A Unifying Framework for Adaptive Radar Detection in Homogeneous plus Structured Interference-Part I: On the Maximal Invariant Statistic

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Abstract—This paper deals with the problem of adaptive multidimensional/multichannel signal detection in homogeneous Gaussian disturbance with unknown covariance matrix and structured deterministic interference. The aforementioned problem corresponds to a generalization of the well-known Generalized Multivariate Analysis of Variance (GMANOVA). In this first part of the work, we formulate the considered problem in canonical form and, after identifying a desirable group of transformations for the considered hypothesis testing, we derive a Maximal Invariant Statistic (MIS) for the problem at hand. Furthermore, we provide the MIS distribution in the form of a stochastic representation. Finally, strong connections to the MIS obtained in the open literature in simpler scenarios are underlined.

Index Terms—Adaptive Radar detection, CFAR, Statistical Invariance, Maximal Invariants, Double-subspace model, GMANOVA, coherent interference.

I. INTRODUCTION

A. Motivation and Related Works

THE PROBLEM of adaptive detection of targets embedded in Gaussian interference is an active research field which has been object of great interest in the last decades. Many works appeared in the open literature, dealing with the design and performance analysis of several detectors handling many specific detection problems (the interested reader is referred to [1] and references therein).

It can be shown that most of the aforementioned models can be seen as special cases of the model considered by Kelly and Forsythe [2], which is very general and encompasses pointlike and extended targets as special instances. The considered model allows for training samples which contain random interference modeled as an unknown covariance matrix that accounts for both clutter and thermal noise, with the implict assumption that signal plus noise and noise-only vector samples share the same covariance matrix, thus determining a socalled *homogeneous environment*.

The signal model considered in the aforementioned report is the well-known Generalized Multivariate Analysis of Variance (GMANOVA) in statistics literature [3], also referred to as a "double-subspace" signal model (see for example [4], [5]). The standard GMANOVA model was first formulated by Potthoff and Roy [6] and consists in a generic patterned mean problem with a data matrix whose columns are normal random vectors with a common unknown covariance matrix. The GMANOVA model was later studied in more detail in [7], where maximum likelihood estimates of unknown parameters were obtained. For a detailed introduction to estimation and detection in GMANOVA model (along with few interesting application examples) the interested reader may refer to the excellent tutorial [8].

Differently, in this paper we will study a modified version of GMANOVA with respect to its classical formulation [8], referred to as I-GMANOVA in what follows. The considered model allows for the presence of a structured (partially known) non-zero mean under both hypotheses. Such disturbance is collectively represented as an unknown deterministic matrix, which determines an additional set of nuisance parameters for the considered hypothesis testing (i.e., other than the covariance matrix). The aforementioned model easily accounts for the presence of structured subspace interference affecting the target detection task. Thus it is clear that taking such interference into consideration enables the application of this model to adaptive radar detection; for instance it may accomodate the presence of multiple pulsed coherent jammers impinging on the radar antenna from some directions.

Although several different decision criteria can be considered to attack composite hypothesis testing problems [9], [10], an elegant and systematic way consists in resorting to the socalled Principle of Invariance [11], [10]. Indeed, the aforementioned principle, when exploited at the design stage, allows to focus on decision rules enjoying some desirable practical features. The preliminary step consists in individuating a suitable group of trasformations which leaves the formal structure of the hypothesis testing problem unaltered. With reference to radar adaptive detection, the mentioned principle represents an effective tool for obtaining a statistic which is invariant with respect to the set of nuisance parameters, therefore constituting the basis for Constant-False Alarm Rate (CFAR) rules. Indeed, every invariant decision rule can be written in terms of the maximal invariant statistic. Therefore, with reference to I-GMANOVA model, the principle of invariance allows for imposing CFARness property with respect to the clutter plus

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noise (disturbance) covariance matrix and the jammer location parameters.

It is worth remarking that the use of the invariance principle for generic composite hypothesis testing problems [11], [10] (and, more specifically, in the context of radar adaptive detection) is not new. Indeed, starting from the seminal paper [12], many works focused on adaptive radar detection problems with the use of *invariance theory*. For example, in [13], [14], [15], invariance theory was exploited to study the problem of single-subspace (adaptive) detection of point-like targets. Later, similar works appeared in the open literature dealing with the case of a target spread among more range cells [16], [17]. More recently, the same statistical tool has been employed to address the problem of adaptive single-subspace detection problem (of point-like targets) in the joint presence of random and subspace structured interference in [18]. In this respect, we build upon the aforementioned results in order to develop an exhaustive study for the considered I-GMANOVA model under the point of view of the invariance.

B. Summary of the contributions and Paper Organization

The main contributions of the first part of the present study are summarized as follows:

- We first show that the problem at hand admits a more intuitive representation, by exploiting a *canonical form* representation. Such representation helps obtaining the maximal invariant statistics and gaining insights for the problem under investigation;
- The group of transformations which leaves the problem invariant is identified, thus allowing the search for a MIS.
- Given the aforementioned group of transformations, the canonical form is exploited in order to obtain the MIS, which, for the I-GMANOVA model is represented by *two matrices* which compress the original data. Such result can be interpreted as the generalization of the two-components scalar MIS obtained in the classical references [12], [14].
- A theoretical performance analysis of the MIS is obtained, in terms of its distribution. Even though in the considered setup the MIS does not generally admit an explicit expression for its probability density function (pdf), a simpler form of the statistic distribution, by means of a suitable stochastic representation, is provided.
- Finally, the obtained MIS expression is compared with similar findings obtained in the literature for simpler scenarios, thus showing that the aforementioned cases can be seen as special instances of the obtained MIS.

The explicit expression of the MIS obtained in this first part is then exploited to show CFARness of all the detectors considered in part II of this work.

The remainder of the paper is organized as follows: in Sec. II we introduce the hypothesis testing problem under investigation; in Sec. III we describe the desirable invariance properties and derive the MIS. Sec. IV is devoted to the statistical characterization of the MIS, while in Sec. V we particularize the MIS to specific instances and compare it with previously obtained results in the open literature. Some concluding remarks and future research directions are given in Sec. VI; finally, proofs and derivations are confined to the Appendices.

Notation - Lower-case (resp. Upper-case) bold letters denote vectors (resp. matrices), with a_n (resp. $A_{n,m}$) representing the *n*th (resp. the (n, m)th) element of the vector \boldsymbol{a} (resp. matrix A); \mathbb{R}^N , \mathbb{C}^N , and $\mathbb{H}^{N \times N}$ are the sets of N-dimensional vectors of real numbers, of complex numbers, and of $N \times N$ Hermitian matrices, respectively; upper-case calligraphic letters and braces denote finite sets; $\mathbb{E}\{\cdot\}$, $\operatorname{Cov}[\cdot], (\cdot)^T, (\cdot)^{\dagger}, \angle \cdot$, $Tr[\cdot]$, denote expectation, covariance, transpose, Hermitian, phase and matrix trace operators, respectively; $\mathbf{0}_{N \times M}$ (resp. I_N) denotes the $N \times M$ null (resp. identity) matrix; $\mathbf{0}_N$ (resp. $\mathbf{1}_N$) denotes the null (resp. ones) column vector of length N; vec(M) stacks the first to the last column of the matrix M one under another to form a long vector; det(A) and $||A||_F$ denote the determinant and Frobenius norm of matrix A; $A \otimes B$ indicates the Kronecker product between A and B matrices; diag(A, B) denotes the blockdiagonal matrix obtained by placing matrices A and B along the main diagonal; the symbol "~" means "distributed as"; $\boldsymbol{x} \sim \mathcal{CN}_N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes a complex (proper) Gaussiandistributed vector $m{x}$ with mean vector $m{\mu} \in \mathbb{C}^{N imes 1}$ and covariance matrix $oldsymbol{\Sigma} \in \mathbb{C}^{N imes N}; \ oldsymbol{X} \sim \mathcal{CN}_{N imes M}(oldsymbol{A}, oldsymbol{B}, oldsymbol{C})$ denotes a complex (proper) Gaussian-distributed matrix X with mean $A \in \hat{\mathbb{C}}^{N \times \hat{M}}$ and $\operatorname{Cov}[\operatorname{vec}(X)] = B \otimes C; S \sim \mathcal{CW}_N(K, A)$ denotes a complex central Wishart distributed matrix S with parameters $K \in \mathbb{N}$ and $A \in \mathbb{C}^{N \times N}$ positive definite matrix; $M \sim \mathcal{CF}_a(\mathbf{A}, \ell, m)$ is a non-central multivariate complex F distributed matrix M with mean A and parameters a, ℓ , and m; P_A denotes the orthogonal projection of the full-columnrank matrix A, that is $P_A \triangleq [A(A^{\dagger}A)^{-1}A^{\dagger}]$, while P_A^{\perp} its complement, that is $P_A^{\perp} \triangleq (I - P_A)$.

II. PROBLEM FORMULATION

We assume that a matrix of complex-valued samples $X \in \mathbb{C}^{N \times K}$ is collected, accounting for both primary (signalbearing) and secondary (signal-free) data. The hypothesis testing problem under investigation can be formulated as:

$$\begin{cases} \mathcal{H}_0: \quad \boldsymbol{X} = \widetilde{\boldsymbol{A}}_t \, \widetilde{\boldsymbol{B}}_t \, \widetilde{\boldsymbol{C}} + \boldsymbol{N}_0 \\ \mathcal{H}_1: \quad \boldsymbol{X} = \left(\widetilde{\boldsymbol{A}}_t \, \widetilde{\boldsymbol{B}}_t + \widetilde{\boldsymbol{A}}_r \, \widetilde{\boldsymbol{B}}_r \right) \widetilde{\boldsymbol{C}} + \boldsymbol{N}_0 \end{cases} \tag{1}$$

where:

- N₀ ∈ C^{N×K} is a matrix whose columns are independent and identically distributed (iid) proper complex normal random vectors with zero mean and (unknown) positive definite covariance matrix R_{*} ∈ C^{N×N}, that is N₀ ~ CN_{N×K}(0_{N×K}, I_K, R_{*});
 B̃_t ∈ C^{t×M} and B̃_r ∈ C^{r×M} denote the (unknown)
- $B_t \in \mathbb{C}^{t \times M}$ and $B_r \in \mathbb{C}^{r \times M}$ denote the (unknown) deterministic matrix coordinates, representing the interference and the useful signal, respectively;
- $\widetilde{A}_t \in \mathbb{C}^{N \times t}$ and $\widetilde{A}_r \in \mathbb{C}^{N \times r}$ represent the (known) left subspace of the interference and the useful signal, respectively. The matrices \widetilde{A}_t and \widetilde{A}_r are both assumed full-column rank, with their columns being linearly independent;

Similarly, *C̃* ∈ ℂ^{M×K} is a known matrix describing the right subspace associated to *both* signal and interference; the matrix *C̃* is assumed full-row-rank.

Additionally, aiming at a compact notation, we define $\boldsymbol{A} \triangleq \begin{bmatrix} \widetilde{\boldsymbol{A}}_t & \widetilde{\boldsymbol{A}}_r \end{bmatrix} \in \mathbb{C}^{N \times J}$ and $\widetilde{\boldsymbol{B}} \triangleq \begin{bmatrix} \widetilde{\boldsymbol{B}}_t^T & \widetilde{\boldsymbol{B}}_r^T \end{bmatrix}^T \in \mathbb{C}^{J \times M}$, where we have denoted $J \triangleq r + t$.

From inspection of Eq. (1), we notice that the considered test has a complicated structure, which is thus difficult to analyze. Therefore, before proceeding further, we first show that a simpler equivalent formulation of the considered problem can be obtained in the so-called *canonical form* [2].

