \left\| \boldsymbol{A} \left(\sum_{i=1}^{2K} \boldsymbol{h}_i \sqrt{p_i} s_i(t) + \boldsymbol{n}_R(t) \right) \right\|^2$
= $\sum_{i=1}^{2K} p_i \|\boldsymbol{A} \boldsymbol{h}_i\|^2 + \operatorname{tr}(\boldsymbol{A} \boldsymbol{\Lambda}_R \boldsymbol{A}^H).$

Suppose that channel reciprocity holds for the uplink and downlink transmission between the relay and users. The received signal at user $i \in \{1, ..., 2K\}$ is given by

$$y_i(t) = \boldsymbol{h}_i^T \boldsymbol{A} \sum_{j=1}^{2K} \boldsymbol{h}_j \sqrt{p_j} s_j(t) + \boldsymbol{h}_i^T \boldsymbol{A} \boldsymbol{n}_R(t) + n_i(t)$$
(2)

where the receive noise $n_i(t) \sim C\mathcal{N}(0, \sigma_i^2)$.

Upon receiving the downlink signal, user (2k-1) intends to detect the signal $s_{2k}(t)$ from user 2k, and the term $\sqrt{p_{2k-1}}h_{2k-1}^T A h_{2k-1}s_{2k-1}(t)$ in (2) is referred to as "self-interference". In the spirit of ANC, this self-interference can be canceled before signal detection. The SINR at the (2k-1)th user is thus

$$\operatorname{SINR}_{2k-1}(\boldsymbol{A}) = \frac{p_{2k} |\boldsymbol{h}_{2k-1}^T \boldsymbol{A} \boldsymbol{h}_{2k}|^2}{\sum_{i \neq 2k-1, 2k} [p_i |\boldsymbol{h}_{2k-1}^T \boldsymbol{A} \boldsymbol{h}_i|^2] + \|\boldsymbol{\Lambda}_R^{1/2} \boldsymbol{A}^H \boldsymbol{h}_{2k-1}^*\|^2 + \sigma_{2k-1}^2};$$
(3)

and, similarly, the SINR at the (2k)th user is

$$\operatorname{SINR}_{2k}(\boldsymbol{A}) = \frac{p_{2k-1} |\boldsymbol{h}_{2k}^{T} \boldsymbol{A} \boldsymbol{h}_{2k-1}|^{2}}{\sum_{i \neq 2k-1, 2k} [p_{i} |\boldsymbol{h}_{2k}^{T} \boldsymbol{A} \boldsymbol{h}_{i}|^{2}] + \|\boldsymbol{\Lambda}_{R}^{1/2} \boldsymbol{A}^{H} \boldsymbol{h}_{2k}^{*}\|^{2} + \sigma_{2k}^{2}}.$$
(4)

Based on the SINRs (3) and (4), we will develop a unified approach for beamforming designs in AF-based TWR under different criteria.

III. SINR BALANCING OPTIMIZATION

In this section, we describe two alternative forms of the SINR balancing problem. The effective solution to this problem will serve as a corner stone of our proposed framework. We start with the first form of SINR balancing, i.e., the max-min SINR problem formulated as

$$\lambda^{\text{opt}} = \max_{\boldsymbol{A}} \min_{i=1,\dots,2K} \frac{\text{SINR}_{i}(\boldsymbol{A})}{\gamma_{i}}$$

s. t.
$$\sum_{i=1}^{2K} p_{i} \|\boldsymbol{A}\boldsymbol{h}_{i}\|^{2} + \text{tr}(\boldsymbol{A}\boldsymbol{\Lambda}_{R}\boldsymbol{A}^{H}) \leq \check{P}_{R}$$
 (5)

where γ_i denotes the SINR target for user *i*, and \check{P}_R denotes the total power budget at the relay.

Relying on a semi-definite programming (SDP) based Dinkelbach-type algorithm, this max-min SINR problem has been solved for one-pair (i.e., K = 1) TWR [19]. The problem has also been approximately solved using bisection search over the SDP relaxation solvers for related power minimization problems for the general case of K-pair users [14]. Here, we generalize the Dinkelbach-type algorithm in [19] to approximately solve (5) for the case of K-pair users. We show that the proposed algorithm is more efficient than the bisection search method.

We start with the following definitions:

$$\boldsymbol{q}_{ji} := \operatorname{vec}(\boldsymbol{h}_j \boldsymbol{h}_i^T) \text{ and } \boldsymbol{B}_i := \operatorname{diag}(\boldsymbol{h}_i^T, \dots, \boldsymbol{h}_i^T) \in \mathbb{C}^{M \times (MM)}$$
 (6)

where \boldsymbol{h}_i^T is repeated by M times in \boldsymbol{B}_i .

Let $\Theta := \sum_{i=1}^{2K} [p_i h_i h_i^H] + \Lambda_R$, and $\Phi := (\Theta^{1/2})^T \otimes I_M$. Further let $a := \operatorname{vec}(A)$, $X := aa^H$, $E_0 := \Phi^H \Phi$. Then we have the relay transmit power:

$$\sum_{i=1}^{2K} p_i \|\boldsymbol{A}\boldsymbol{h}_i\|^2 + \operatorname{tr}(\boldsymbol{A}\boldsymbol{\Lambda}_R\boldsymbol{A}^H) = \operatorname{tr}(\boldsymbol{A}(\sum_{i=1}^{2K} p_i\boldsymbol{h}_i\boldsymbol{h}_i^H + \boldsymbol{\Lambda}_R)\boldsymbol{A}^H) = \|\boldsymbol{\Phi}\boldsymbol{a}\|^2 = \operatorname{tr}(\boldsymbol{E}_0\boldsymbol{X}).$$

With (6), we also have $|\boldsymbol{h}_{j}^{T}\boldsymbol{A}\boldsymbol{h}_{i}|^{2} = |\boldsymbol{q}_{ji}^{T}\boldsymbol{a}|^{2}$ and $\boldsymbol{h}_{i}^{T}\boldsymbol{A}\boldsymbol{\Lambda}_{R}\boldsymbol{A}^{H}\boldsymbol{h}_{i}^{*} = \|\boldsymbol{\Lambda}_{R}^{1/2}\boldsymbol{A}^{H}\boldsymbol{h}_{i}^{*}\|^{2} = \|\boldsymbol{\Lambda}_{R}^{1/2}\boldsymbol{B}_{i}\boldsymbol{a}\|^{2}$. Hence,

we have

$$\frac{\mathrm{SINR}_{2k-1}(\boldsymbol{a})}{\gamma_{2k-1}} = \frac{p_{2k}|\boldsymbol{q}_{2k-1,2k}^T\boldsymbol{a}|^2}{\gamma_{2k-1}(\sum_{i\neq 2k-1,2k}p_i|\boldsymbol{q}_{2k-1,i}^T\boldsymbol{a}|^2 + \|\boldsymbol{\Lambda}_R^{1/2}\boldsymbol{B}_{2k-1}\boldsymbol{a}\|^2 + \sigma_{2k-1}^2)},$$

and

$$\frac{\text{SINR}_{2k}(\boldsymbol{a})}{\gamma_{2k}} = \frac{p_{2k-1} |\boldsymbol{q}_{2k,2k-1}^T \boldsymbol{a}|^2}{\gamma_{2k} (\sum_{i \neq 2k-1,2k} p_i |\boldsymbol{q}_{2k,i}^T \boldsymbol{a}|^2 + \|\boldsymbol{\Lambda}_R^{1/2} \boldsymbol{B}_{2k} \boldsymbol{a}\|^2 + \sigma_{2k}^2)}$$

Define $E_{2k-1}^{(1)} := p_{2k} q_{2k-1,2k}^* q_{2k-1,2k}^T$, $E_{2k-1}^{(2)} := \sum_{i \neq 2k-1,2k} [p_i q_{2k-1,i}^* q_{2k-1,i}^T] + B_{2k-1}^H \Lambda_R B_{2k-1}$, $E_{2k}^{(1)} := p_{2k-1} q_{2k,2k-1}^* q_{2k,2k-1}^T$, and $E_{2k}^{(2)} := \sum_{i \neq 2k-1,2k} [p_i q_{2k,i}^* q_{2k,i}^T] + B_{2k}^H \Lambda_R B_{2k}$, for $k = 1, \dots, K$. In terms of X, let

$$f_i(\boldsymbol{X}) := \operatorname{tr}(\boldsymbol{E}_i^{(1)}\boldsymbol{X}) \text{ and } g_i(\boldsymbol{X}) := \operatorname{tr}(\boldsymbol{E}_i^{(2)}\boldsymbol{X}) + \sigma_i^2, \text{ for } i = 1, \dots, 2K.$$
(7)