With this intent, we first consider the QR-decomposition of $\widetilde{A} = Q_{\alpha} R_{\alpha}$, where $Q_{\alpha} \in \mathbb{C}^{N \times J}$ is a slice of a unitary matrix (i.e., $Q_{\alpha}^{\dagger} Q_{\alpha} = I_{J}$) and $R_{\alpha} \in \mathbb{C}^{J \times J}$ a non-singular upper triangular matrix. It can be readily shown that Q_{α} and R_{α} can be conveniently partitioned as:

$$\boldsymbol{Q}_{\alpha} = \begin{bmatrix} \boldsymbol{Q}_{\alpha,t} & \boldsymbol{Q}_{\alpha,r} \end{bmatrix} \qquad \boldsymbol{R}_{\alpha} = \begin{bmatrix} \boldsymbol{R}_{\alpha,t} & \boldsymbol{R}_{\alpha,x} \\ \boldsymbol{0}_{r \times t} & \boldsymbol{R}_{\alpha,r} \end{bmatrix}$$
(2)

where $Q_{\alpha,t} \in \mathbb{C}^{N \times t}$ and $R_{\alpha,t} \in \mathbb{C}^{t \times t}$ arise from the QRdecomposition of \widetilde{A}_t , namely $\widetilde{A}_t = Q_{\alpha,t} R_{\alpha,t}$, with $Q_{\alpha,t}$ such that $Q_{\alpha,t}^{\dagger} Q_{\alpha,t} = I_t$ and $R_{\alpha,t}$ a non-singular upper triangular matrix. Furthermore, $R_{\alpha,x} \in \mathbb{C}^{t \times r}$, and $R_{\alpha,r} \in \mathbb{C}^{r \times r}$ is another non-singular upper triangular matrix. Similarly, $Q_{\alpha,r} \in \mathbb{C}^{N \times r}$ is such that $Q_{\alpha,r}^{\dagger} Q_{\alpha,r} = I_r$. Equalities in Eq. (2) are almost evident consequences of the well-known Gram-Schmidt procedure [19]. Now, let us define a unitary matrix $U_{\alpha} \in \mathbb{C}^{N \times N}$ whose first J columns are collectively equal to Q_{α} . Then, it follows that:

$$\boldsymbol{A} \triangleq \underbrace{\boldsymbol{U}_{\alpha}^{\dagger} \boldsymbol{Q}_{\alpha}}_{\in \mathbb{C}^{N \times J}} = \begin{bmatrix} \boldsymbol{I}_{t} & \boldsymbol{0}_{t \times r} \\ \boldsymbol{0}_{r \times t} & \boldsymbol{I}_{r} \\ \boldsymbol{0}_{(N-J) \times t} & \boldsymbol{0}_{(N-J) \times r} \end{bmatrix} = \begin{bmatrix} \boldsymbol{E}_{t} & \boldsymbol{E}_{r} \end{bmatrix}$$
(3)

where $\boldsymbol{E}_t \triangleq \begin{bmatrix} \boldsymbol{I}_t & \boldsymbol{0}_{t \times r} & \boldsymbol{0}_{t \times (N-J)} \end{bmatrix}^T$ and $\boldsymbol{E}_r \triangleq \begin{bmatrix} \boldsymbol{0}_{r \times t} & \boldsymbol{I}_r & \boldsymbol{0}_{r \times (N-J)} \end{bmatrix}^T$, respectively. Also, let $\tilde{\boldsymbol{C}}$ be expressed in terms of its Singular Value Decomposition (SVD) as

$$\widetilde{\boldsymbol{C}} = \boldsymbol{U}_{\gamma} \, \boldsymbol{\Lambda}_{\gamma} \, \boldsymbol{V}_{\gamma}^{\dagger} \,, \tag{4}$$

where $U_{\gamma} \in \mathbb{C}^{M \times M}$ and $V_{\gamma} \in \mathbb{C}^{K \times K}$ are both unitary matrices, and the matrix of the singular values $\Lambda_{\gamma} \in \mathbb{C}^{M \times K}$ has the following noteworthy form:

$$\mathbf{\Lambda}_{\gamma} = \begin{bmatrix} \widetilde{\mathbf{\Lambda}}_{\gamma} & \mathbf{0}_{M \times (K-M)} \end{bmatrix}, \qquad (5)$$

with $\widetilde{\mathbf{A}}_{\gamma} \in \mathbb{C}^{M \times M}$ being a diagonal matrix. Therefore

$$\widetilde{C}V_{\gamma} = M_{\gamma} \begin{bmatrix} I_M & \mathbf{0}_{M \times (K-M)} \end{bmatrix}$$
(6)

holds, where $M_{\gamma} \triangleq U_{\gamma} \Lambda_{\gamma}$.

Given the aforementioned definitions, without loss of generality we will consider the transformed data matrix $Z \triangleq (U_{\alpha}^{\dagger} X V_{\gamma}) \in \mathbb{C}^{N \times K}$ in what follows. Such transformation does not alter the hypothesis testing problem being considered, as it simply applies left and right rotations to data matrix X

(viz. multiplications by unitary matrices). The new data matrix, when \mathcal{H}_1 is in force, can be expressed as:

$$Z = U_{\alpha}^{\dagger} \left(Q_{\alpha} R_{\alpha} \widetilde{B} U_{\gamma} \Lambda_{\gamma} V_{\gamma}^{\dagger} \right) V_{\gamma} + N$$

$$\widetilde{}$$

$$(7)$$

$$= \boldsymbol{A} \left(\boldsymbol{R}_{\alpha} \boldsymbol{B} \boldsymbol{M}_{\gamma} \right) \left[\boldsymbol{I}_{M} \quad \boldsymbol{0}_{M \times (K-M)} \right] + \boldsymbol{N} \qquad (8)$$

$$= \mathbf{A} \begin{bmatrix} \mathbf{B}_{t,1} \\ \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{I}_M & \mathbf{0}_{M \times (K-M)} \end{bmatrix} + \mathbf{N}$$
(9)

where we have defined $\boldsymbol{B}_{t,1} \triangleq ((\boldsymbol{R}_{\alpha,t} \, \boldsymbol{\tilde{B}}_t + \boldsymbol{R}_{\alpha,x} \, \boldsymbol{\tilde{B}}_r) \, \boldsymbol{M}_{\gamma}) \in \mathbb{C}^{t \times M}$, $\boldsymbol{B} \triangleq (\boldsymbol{R}_{\alpha,r} \, \boldsymbol{\tilde{B}}_r \, \boldsymbol{M}_{\gamma}) \in \mathbb{C}^{r \times M}$ and $\boldsymbol{N} \triangleq (\boldsymbol{U}_{\alpha}^{\dagger} \, \boldsymbol{N}_0 \, \boldsymbol{V}_{\gamma}) \in \mathbb{C}^{N \times K}$, respectively. Furthermore, for the sake of notational convenience, we define $\boldsymbol{B}_s \triangleq [\boldsymbol{B}_{t,1}^T \, \boldsymbol{B}^T]^T$. On the other hand, when \mathcal{H}_0 holds true, the matrix \boldsymbol{Z} can be expressed as:

$$\boldsymbol{Z} = \boldsymbol{U}_{\alpha}^{\dagger} \left(\boldsymbol{Q}_{\alpha} \, \boldsymbol{R}_{\alpha} \, \begin{bmatrix} \boldsymbol{\widetilde{B}}_{t} \\ \boldsymbol{0}_{r \times M} \end{bmatrix} \, \boldsymbol{U}_{\gamma} \, \boldsymbol{\Lambda}_{\gamma} \, \boldsymbol{V}_{\gamma}^{\dagger} \right) \boldsymbol{V}_{\gamma} + \boldsymbol{N} \quad (10)$$

$$= \boldsymbol{A} \begin{bmatrix} \boldsymbol{B}_{t,0} \\ \boldsymbol{0}_{r \times M} \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_M & \boldsymbol{0}_{M \times (K-M)} \end{bmatrix} + \boldsymbol{N}$$
(11)

where $B_{t,0} \triangleq (\mathbf{R}_{\alpha,t} \, \tilde{B}_t \, \mathbf{M}_{\gamma}) \in \mathbb{C}^{t \times M}$. Furthermore, aiming at keeping a compact notation, we will employ the definition $C \triangleq \begin{bmatrix} \mathbf{I}_M & \mathbf{0}_{M \times (K-M)} \end{bmatrix}$ in what follows. Gathering all the above results, the problem in Eq. (1) can be equivalently rewritten in terms of Z as:

$$\begin{cases} \mathcal{H}_{0}: \quad \boldsymbol{Z} = \boldsymbol{A} \begin{bmatrix} \boldsymbol{B}_{t,0} \\ \boldsymbol{0}_{r \times M} \end{bmatrix} \boldsymbol{C} + \boldsymbol{N} \\ \mathcal{H}_{1}: \quad \boldsymbol{Z} = \boldsymbol{A} \boldsymbol{B}_{s} \boldsymbol{C} + \boldsymbol{N} \end{cases}$$
(12)

Finally we recall that, since $N_0 \sim C\mathcal{N}_{N \times K}(\mathbf{0}_{N \times K}, \mathbf{I}_K, \mathbf{R}_*)$, N is distributed as $N \sim C\mathcal{N}_{N \times K}(\mathbf{0}_{N \times K}, \mathbf{I}_K, \mathbf{R})$, where $\mathbf{R} \triangleq (\mathbf{U}_{\alpha}^{\dagger} \mathbf{R}_* \mathbf{U}_{\alpha})$ [2].

An important remark is now in order. Specifically, for the problem in Eq. (1), the relevant parameter to decide for the presence of a target is \tilde{B}_r . Otherwise stated, if the hypothesis \mathcal{H}_1 holds true, then $||\tilde{B}_r||_F > 0$, while $||\tilde{B}_r||_F = 0$ under the target-absent hypothesis (\mathcal{H}_0). As a consequence, since $R_{\alpha,r}$ is non-singular, problem in Eq. (12) is equivalent to:

$$\begin{cases} \mathcal{H}_0 : & ||B||_F = 0, \\ \mathcal{H}_1 : & ||B||_F > 0, \end{cases}$$
(13)

which partitions the relevant-signal parameter space, say Θ_r , as:

$$\Theta_r = \underbrace{\{\mathbf{0}_{r \times M}\}}_{\Theta_{r,0}} \cup \underbrace{\{\mathbf{B} \in \mathbb{C}^{r \times M} : \|\mathbf{B}\|_F > 0\}}_{\Theta_{r,1}}.$$
 (14)

The canonical form in Eq. (12) will be exploited hereinafter in our analysis.

In the following, our analysis is carried out assuming that $(K-M) \ge N$. Such condition is typically satisfied in practical adaptive detection setups [2].

III. MAXIMAL INVARIANT STATISTIC

In what follows, we will search for decision rules sharing invariance with respect to those parameters (namely the nuisance parameters, \mathbf{R} , $\mathbf{B}_{t,1}$, and $\mathbf{B}_{t,0}$) which are irrelevant for

the specific decision problem. To this end, we resort to the socalled "Principle of Invariance" [10], whose main idea consists in finding transformations that properly cluster data without altering

- the formal structure of the hypothesis testing problem given by (14);
- the Gaussian assumption for the received data matrix under each hypothesis;
- the double-subspace structure containing the useful signal components.

The following subsection is thus devoted to the definition of a suitable group which fullfils the above requirements.

A. Desired invariance properties

Let

$$\boldsymbol{V}_{c,1} \triangleq \begin{bmatrix} \boldsymbol{I}_{M} \\ \boldsymbol{0}_{(K-M) \times M} \end{bmatrix}, \quad \boldsymbol{V}_{c,2} \triangleq \begin{bmatrix} \boldsymbol{0}_{M \times (K-M)} \\ \boldsymbol{I}_{K-M} \end{bmatrix}, \quad (15)$$

and observe that $P_{C^{\dagger}} = (V_{c,1}V_{c,1}^{\dagger})$ and $P_{C^{\dagger}}^{\perp} = (V_{c,2}V_{c,2}^{\dagger})$. Also, let us consider the sufficient statistic¹ $\{Z_c, S_c\}$, where

Also, let us consider the sufficient statistic¹ $\{Z_c, S_c\}$, where the mentioned quantities are defined as

$$\boldsymbol{Z}_{c} \triangleq (\boldsymbol{Z}\boldsymbol{V}_{c,1}) \in \mathbb{C}^{N \times M}$$
(16)

$$\boldsymbol{Z}_{c,\perp} \triangleq (\boldsymbol{Z} \, \boldsymbol{V}_{c,2}) \in \mathbb{C}^{N \times (K-M)}$$
(17)

$$\boldsymbol{S}_{c} \triangleq (\boldsymbol{Z}_{c,\perp} \boldsymbol{Z}_{c,\perp}^{\dagger}) = (\boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}^{\perp} \boldsymbol{Z}^{\dagger}) \in \mathbb{C}^{N \times N}$$
(18)

Clearly, given the simplified structure of C, Z_c (resp. $Z_{c,\perp}$) is simply obtained by taking the first M (resp. the last K - M) columns of the transformed data matrix Z.

Now, denote by $\mathcal{GL}(N)$ the linear group of $N \times N$ nonsingular matrices and introduce the following sets

$$\mathcal{G} \triangleq \begin{cases} \mathbf{G} \triangleq \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} & \mathbf{G}_{13} \\ \mathbf{0}_{r \times t} & \mathbf{G}_{22} & \mathbf{G}_{23} \\ \mathbf{0}_{(N-J) \times t} & \mathbf{0}_{(N-J) \times r} & \mathbf{G}_{33} \end{bmatrix} \in \mathcal{GL}(N) (19) \\ \mathbf{G} : \mathbf{G}_{11} \in \mathcal{GL}(t), \ \mathbf{G}_{22} \in \mathcal{GL}(r), \ \mathbf{G}_{33} \in \mathcal{GL}(N-J) \} \end{cases}$$

$$\mathcal{F} \triangleq \left\{ \boldsymbol{F} \triangleq \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{0}_{r \times M} \\ \boldsymbol{0}_{(N-J) \times M} \end{bmatrix} \in \mathbb{C}^{N \times M} : \boldsymbol{F}_1 \in \mathbb{C}^{t \times M} \right\}$$
(20)

along with the composition operator "o", defined as:

$$(\boldsymbol{G}_a, \boldsymbol{F}_a) \circ (\boldsymbol{G}_b, \boldsymbol{F}_b) = (\boldsymbol{G}_b \boldsymbol{G}_a, \boldsymbol{G}_b \boldsymbol{F}_a + \boldsymbol{F}_b)$$
(21)

The sets and the composition operator are here represented compactly as $\mathcal{L} \triangleq (\mathcal{G} \times \mathcal{F}, \circ)$. Then, it is not difficult to show that \mathcal{L} constitutes a *group*, since it satisfies the following elementary axioms:

- *L* is *closed* with respect to the operation "∘", defined in Eq. (21);
- $\forall (\boldsymbol{G}_a, \boldsymbol{F}_a), \ (\boldsymbol{G}_b, \boldsymbol{F}_b), \text{ and } (\boldsymbol{G}_c, \boldsymbol{F}_c) \in \mathcal{L}: \ [(\boldsymbol{G}_a, \boldsymbol{F}_a) \circ (\boldsymbol{G}_b, \boldsymbol{F}_b)] \circ (\boldsymbol{G}_c, \boldsymbol{F}_c) = (\boldsymbol{G}_a, \boldsymbol{F}_a) \circ [(\boldsymbol{G}_b, \boldsymbol{F}_b) \circ (\boldsymbol{G}_c, \boldsymbol{F}_c)]$ (Associative property);
- there exists a unique $(G_I, F_I) \in \mathcal{L}$ such that $\forall (G, F) \in \mathcal{L}$: $(G_I, F_I) \circ (G, F) = (G, F) \circ (G_I, F_I) = (G, F)$ (Existence of Identity element);

¹Indeed, Fisher-Neyman factorization theorem ensures that deciding from $\{\mathbf{Z}_c, \mathbf{S}_c\}$ is tantamount to deciding from raw data \mathbf{Z} [3].

• $\forall (G, F) \in \mathcal{L}$, there exists $(G_{-1}, F_{-1}) \in \mathcal{L}$ such that $(G_{-1}, F_{-1}) \circ (G, F) = (G, F) \circ (G_{-1}, F_{-1}) = (G_I, F_I)$ (*Existence of Inverse element*).

Also, the aforementioned group leaves the hypothesis testing problem in Eq. (12) invariant under the action ℓ defined by:

$$\ell(\boldsymbol{Z}_c, \boldsymbol{S}_c) = \left(\boldsymbol{G}\boldsymbol{Z}_c + \boldsymbol{F}, \boldsymbol{G}\boldsymbol{S}_c\,\boldsymbol{G}^{\dagger}\right) \quad \forall (\boldsymbol{G}, \boldsymbol{F}) \in \mathcal{L} \,. \tag{22}$$

The proof of the aforementioned statement is given in Appendix A. Moreover, it is important to point out that \mathcal{L} preserves the family of distributions, and, at the same time, includes those transformations which are relevant from a practical point of view, as they allow claiming the CFAR property (with respect to \mathbf{R} and $\mathbf{B}_{t,i}$) as a consequence of the invariance.

B. Derivation of the MIS

In Sec. III-A we have identified a group \mathcal{L} which leaves unaltered the problem under investigation. It is thus reasonable finding decision rules that are invariant under \mathcal{L} . Toward this goal, the Principle of Invariance is invoked because it allows to construct statistics that organize data into distinguishable equivalence classes. Such functions of the data are called *Maximal Invariant Statistics* and, given the group of transformations, every invariant test may be written as a function of the maximal invariant [11].

Before presenting the explicit expression of the MIS, we give the following preliminary definitions based on the partitioning of matrices Z_c and S_c :

$$\boldsymbol{Z}_{c} = \begin{bmatrix} \boldsymbol{Z}_{1} \\ \boldsymbol{Z}_{2} \\ \boldsymbol{Z}_{3} \end{bmatrix}; \quad \boldsymbol{S}_{c} = \begin{bmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} & \boldsymbol{S}_{13} \\ \boldsymbol{S}_{21} & \boldsymbol{S}_{22} & \boldsymbol{S}_{23} \\ \boldsymbol{S}_{31} & \boldsymbol{S}_{32} & \boldsymbol{S}_{33} \end{bmatrix}.$$
(23)

where $Z_1 \in \mathbb{C}^{t \times M}$, $Z_2 \in \mathbb{C}^{r \times M}$, and $Z_3 \in \mathbb{C}^{(N-J) \times M}$, respectively; S_{ij} , $(i, j) \in \{1, 2, 3\} \times \{1, 2, 3\}$, is a sub-matrix whose dimensions can be obtained replacing 1, 2, and 3 with t, r, and (N-J), respectively². Furthermore, we also define the following participation for $Z_{c,\perp}$, which will be used throughout the manuscript:

$$\boldsymbol{Z}_{c,\perp} = \begin{bmatrix} \boldsymbol{Z}_{\perp,1}^T & \boldsymbol{Z}_{\perp,2}^T & \boldsymbol{Z}_{\perp,3}^T \end{bmatrix}^T$$
(24)

where $\mathbf{Z}_{\perp,1} \in \mathbb{C}^{t \times (K-M)}$, $\mathbf{Z}_{\perp,2} \in \mathbb{C}^{r \times (K-M)}$ and $\mathbf{Z}_{\perp,3} \in \mathbb{C}^{(N-J) \times (K-M)}$, respectively. Observe that each sub-matrix of \mathbf{S}_c in Eq. (23) can be expressed in terms of Eq. (24), that is, $\mathbf{S}_{ij} = (\mathbf{Z}_{\perp,i}\mathbf{Z}_{\perp,j}^{\dagger})$. We are thus ready to present the proposition providing the expression of a maximal invariant for the problem at hand.

Proposition 1. A MIS with respect to \mathcal{L} for the problem in Eq. (12) is given by:

$$\boldsymbol{T}(\boldsymbol{Z}_{c},\boldsymbol{S}_{c}) = \begin{cases} \begin{bmatrix} \boldsymbol{T}_{a} \triangleq \left\{ \boldsymbol{Z}_{2,3}^{\dagger} \, \boldsymbol{S}_{2,3}^{-1} \, \boldsymbol{Z}_{2,3} \right\} \\ \boldsymbol{T}_{b} \triangleq \left\{ \boldsymbol{Z}_{3}^{\dagger} \, \boldsymbol{S}_{33}^{-1} \boldsymbol{Z}_{3} \right\} \end{bmatrix} & J < N \\ \boldsymbol{Z}_{2}^{\dagger} \, \boldsymbol{S}_{22}^{-1} \boldsymbol{Z}_{2} & J = N \end{cases}$$
(25)

²Hereinafter, in the case J = N, the "3-components" are no longer present in the partitioning.



Figure 1. Block diagram of processing leading to a generic invariant test.

where $Z_{2,3} \triangleq (Z_2 - S_{23}S_{33}^{-1}Z_3)$ and $S_{2,3} \triangleq (S_{22} - \text{as in Eq. (23) and the pair } (G^\circ, F^\circ)$ is suitably defined as: $S_{23}S_{33}^{-1}S_{32}^{-1}$).

Proof: The proof is given in Appendix B. Some important remarks are now in order.

- In the case J < N, the MIS is given by a pair of matrices (namely T_a and T_b) where the second component (T_b) represents an ancillary part, that is, its distribution does not depend on the hypothesis in force;
- In the case J < N, the matrices $T_a \in \mathbb{C}^{M imes M}$ and $T_h \in$ $\mathbb{C}^{M \times M}$ have rank equal to $\min\{M, r\}$ and $\min\{M, N-$ J, respectively;
- It is of certain interest comparing the general expression in Eq. (25) with the MIS instances obtained in [14], [18], [12], [16] for specific adaptive detection scenarios. Accordingly, Sec. V will be devoted to comparisons and exhaustive discussion of the specialized forms in some relevant scenarios;
- Finally, exploiting [10, Thm. 6.2.1], every invariant test may be written as a function of Eq. (25) (see. Fig. 1 for a schematic representation). Therefore, it naturally follows that every CFAR test can be expressed in terms of the MIS. Part II of this study will be devoted to the design of theoretically-founded detectors whose CFARness will be proved by showing their dependence on the data solely through the obtained MIS.

IV. STATISTICAL CHARACTERIZATION OF THE MIS

In this section, we provide the statistical characterization of the MIS for the case J < N and then, we will give a corollary referring to J = N. To this end, we show that the MIS can be written as a function of whitened random vectors and matrices and then we find a suitable stochastic representation by means of a one-to-one transformation.

First, we consider the following transformation $(G^{\circ}, F^{\circ}) \in$ \mathcal{L} , which leads to:

$$\boldsymbol{Z}_{c}^{\circ} = \boldsymbol{G}^{\circ}\boldsymbol{Z}_{c} + \boldsymbol{F}^{\circ} = \begin{bmatrix} \boldsymbol{Z}_{1}^{\circ T} & \boldsymbol{Z}_{2}^{\circ T} & \boldsymbol{Z}_{3}^{\circ T} \end{bmatrix}^{T}$$
(26)

and

$$\mathbf{S}_{c}^{\circ} = \mathbf{G}^{\circ} \mathbf{S}_{c} \, \mathbf{G}^{\circ \dagger} = \begin{bmatrix} \mathbf{S}_{11}^{\circ} & \mathbf{S}_{12}^{\circ} & \mathbf{S}_{13}^{\circ} \\ \mathbf{S}_{21}^{\circ} & \mathbf{S}_{22}^{\circ} & \mathbf{S}_{23}^{\circ} \\ \mathbf{S}_{31}^{\circ} & \mathbf{S}_{32}^{\circ} & \mathbf{S}_{33}^{\circ} \end{bmatrix}$$
(27)

where Z_i° and $S_{\ell m}^{\circ}$ $(i, \ell, m \in \{1, 2, 3\})$ are similarly defined

$$G^{\circ} \triangleq \begin{bmatrix} G_{11}^{\circ} & G_{12}^{\circ} & G_{13}^{\circ} \\ \mathbf{0}_{r \times t} & \mathbf{R}_{2.3}^{-1/2} & -\mathbf{R}_{2.3}^{-1/2} \mathbf{R}_{23} \mathbf{R}_{33}^{-1} \\ \mathbf{0}_{(N-J) \times t} & \mathbf{0}_{(N-J) \times r} & \mathbf{R}_{33}^{-1/2} \end{bmatrix}; \quad (28)$$

$$\mathbf{F}^{\circ} \triangleq \begin{bmatrix} \mathbf{F}_{1}^{\circ} \\ \mathbf{0}_{r \times M} \\ \mathbf{0}_{(N-J) \times M} \end{bmatrix}; \quad (29)$$

where $G_{i,j}^{\circ}$, $i = 1, j \in \{1, 2, 3\}$, and F_1° are generic matrices of proper dimensions, while $\mathbf{R}_{22} \in \mathbb{C}^{r \times r}$, $\mathbf{R}_{23} \in \mathbb{C}^{r \times (N-J)}$, and $\mathbf{R}_{33} \in \mathbb{C}^{(N-J) \times (N-J)}$ are obtained partitioning the true covariance matrix R in the same way as done for S in Eq. (23). Finally, we have defined $R_{2.3} \triangleq R_{22}$ – $R_{23} R_{33}^{-1} R_{23}^{\dagger}$.

Hereinafter, we will study MIS statistical characterization after the trasformation (G°, F°) . This will simplify the subsequent analysis and does not affect the obtained results since the MIS is (by definition) invariant with respect to every trasformation belonging to \mathcal{L} .

Now observe that, under \mathcal{H}_i , $i \in \{0, 1\}$, it holds:

$$\begin{bmatrix} \mathbf{Z}_{2}^{\circ} \\ \mathbf{Z}_{3}^{\circ} \end{bmatrix} | \mathcal{H}_{i} = \mathbf{G}_{3}^{\circ} \begin{bmatrix} \mathbf{Z}_{2} \\ \mathbf{Z}_{3}^{\circ} \end{bmatrix} | \mathcal{H}_{i}$$

$$\sim \mathcal{CN}_{(N-t) \times M} \left(i \, \mathbf{G}_{3}^{\circ} \begin{bmatrix} \mathbf{I}_{r} \\ \mathbf{0}_{(N-J) \times r} \end{bmatrix} \mathbf{B}, \, \mathbf{I}_{M}, \, \mathbf{I}_{N-t} \right) \quad (30)$$

$$\begin{bmatrix} \mathbf{S}_{22}^{\circ} & \mathbf{S}_{23}^{\circ} \\ \mathbf{S}_{32}^{\circ} & \mathbf{S}_{33}^{\circ} \end{bmatrix} = \left\{ \mathbf{G}_{3}^{\circ} \begin{bmatrix} \mathbf{S}_{22} & \mathbf{S}_{23} \\ \mathbf{S}_{32} & \mathbf{S}_{33}^{\circ} \end{bmatrix} = \left\{ \mathbf{G}_{3}^{\circ} \begin{bmatrix} \mathbf{S}_{22} & \mathbf{S}_{23} \\ \mathbf{S}_{32} & \mathbf{S}_{33}^{\circ} \end{bmatrix} (\mathbf{G}_{3}^{\circ})^{\dagger} \right\}$$

$$\sim \mathcal{CW}_{N-t}(K-M, \, \mathbf{I}_{N-t}) \quad (31)$$

where we have defined $G_3^{\circ} \in \mathbb{C}^{(N-t) \times (N-t)}$ as

$$\boldsymbol{G}_{3}^{\circ} \triangleq \begin{bmatrix} \boldsymbol{R}_{2.3}^{-1/2} & -\boldsymbol{R}_{2.3}^{-1/2} \boldsymbol{R}_{23} \boldsymbol{R}_{33}^{-1} \\ \boldsymbol{0}_{(N-J)\times r} & \boldsymbol{R}_{33}^{-1/2} \end{bmatrix} .$$
(32)