Using X as the optimization variable and dropping the constraint of rank(X) = 1, we can relax (5) to

$$\tilde{\lambda}^{\text{opt}} = \max_{\mathbf{X}} \min_{i=1,\dots,2K} \frac{f_i(\mathbf{X})}{\gamma_i g_i(\mathbf{X})}$$
s. t. $\mathbf{X} \succeq 0$, $\operatorname{tr}(\mathbf{E}_0 \mathbf{X}) \leq \check{P}_R$.
(8)

The problem (8) is a max-min fractional program, and can be solved using a primal Dinkelbach-type algorithm [18]. This algorithm is based on solving a sequence of the following parametric optimization problems for $\lambda \leq \lambda^{\text{opt}}$:

$$\max_{\boldsymbol{X}} \min_{i=1,\dots,2K} f_i(\boldsymbol{X}) - \lambda \gamma_i g_i(\boldsymbol{X})$$
s. t. $\boldsymbol{X} \succeq 0$, $\operatorname{tr}(\boldsymbol{E}_0 \boldsymbol{X}) \leq \check{P}_R$.
(9)

Let $E_i := E_i^{(1)} - \lambda \gamma_i E_i^{(2)}$, i = 1, ..., 2K. The problem (9) becomes a convex SDP as

$$\min_{\mathbf{X},\tau} - \tau$$
(10)

t. $\mathbf{X} \succeq 0$, $\operatorname{tr}(\mathbf{E}_0 \mathbf{X}) \leq \check{P}_R$, $\operatorname{tr}(\mathbf{E}_i \mathbf{X}) - \lambda \gamma_i \sigma_i^2 \geq \tau$, $i = 1, \dots, 2K$.

This SDP can be solved by the interior point method in polynomial time [20].

Relying on this SDP solution, we propose the following algorithm to solve (8):

Algorithm 1: for max-min SINR problem

s.

Initialize:
$$A^0 = \left(\frac{\hat{P}_R}{\sum_{i=1}^{2K} (p_i || h_i ||^2) + \operatorname{tr}(\Lambda_R)}\right)^{1/2} I$$
, $X^{(0)} = \operatorname{vec}(A^0)\operatorname{vec}(A^0)^H$, and $j = 0$.
Repeat: $j = j + 1$,
given $X^{(j-1)}$, find $\lambda^{(j)} = \min_{i=1,...,2K} \frac{f_i(X^{(j-1)})}{\gamma_i g_i(X^{(j-1)})}$;
given $\lambda^{(j)}$, solve (10) with SDP to obtain: $X^{(j)} = \arg \max_X \min_{i=1,...,2K} [f_i(X) - \lambda^{(j)} \gamma_i g_i(X)]$;
until $\min_{i=1,...,2K} [f_i(X^{(j)}) - \lambda^{(j)} \gamma_i g_i(X^{(j)})] \leq 0$.
Output: $\tilde{\lambda}^{\operatorname{opt}} = \lambda^{(j)}$, and $X^{(j)}$ as the solution.

Algorithm 1 is a classic Dinkelbach-type algorithm [18]. In Problem (8), it is clear that $0 < g_i(\mathbf{X}) < \infty, \forall \mathbf{X}$, and $\tilde{\lambda}^{\text{opt}}$ is finite. Hence, Condition 8.5 in [18] holds. According to [18, Theorem 8.7], we immediately have the following result.

Lemma 1: Algorithm 1 converges Q-superlinearly¹ to the global optimal solution X^{opt} for (8).

Remark 1: We note that the max-min SINR problem can be alternatively solved with the bisection search method in [14]. It is known that the bisectional search has a linear, i.e., geometrically fast convergence speed. In contrast, the proposed Dinkelbach-type algorithm has quotient-superlinear convergence. Therefore, the proposed algorithm in general exhibits a faster convergence speed than the bisection search method in [14]. We further remark that, the proposed Algorithm 1 is guaranteed to converge to the optimal solution of (8) from any feasible initial A^0 per Lemma 1. Here, we set A^0 to be a scaled identity matrix for simplicity; we may also use the existing beamforming solutions, such as the ZF or MMSE beamforming in [9] as A^0 , for initialization. The choice of A^0 does not significantly affect the convergence speed.

Remark 2: The optimality of the solution given by Algorithm 1 to the original problem in (5) depends on the rank of the solution matrix X^{opt} . If Algorithm 1 yields a rank-one X^{opt} for (8), then we find the optimal a^{opt} as the (scaled) eigenvector with respect to the only positive eigenvalue of X^{opt} , and obtain optimal beamforming matrix A^{opt} for the original problem (5) by "de-stacking" the $MM \times 1$ vector a^{opt} into a $M \times M$ matrix. In fact, for the two-user case, it was shown in [12], [19] that the problem (10), and consequently (8), always has a rank-one optimal solution X^{opt} . However, for the general K > 1 case, the existence of a rank-one optimal solution for (10) cannot be provably guaranteed; see also [14]. Hence, the exact optimal solution for the original problem (5) may not be constructed from the optimal X^{opt} for its relaxed problem (8), the solution to which possibly has a rank greater than one. Randomized rounding is a widely adopted method to obtain a feasible rank-one approximate solution from the SDP relaxation; specifically, a Gaussian randomized rounding strategy [20] can be applied to get a vector a^{opt} from X^{opt}

¹Let $\tilde{\lambda}^{\text{opt}}$ denote the optimal value of problem (8), and $\lambda^{(j)}$ the output value of the *j*-th iteration of Algorithm 1. We say that the sequence $\lambda^{(j)}$ converges Q-superlinearly to $\tilde{\lambda}^{\text{opt}}$ if $\lim_{j \to \infty} \frac{|\lambda^{(j+1)} - \tilde{\lambda}^{\text{opt}}|}{|\lambda^{(j)} - \tilde{\lambda}^{\text{opt}}|} = 0.$

to nicely approximate the solution of the original problem (5).

It is worth mentioning that, for the case of K > 1, the output value of Algorithm 1, obtained by dropping the rank constraint, is an upper bound of the solution to the original max-min SINR problem in (5). This upper bound can be used as a benchmark to assess the approximate solution obtained by randomized rounding.

B. Power Minimization Problem

We next describe the SINR balancing problem in the form of power minimization. We show that, for the two alternative forms of the SINR balancing problem, the solution to one can be obtained through solving the other.

The power minimization problem is formulated as follows:

$$\min_{\boldsymbol{A}} \sum_{i=1}^{2K} p_i \|\boldsymbol{A}\boldsymbol{h}_i\|^2 + \operatorname{tr}(\boldsymbol{A}\boldsymbol{\Lambda}_R \boldsymbol{A}^H)$$
s. t. $\operatorname{SINR}_i(\boldsymbol{A}) \ge \gamma_i, \quad i = 1, \dots, 2K.$
(11)

Noting a = vec(A) and $X = aa^{H}$, and dropping the rank constraint of X, we can rewrite (11) as

$$P_{R}(\lambda) = \min_{\mathbf{X} \succeq 0} \operatorname{tr}(\mathbf{E}_{0}\mathbf{X})$$

s. t. $\frac{f_{i}(\mathbf{X})}{g_{i}(\mathbf{X})} \ge \lambda \gamma_{i}, \quad i = 1, \dots, 2K.$ (12)

Clearly, setting the parameter λ to 1 reduces (12) to (11). Here, we allow λ to be an arbitrary positive number for ease of further discussions. We note that the power minimization in (12) can be efficiently solved with a single SDP [14].

We next establish a close relation between the max-min SINR problem in (5) and the power minimization problem in (12). We first show that (12) can be solved via solving (5). Let $\tilde{\lambda}^{opt}(\check{P}_R)$ denote the optimal value of (8) for a given power budget \check{P}_R . It can be shown that $\tilde{\lambda}^{opt}(\check{P}_R)$ is a strictly increasing function of \check{P}_R , and the optimal solution to (11) is the same as that to (5) with the power budget P_R satisfying $\tilde{\lambda}^{opt}(P_R) = 1$. (See the Appendix for proof.) As a result, the optimal solution to (11) can be obtained by solving the equation $\tilde{\lambda}^{opt}(P_R) = 1$, which simply requires a one-dimensional bisection search. What remains is to show that (5) can be solved via solving (12). It can be similarly shown that $P_R(\lambda)$ in (12) is a strictly increasing function of λ . Together with the fact that, for an arbitrary $\lambda > 0$, (12) is readily solvable using a single SDP, we conclude that (5) is solvable by a bisection search over λ satisfying $P_R(\lambda) = \check{P}_R$.