Thus, exploiting the invariance property, we can equivalently rewrite the two components of $T(Z_c, S_c)$ (cf. Eq. (25)) in terms of the whitened quantities

$$\boldsymbol{T}_{a} = (\boldsymbol{Z}_{2.3}^{\circ})^{\dagger} \, (\boldsymbol{S}_{2.3}^{\circ})^{-1} \, \boldsymbol{Z}_{2.3}^{\circ}, \tag{33}$$

$$\boldsymbol{T}_{b} = (\boldsymbol{Z}_{3}^{\circ})^{\dagger} \, (\boldsymbol{S}_{33}^{\circ})^{-1} \, \boldsymbol{Z}_{3}^{\circ}, \tag{34}$$

where $Z_{23}^{\circ} \triangleq Z_2^{\circ} - S_{23}^{\circ} (S_{33}^{\circ})^{-1} Z_3^{\circ}$ and $S_{23}^{\circ} \triangleq S_{22}^{\circ} - S_{23}^{\circ} (S_{33}^{\circ})^{-1} S_{32}^{\circ}$, respectively. Let us focus on Z_{23}° and rewrite

$$Z_2^\circ = \begin{bmatrix} \boldsymbol{z}_{2,1}^\circ \cdots & \boldsymbol{z}_{2,M}^\circ \end{bmatrix}$$
(35)

$$\boldsymbol{S}_{23}^{\circ} (\boldsymbol{S}_{33}^{\circ})^{-1} \boldsymbol{Z}_{3}^{\circ} = \sum_{k=1}^{(N-M)} \boldsymbol{r}_{2,k} \, \boldsymbol{r}_{3,k}^{\dagger} \, (\boldsymbol{S}_{33}^{\circ})^{-1} \, \boldsymbol{Z}_{3}^{\circ} \, (36)$$

$$\boldsymbol{Z}_{2,3}^{\circ} = \begin{bmatrix} \boldsymbol{q}_{1} & \cdots & \boldsymbol{q}_{M} \end{bmatrix} = \begin{bmatrix} \left(\boldsymbol{z}_{2,1}^{\circ} - \sum_{k=1}^{K-M} \boldsymbol{r}_{2,k} \boldsymbol{r}_{3,k}^{\dagger} \boldsymbol{a}_{1} \right) & \cdots & \left(\boldsymbol{z}_{2,M}^{\circ} - \sum_{k=1}^{K-M} \boldsymbol{r}_{2,k} \boldsymbol{r}_{3,k}^{\dagger} \boldsymbol{a}_{M} \right) \end{bmatrix}$$
(37)

where $r_{j,k}$ generically denotes the k-th column of $Z_{\perp,j}$. Given the aforementioned definitions, we obtain the explicit form given in Eq. (37) (at the top of next page) for $Z_{2,3}^{\circ}$ in terms of its columns q_{ℓ} , $\ell = 1, \ldots M$, where we have further defined $a_{\ell} \triangleq \left\{ (S_{33}^{\circ})^{-1} z_{3,\ell}^{\circ} \right\}$ and $z_{3,\ell}$ similarly represents the ℓ -th column of Z_{3}° , that is $Z_{3}^{\circ} = [z_{3,1}^{\circ} \cdots z_{3,M}^{\circ}]$.

First, we observe that $\mathbf{r}_{2,k} \sim C\mathcal{N}_r(\mathbf{0}_r, \mathbf{I}_r)$ and $\mathbf{r}_{3,k} \sim C\mathcal{N}_{N-J}(\mathbf{0}_{N-J}, \mathbf{I}_{N-J})$; also it is apparent that these vectors are all mutually independent. Before proceeding, we define the short-hand notation "#3" to denote the conditioning with respect to all the terms with subscript "3". Then, it can be shown that $q_\ell|(\#3, \mathcal{H}_0)$ is Gaussian distributed (recall that $\mathbf{z}_{2,\ell}|\mathcal{H}_0 \sim C\mathcal{N}_r(\mathbf{0}_r, \mathbf{I}_r)$) with mean vector $\mathbf{0}_r$ and covariance:

$$\mathbb{E}\left\{ (\boldsymbol{z}_{2,\ell}^{\circ} - \sum_{k=1}^{K-M} \boldsymbol{r}_{2,k} \boldsymbol{r}_{3,k}^{\dagger} \boldsymbol{a}_{\ell}) (\boldsymbol{z}_{2,\ell}^{\circ} - \sum_{k=1}^{K-M} \boldsymbol{r}_{2,k} \boldsymbol{r}_{3,k}^{\dagger} \boldsymbol{a}_{\ell})^{\dagger} \right\} = \boldsymbol{I}_{r} \left(1 + \boldsymbol{z}_{3,\ell}^{\dagger} (\boldsymbol{S}_{33}^{\circ})^{-1} \boldsymbol{z}_{3,\ell} \right)$$
(38)

Similarly, the cross-covariance between $q_{\ell}|(\#3, \mathcal{H}_0)$ and $q_m|(\#3, \mathcal{H}_0)$ is given by:

$$\mathbb{E}\left\{ (\boldsymbol{z}_{2,m}^{\circ} - \sum_{k=1}^{K-M} \boldsymbol{r}_{2,k} \boldsymbol{r}_{3,k}^{\dagger} \boldsymbol{a}_{m}) (\boldsymbol{z}_{2,\ell}^{\circ} - \sum_{k=1}^{K-M} \boldsymbol{r}_{2,k} \boldsymbol{r}_{3,k}^{\dagger} \boldsymbol{a}_{\ell})^{\dagger} \right\} = \boldsymbol{I}_{r} \left(\boldsymbol{z}_{3,m}^{\dagger} (\boldsymbol{S}_{33}^{\circ})^{-1} \boldsymbol{z}_{3,\ell} \right)$$
(39)

Therefore, in view the aforementioned results, it follows that $\zeta_{2,3} \triangleq \operatorname{vec}(\mathbf{Z}_{2,3}^\circ)$ is conditionally distributed as:

$$\boldsymbol{\zeta}_{2.3}|(\#3,\mathcal{H}_0) \sim \mathcal{CN}_{rM} \left(\boldsymbol{0}_{rM}, (\boldsymbol{I}_M + (\boldsymbol{Z}_3^\circ)^{\dagger} (\boldsymbol{S}_{33}^\circ)^{-1} \boldsymbol{Z}_3^\circ) \otimes \boldsymbol{I}_r \right)$$
(40)

Then, we whiten $\zeta_{2.3}$, that is, we define:

$$\boldsymbol{x} \triangleq \left[(\boldsymbol{I}_M + \boldsymbol{Z}_3^{\circ\dagger} (\boldsymbol{S}_{33}^{\circ})^{-1} \boldsymbol{Z}_3^{\circ}) \otimes \boldsymbol{I}_r \right]^{-1/2} \boldsymbol{\xi}_{2.3}$$
(41)

which evidently gives $\boldsymbol{x}|(\#3, \mathcal{H}_0) \sim \mathcal{CN}_{rM}(\boldsymbol{0}_{rM}, \boldsymbol{I}_{rM})$. Also, we observe that:

$$\begin{bmatrix} \left(\boldsymbol{I}_{M} + \boldsymbol{Z}_{3}^{\circ\dagger}(\boldsymbol{S}_{33}^{\circ})^{-1}\boldsymbol{Z}_{3}^{\circ} \right) \otimes \boldsymbol{I}_{r} \end{bmatrix}^{-1/2} = \qquad (42)$$
$$\begin{bmatrix} \left(\boldsymbol{I}_{M} + \boldsymbol{Z}_{3}^{\circ\dagger}(\boldsymbol{S}_{33}^{\circ})^{-1}\boldsymbol{Z}_{3}^{\circ} \right)^{-1/2} \end{bmatrix} \otimes \boldsymbol{I}_{r}$$

which readily follows from the distributive property of Kronecker product. As a consequence, we have that $\boldsymbol{x} = \text{vec}(\boldsymbol{X})$, where we have defined

$$\boldsymbol{X} \triangleq \left(\boldsymbol{Z}_{2.3}^{\circ} \, \boldsymbol{K}_{s3} \right), \tag{43}$$

and $\mathbf{K}_{s3} \triangleq \left[(\mathbf{I}_M + \mathbf{Z}_3^{\circ\dagger} (\mathbf{S}_{33}^{\circ})^{-1} \mathbf{Z}_3^{\circ})^{-1/2} \right]^T$. Also, it is straightforward to show that $\mathbf{X} | (\mathcal{H}_0, \#3) = \mathbf{X} | \mathcal{H}_0 \sim C \mathcal{N}_{r \times M} (\mathbf{0}_{r \times M}, \mathbf{I}_M, \mathbf{I}_r)$ (i.e., it does not depend on the

components with subscript "3"). We then consider a one-toone transformation of $T(Z_c^{\circ}, S_c^{\circ})$, defined as:

$$\boldsymbol{T}_{1}(\boldsymbol{Z}_{c}^{\circ},\,\boldsymbol{S}_{c}^{\circ}) \triangleq \begin{bmatrix} \boldsymbol{K}_{s3}^{\dagger}\,(\boldsymbol{Z}_{2.3}^{\circ})^{\dagger}\,(\boldsymbol{S}_{2.3}^{\circ})^{-1}\,\boldsymbol{Z}_{2.3}^{\circ}\,\boldsymbol{K}_{s3} \\ (\boldsymbol{Z}_{3}^{\circ})^{\dagger}\,(\boldsymbol{S}_{33}^{\circ})^{-1}\,\boldsymbol{Z}_{3}^{\circ} \end{bmatrix}$$
(44)

$$= \begin{bmatrix} \boldsymbol{X}^{\dagger} \left(\boldsymbol{S}_{2.3}^{\circ}\right)^{-1} \boldsymbol{X} \\ \left(\boldsymbol{Z}_{3}^{\circ}\right)^{\dagger} \left(\boldsymbol{S}_{33}^{\circ}\right)^{-1} \boldsymbol{Z}_{3}^{\circ} \end{bmatrix} \triangleq \begin{bmatrix} \boldsymbol{T}_{1,a} \\ \boldsymbol{T}_{1,b} \end{bmatrix}$$
(45)

It is clear that, since $T_1(\cdot)$ is a one-to-one transformation of the MIS, it is a MIS itself [10]. Therefore, without loss of generality, we will concentrate on the statistical characterization of $T_1(\cdot)$.

We start by recalling that $S_{2.3}^{\circ}$ is independent of $\{S_{22}^{\circ}, S_{33}^{\circ}\}$ [20, Thm. A.11]. Also, we notice that $S_{33}^{\circ} \sim CW_{N-J}(K - M, I_{N-J})$ and $S_{2.3}^{\circ} \sim CW_r((K - M) - (N - J), I_r)$. These results hold under both the hypotheses. Furthermore, conditioned on \mathcal{H}_0 , X is independent on $T_{1,b}$ (as it is independent on terms with subscript "3").

Therefore, it follows that conditioned on \mathcal{H}_0 , $T_{1,a}$ and $T_{1,b}$ are *statistically independent matrices*, which means that the joint pdf can be written as $f_0(T_{1,a}, T_{1,b}) = f_0(T_{1,a}) f(T_{1,b})$ (as $T_{1,b}$ denotes the ancillary part of the MIS and thus its pdf is independent on the specific hypothesis). Finally, it is worth noticing that in the case $M \leq r$ we obtain the explicit pdf of $T_{1,a}|\mathcal{H}_0 \sim C\mathcal{F}_M(\mathbf{0}_{M\times M}, r, (K-M) - (N-J))$ and, for $M \leq (N-J), T_{1,b} \sim C\mathcal{F}_M(\mathbf{0}_{M\times M}, N-J, (K-M) - (N-J))$, following [21].

On the other hand, when \mathcal{H}_1 holds true, it is not difficult to show that $\mathbf{Z}_{2,3}^{\circ}|(\mathcal{H}_1, \#3) \sim \mathcal{CN}_{r \times M}(\mathbf{R}_{2,3}^{-1/2}\mathbf{B}, (\mathbf{I}_M + (\mathbf{Z}_3^{\circ})^{\dagger}(\mathbf{S}_{33}^{\circ})^{-1}\mathbf{Z}_3^{\circ}), \mathbf{I}_r)$ and consequently $\mathbf{X}|(\mathcal{H}_1, \#3) \sim \mathcal{CN}_{r \times M}(\mathbf{R}_{2,3}^{-1/2}\mathbf{B}\mathbf{K}_{s3}, \mathbf{I}_M, \mathbf{I}_r)$. A direct inspection of the last result reveals that

$$\boldsymbol{X}|(\#3,\mathcal{H}_1) = \boldsymbol{X}|(\boldsymbol{T}_{1,b},\mathcal{H}_1) \sim \mathcal{CN}_{r \times M} \left(\boldsymbol{R}_{2.3}^{-1/2} \boldsymbol{B} \left((\boldsymbol{I}_M + \boldsymbol{T}_{1,b})^{-1/2} \right)^T, \, \boldsymbol{I}_M, \, \boldsymbol{I}_r \right)$$
(46)

which underlines that $T_{1,a}$ and $T_{1,b}$ are statistically dependent under \mathcal{H}_1 , thus leading to $f_1(T_{1,a}, T_{1,b}) = f_1(T_{1,a}|T_{1,b}) f(T_b)$. Again, in the specific case $M \leq r$ it holds $T_{1,a}|(\mathcal{H}_1, T_{1,b}) \sim C\mathcal{F}_M(\Omega, r, (K-M) - (N-J))$, where we have denoted $\Omega \triangleq (K_{s3}^{\dagger} B^{\dagger} R_{2,3}^{-1} B K_{s3})$, following [21].

Finally, we conclude the section with a discussion on the induced maximal invariant in the parameter space [10]. The induced maximal invariant represents the reduced set of unknown parameters on which the hypothesis testing in the invariant domain depends. It is not difficult to show that for I-GMANOVA model this equals to $T_p \triangleq B^{\dagger} R_{2,3}^{-1} B \in \mathbb{C}^{M \times M}$. In addition, the induced maximal invariant is not full rank in the general case, with corresponding rank being equal to $\min\{r, M\}$. It is worth remarking that such result applies in general, that is, the distribution of the MIS will depend on the parameter space only through T_p , following classic results from [10].

V. MIS IN SPECIAL CASES

A. Adaptive detection of a point-like target

In the present case we start from general formulation in Eq. (1) and assume that: (i) t = 0 (i.e., there is no interference); (ii) r = 1 (thus J = 1) i.e., the matrix \tilde{A}_r collapses to $\tilde{a}_r \in \mathbb{C}^{N \times 1}$; (iii) M = 1, i.e. the matrix \tilde{B}_r collapses to a scalar $\tilde{b}_r \in \mathbb{C}$ and (iv) $\tilde{c} \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{1 \times K}$ (i.e. a row vector). Such case has been dealt in [12]. Therefore, the hypothesis testing in canonical form is given by:

$$\begin{cases} \mathcal{H}_0: \quad \boldsymbol{Z} = \boldsymbol{N} \\ \mathcal{H}_1: \quad \boldsymbol{Z} = \boldsymbol{a} \, \boldsymbol{b} \, \boldsymbol{c} + \boldsymbol{N} \end{cases}$$
(47)

where $\boldsymbol{a} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T \in \mathbb{C}^{N \times 1}$ and $\boldsymbol{c} = \tilde{\boldsymbol{c}}$. By looking at the general MIS statistic expression in Eq. (25), it is not difficult to show that the present problem admits the following simplified partioning:

$$\boldsymbol{z}_{c} = \begin{bmatrix} z_{2} \\ \boldsymbol{z}_{3} \end{bmatrix}, \quad \boldsymbol{Z}_{c,\perp} = \begin{bmatrix} \boldsymbol{z}_{\perp,2} \\ \boldsymbol{Z}_{\perp,3} \end{bmatrix}, \quad \boldsymbol{S}_{c} = \begin{bmatrix} s_{22} & \boldsymbol{s}_{23} \\ \boldsymbol{s}_{32} & \boldsymbol{S}_{33} \end{bmatrix}, \quad (48)$$

where $z_2 \in \mathbb{C}$, $z_3 \in \mathbb{C}^{(N-1)\times 1}$, $z_{\perp,2} \in \mathbb{C}^{1\times(K-1)}$ (i.e., a row vector), $Z_{\perp,3} \in \mathbb{C}^{(N-1)\times(K-1)}$, $s_{22} \in \mathbb{C}$, $s_{23} \in \mathbb{C}^{1\times(N-1)}$ (i.e. a row vector), $s_{32} \in \mathbb{C}^{(N-1)\times 1}$ and $S_{33} \in \mathbb{C}^{(N-1)\times(N-1)}$, respectively. Exploiting the above particles, gives the simplified expressions:

$$s_{2.3} = (s_{22} - s_{23} S_{33}^{-1} s_{32})$$
 (49)

$$z_{2.3} = (z_2 - s_{23} S_{33}^{-1} z_3)$$
 (50)

which are both scalar valued. Therefore, it is not difficult to show that the two components of the MIS are both *scalar* valued and equal to:

$$t_a = \frac{|z_2 - s_{23} S_{33}^{-1} z_3|^2}{[s_{22} - s_{22} S_{23}^{-1} s_{22}]}$$
(51)

$$t_b = \boldsymbol{z}_3^{\dagger} \, \boldsymbol{S}_{33}^{-1} \, \boldsymbol{z}_3 \tag{52}$$

which can be rewritten in the more familiar form:

$$t_{a} = \frac{\left|z_{2} - \boldsymbol{z}_{\perp,2} \boldsymbol{Z}_{\perp,3}^{\dagger} \left(\boldsymbol{Z}_{\perp,3} \boldsymbol{Z}_{\perp,3}^{\dagger}\right)^{-1} \boldsymbol{z}_{3}\right|^{2}}{\boldsymbol{z}_{\perp,2} \left[\boldsymbol{I}_{K-1} - \boldsymbol{Z}_{\perp,3}^{\dagger} \left(\boldsymbol{Z}_{\perp,3} \boldsymbol{Z}_{\perp,3}^{\dagger}\right)^{-1} \boldsymbol{Z}_{\perp,3}\right] \boldsymbol{z}_{\perp,2}^{\dagger}} \quad (53)$$

$$t_b = \boldsymbol{z}_3^{\dagger} (\boldsymbol{Z}_{\perp,3} \boldsymbol{Z}_{\perp,3}^{\dagger})^{-1} \boldsymbol{z}_3$$
(54)

which are those obtained in [12].

Finally, it is not difficult to show that in such case the (scalar-valued) induced maximal invariant is $t_p = (|b|^2/r_{2.3})$, where $r_{2.3} \triangleq (r_{22} - r_{23} R_{33}^{-1} r_{32})$, where we exploited similar simplified partitioning for R as for S_c . The induced maximal invariant clearly coincides with the Signal-to-Noise plus Interference Ratio (SINR).

B. Adaptive vector subspace detection

In the present case we start from the general formulation in Eq. (1) and assume that: (i) t = 0 (i.e. there is no interference, thus J = r); (ii) M = 1, i.e. the matrix \tilde{B}_r collapses to a vector $\tilde{b}_r \in \mathbb{C}^{J \times 1}$ and (iii) $\tilde{c} \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{1 \times K}$ (i.e., a row vector). Such case has been dealt in [13], [14].

Therefore, the hypothesis testing in canonical form is given by:

$$\begin{cases} \mathcal{H}_0: \quad \boldsymbol{Z} = \boldsymbol{N} \\ \mathcal{H}_1: \quad \boldsymbol{Z} = \boldsymbol{A} \, \boldsymbol{b} \, \boldsymbol{c} + \boldsymbol{N} \end{cases}$$
(55)

where $\mathbf{A} = \begin{bmatrix} \mathbf{I}_r & \mathbf{0}_{r \times (N-r)} \end{bmatrix}^T \in \mathbb{C}^{N \times r}$ and $\mathbf{c} = \tilde{\mathbf{c}}$. By looking at the general MIS statistic expression in Eq. (25) it is not difficult to show that the present problem admits the following simplified participation:

$$\boldsymbol{z}_{c} = \begin{bmatrix} \boldsymbol{z}_{2} \\ \boldsymbol{z}_{3} \end{bmatrix}, \quad \boldsymbol{Z}_{c,\perp} = \begin{bmatrix} \boldsymbol{Z}_{\perp,2} \\ \boldsymbol{Z}_{\perp,3} \end{bmatrix}, \quad \boldsymbol{S}_{c} = \begin{bmatrix} \boldsymbol{S}_{22} & \boldsymbol{S}_{23} \\ \boldsymbol{S}_{32} & \boldsymbol{S}_{33} \end{bmatrix},$$
 (56)

where $\boldsymbol{z}_2 \in \mathbb{C}^{J \times 1}$, $\boldsymbol{z}_3 \in \mathbb{C}^{(N-J) \times 1}$, $\boldsymbol{Z}_{\perp,2} \in \mathbb{C}^{J \times (K-1)}$, $\boldsymbol{Z}_{\perp,3} \in \mathbb{C}^{(N-J) \times (K-1)}$, $\boldsymbol{S}_{22} \in \mathbb{C}^{J \times J}$, $\boldsymbol{S}_{23} \in \mathbb{C}^{J \times (N-J)}$, $\boldsymbol{S}_{32} \in \mathbb{C}^{(N-J) \times J}$ and $\boldsymbol{S}_{33} \in \mathbb{C}^{(N-J) \times (N-J)}$, respectively.

Exploiting the above partitioning, gives $S_{2,3} \in \mathbb{C}^{J \times J}$ and the simplified expression:

$$\boldsymbol{z}_{2.3} = (\boldsymbol{z}_2 - \boldsymbol{S}_{23} \, \boldsymbol{S}_{33}^{-1} \, \boldsymbol{z}_3) \in \mathbb{C}^{J \times 1}$$
 (57)

Since $z_{2,3}$ and z_3 are both column vectors, it is not difficult to show that the two components of the MIS are both *scalar* valued and equal to:

$$t_a = \boldsymbol{z}_{2.3}^{\dagger} \, \boldsymbol{S}_{2.3}^{-1} \, \boldsymbol{z}_{2.3}, \qquad t_b = \boldsymbol{z}_3^{\dagger} \, \boldsymbol{S}_{33}^{-1} \, \boldsymbol{z}_3, \qquad (58)$$

which is the classic result obtained in [14].

Finally, the (scalar-valued) induced maximal invariant equals $t_{\rm p} = b^{\dagger} R_{2.3}^{-1} b$ which is the result obtained in [14], being equal to the SINR.

C. Adaptive vector subspace detection with structured interference

In the present case we start from general formulation in Eq. (1) and assume that: (i) M = 1, i.e., the matrices \tilde{B}_r and \tilde{B}_t collapse to the vectors $\tilde{b}_r \in \mathbb{C}^{r \times 1}$ and $\tilde{b}_t \in \mathbb{C}^{t \times 1}$, respectively; (ii) $\tilde{c} \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{1 \times K}$ (i.e. a row vector). Such case has been dealt in [18]. Given the aforementioned assumptions, the problem in canonical form is given as:

$$\begin{cases} \mathcal{H}_{0}: \quad \boldsymbol{Z} = \boldsymbol{A} \begin{bmatrix} \boldsymbol{b}_{t,0}^{T} & \boldsymbol{0}_{r}^{T} \end{bmatrix}^{T} \boldsymbol{c} + \boldsymbol{N} \\ \mathcal{H}_{1}: \quad \boldsymbol{Z} = \boldsymbol{A} \begin{bmatrix} \boldsymbol{b}_{t,1}^{T} & \boldsymbol{b}^{T} \end{bmatrix}^{T} \boldsymbol{c} + \boldsymbol{N} \end{cases}$$
(59)

where $b_{t,i} \in \mathbb{C}^{t \times 1}$, $b \in \mathbb{C}^{r \times 1}$ and $c = \tilde{c}$. By looking at the general MIS statistic expression in Eq. (25) it is not difficult to show that the present problem admits the following simplified expression:

$$\boldsymbol{z}_{c}^{T} = \begin{bmatrix} \boldsymbol{z}_{1}^{T} & \boldsymbol{z}_{2}^{T} & \boldsymbol{z}_{3}^{T} \end{bmatrix}^{T}$$
(60)

where $z_1 \in \mathbb{C}^{t \times 1}$, $z_2 \in \mathbb{C}^{r \times 1}$ and $z_3 \in \mathbb{C}^{(N-J) \times 1}$, respectively. Therefore, it is readily shown that $S_{2.3} \in \mathbb{C}^{r \times r}$ and $z_{2.3} = (z_2 - S_{23} S_{33}^{-1} z_3) \in \mathbb{C}^{r \times 1}$. Since $z_{2.3}$ and z_3 are both column vectors, it is not difficult to show that both components of the MIS are scalar-valued (similarly to the "nointerference" case) and equal to:

$$t_a = \boldsymbol{z}_{2.3}^{\dagger} \, \boldsymbol{S}_{2.3}^{-1} \, \boldsymbol{z}_{2.3} \,; \qquad t_b = \boldsymbol{z}_3^{\dagger} \, \boldsymbol{S}_{33}^{-1} \, \boldsymbol{z}_3 \,; \qquad (61)$$

which can be recognized as the result obtained in [18]. It is worth noticing that this result seems identical to that obtained in the previous sub-section (i.e., the interference-free case). However, we observe that, as opposed to the expression in (58), definition of constituents of MIS in Eq. (61) is obtained by discarding terms with subscript "1". In other terms, Eq. (61) is analogous to (58) *only* after projection in the complementary subspace of the interference.

Finally, the (scalar-valued) induced maximal invariant is $t_{\rm p} = b^{\dagger} R_{2.3}^{-1} b$, which coincides with the result obtained in [18], being equal to the SINR in the complementary interference subspace.

D. Multidimensional signals

In the present case we start from general formulation in Eq. (1) and assume that: (i) t = 0 (i.e., there is no interference, thus J = r), (ii) $\widetilde{A}_r = I_N$ (thus J = r = N) and (iii) $\widetilde{C} \triangleq \begin{bmatrix} I_M & \mathbf{0}_{M \times (K-M)} \end{bmatrix}$. Such case has been dealt in [16]. Therefore, the hypothesis testing problem in canonical form³ is given by:

$$\begin{cases} \mathcal{H}_0: \quad Z = N\\ \mathcal{H}_1: \quad Z = BC + N \end{cases}$$
(62)

By looking at the general MIS statistic expression in Eq. (25) it is not difficult to show that the present problem admits the following simplified partitioning:

$$\boldsymbol{Z}_c = \boldsymbol{Z}_2, \quad \boldsymbol{Z}_{c,\perp} = \boldsymbol{Z}_{\perp,2}, \quad \boldsymbol{S}_c = \boldsymbol{S}_{22}, \quad (63)$$

where $Z_2 \in \mathbb{C}^{N \times M}$, $Z_{\perp,2} \in \mathbb{C}^{N \times (K-M)}$ and $S_{22} \in \mathbb{C}^{N \times N}$, respectively. Since in this particular setting J = N holds, we exploit the alternative expression for the MIS in Eq. (25), which shows that the MIS reduces to a single matrix, being equal to:

$$T(Z_2, S_{22}) = Z_2^{\dagger} S_{22}^{-1} Z_2$$

= $Z_2^{\dagger} (Z_{\perp,2} Z_{\perp,2}^{\dagger})^{-1} Z_2$ (64)

In the latter case, it is not difficult to show that the maximal invariant induced in the parameter space reduces to $T_{\rm p} = B^{\dagger} R^{-1} B$.