So far, we have shown that the power minimization and max-min SINR problems are two alternative forms of the SINR balancing problem. This allows us to freely choose a more tractable form, i.e., a form that is more efficiently solvable, as the corner stone to pursue the optimal beamforming designs under various important optimization criteria, as detailed in what follows.

IV. A UNIFIED APPROACH VIA MONOTONIC PROGRAM

In this section, using the max-min SINR or power minimization solution as a corner stone, we propose a unified approach to find the relay beamforming designs for sum rate maximization, sum MSE minimization, and average BER minimization, etc.

A. Some Useful Definitions

We start with some commonly used terminologies in monotonic programming [21]:

Definition 1 (Box): A box [0, b] is defined as the set of all z such that $0 \le z \le b$.

Definition 2 (Normal): A set S is called *normal* if $z' \leq z$ and $z \in S$ implies $z' \in S$.

Definition 3 (Reverse Normal): A set S is called *reverse normal* if $z' \ge z$ and $z \in S$ implies $z' \in S$. **Definition 4 (Polyblock):** For any finite vector set $T := \{v_j | j = 1, ..., J\}$, the union of all the boxes $[0, v_j], \forall j$, is a *polyblock* with vertex set T.

Definition 5 (Proper): A vertex $v_j \in \mathcal{T}$ is called *proper* if there does not exist another $v_{j'} \in \mathcal{T}$ such that $v_{j'} \geq v_j$. A polyblock is fully determined by its proper vertices.

Definition 6 (Projection): For any $z \in \mathbb{R}^{2K} \setminus \{0\}$ and a normal set \mathcal{G} , $\pi_{\mathcal{G}}(z)$ is a *projection* of z on \mathcal{G} if $\pi_{\mathcal{G}}(z) = \lambda z$ where $\lambda = \max\{\alpha \mid \alpha z \in \mathcal{G}\}$; i.e., $\pi_{\mathcal{G}}(z)$ is the unique point where the halfline from 0 through z meets the upperboundary of \mathcal{G} .

B. Weighted Sum-Rate Maximization

Now consider the beamforming design for weighted sum-rate maximization. Treat the inter-user interference as noise. For the $SINR_i(A)$ in (3) and (4), we adopt a Shannon-capacity rate formula $r_i(A) = 0.5 \log_2(1 + SINR_i(A))$ due to its wide applications in communication systems. The results will be generalized to other utility functions in the sequel. Let w_i denote the priority weight for user *i*. We aim to solve the weighted sum-rate maximization problem formulated as

$$\max_{\boldsymbol{A}} \sum_{i=1}^{2K} 0.5 w_i \log_2(1 + \text{SINR}_i(\boldsymbol{A}))$$

s. t.
$$\sum_{i=1}^{2K} p_i \|\boldsymbol{A}\boldsymbol{h}_i\|^2 + \text{tr}(\boldsymbol{A}\boldsymbol{\Lambda}_R \boldsymbol{A}^H) \leq \check{P}_R.$$
 (13)

In terms of $\mathbf{X} = \text{vec}\mathbf{A}(\text{vec}\mathbf{A})^H$, we rewrite (13) as

$$\max_{\boldsymbol{X} \succeq 0} \sum_{i=1}^{2K} 0.5 w_i \log_2(1 + \operatorname{SINR}_i(\boldsymbol{X})), \quad \text{s. t. } \operatorname{tr}(\boldsymbol{E}_0 \boldsymbol{X}) \leq \check{P}_R$$
(14)

where $\text{SINR}_i(X) = f_i(X)/g_i(X)$. Note that the rank constraint of X is dropped in (14), and thus (14) is in fact a relaxation of (13).

Define the set $\mathcal{X} := \{ \mathbf{X} \mid \text{tr}(\mathbf{E}_0 \mathbf{X}) \leq \check{P}_R \}$. Introducing an auxiliary vector $\mathbf{z} = [z_1, \dots, z_{2K}]^T$, we can reformulate (14) into

$$\max_{\boldsymbol{z}\in\mathcal{Z}} \Phi(\boldsymbol{z}) := \sum_{i=1}^{2K} 0.5 w_i \log_2(z_i),$$
(15)

where the feasible set $\mathcal{Z} := \{ \boldsymbol{z} \mid 1 \leq z_i \leq 1 + \text{SINR}_i(\boldsymbol{X}), i = 1, ..., 2K, \forall \boldsymbol{X} \in \mathcal{X} \}$. Let $\boldsymbol{z}^{\text{opt}}$ be the optimal solution to (15). Then, $\boldsymbol{X}^{\text{opt}} \in \mathcal{X}$ satisfying $z_i^{\text{opt}} = 1 + \text{SINR}_i(\boldsymbol{X}^{\text{opt}})$ for all i is clearly the optimal solution to the original problem (14).

Now let

$$\mathcal{G} := \{ \boldsymbol{z} \mid 0 \le z_i \le 1 + \mathrm{SINR}_i(\boldsymbol{X}), \forall i, \ \forall \boldsymbol{X} \in \mathcal{X} \}.$$
(16)

Also let $\boldsymbol{b}(\boldsymbol{X}) := [1 + \text{SINR}_1(\boldsymbol{X}), \dots, 1 + \text{SINR}_{2K}(\boldsymbol{X})]^T$, for any $\boldsymbol{X} \in \mathcal{X}$. Then $\mathcal{G} = \bigcup_{\boldsymbol{X} \in \mathcal{X}} [\boldsymbol{0}, \boldsymbol{b}(\boldsymbol{X})]$, implying that \mathcal{G} can be represented as the union of an infinite number of normal boxes; hence, \mathcal{G} is also normal [21]. Let $\boldsymbol{d} := [d_1, \ldots, d_{2K}]^T$, with

$$d_{2k-1} = 1 + \frac{p_{2k}\check{P}_R \|\boldsymbol{h}_{2k-1}\|^2 \|\boldsymbol{h}_{2k}\|^2}{\sigma_{2k-1}^2}, \qquad d_{2k} = 1 + \frac{p_{2k-1}\check{P}_R \|\boldsymbol{h}_{2k-1}\|^2 \|\boldsymbol{h}_{2k}\|^2}{\sigma_{2k}^2}.$$
 (17)

It clearly holds: $1 + \text{SINR}_i(\mathbf{X}) \le d_i, \forall i, \forall \mathbf{X} \in \mathcal{X}$. Therefore, $\mathcal{G} \subset [\mathbf{0}, \mathbf{d}]$ is a compact normal set with nonempty interior. Further define $\mathcal{H} := \{\mathbf{z} \mid z_i \ge 1, \forall i\}$. Clearly, \mathcal{H} is a reverse normal set. Then (15) can be written in the form of a standard MP [21] as

$$\max_{\boldsymbol{z}} \Phi(\boldsymbol{z}), \quad \text{s. t.} \quad \boldsymbol{z} \in \mathcal{G} \cap \mathcal{H}.$$
(18)

For the MP (18), a polyblock outer approximation method can be employed to efficiently find its global optimal solution [21]. Specifically, we target at constructing a nested sequence of polyblocks \mathcal{P}_n , $n = 1, 2, \ldots$, approximating $\mathcal{G} \cap \mathcal{H}$: $\mathcal{P}_1 \supset \mathcal{P}_2 \supset \cdots \supset \mathcal{G} \cap \mathcal{H}$ in such a way that $\max_{z \in \mathcal{P}_n} \Phi(z) \searrow \max_{z \in \mathcal{G} \cap \mathcal{H}} \Phi(z)$. Denote the maximizer at iteration n as

$$\boldsymbol{z}^{n} = \arg \max_{\boldsymbol{z} \in \mathcal{T}_{n}} \Phi(\boldsymbol{z}), \tag{19}$$

where \mathcal{T}_n is the (finite) proper vertex set of \mathcal{P}_n . Note that z^n can be obtained by exhaustively searching over the finite set \mathcal{T}_n . If $z^n \in \mathcal{G} \cap \mathcal{H}$, then it solves the MP in (18). Otherwise, we find the next polyblock \mathcal{P}_{n+1} contained in \mathcal{P}_n but still containing $\mathcal{G} \cap \mathcal{H}$, and continue the process.