It is now interesting to compare the result in Eq. (64) with that obtained in [16]. Indeed, in the aforementioned work, the elementary action $\ell(\cdot)$ is defined as:

$$\ell_2(\boldsymbol{Z}_2, \boldsymbol{S}_{22}) = \left(\boldsymbol{G}_{22} \, \boldsymbol{Z}_2 \, \boldsymbol{U}_d, \boldsymbol{G}_{22} \, \boldsymbol{S}_{22} \, \boldsymbol{G}_{22}^{\dagger}\right) \qquad (65)$$
$$\forall \boldsymbol{G}_{22} \in \mathcal{GL}(N), \forall \boldsymbol{U}_d \in \mathcal{U}(M).$$

which, compared to Eq. (22), enforces an *additional invariance* with respect to a right subspace rotation, via the unitary matrix U_d . Clearly, this restricts further the class of invariant tests. Moreover, in Eq. (65) we have used $\mathcal{U}(M)$ to denote the group of unitary $M \times M$ matrices. It was shown in [16] that the MIS for the elementary action defined in Eq. (65) is given by the non-zero eigenvalues of the matrix

$$T_{c} \triangleq S_{22}^{-1/2} (Z_{2} Z_{2}^{\dagger}) S_{22}^{-1/2} = (Z_{\perp,2} Z_{\perp,2}^{\dagger})^{-1/2} (Z_{2} Z_{2}^{\dagger}) (Z_{\perp,2} Z_{\perp,2}^{\dagger})^{-1/2}, \quad (66)$$

 3 It is worth noticing that in this case the original formulation in Eq. (1) is in canonical form already.

denoted with $\operatorname{eig}(\mathbf{T}_c)$ in what follows. Remarkably, we show hereinafter that the MIS in Eq. (66) can be directly linked to the expression in Eq. (64). We first notice that, after defining $\mathbf{Z}_m \triangleq \left\{ (\mathbf{Z}_{\perp,2} \mathbf{Z}_{\perp,2}^{\dagger})^{-1/2} \mathbf{Z}_2 \right\} \in \mathbb{C}^{N \times M}$, the following equalities hold:

$$\boldsymbol{T} = (\boldsymbol{Z}_m^{\dagger} \, \boldsymbol{Z}_m) \qquad \quad \boldsymbol{T}_c = (\boldsymbol{Z}_m \, \boldsymbol{Z}_m^{\dagger}) \qquad (67)$$

Therefore, by construction, the matrices $T \in \mathbb{C}^{M \times M}$ and $T_c \in \mathbb{C}^{N \times N}$ are such that $\operatorname{eig}(T_c) = \operatorname{eig}(T)$ holds (and the vector length equals $\min\{M, N\}$), where we have expressed the non-zero eigenvalues through the implicit vector-valued function $\operatorname{eig}(\cdot)$. Then, we notice that the action $\ell_2(\cdot)$ can be re-interpreted as the composition of the following sub-actions:

$$\ell_{2,a}(\boldsymbol{Z}_{2}, \boldsymbol{S}_{22}) = (\boldsymbol{G}_{22} \, \boldsymbol{Z}_{2}, \boldsymbol{G}_{22} \, \boldsymbol{S}_{22} \, \boldsymbol{G}_{22}^{\dagger}) \quad \forall \boldsymbol{G}_{22} \in \mathcal{GL}(N) \\ \ell_{2,b}(\boldsymbol{Z}_{2}, \boldsymbol{S}_{22}) = (\boldsymbol{Z}_{2} \boldsymbol{U}_{d}, \boldsymbol{S}_{22}) \quad \forall \boldsymbol{U}_{d} \in \mathcal{U}(M).$$
(68)

It is then recognized that $\ell_{2,a}(\cdot) = \ell(\cdot)$ for the case of multidimensional signals. Previously, we have shown that the MIS for the elementary action $\ell_{2,a}(\cdot) = \ell(\cdot)$ is simply given by the matrix T in Eq. (64).

Additionally, we notice that, for each $U_d \in \mathcal{U}(M)$,

$$T(\bar{Z}_2, \bar{S}_{22}) = T(Z_2, S_{22}) \Rightarrow$$

$$T(\bar{Z}_2 U_d, \bar{S}_{22}) = T(Z_2 U_d, S_{22})$$
(69)

Now, define the action $\ell_{2,b}^{\star}(\cdot)$ as:

$$\ell_{2,b}^{\star}(\boldsymbol{T}) = \left(\boldsymbol{U}_{d}^{\dagger} \, \boldsymbol{T} \, \boldsymbol{U}_{d}\right) \quad \forall \, \boldsymbol{U}_{d} \in \mathcal{U}(M).$$
(70)

where $T \in \mathbb{H}^{M \times M}$. It is not difficult to show that a MIS for the elementary operation $\ell_{2,b}^{\star}(\cdot)$ in Eq. (70) is given by $\operatorname{eig}(T)$. Therefore, exploiting [10, p. 217, Thm. 6.2.2], it follows that a MIS for the action $\ell_2(\cdot)$ is the composite function $\operatorname{eig}(T(Z_2, S_{22}))$. However, since as underlined in Eq. (67), we have $\operatorname{eig}(T) = \operatorname{eig}(T_c)$, this clearly coincides with the result in [16].

Finally, by similar reasoning it is not difficult to show that, in such a case, the induced maximal invariant is given by $\operatorname{eig}(\mathbf{T}_{p}) = \operatorname{eig}(\mathbf{B}^{\dagger}\mathbf{R}^{-1}\mathbf{B}) = \operatorname{eig}(\mathbf{R}^{-1/2}\mathbf{B}\mathbf{B}^{\dagger}\mathbf{R}^{-1/2})$, thus obtaining the result in [16].

E. Range-spread Targets

In the present case we start from general formulation in Eq. (1) and assume that: (i) t = 0 (i.e., there is no interference, thus J = r); (ii) r = 1, thus the matrices \widetilde{A}_r and \widetilde{B}_r collapse to $\widetilde{a}_r \in \mathbb{C}^{N\times 1}$ and $\widetilde{b}_r \in \mathbb{C}^{1\times M}$ (i.e. a row vector), respectively; (iii) $\widetilde{C} \triangleq [I_M \ \mathbf{0}_{M \times K - M}]$. Such case has been dealt in [22], [17]. Therefore, the hypothesis testing in canonical form is given by:

$$\begin{cases}
\mathcal{H}_0: \quad \boldsymbol{Z} = \boldsymbol{N} \\
\mathcal{H}_1: \quad \boldsymbol{Z} = \boldsymbol{a} \, \boldsymbol{b} \, \boldsymbol{C} + \boldsymbol{N}
\end{cases}$$
(71)

where $\boldsymbol{a} \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T \in \mathbb{C}^{N \times 1}$, $\boldsymbol{b} \in \mathbb{C}^{1 \times M}$ and $\boldsymbol{C} = \widetilde{\boldsymbol{C}}$, respectively. By looking at the general MIS expression in Eq. (25), it is not difficult to show that the present problem admits the following simplified partitioning:

$$\boldsymbol{Z}_{c} = \begin{bmatrix} \boldsymbol{z}_{2} \\ \boldsymbol{Z}_{3} \end{bmatrix}, \quad \boldsymbol{Z}_{c,\perp} = \begin{bmatrix} \boldsymbol{z}_{\perp,2} \\ \boldsymbol{Z}_{\perp,3} \end{bmatrix}, \quad \boldsymbol{S}_{c} = \begin{bmatrix} s_{22} & \boldsymbol{s}_{23} \\ \boldsymbol{s}_{32} & \boldsymbol{S}_{33} \end{bmatrix},$$
(72)

where $\boldsymbol{z}_2 \in \mathbb{C}^{1 \times M}$ (i.e., a row vector), $\boldsymbol{Z}_3 \in \mathbb{C}^{(N-1) \times M}$, $\boldsymbol{Z}_{\perp,2} \in \mathbb{C}^{1 \times (K-M)}$ (i.e., a row vector), $\boldsymbol{Z}_{\perp,3} \in \mathbb{C}^{(N-1) \times (K-M)}$, $\boldsymbol{s}_{22} \in \mathbb{C}$, $\boldsymbol{s}_{23} \in \mathbb{C}^{1 \times (N-1)}$ (i.e., a row vector), $\boldsymbol{s}_{32} \in \mathbb{C}^{(N-1) \times 1}$ and $\boldsymbol{S}_{33} \in \mathbb{C}^{(N-1) \times (N-1)}$, respectively. Exploiting the above partitioning, gives $\boldsymbol{s}_{2.3} = (\boldsymbol{s}_{22} - \boldsymbol{s}_{23} \boldsymbol{S}_{33}^{-1} \boldsymbol{s}_{32}) \in \mathbb{C}$ (i.e., a scalar) and the simplified expression:

$$\boldsymbol{z}_{2.3} = (\boldsymbol{z}_2 - \boldsymbol{s}_{23} \, \boldsymbol{S}_{33}^{-1} \, \boldsymbol{Z}_3) \in \mathbb{C}^{1 \times M}$$
 (73)

Given the simplified expressions for $z_{2,3}$ (row vector) and $s_{2,3}$ (scalar), it is not difficult to show that the two matrix components of the MIS are given by:

$$T_a = \left(\frac{1}{s_{2.3}}\right) \boldsymbol{z}_{2.3}^{\dagger} \boldsymbol{z}_{2.3} \qquad T_b = \boldsymbol{Z}_3^{\dagger} \, \boldsymbol{S}_{33}^{-1} \, \boldsymbol{Z}_3 \qquad (74)$$

where the matrix T_a is rank-one in this specific case (as it is the output of a dyadic product). Also, the induced maximal invariant in the parameter space equals $T_p = (\frac{1}{r_{2.3}}) b^{\dagger} b$, i.e., a rank-one matrix.

It is now of interest comparing the MIS represented by Eq. (74) with that obtained in [17]. The approach taken in the following is similar to that used for multidimensional signals in Sec. V-D. However, due to the more tedious mathematics involved, we confine the proof to Appendix C and we only state the results hereinafter.

Indeed, in the aforementioned work, the elementary action $\ell(\cdot)$ is defined as:

$$\ell_2(\boldsymbol{Z}_c, \boldsymbol{S}_c) = \left(\boldsymbol{G} \, \boldsymbol{Z}_c \, \boldsymbol{U}_d, \boldsymbol{G} \, \boldsymbol{S}_c \, \boldsymbol{G}^{\dagger}\right) \,, \tag{75}$$
$$\forall \, \boldsymbol{G} \in \mathcal{G} \,, \quad \forall \, \boldsymbol{U}_d \in \mathcal{U}(M),$$

which, compared to Eq. (22), enforces an additional invariance with respect to a right subspace rotation of primary data, via the unitary matrix U_d . It was shown in [17] that the MIS for the elementary action defined in Eq. (75) is given by the eigenvalues of the matrices

$$(\boldsymbol{T}_a + \boldsymbol{T}_b), \qquad \boldsymbol{T}_b, \tag{76}$$

denoted with $\operatorname{eig}(T_a + T_b)$ and $\operatorname{eig}(T_b)$ in what follows. Remarkably, we show hereinafter that the MIS in Eq. (76) can be directly linked to the expression in Eq. (74). We first notice that the action $\ell_2(\cdot)$ can be re-interpreted as the composition of the following sub-actions:

$$\ell_{2,a}(\boldsymbol{Z}_c, \boldsymbol{S}_c) = (\boldsymbol{G}\,\boldsymbol{Z}_c, \boldsymbol{G}\,\boldsymbol{S}_c\,\boldsymbol{G}^{\dagger}), \quad \forall \boldsymbol{G} \in \mathcal{G}, \\ \ell_{2,b}(\boldsymbol{Z}_c, \boldsymbol{S}_c) = (\boldsymbol{Z}_c\,\boldsymbol{U}_d, \boldsymbol{S}_c), \quad \forall \boldsymbol{U}_d \in \mathcal{U}(M).$$
(77)

It is then recognized that $\ell_{2,a}(\cdot) = \ell(\cdot)$ for the case of rangespread signals. Also, we have previously shown that a MIS for the elementary action $\ell_{2,a}(\cdot) = \ell(\cdot)$ is given by Eq. (74).