We next find \mathcal{P}_{n+1} from \mathcal{P}_n . Let y^n be the projection of z^n on \mathcal{G} , i.e., $y^n = \pi_{\mathcal{G}}(z^n)$, and denote

$$\boldsymbol{z}^{n}(i) = \boldsymbol{z}^{n} - (z_{i}^{n} - y_{i}^{n})\boldsymbol{e}_{i}, \quad i = 1, \dots 2K,$$

$$(20)$$

where e_i is a unit vector with the only non-zero (i.e., "1") in the *i*-th entry. Note that $z^n(i)$ is obtained by replacing the *i*-th entry of z^n by y_i^n . Clearly, $y^n \leq z^n(i) \leq z^n$. Let \mathcal{T}_{n+1} be the set obtained from \mathcal{T}_n by replacing the vertex z^n with 2K new vertices $z^n(i)$ and then remove the improper vertices; i.e., $\mathcal{T}_{n+1} = (\mathcal{T}_n \setminus \{z^n\}) \cup \{z^n(i) \mid z^n(i) \text{ is proper}\}$. Since $z^{\text{opt}} \in \mathcal{H}$, we can further reduce the vertex set $\mathcal{T}_{n+1} = \mathcal{T}_{n+1} \cap \mathcal{H}$. From [21, Proposition 17], we immediately have

Lemma 2: The polyblock \mathcal{P}_{n+1} with vertex set \mathcal{T}_{n+1} satisfies $(\mathcal{G} \cap \mathcal{H}) \subset \mathcal{P}_{n+1} \subset \mathcal{P}_n$.

Lemma 2 guarantees the validity of the above constructed \mathcal{P}_{n+1} to continue the polyblock outer approximation process. A key step in the above construction of \mathcal{P}_{n+1} is to find the projection $\boldsymbol{y}^n = \pi_{\mathcal{G}}(\boldsymbol{z}^n) = \lambda^n \boldsymbol{z}^n$, which can be determined by solving

$$\lambda^{n} = \max\{\alpha \mid \alpha \boldsymbol{z}^{n} \in \mathcal{G}\}$$

= $\max\{\alpha \mid \alpha \leq \min_{i=1,\dots,2K} \frac{1 + \operatorname{SINR}_{i}(\boldsymbol{X})}{z_{i}^{n}}, \forall \boldsymbol{X} \in \mathcal{X}\}$
= $\max_{\boldsymbol{X} \in \mathcal{X}} \min_{i=1,\dots,2K} \frac{1 + \operatorname{SINR}_{i}(\boldsymbol{X})}{z_{i}^{n}},$ (21)

where the second step utilizes the definition of \mathcal{G} in (16). The above is an extended max-min SINR balancing problem written as

$$\lambda^{n} = \max_{\boldsymbol{X}} \min_{i=1,\dots,2K} \frac{1 + \operatorname{SINR}_{i}(\boldsymbol{X})}{z_{i}^{n}}$$
s. t. $\boldsymbol{X} \succeq 0, \quad \operatorname{tr}(\boldsymbol{E}_{0}\boldsymbol{X}) \leq \check{P}_{R}.$
(22)

This problem can be solved using the Dinkelbach-type Algorithm 1 with minor modifications. Use the definitions in Section II (such as Φ , q_{ji} , B_i , and $g_i(X)$), except that $f_i(X)$ is redefined as $f_i(X) := tr(\boldsymbol{E}_i^{(1)}\boldsymbol{X}) + tr(\boldsymbol{E}_i^{(2)}\boldsymbol{X}) + \sigma_i^2$. Then the solution of (22) can be obtained by solving a series of (9).

We are now ready to implement polyblock outer approximation method for (13). For a given accuracy tolerance level $\epsilon > 0$, we say that a feasible \bar{z} is an ϵ -optimal solution if $(1 + \epsilon)\Phi(\bar{z}) \ge \Phi(z^{\text{opt}})$. The following algorithm is proposed to find an ϵ -optimal solution for (14).

Algorithm 2: for weighted sum-rate maximization

Initialize: select an accuracy level $\epsilon > 0$, let n = 0, $\mathcal{T}_0 = \{d\}$, and $CBV = -\infty$.

Repeat:

1). let $z^n = \arg \max_{z \in T_n} \Phi(z)$, For z^n , use Algorithm 1 to solve (22) to obtain λ^n , and the corresponding X^{opt} , as well as $y^n = \lambda^n z^n$.

2). If $y^n \in \mathcal{H}$ and $\Phi(y^n) > \text{CBV}$, then $\text{CBV} = \Phi(y^n)$, $\bar{z} = y^n$ and $\bar{X} = X^{\text{opt}}$.

3). Let
$$\boldsymbol{z}^n(i) = \boldsymbol{z}^n - (z_i^n - y_i^n)\boldsymbol{e}_i, \forall i, \text{ and } \mathcal{T}_{n+1} = [(\mathcal{T}_n \setminus \{\boldsymbol{z}^n\}) \cup \{\text{proper } \boldsymbol{z}^n(i)\}] \cap \mathcal{H}.$$

4). Further remove from \mathcal{T}_{n+1} any $v_j \in \mathcal{T}_{n+1}$ satisfying $\Phi(v_j) \leq \text{CBV}(1+\epsilon)$.

5). Set
$$n = n + 1$$
.

until $\mathcal{T}_n = \phi$.

Output: \bar{z} as the ϵ -optimal solution for (15) and X the solution for (14).

Per iteration *n* of Algorithm 2, we have $y^n = \pi_{\mathcal{G}}(z^n) \in \mathcal{G}$. If $y^n \in \mathcal{H}$ is also true, we obtain a feasible point $y^n \in \mathcal{G} \cap \mathcal{H}$. In this case, we update $\text{CBV} = \max\{\text{CBV}, \Phi(y^n)\}$. This implies that CBV is the current best value so far, and the corresponding $\bar{z} = \arg \max_{\{y^m \mid y^m \in \mathcal{H}, m \leq n\}} \Phi(y^m)$ is the current best solution for (15). Observe that for any $v_j \in \mathcal{T}_{n+1}$ satisfying $\Phi(v_j) \leq \text{CBV}(1 + \epsilon)$, we have $(1 + \epsilon)\text{CBV} \geq \Phi(y)$, $\forall y \in [0, v_j]$, due to monotonicity of Φ . Hence, v_j can be removed from \mathcal{T}_{n+1} for further consideration since \bar{z} will be the desired ϵ -optimal solution if $z^{\text{opt}} \in [0, v_j]$.

Remark 3: We remark that Algorithm 2 yields the ϵ -optimal solution to (13) for the case of K = 1. However, for the general case of K > 1, the output value of Algorithm 2, obtained by dropping the rank constraint, only provides an upper bound of the maximum weighted sum-rate of (13). Again, randomized rounding is used to obtain a good approximate solution to (13).

An illustration of Algorithm 2 for K = 1 is given in Fig. 2. With a vertex set \mathcal{T}_n , the upperboundary of polyblock \mathcal{P}_n is depicted by the black dotted-dashed line. Among the three entries of \mathcal{T}_n , the third one is the maximizer: $\mathbf{z}^n = \arg \max_{\mathbf{z} \in \mathcal{T}_n} \Phi(\mathbf{z})$, which is marked with a blue dot. After finding its projection \mathbf{y}^n (marked with a blue cross) on the achievable SINR boundary, two new vertices $\mathbf{z}^{n,1}$ and $\mathbf{z}^{n,2}$ are then obtained through (20). By replacing \mathbf{z}^n with these two vertices, we determine the new polyblock \mathcal{P}_{n+1} with its upperboundary given by the red dashed line.

Similar polyblock outer approximation approaches have been adopted to solve the linear fractional programming and non-convex wireless power control problems in [22], [23]. A key requirement for provable convergence of Algorithm 2 is that z is lower bounded by a strictly positive vector. Since $z \ge 1 > 0$ in (15), it readily follows from [21, Theorem 1] that

Proposition 1: Algorithm 2 globally converges to an ϵ -optimal solution for (15) and (14).

The proposed Algorithm 2 can yield optimal TWR beamforming solution for the relaxed weighted throughput maximization (14) with guaranteed convergence and global optimality. For the two-user case,



Fig. 2. The polyblock outer approximation procedure.

the algorithm can also yield the globally optimal solution for the original problem (13); for the general K-pair case, it can provide a good approximate solution for (13). Hence, the proposed approach provides a good benchmark for all the beamforming (or precoding) schemes that are designed to maximize the user rates in AF-based TWR.

Note that the outer polyblock approximation is in fact a branch-and-bound method. For coordinated beamforming designs in multicell networks, a branch-reduce-and-bound (BRB) algorithm was proposed. It was shown that this BRB algorithm can have faster convergence for weighted sum-rate maximization problems, whereas the polyblock approximation has faster convergence for many other utility functions [24]. The key in the BRB algorithm is again finding the projection of an outer vertex on the upperboundary of the achievable SINR region. Using the max-min SINR solution for (5), a BRB algorithm similar to Algorithm 2 can be also developed to find the optimal TWR beamforming design for the weighted throughput maximization (13), probably with a faster convergence speed.

C. General Design Criteria

The proposed MP approach only relies on the monotonicity of the objective function and the normality of the feasible set. Thus, it can apply to beamforming designs under more general criteria. Consider maximizing a general increasing function F_i of SINRs

$$\max_{\boldsymbol{A}} \sum_{i=1}^{2K} F_i(\operatorname{SINR}_i(\boldsymbol{A}))$$
s. t.
$$\sum_{i=1}^{2K} p_i \|\boldsymbol{A}\boldsymbol{h}_i\|^2 + \operatorname{tr}(\boldsymbol{A}\boldsymbol{\Lambda}_R\boldsymbol{A}^H) \leq \check{P}_R.$$
(23)

The function F_i can be a specific rate function (different from the Shannon capacity formula) $r_i(\text{SINR}_i(\mathbf{A}))$ for practical modulation and coding schemes. Maximization of the utility of user rates has gained a growing interest in the communication and networking context, where different types of utility functions are proposed to trade off the throughput and fairness, or to capture the "happiness" of the user links [25]. The function F_i here can also be the composition of an increasing (not necessarily concave) utility function with that particular rate function $U_i(r_i(\text{SINR}_i(\mathbf{A})))$.

In addition, the formulation (23) includes the following two important cases:

 MSE minimization: Assume that all the user receivers use the linear-minimum-mean-square-error (LMMSE) filters for estimating the received symbols. The weighted sum-MSEs at the output of the LMMSE receivers is given by [26]:

$$\sum_{i=1}^{2K} w_i \mathsf{MSE}_i = \sum_{i=1}^{2K} \frac{w_i}{1 + \mathsf{SINR}_i}$$

With $F_i(SINR_i(A)) := -\frac{w_i}{1+SINR_i(A)}$, (23) specializes to weighted sum-MSE minimization.

2) SER or BER minimization: Using a Q-function: Q(x) := 1/√2π ∫_x[∞] exp(-u²/2)du, the SER and BER of practical modulation schemes can be calculated or approximated in closed-form [27]. Clearly all these SER or BER functions, say ε_i(SINR_i), are strictly decreasing in SINR. With F_i(SINR_i(A)) := -w_iε_i(SINR_i(A)), the problem (23) specializes to weighted sum-SER (or BER) minimization.

It is clear that (23) also carries over to minimization of increasing (not necessarily convex) cost functions of MSE, SER or BER.

For all these $F_i(SINR_i(A))$ functions, we can redefine $\Phi(z) := \sum_{i=1}^{2K} F_i(z_i - 1)$, and consider

$$\max_{\boldsymbol{z}\in\mathcal{Z}} \Phi(\boldsymbol{z}) := \sum_{i=1}^{2K} F_i(z_i - 1).$$
(24)

Algorithm 2 can be used to approximately solve this MP, and, subsequently, provide the solution for (23). It provides a benchmark for the beamforming designs in AF-based TWR under many important criteria.

V. COLLABORATIVE TWR BEAMFORMING

A. Collaborative TWR Model

The proposed unified framework also applies to collaborative TWR where a cluster of M single-antenna relay nodes $\{R_m \mid m = 1, ..., M\}$ cooperatively assist the bidirectional communications between multiple users. Such a collaborative TWR scheme was previously considered in [16], [17] and [28], where the beamforming coefficients for the relays are designed under a total relay power constraint, i.e., the relays share a total power budget. This total relay power constraint is usually not realistic in practical scenarios. Therefore, we consider collaborative beamforming design with individual relay power constraints.

The system model for collaborative TWR can be viewed as a special case of the TWR model described in Section II. The only difference is that in collaborative TWR, the signals received by different antennas at relays cannot be jointly processed. Assume that the (2k - 1)th user and the (2k)th user communicate with each other, k = 1, ..., K, and that data exchange consists of two phases. In the first phase, each user transmits its signal $s_i(t)$ to the relays, and the received signal $y_{R_m}(t)$ at the relay R_m is

$$y_{R_m}(t) = \sum_{i=1}^{2K} h_{i,m} \sqrt{p_i} s_i(t) + n_{R_m}(t),$$
(25)

where $h_{i,m}$ denotes the channel coefficient from user *i* to relay R_m , and $z_{R_m}(t) \sim C\mathcal{N}(0, \sigma_{R_m}^2)$ denotes the additive noise at relay R_m . Let $\mathbf{h}_i := [h_{i,1}, \ldots, h_{i,M}]^T$, $\mathbf{y}_R(t) := [y_{R_1}(t), \ldots, y_{R_M}(t)]^T$, and $\mathbf{z}_R(t) := [z_{R_1}(t), \ldots, z_{R_M}(t)]^T$. Then the received signal vector $\mathbf{y}_R(t)$ at all relays is again given by (1).

Upon receiving $y_{R_m}(t)$, the relay collaboratively amplifies and forwards its signal $x_{R_m}(t) = \tilde{a}_m y_{R_m}(t)$ to all users in the next phase. Let $\tilde{\boldsymbol{a}} := [\tilde{a}_1, \ldots, \tilde{a}_M]$ collect the (complex) AF gains for all relays. The signal vector $\boldsymbol{x}_R(t) := [x_{R_1}(t), \ldots, x_{R_M}(t)]^T$ can be written as $\boldsymbol{x}_R(t) = \tilde{\boldsymbol{A}}\boldsymbol{y}_R(t)$, where $\tilde{\boldsymbol{A}} := \text{diag}(\tilde{\boldsymbol{a}})$. Different from the TWR model with a multi-antenna relay in Section II, the beamforming matrix for collaborative TWR is restricted to be diagonal. The transmit power of the relay R_m is given by

$$p_{R_m}(\tilde{\boldsymbol{a}}) = \sum_{i=1}^{2K} p_i \|\tilde{a}_m \boldsymbol{h}_{i,m}\|^2 + \sigma_{R_m}^2 |\tilde{a}_m|^2.$$
(26)

Assuming channel reciprocity, the received signal at user i = 1, ..., 2K, is then given by

$$y_i(t) = \boldsymbol{h}_i^T \tilde{\boldsymbol{A}} \sum_{j=1}^{2K} \boldsymbol{h}_j \sqrt{p_j} s_j(t) + \boldsymbol{h}_i^T \tilde{\boldsymbol{A}} \boldsymbol{n}_R(t) + n_i(t)$$
(27)

where the noise $n_i(t) \sim C\mathcal{N}(0, \sigma_i^2)$. Clearly, (27) is equivalent to (2) by replacing \tilde{A} with A. Therefore, after removing the self-interference, the SINR at the (2k-1)th user and at the (2k)th user are respectively given by (3) and (4) (with A replaced by \tilde{A}).

B. Algorithm Design

Based on these SINRs, the max-min SINR problem for collaborative TWR can be formulated as

$$\lambda^{\text{opt}} = \max_{\tilde{\boldsymbol{A}}} \min_{i=1,\dots,2K} \frac{\text{SINR}_{i}(\boldsymbol{A})}{\gamma_{i}}$$

s. t.
$$\sum_{i=1}^{2K} p_{i} \|\tilde{a}_{m}\boldsymbol{h}_{i,m}\|^{2} + \sigma_{R_{m}}^{2} |\tilde{a}_{m}|^{2} \leq \check{P}_{R_{m}}, m = 1,\dots,M.$$
 (28)

Problem (28) is similar to (5) except that \tilde{A} in (28) is constrained to be diagonal and there are M transmit power constraints. Thus, (28) can be solved in a similar way as (5) is. Let $X = \tilde{a}\tilde{a}^{H}$, $\theta_{m} := \sum_{i=1}^{2K} p_{i} || h_{i,m} ||^{2} + \sigma_{R_{m}}^{2}$, $\Phi_{m} := [\mathbf{0}_{1\times(m-1)}, \theta_{m}, \mathbf{0}_{1\times(M-m)}]$, and $E_{0,m} := \Phi_{m}^{H}\Phi_{m}$. Then the transmit power constraint of relay R_{m} can be expressed as tr $(E_{0,m}X) \leq \check{P}_{R_{m}}$. Upon defining $f_{i}(X)$ and $g_{i}(X)$ as with (7), the problem (28) can be relaxed to a max-min fractional program similar to (8). Consequently, it can be efficiently solved by the Dinkelbach-type Algorithm 1 with minor modifications.