Additionally, we notice that, for each $U_d \in \mathcal{U}(M)$,

$$T(\mathbf{Z}_c, \mathbf{S}_c) = T(\mathbf{Z}_c, \mathbf{S}_c) \Rightarrow$$
$$T(\bar{\mathbf{Z}}_c \mathbf{U}_d, \bar{\mathbf{S}}_c) = T(\mathbf{Z}_c \mathbf{U}_d, \mathbf{S}_c)$$
(78)

Now, define the action $\ell_{2,b}^{\star}(\cdot)$ as:

$$\ell_{2,b}^{\star}(\boldsymbol{T}_{a},\boldsymbol{T}_{b}) = \left(\boldsymbol{U}_{d}^{\dagger}\,\boldsymbol{T}_{a}\,\boldsymbol{U}_{d},\boldsymbol{U}_{d}^{\dagger}\,\boldsymbol{T}_{b}\,\boldsymbol{U}_{d}\right),\tag{79}$$

 $\forall U_d \in \mathcal{U}(M)$, where $T_b \in \mathbb{H}^{M \times M}$ and $T_a = (aa^{\dagger})$ (that is, a rank-one matrix). It is shown in Appendix C that the MIS for the elementary operation $\ell_{2,b}^{\star}(\cdot)$ in Eq. (79) is given by $\{\operatorname{eig}(T_b), \operatorname{eig}(T_a + T_b)\}$. Therefore, by exploiting [10, p. 217, Thm. 6.2.2], it follows that the MIS for the action $\ell_2(\cdot)$ is the composite function

$$\begin{cases} \operatorname{eig} \left(\boldsymbol{Z}_{3}^{\dagger} \, \boldsymbol{S}_{33}^{-1} \, \boldsymbol{Z}_{3} \right) \\ \operatorname{eig} \left(\boldsymbol{Z}_{3}^{\dagger} \, \boldsymbol{S}_{33}^{-1} \, \boldsymbol{Z}_{3} + \left(\frac{1}{s_{2.3}} \right) \boldsymbol{z}_{2.3}^{\dagger} \boldsymbol{z}_{2.3} \right) \quad , \qquad (80) \end{cases}$$

which clearly coincides with the result in [17].

Finally, it is not difficult to show that the induced maximal invariant in such a case can be obtained as $\operatorname{eig}(T_p) = (\frac{\|\boldsymbol{b}\|^2}{r_{2.3}}) = \|\boldsymbol{b}\|^2 (\boldsymbol{a}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{a})$ (since the rank-one induced maximal invariant has only one non-zero eigenvalue), which represents the overall SINR over the *M* cells, as defined in [17].

F. Standard GMANOVA

Finally, in the present case we start from general formulation in Eq. (1) and assume that: (i) t = 0 (i.e., there is no interference, thus J = r). Such model clearly coincides with that analyzed in [2], [4], *unfortunately not dealing with the derivation of the MIS*. Therefore, the hypothesis testing in canonical form is given by:

$$\begin{array}{ll} \mathcal{H}_0: & \boldsymbol{Z} = \boldsymbol{N} \\ \mathcal{H}_1: & \boldsymbol{Z} = \boldsymbol{A} \, \boldsymbol{B} \, \boldsymbol{C} + \boldsymbol{N} \end{array}$$

$$\tag{81}$$

where $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{I}_J & \boldsymbol{0}_{J \times (N-J)} \end{bmatrix}^T$ and $\boldsymbol{B} \in \mathbb{C}^{J \times M}$, respectively. By looking at the general MIS statistic expression in Eq. (25) it is not difficult to show that the present problem admits the following simplified partitioning:

$$\boldsymbol{Z}_{c} = \begin{bmatrix} \boldsymbol{Z}_{2} \\ \boldsymbol{Z}_{3} \end{bmatrix}, \ \boldsymbol{Z}_{c,\perp} = \begin{bmatrix} \boldsymbol{Z}_{\perp,2} \\ \boldsymbol{Z}_{\perp,3} \end{bmatrix}, \ \boldsymbol{S}_{c} = \begin{bmatrix} \boldsymbol{S}_{22} & \boldsymbol{S}_{23} \\ \boldsymbol{S}_{32} & \boldsymbol{S}_{33} \end{bmatrix}, \quad (82)$$

where $Z_2 \in \mathbb{C}^{J \times M}$, $Z_3 \in \mathbb{C}^{(N-J) \times M}$, $Z_{\perp,2} \in \mathbb{C}^{J \times (K-M)}$, $Z_{\perp,3} \in \mathbb{C}^{(N-J) \times (K-M)}$, $S_{22} \in \mathbb{C}^{J \times J}$, $S_{23} \in \mathbb{C}^{J \times (N-J)}$, $S_{32} \in \mathbb{C}^{(N-J) \times J}$ and $S_{33} \in \mathbb{C}^{(N-J) \times (N-J)}$, respectively. Given the simplified definitions in Eq. (82), the MIS is readily obtained via the standard formula in Eq. (25).

Finally, the induced maximal invariant is obtained through the standard formula $T_{\rm p} = (B^{\dagger} R_{2,3}^{-1} B)$. The sole difference consists in the rank of matrix $T_{\rm p}$, being equal to $\min\{J, M\}$, i.e., there is no reduction in the observation space due to structured interference.

VI. CONCLUSIONS

In the first part of this work, we have studied a generalization of GMANOVA model (denoted as I-GMANOVA) which comprises additional (deterministic) structured interference, modeling possible jamming interference. The study has been conducted with the help of the statistical theory of invariance. For the present problem, the group of trasformations leaving the hypothesis testing problem invariant was derived, thus allowing identification of trasformations which enforce CFARness. Then, a MIS was derived for the aforementioned group, thus explicitly underlining the basic structure of a generic CFAR receiver (several examples of CFAR receivers, based on theoretically-founded criteria, will be derived in part II of the present work).

Furthermore, a statistical characterization of the considered MIS under both hypotheses was obtained, thus allowing for an efficient stochastic representation. As a byproduct, the general form of the induced maximal invariant in the parameter space was obtained for the considered hypothesis testing. Finally, the general MIS expression was particularized and compared with MIS obtained in specific instances found in the open literature. Analogies to other expressions of the MIS, obtained by enforcing invariance to a wider class of transformations (cf. Sec. V-E and V-D), were underlined and discussed.

APPENDIX A INVARIANCE OF THE PROBLEM WITH RESPECT TO THE GROUP \mathcal{L}

In this appendix we prove the invariance of the hypothesis testing problem in (12) with respect to the group of trasformations \mathcal{L} defined in Sec. III-A. Let $(G, F) \in \mathcal{L}$ and observe that, under \mathcal{H}_1 , the columns of $GZ_c + F$ are independent complex normal vectors with covariance matrix GRG^{\dagger} and mean:

$$GAB_{s} + F = \begin{bmatrix} G_{11}B_{t,1} + G_{12}B + F_{1} \\ G_{22}B \\ \mathbf{0}_{(N-J)\times M} \end{bmatrix}$$
(83)
$$= \begin{bmatrix} B'_{t,1} \\ B' \\ \mathbf{0}_{(N-J)\times M} \end{bmatrix} = AB'_{s}$$
(84)

where we have employed the definitions $\boldsymbol{B}_{t,1}' \triangleq (\boldsymbol{G}_{11}\boldsymbol{B}_{t,1} + \boldsymbol{G}_{12}\boldsymbol{B} + \boldsymbol{F}_{11}) \in \mathbb{C}^{t \times M}$ and $\boldsymbol{B}' \triangleq (\boldsymbol{G}_{22}\boldsymbol{B}) \in \mathbb{C}^{r \times M}$, respectively. Also, aiming at compact notation, we have denoted $\boldsymbol{B}_{s}' \triangleq \left[(\boldsymbol{B}_{t,1}')^{T} \quad (\boldsymbol{B}')^{T} \right]^{T}$. Furthermore, it is not difficult to show that $\boldsymbol{GS}_{c}\boldsymbol{G}^{\dagger} = (\boldsymbol{GZ}_{c,\perp})(\boldsymbol{GZ}_{c,\perp})^{\dagger}$, with $(\boldsymbol{GZ}_{c,\perp}) \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}_{N \times (K-M)}, \boldsymbol{I}_{K-M}, \boldsymbol{GRG}^{\dagger})$.

On the other hand, when \mathcal{H}_0 holds true, $GZ_c + F$ shares the same covariance structure as in the case of \mathcal{H}_1 , except for the mean, which becomes

$$\boldsymbol{G}\boldsymbol{A}\begin{bmatrix}\boldsymbol{B}_{t,0}\\\boldsymbol{0}_{r\times M}\end{bmatrix} + \boldsymbol{F} = \begin{bmatrix}\boldsymbol{G}_{11}\boldsymbol{B}_{t,0} + \boldsymbol{F}_{1}\\\boldsymbol{0}_{r\times M}\\\boldsymbol{0}_{(N-J)\times M}\end{bmatrix}$$
(85)

$$= \boldsymbol{A} \begin{bmatrix} \boldsymbol{B}_{t,0}'\\ \boldsymbol{0}_{r\times M} \end{bmatrix}$$
(86)

where $\boldsymbol{B}_{t,0}^{'} \triangleq (\boldsymbol{G}_{11}\boldsymbol{B}_{t,0} + \boldsymbol{F}_{1}) \in \mathbb{C}^{t \times M}$. Again, it is not difficult to show that $(\boldsymbol{G}\boldsymbol{Z}_{c,\perp}) \sim \mathcal{CN}_{N \times (K-M)}(\boldsymbol{0}_{N \times (K-M)}, \boldsymbol{I}_{K-M}, \boldsymbol{GRG}^{\dagger})$.

Therefore, it is apparent that the original partition of the parameter space, the data distribution, and the structure of the subspace containing the useful signal components are preserved after the transformation (G, F). Indeed, the following equivalence holds between the original and the transformed test:

$$\begin{cases} \mathcal{H}_0 : & ||\boldsymbol{B}||_F = 0 \iff ||\boldsymbol{B}'||_F = 0, \\ \mathcal{H}_1 : & ||\boldsymbol{B}||_F > 0 \iff ||\boldsymbol{B}'||_F > 0, \end{cases}$$
(87)

where the nuisance parameters in the transformed space are $B'_{t,i}$ and (GRG^{\dagger}) .

APPENDIX B

DERIVATION OF THE MAXIMAL INVARIANT STATISTIC

In the present appendix we provide a proof for Prop. 1. In particular, hereinafter we will focus on the case J < N, as to the derivation for J = N can be obtained through identical steps. Before proceeding further, we recall that a statistic $T(Z_c, S_c)$ is said to be a maximal invariant with respect to the group of transformations \mathcal{L} iff

(a)
$$T(Z_c, S_c) = T[\ell(Z_c, S_c)], \quad \forall \ell \in \mathcal{L};$$
 (88)

(b)
$$T(Z_c, S_c) = T(Z_c, \bar{S}_c) \Rightarrow$$

 $\exists \ell \in \mathcal{L} : (Z_c, S_c) = \ell(\bar{Z}_c, \bar{S}_c).$ (89)

Conditions (a) and (b) correspond to the so-called *invariance* and *maximality* properties, respectively. In order to prove (a), we first consider the following partitioning of matrix G and sub-matrix of S_c :

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{G}_1 & \boldsymbol{G}_2 \\ \boldsymbol{0}_{(N-t)\times t} & \boldsymbol{G}_3 \end{bmatrix}, \quad \boldsymbol{S}_2 \triangleq \begin{bmatrix} \boldsymbol{S}_{22} & \boldsymbol{S}_{23} \\ \boldsymbol{S}_{32} & \boldsymbol{S}_{33} \end{bmatrix}, \quad (90)$$

where $S_2 \in \mathbb{C}^{(N-t)\times(N-t)}$, $G_1 \triangleq G_{11} \in \mathbb{C}^{t\times t}$, $G_2 \triangleq [G_{12} \quad G_{13}] \in \mathbb{C}^{t\times(N-t)}$ and

$$\boldsymbol{G}_{3} \triangleq \begin{bmatrix} \boldsymbol{G}_{22} & \boldsymbol{G}_{23} \\ \boldsymbol{0}_{(N-J)\times r} & \boldsymbol{G}_{33} \end{bmatrix} \in \mathbb{C}^{(N-t)\times(N-t)}.$$
(91)

Then, let $(\bar{Z}_c, \bar{S}_c) \triangleq \ell(Z_c, S_c)$, with

$$\bar{\boldsymbol{Z}}_c = \boldsymbol{G}\boldsymbol{Z}_c + \boldsymbol{F}, \qquad \bar{\boldsymbol{S}}_c = \boldsymbol{G}\,\boldsymbol{S}_c\,\boldsymbol{G}^{\dagger}.$$
 (92)

It is apparent that the following equalities hold, when exploiting the specific structure of G and F:

$$\bar{Z}_2 = G_{22} Z_2 + G_{23} Z_3,$$
 (93)

$$\bar{Z}_3 = G_{33} Z_3,$$
 (94)

and

$$\bar{\boldsymbol{S}}_2 = \boldsymbol{G}_3 \, \boldsymbol{S}_2 \, \boldsymbol{G}_3^{\dagger} \,, \tag{95}$$

where \bar{S}_2 is similarly defined as in Eq. (90). From Eq. (95), it can be inferred that:

$$\bar{\boldsymbol{S}}_{22} = \begin{bmatrix} \boldsymbol{G}_{22} & \boldsymbol{G}_{23} \end{bmatrix} \boldsymbol{S}_2 \begin{bmatrix} \boldsymbol{G}_{22} & \boldsymbol{G}_{23} \end{bmatrix}^{\dagger}$$
 (96)

$$\bar{\boldsymbol{S}}_{23} = \boldsymbol{G}_{22} \, \boldsymbol{S}_{23} \, \boldsymbol{G}_{33}^{\dagger} + \boldsymbol{G}_{23} \, \boldsymbol{S}_{33} \, \boldsymbol{G}_{33}^{\dagger} \tag{97}$$

$$\bar{S}_{33} = G_{33} S_{33} G_{33}^{\dagger}$$
 (98)