Using the max-min SINR solution as the corner stone, the beamforming designs for the collaborative TWR under the various criteria considered in Section IV can be done with minor modifications of Algorithm 2. For example, the weighted sum-rate maximization problem for collaborative TWR is the same as (18) except that the set \mathcal{X} is now given by $\mathcal{X} := \{ \mathbf{X} \mid \text{tr}(\mathbf{E}_{0,m}\mathbf{X}) \leq \check{P}_{R_m}, m = 1, \dots, M \}$. It is clear that the corresponding set \mathcal{G} for collaborative TWR is still normal. Hence, the optimization problem can be still formulated as an MP, and the optimal beamforming matrix can be obtained using the polyblock outer approximation method in Algorithm 2.

A. MIMO TWR Model

The performance of TWR can be enhanced when both the relay and the users are equipped with multiple antennas [19]. In what follows, we consider the joint optimization of users' transmit and receive beamforming vectors and the relay's beamforming matrix.

Let M_i denote the number of antennas at user i = 1, ..., 2K, and $s_i(t)$ denote the data signal. In the first phase, user i performs transmit beamforming with vector $u_i \in \mathbb{C}^{M_i \times 1}$ as $x_i(t) = u_i s_i(t)$, where $||u_i||^2 \leq p_i$, and p_i is the transmit power budget of user i. The received signal at the relay is

$$\boldsymbol{y}_{R}(t) = \sum_{i=1}^{2K} \boldsymbol{H}_{i} \boldsymbol{x}_{i}(t) + \boldsymbol{n}_{R}(t), \qquad (29)$$

where $\boldsymbol{H}_i \in \mathbb{C}^{M \times M_i}$ is the channel matrix from user i to the relay.

In the second phase, the relay amplifies and forwards the signal $\boldsymbol{x}_R(t) = \boldsymbol{A}\boldsymbol{y}_R(t)$ to both users. The transmit power at the relay is given by

$$p_R(\boldsymbol{A}) = \sum_{i=1}^{2K} \operatorname{tr}(\boldsymbol{A}\boldsymbol{H}_i \boldsymbol{u}_i \boldsymbol{u}_i^H \boldsymbol{H}_i^H \boldsymbol{A}^H) + \operatorname{tr}(\boldsymbol{A}\boldsymbol{\Lambda}_R \boldsymbol{A}^H).$$
(30)

The received signal at user i is given by

$$\boldsymbol{y}_{i}(t) = \boldsymbol{H}_{i}^{T} \boldsymbol{A} \sum_{j=1}^{2K} \boldsymbol{H}_{j} \boldsymbol{x}_{j}(t) + \boldsymbol{H}_{i}^{T} \boldsymbol{A} \boldsymbol{n}_{R}(t) + \boldsymbol{n}_{i}(t), \qquad (31)$$

where $n_i(t) \sim C\mathcal{N}(0, \Lambda_i)$ is the additive noise at user *i*.

The user *i* first combines its received signal with a vector $v_i \in \mathbb{C}^{M_i \times 1}$ to obtain $y'_i(t) = v_i^H y_i(t)$, which can be expressed as

$$\boldsymbol{y}_{i}'(t) = \boldsymbol{v}_{i}^{H} [\boldsymbol{H}_{i}^{T} \boldsymbol{A} \sum_{j=1}^{2K} \boldsymbol{H}_{j} \boldsymbol{u}_{j} s_{j}(t) + \boldsymbol{H}_{i}^{T} \boldsymbol{A} \boldsymbol{n}_{R}(t) + \boldsymbol{n}_{i}(t)].$$
(32)

Clearly, the output SINR of each user depends on the relay precoding matrix A, the users' transmit precoding vectors, and the receive combining vectors. The SINR at the user i is

$$SINR_{i}(\boldsymbol{A}, \{\boldsymbol{u}_{i}\}, \{\boldsymbol{v}_{i}\}) = \frac{|\boldsymbol{v}_{i}^{H}\boldsymbol{H}_{i}^{T}\boldsymbol{A}\boldsymbol{H}_{\pi(i)}\boldsymbol{u}_{\pi(i)}|^{2}}{\sum_{j\neq i,\pi(i)}|\boldsymbol{v}_{i}^{H}\boldsymbol{H}_{i}^{T}\boldsymbol{A}\boldsymbol{H}_{j}\boldsymbol{u}_{j}|^{2} + \|\boldsymbol{\Lambda}_{\boldsymbol{R}}^{1/2}\boldsymbol{A}^{H}\boldsymbol{H}_{i}^{*}\boldsymbol{v}_{i}\|^{2} + \|\boldsymbol{\Lambda}_{i}^{1/2}\boldsymbol{v}_{i}\|^{2}},$$
(33)

where $\pi(i)$ denotes the partner of user *i*, i.e., $\pi(2k-1) = 2k$ and $\pi(2k) = 2k - 1, \forall k$.

B. Algorithm Design

The max-min SINR problem of the considered multi-pair MIMO TWR can be formulated as

$$\lambda^{\text{opt}} = \max_{\boldsymbol{A}, \{\boldsymbol{u}_i\}, \{\boldsymbol{v}_i\}} \min_{i=1,\dots,2K} \frac{\text{SINR}_i(\boldsymbol{A}, \{\boldsymbol{u}_i\}, \{\boldsymbol{v}_i\})}{\gamma_i}$$

s. t. $p_R(\boldsymbol{A}) = \sum_{i=1}^{2K} \text{tr}(\boldsymbol{A}\boldsymbol{H}_i\boldsymbol{u}_i\boldsymbol{u}_i^H\boldsymbol{H}_i^H\boldsymbol{A}^H) + \text{tr}(\boldsymbol{A}\boldsymbol{\Lambda}_R\boldsymbol{A}^H) \leq \check{P}_R,$ (34)
 $\|\boldsymbol{u}_i\|^2 \leq p_i, i = 1,\dots,2K.$

This optimization problem is in general difficult to solve. We next propose an iterative algorithm to optimize A, $\{u_i\}$, and $\{v_i\}$ in an alternating fashion.

1) User Receive Combining: Given the relay beamforming matrix A and users' transmit precoding vectors $u_i, i = 1, ..., 2K$, the well-known MMSE combining can be employed at user i to detect the transmit signal from its partner user. Let $\alpha_{i,j} := H_i^T A H_j u_j \in \mathbb{C}^{M_i \times 1}$, and $R_i := \sum_{j \neq i} \alpha_{i,j} \alpha_{i,j}^H + H_i^T A \Lambda_R A^H H_i^* + \Lambda_i$. Then the combining vector v_i is given by

$$\boldsymbol{v}_i = \boldsymbol{R}_i^{-1} \boldsymbol{\alpha}_{i,\pi(i)}. \tag{35}$$

2) Optimal Relay Precoding: Now consider the relay beamforming design with fixed transmit and receive beamforming vectors at the users. Let $h_i := H_i u_i \in \mathbb{C}^{M \times 1}$, $g_i := H_i^* v_i \in \mathbb{C}^{M \times 1}$. The max-min optimization problem in (34) becomes

$$\lambda_{\boldsymbol{A}}^{\text{opt}} = \max_{\boldsymbol{A}} \min_{i=1,\dots,2K} \frac{|\boldsymbol{g}_{i}^{H} \boldsymbol{A} \boldsymbol{h}_{\pi(i)}|^{2}}{\gamma_{i}(\sum_{j\neq i,\pi(i)} |\boldsymbol{g}_{i}^{H} \boldsymbol{A} \boldsymbol{h}_{j}|^{2} + \|\boldsymbol{\Lambda}_{R}^{1/2} \boldsymbol{A}^{H} \boldsymbol{g}_{i}\|^{2} + \|\boldsymbol{\Lambda}_{i}^{1/2} \boldsymbol{v}_{i}\|^{2})}$$

s. t.
$$\sum_{i=1}^{2K} \|\boldsymbol{A} \boldsymbol{h}_{i}\|^{2} + \operatorname{tr}(\boldsymbol{A} \boldsymbol{\Lambda}_{R} \boldsymbol{A}^{H}) \leq \check{P}_{R}.$$
 (36)

This problem has almost the same form with (5); hence, it can be efficiently solved by Algorithm 1.

3) Optimal Transmit Precoding: The users' transmit precoding vectors u_i , i = 1, ..., 2K, are also designed to maximize the minimum SINR, and the optimization problem can be formulated as

$$\lambda_{\boldsymbol{u}}^{\text{opt}} = \max_{\{\boldsymbol{u}_j\}_{j=1}^{2K}} \min_{i=1,\dots,2K} \frac{\text{SINR}_i(\boldsymbol{u})}{\gamma_i}$$

s. t. $\|\boldsymbol{u}_i\|^2 \le p_i, i = 1,\dots,2K.$ (37)

Let
$$\beta_{i,j} := H_j^H A^H H_i^* v_i$$
, and $d_i := \|\Lambda_R^{1/2} A^H g_i\|^2 + \|\Lambda_i^{1/2} v_i\|^2$. Define: $E_{i,j} := \beta_{i,j} \beta_{i,j}^H$, and $X_i = u_i u_i^H$.
The SINR of user *i* can be expressed as

$$\operatorname{SINR}_{i}(\boldsymbol{u}) = \frac{\operatorname{tr}(\boldsymbol{E}_{i,\pi(i)}\boldsymbol{X}_{\pi(i)})}{\sum_{j\neq i,\pi(i)}\operatorname{tr}(\boldsymbol{E}_{i,j}\boldsymbol{X}_{j}) + d_{i}}.$$
(38)

Using X_i , i = 1, ..., 2K, as the optimization variables and dropping the constraint of rank $(X_i) = 1$, i = 1, ..., 2K, the problem (37) becomes a max-min fractional program

$$\lambda_{\boldsymbol{u}}^{\text{opt}} = \max_{\{\boldsymbol{X}_j\}_{j=1}^{2K}} \min_{i=1,\dots,2K} \frac{\operatorname{tr}(\boldsymbol{E}_{i,\pi(i)}\boldsymbol{X}_{\pi(i)})}{\gamma_i(\sum_{j\neq i,\pi(i)} \operatorname{tr}(\boldsymbol{E}_{i,j}\boldsymbol{X}_j) + d_i)}$$

s. t. $\boldsymbol{X}_i \succeq 0, \quad i = 1,\dots,2K,$
 $\operatorname{tr}(\boldsymbol{E}_i\boldsymbol{X}_i) \leq p_i, i = 1,\dots,2K.$ (39)

Again, the problem is similar to (8); it can be efficiently solved using the Dinkelbach-type Algorithm 1 with minor modifications.

4) Overall Iterative Algorithm: We are now ready to present the overall iterative algorithm to alternatingly optimize the users' transmit precoding vectors, the relay's beamforming matrix, and the users' receive combining vectors.

Algorithm 3: Iterative optimization for multi-pair MIMO TWR

Initialize: u_i^0, A^0 , and $v_i^0, i = 1, ..., 2K$. Select an accuracy level $\epsilon > 0$. Let n = 0.

Repeat:

1). Given u_i^n, A^n , update the receive combining vectors $v_i^{n+1}, i = 1, \dots, 2K$, via (35).

2). With u_i^n and v_i^{n+1} fixed, use Algorithm 1 to solve the max-min SINR problem (36) to obtain the relay beamforming matrix A^{n+1} .

3). With A^{n+1} and v_i^{n+1} fixed, solve the max-min SINR problem (39) to compute its optimal value λ_u^n and the corresponding users' transmit precoding vectors u_i^{n+1} , i = 1, ..., 2K, via Algorithm 1 (with minor modification).

4). Set n = n + 1.

 $\textbf{until} \ |\lambda_u^n - \lambda_u^{n-1}| < \epsilon.$

Since the objective of the intended problem (34) is clearly upper-bounded and it is increased in each iteration of Algorithm 3, the convergence of the proposed alternative optimization approach readily follows. Note that Algorithm 3 in general converges to a local optimum point. Nevertheless, as will be shown in the next section, the beamforming design with the proposed iterative algorithm can significantly outperform the existing methods.

For weighted sum-rate maximization and other criteria, a similar iterative optimization algorithm can be developed to find the users' transmit precoding vectors, the relay's beamforming matrix, and the users' receive combining vectors. Consider the beamforming designs for weighted sum-rate maximization. The joint design problem can be again decoupled into three sub-problems and an iterative method can be used to alternatively solve the three sub-problems. Specifically, during the *n*-th iteration, we first update the users' receive combining vectors v_i^{n+1} , i = 1, ..., 2K, via (35) with fixed u_i^n , i = 1, ..., 2K, and A^n . Given u_i^n and v_i^{n+1} , we next find the optimal relay beamforming matrix A^{n+1} . This sub-optimization problem is an MP. Building on the max-min SINR solution to (36), Algorithm 2 can be used to obtain A^{n+1} . With A^{n+1} and v_i^{n+1} fixed, the optimal precoding vectors u_i^{n+1} , i = 1, ..., 2K, for weighted sum-rate maximization can also be found by the polyblock outer approximation method in Algorithm 2 building on the max-min SINR solution to (39). It is guaranteed that the proposed MP based alternative optimization approach converges to, at least, a local optimum.

VII. NUMERICAL RESULTS

In this section, numerical results are presented to test the proposed beamforming designs. The simulation settings are as follows. We consider uncorrelated Rayleigh flat fading channels, i.e., each element in h_i or H_i is independent complex Gaussian distributed with zero mean and unit variance. Unless otherwise specified, each user is equipped with a single antenna; the noise components are complex white Gaussian with $n_R(t) \sim C\mathcal{N}(0, N_0 I_M)$, and $n_i(t) \sim C\mathcal{N}(0, N_0)$; assume $p_i = p, \forall i$, and define $SNR = p/N_0$.



Fig. 3. Weighted sum-rate of two-user two-way relaying with various schemes, $p_1 = p_2 = \check{P}_R, w_1 = 0.2, w_2 = 0.8$, and M = 2.



Fig. 4. Evolution of CBV in Algorithm 2.

A. One-pair TWR

In Fig. 3, we check the optimality of the proposed monotonic program based weighted sum-rate maximization beamforming design method for K = 1 user pair, by comparing with the optimal beamforming scheme in [12], and the antenna selection relaying scheme, where the best antenna is selected for signal relaying. There are M = 2 antennas at the relay, and the transmit power of the relay and the two users are the same : $p_1 = p_2 = \check{P}_R$. The weights are chosen as $w_1 = 0.2$ and $w_2 = 0.8$, and $\epsilon = 0.01$ for Algorithm 2. It is seen that the proposed monotonic program based design method achieves the same performance as



Fig. 5. Comparison between the proposed Algorithm 1 and the bisection search method in [14] for $\epsilon = 0.01$ and K = 2.

the scheme in [12], which confirms that the beamforming matrix obtained by Algorithm 2 is optimal. (The slight differences between the two are due to numerical errors.) To illustrate the convergence behavior of the proposed method, the CBV in Algorithm 2 is shown in Fig. 4. The weighted sum-rate upper bound is obtained as follows: we ignore the rank-one constrain when solving the problem (22), and find the minimal of $\Phi(z^n)$ in Algorithm 2 as the upper bound. We see that Algorithm 2 converges fast. In this particular example, three iterations is sufficient to determine the optimal beamforming matrix.

B. Multi-pair TWR

Now consider a two-pair TWR with a four-antenna relay, i.e., K = 2 and M = 4. We assume equal power allocation among the four users and the relay. Fig. 5 compares the number of iterations of the proposed Dinkelbach-type Algorithm 1 with the bisection search method in [14] for a given solution accuracy $\epsilon = 0.01$. For the bisection method in [14], the number of iterations is $\lceil \log_2(t/\epsilon) \rceil$, where t and ϵ are the search bound and error precision, respectively. The search bound t depends on the SNR and the channel coefficients [14]. Hence, the number of iterations of the bisection method increases as the SNR increases or the number of antennas increases as shown in the figure. On the other hand, the number of iterations for the proposed Dinkelbach-type Algorithm 1 remains almost unchanged. Using the zero-forcing beamforming matrix in [9] as the initial A^0 , it can be seen that the proposed Algorithm 1



Fig. 6. Weighted sum-rate of four-user TWR with different beamforming schemes, $w_1 = 0.2$, $w_2 = 0.8$, $w_3 = 0.5$, $w_4 = 0.5$, and M = 2.



Fig. 7. Weighted sum-rate of four-user TWR with different beamforming schemes, $w_1 = 0.2$, $w_2 = 0.8$, $w_3 = 0.5$, $w_4 = 0.5$, and M = 4. converges much faster than the bisection method. About 5 or 6 iterations are sufficient for the convergence of Algorithm 1 in the whole SNR region.

Fig. 6 and Fig. 7 show the achievable weighted sum-rate of various beamforming schemes with M = 2and 4 antennas at the relay, respectively. The weights are chosen as $w_1 = 0.2$, $w_2 = 0.8$, and $w_3 = w_4 = 0.5$. For the proposed weighted sum-rate maximization (Max WSR) beamforming, the optimal beamforming matrix A^{opt} is obtained by the monotonic program method in Algorithm 2 with $\epsilon = 0.01$. The weighted sum-rate performance upper bound is obtained as in Fig. 4. We compare the proposed design with the following methods: 1) max-min beamforming in [14], 2) minimum mean-square-error



Fig. 8. Weighted sum-rate of multi-pair collaborative TWR with $p_i = \check{P}_R, \forall i$.

(MMSE) beamforming in [9], 3) zero-forcing based network coding (ZFNC) in [10], and 4) ProBaSeMO scheme in [15]. Note that for the ZFNC scheme, the number of antennas at the relay should be no less than the number of users, hence it is only applicable when M = 4. From both figures, it is shown that the performance of the proposed beamforming design is close to the performance upper bound, and it outperforms all other alternatives for all SNR values. In particular, the MP approach building on the maxmin SINR solution can significantly improve the sum-rate performance, when there is only two antennas at the relay.

C. Collaborative Multi-pair TWR

Now consider a collaborative four-user TWR with four single-antenna relays. Fig. 8 shows the performance of the proposed collaborative beamforming design and the zero-forcing distributed beamforming (ZFDBF) scheme in [28]. The simulation parameters are the same as in Fig. 6. We consider two transmit power constraints: 1) the relays have a total transmit power constraint that $\sum_{m=1}^{M} \check{P}_{R_m} = p$, and 2) each relay has individual transmit power constraint that $\check{P}_{R_m} = p/M, \forall m$. For the considered two transmit power constraints, it is shown that the collaborative TWR with total transmit power constraint slightly outperforms that with individual transmit power constraint in the high SNR region. Compared with the ZFDBF scheme, significant performance gains can be achieved with the proposed beamforming designs.



It can be also seen that the achievable weighted sum-rate of collaborative TWR with four single-antenna relays is much lower than that of TWR with a single four-antenna relay. This is due to the fact that the beamforming matrix \tilde{A} for collaborative TWR is restricted to be diagonal. Hence certain multiplexing gain is lost as compared with the single multi-antenna relay case.

Average BER of multiuser MIMO TWR with the proposed beamforming design. $p_i = \check{P}_R, \forall i \text{ and } M_1 = M_2 = ... = M_{2K}$.

D. MIMO Multi-pair TWR

Fig. 9.

Finally, Fig. 9 presents the BER performance of a four-user MIMO TWR system with QPSK modulation, where both the users and the relay are equipped with multiple antennas. The number of antennas for one user varies from 1 to 2, and there are 4 antennas at the relay. It is shown that the BER performance improves as the number of antennas at each user increases. Also, significant performance improvement is observed for the proposed optimal beamforming as compared with the MMSE beamforming scheme in [9] and the interference alignment (IA) scheme in [29]. For instance, there is more than 10dB gain at a BER of 10^{-3} for the proposed design when there are two antennas at each user.

VIII. CONCLUSION

We developed a unified framework of beamforming designs for non-regenerative two-way relaying. Using the max-min SINR solution as a corner stone, we proposed efficient algorithms to find the near-optimal beamforming designs under various important criteria such as power minimization, rate maximization, MSE minimization, and BER minimization. We further extended the proposed framework to distributed beamforming for TWR, as well as to MIMO TWR. The proposed unified approach can provide important insights for tackling the optimal beamforming designs in other emerging network models and settings.

APPENDIX

We first show that

Lemma 3: $\tilde{\lambda}^{\text{opt}}(\check{P}_R)$ is a strictly increasing function of \check{P}_R .

Proof: Let \mathbf{X}^{opt} denote the optimal solution for (8) with power budget $\check{P}_R > 0$. For a $\check{P}'_R > \check{P}_R$, let $\alpha = \check{P}'_R / \check{P}_R > 1$, and $\mathbf{X}' = \alpha \mathbf{X}^{\text{opt}}$. Then \mathbf{X}' is feasible for (8) with power budget \check{P}'_R , since $\operatorname{tr}(\mathbf{E}_0 \mathbf{X}') = \alpha \operatorname{tr}(\mathbf{E}_0 \mathbf{X}^{\text{opt}}) \leq \alpha \check{P}_R = \check{P}'_R$.

On the other hand,

$$\begin{aligned} \operatorname{SINR}_{i}(\boldsymbol{X}') &= \frac{f_{i}(\boldsymbol{X}')}{g_{i}(\boldsymbol{X}')} = \frac{\operatorname{tr}(\boldsymbol{E}_{i}^{(1)}\boldsymbol{X}')}{\operatorname{tr}(\boldsymbol{E}_{i}^{(2)}\boldsymbol{X}') + \sigma_{i}^{2}} = \frac{\alpha \operatorname{tr}(\boldsymbol{E}_{i}^{(1)}\boldsymbol{X}^{\operatorname{opt}})}{\alpha \operatorname{tr}(\boldsymbol{E}_{i}^{(2)}\boldsymbol{X}^{\operatorname{opt}}) + \sigma_{i}^{2}} \\ &> \frac{\operatorname{tr}(\boldsymbol{E}_{i}^{(1)}\boldsymbol{X}^{\operatorname{opt}})}{\operatorname{tr}(\boldsymbol{E}_{i}^{(2)}\boldsymbol{X}^{\operatorname{opt}}) + \sigma_{i}^{2}} = \operatorname{SINR}_{i}(\boldsymbol{X}^{\operatorname{opt}}). \end{aligned}$$

Therefore, $\tilde{\lambda}^{\text{opt}}(\check{P}'_R) \ge \min_{i=1,\dots,2K} \frac{\text{SINR}_i(X')}{\gamma_i} > \min_{i=1,\dots,2K} \frac{\text{SINR}_i(X^{\text{opt}})}{\gamma_i} = \tilde{\lambda}^{\text{opt}}(\check{P}_R).$

Relying on the monotonicity of $\tilde{\lambda}^{\text{opt}}(\check{P}_R)$ stated in Lemma 3, we can further show that:

Lemma 4: The optimal solution for (12) is the same as the matrix X^{opt} for (8) with the power budget P_R that satisfies $\tilde{\lambda}^{\text{opt}}(P_R) = 1$.

Proof: Let \mathbf{X}^{opt} denote the optimal solution for (8) with the power budget P_R that satisfies $\tilde{\lambda}^{\text{opt}}(P_R) = 1$. Since $\tilde{\lambda}^{\text{opt}}(P_R) = 1$ implies $\text{SINR}_i(\mathbf{X}^{\text{opt}}) \ge \gamma_i$, i = 1, ..., 2K, \mathbf{X}^{opt} is in the feasible set of (12). Upon denoting P_R^{opt} as the optimal value for (12), this in turn implies that $P_R^{\text{opt}} \le \text{tr}(\mathbf{E}_0 \mathbf{X}^{\text{opt}}) \le P_R$. Consider (8) with the power budget P_R^{opt} . By Lemma 3, we must have

$$\tilde{\lambda}^{\text{opt}}(P_R^{\text{opt}}) \le \tilde{\lambda}^{\text{opt}}(P_R) = 1 \tag{40}$$

due to $P_R^{\text{opt}} \leq P_R$.

On the other hand, let $\tilde{\boldsymbol{X}}^{\text{opt}}$ denote the optimal solution for (12), which is the feasible set of (8) with the power budget P_R^{opt} since $\text{tr}(\boldsymbol{E}_0 \tilde{\boldsymbol{X}}^{\text{opt}}) = P_R^{\text{opt}}$. For this $\tilde{\boldsymbol{X}}^{\text{opt}}$, we have $\min_{i=1,...,2K} \frac{\text{SINR}_i(\tilde{\boldsymbol{X}}^{\text{opt}})}{\gamma_i} \ge 1$ since $\text{SINR}_i(\tilde{\boldsymbol{X}}^{\text{opt}}) \ge \gamma_i$, i = 1, ..., 2K. This together with the feasibility of $\tilde{\boldsymbol{X}}^{\text{opt}}$ implies that $\tilde{\lambda}^{\text{opt}}(P_R^{\text{opt}}) \ge 1$. Clearly, we have both the latter and (40) satisfied, only when all the inequalities are satisfied with equalities; i.e., $P_R^{\text{opt}} = P_R$, and it is achieved by the beamforming matrix $\boldsymbol{X}^{\text{opt}}$.

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