Additionally, exploiting the appropriate substitutions, it can be shown that:

$$\bar{\boldsymbol{Z}}_{2.3} = \left(\bar{\boldsymbol{Z}}_2 - \bar{\boldsymbol{S}}_{23}\,\bar{\boldsymbol{S}}_{33}^{-1}\,\bar{\boldsymbol{Z}}_3\right) = \boldsymbol{G}_{22}\,\boldsymbol{Z}_{2.3} \tag{99}$$

$$\mathbf{S}_{2.3} = \left(\mathbf{S}_{22} - \mathbf{S}_{23} \, \mathbf{S}_{33}^{-1} \, \mathbf{S}_{32}\right) = \mathbf{G}_{22} \, \mathbf{S}_{2.3} \, \mathbf{G}_{22}^{\dagger} \qquad (100)$$

Finally, substituting Eqs. (94), (98), (99), and (100) into (25), we obtain:

$$\boldsymbol{T}(\ell(\boldsymbol{Z}_{c},\boldsymbol{S}_{c})) = \begin{bmatrix} \bar{\boldsymbol{Z}}_{2.3}^{\dagger} \ \bar{\boldsymbol{S}}_{2.3}^{-1} \ \bar{\boldsymbol{Z}}_{2.3} \\ \bar{\boldsymbol{Z}}_{3}^{\dagger} \ \bar{\boldsymbol{S}}_{33}^{-1} \ \bar{\boldsymbol{Z}}_{3} \end{bmatrix}$$
(101)

$$= \begin{bmatrix} \mathbf{Z}_{2,3}^{\dagger} \, \mathbf{G}_{22}^{\dagger} \, (\mathbf{G}_{22}^{\dagger})^{-1} \, \mathbf{S}_{2.3}^{-1} \, \mathbf{G}_{22}^{-1} \, \mathbf{G}_{22} \, \mathbf{Z}_{2.3} \\ \mathbf{Z}_{3}^{\dagger} \, \mathbf{G}_{33}^{\dagger} \, (\mathbf{G}_{33}^{\dagger})^{-1} \, \mathbf{S}_{33}^{-1} \, (\mathbf{G}_{33}^{-1}) \, \mathbf{G}_{33} \, \mathbf{Z}_{3} \end{bmatrix}$$
(102)

$$= \begin{bmatrix} \mathbf{Z}_{2.3}^{\dagger} \, \mathbf{S}_{2.3}^{-1} \, \mathbf{Z}_{2.3} \\ \mathbf{Z}_{3}^{\dagger} \, \mathbf{S}_{33}^{-1} \, \mathbf{Z}_{3} \end{bmatrix}$$
(103)

which thus proves (a).

Now, in order to prove (b), assume that:

$$T(Z_c, S_c) = T(\bar{Z}_c, \bar{S}_c)$$
(104)
$$[Z_{23}^{\dagger} S_{23}^{-1} Z_{23}] [\bar{Z}_{23}^{\dagger} \bar{S}_{23}^{-1} \bar{Z}_{23}]$$
(107)

$$\begin{bmatrix} 2_{23} & 2_{23} & 2_{23} \\ Z_3^{\dagger} \, S_{33}^{-1} \, Z_3 \end{bmatrix} = \begin{bmatrix} 2_{23} & 2_{23} & 2_{23} \\ \bar{Z}_3^{\dagger} \, \bar{S}_{33}^{-1} \, \bar{Z}_3 \end{bmatrix}$$
(105)

The last equality can be recast as the following pair of equalities

$$Y_{2.3} Y_{2.3}^{\dagger} = \bar{Y}_{2.3} \bar{Y}_{2.3}^{\dagger}, \qquad Y_3 Y_3^{\dagger} = \bar{Y}_3 \bar{Y}_3^{\dagger}, \qquad (106)$$

where $Y_{2.3} \triangleq (S_{2.3}^{-1/2} Z_{2.3})^{\dagger}, \bar{Y}_{2.3} \triangleq (\bar{S}_{2.3}^{-1/2} \bar{Z}_{2.3})^{\dagger}, Y_3 \triangleq (S_{33}^{-1/2} Z_3)^{\dagger}$ and $\bar{Y}_3 = (\bar{S}_{33}^{-1/2} \bar{Z}_3)^{\dagger}$. It follows from direct inspection of Eq. (106) that there exist unitary matrices⁴ $U_{2.3} \in \mathbb{C}^{r \times r}$ and $U_3 \in \mathbb{C}^{(N-J) \times (N-J)}$ such that $Y_{2.3} = \bar{Y}_{2.3} U_{2.3}$ and $Y_3 = \bar{Y}_3 U_3$.

First, let us define the following block-triangular decompositions for matrices $S_2 = L_2^{\dagger} L_2$ and $\bar{S}_2 = \bar{L}_2^{\dagger} \bar{L}_2$, where:

$$\boldsymbol{L}_{2} \triangleq \begin{bmatrix} \boldsymbol{S}_{2.3}^{1/2} & \boldsymbol{0}_{r \times (N-J)} \\ \boldsymbol{S}_{33}^{-1/2} \boldsymbol{S}_{32} & \boldsymbol{S}_{33}^{1/2} \end{bmatrix}$$
(107)

$$\bar{\boldsymbol{L}}_{2} \triangleq \begin{bmatrix} \bar{\boldsymbol{S}}_{2.3}^{1/2} & \boldsymbol{0}_{r \times (N-J)} \\ \bar{\boldsymbol{S}}_{33}^{-1/2} \bar{\boldsymbol{S}}_{32} & \bar{\boldsymbol{S}}_{33}^{1/2} \end{bmatrix}$$
(108)

Therefore, given the aforementioned definitions, it can be shown that:

$$\begin{bmatrix} \mathbf{Y}_{2,3}^{\dagger} \\ \mathbf{Y}_{3}^{\dagger} \end{bmatrix} = (\mathbf{L}_{2}^{\dagger})^{-1} \mathbf{Z}_{23} =$$
(109)

$$\begin{bmatrix} U_{2,3}^{\dagger} \bar{Y}_{2,3}^{\dagger} \\ U_{3}^{\dagger} \bar{Y}_{3}^{\dagger} \end{bmatrix} = U_{1} (\bar{L}_{2}^{\dagger})^{-1} \bar{Z}_{23}$$
(110)

where $Z_{23} \triangleq \begin{bmatrix} Z_2^T & Z_3^T \end{bmatrix}^T$ and $U_1 \triangleq \text{diag}(U_{2.3}^{\dagger}, U_3^{\dagger})$, respectively. From comparison of right hand side. of Eqs. (109) and (110), it readily follows that

$$\boldsymbol{Z}_{23} = \boldsymbol{L}_{2}^{\dagger} \boldsymbol{U}_{1} (\bar{\boldsymbol{L}}_{2}^{\dagger})^{-1} \bar{\boldsymbol{Z}}_{23} \,. \tag{111}$$

From inspection of Eq. (111), it is apparent that selecting the transformation $G_3 = L_2^{\dagger} U_1(\bar{L}_2^{\dagger})^{-1}$ (which is blocktriangular as dictated by Eq. (91)) automatically verifies the set of equations:

$$\begin{cases} (i) & G_3 \bar{Z}_{23} = Z_{23} \\ (ii) & G_3 \bar{S}_2 G_3^{\dagger} = S_2 \end{cases}$$
(112)

since it also holds

$$(\boldsymbol{L}_{2}^{\dagger})\boldsymbol{U}_{1}(\bar{\boldsymbol{L}}_{2}^{\dagger})^{-1}\bar{\boldsymbol{S}}_{2}(\bar{\boldsymbol{L}}_{2})^{-1}\boldsymbol{U}_{1}^{\dagger}\boldsymbol{L}_{2} = (\boldsymbol{L}_{2}^{\dagger})\boldsymbol{U}_{1}(\bar{\boldsymbol{L}}_{2}^{\dagger})^{-1}\bar{\boldsymbol{L}}_{2}^{\dagger}\bar{\boldsymbol{L}}_{2}(\bar{\boldsymbol{L}}_{2})^{-1}\boldsymbol{U}_{1}^{\dagger}\boldsymbol{L}_{2} = \boldsymbol{L}_{2}^{\dagger}\boldsymbol{L}_{2} = \boldsymbol{S}_{2} \quad (113)$$

⁴Such property can be verified as follows: given the equality $AA^{\dagger} = BB^{\dagger}$ between two generic matrices A and B and, after defining the eigenvalue decompositions $(AA^{\dagger}) = U_A \Lambda_A U_A^{\dagger}$ and $(BB^{\dagger}) = U_B \Lambda_B U_B^{\dagger}$, and the SVDs $A = U_A \Sigma_A V_A^{\dagger}$ and $B = U_B \Sigma_B V_B^{\dagger}$, it is apparent that such equality implies: (i) $\Sigma_A = \Sigma_B \widehat{D}$; (ii) $U_A = U_B D$, where D and \widehat{D} denote diagonal matrices of phasors (recall that $\Lambda_A =$ $\Sigma_A \Sigma_A^*$ and $\Lambda_B = \Sigma_B \Sigma_B^*$). Thus, it follows from substitution that A = $U_B D \Sigma_B \widetilde{D} V_A^{\dagger} = U_B \Sigma_B \widetilde{D} D V_A^{\dagger}$. Therefore, after defining the unitary matrix $U^* \triangleq V_A (\widetilde{D} D)^* U_B^{\dagger}$, it is finally demonstrated that $AU^* = B$. Finally, we shown how to build the remaining blocks of G. To this end, let us denote $S_3 \triangleq \begin{bmatrix} S_{12} & S_{13} \end{bmatrix}$, $\bar{S}_3 \triangleq \begin{bmatrix} \bar{S}_{12} & \bar{S}_{13} \end{bmatrix}$, $S_1 \triangleq S_{11}$, and $\bar{S}_1 \triangleq \bar{S}_{11}$, and consider the block-triangular decompositions for matrices $S_c = L_c^{\dagger} L_c$ and $\bar{S}_c = \bar{L}_c^{\dagger} \bar{L}_c$ as:

$$\boldsymbol{L}_{c} \triangleq \begin{bmatrix} \boldsymbol{S}_{.1}^{1/2} & \boldsymbol{0}_{t \times (N-t)} \\ (\boldsymbol{L}_{2}^{\dagger} \boldsymbol{U}_{1})^{-1} \boldsymbol{S}_{3}^{\dagger} & (\boldsymbol{U}_{1}^{\dagger} \boldsymbol{L}_{2}) \end{bmatrix}$$
(114)

$$\bar{\boldsymbol{L}}_{c} \triangleq \begin{bmatrix} \bar{\boldsymbol{S}}_{.1}^{1/2} & \boldsymbol{0}_{t \times (N-t)} \\ (\bar{\boldsymbol{L}}_{2}^{\dagger})^{-1} \bar{\boldsymbol{S}}_{3}^{\dagger} & \bar{\boldsymbol{L}}_{2} \end{bmatrix}$$
(115)

where $S_{.1} \triangleq S_1 - S_3 S_2^{-1} S_3^{\dagger} \in \mathbb{C}^{t \times t}$ and $\bar{S}_{.1} \triangleq \bar{S}_1 - \bar{S}_3 \bar{S}_2^{-1} \bar{S}_3^{\dagger} \in \mathbb{C}^{t \times t}$. Also, since we need to ensure $(G \bar{S}_c G^{\dagger}) = S_c$, it suffices that

$$\bar{\boldsymbol{L}}_c \, \boldsymbol{G}^{\dagger} = \boldsymbol{L}_c \tag{116}$$

$$\begin{bmatrix} \bar{\boldsymbol{S}}_{.1}^{1/2} & \boldsymbol{0}_{t\times(N-t)} \\ (\bar{\boldsymbol{L}}_{2}^{\dagger})^{-1} \bar{\boldsymbol{S}}_{3}^{\dagger} & \bar{\boldsymbol{L}}_{2} \end{bmatrix} \begin{bmatrix} \boldsymbol{G}_{1}^{\dagger} & \boldsymbol{0}_{t\times(N-t)} \\ \boldsymbol{G}_{2}^{\dagger} & \boldsymbol{G}_{3}^{\dagger} \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_{.1}^{1/2} & \boldsymbol{0}_{t\times(N-t)} \\ (\boldsymbol{L}_{2}^{\dagger} \boldsymbol{U}_{1})^{-1} \boldsymbol{S}_{3}^{\dagger} & (\boldsymbol{U}_{1}^{\dagger} \boldsymbol{L}_{2}) \end{bmatrix}$$
(117)

from which the following set of independent equations arises:

$$\begin{cases} (i) & \bar{\mathbf{S}}_{.1}^{1/2} \, \mathbf{G}_{1}^{\dagger} = \mathbf{S}_{.1}^{1/2} \\ (ii) & (\bar{\mathbf{L}}_{2}^{\dagger})^{-1} \, \bar{\mathbf{S}}_{3}^{\dagger} \, \mathbf{G}_{1}^{\dagger} + \bar{\mathbf{L}}_{2} \, \mathbf{G}_{2}^{\dagger} = (\mathbf{L}_{2}^{\dagger} \, \mathbf{U}_{1})^{-1} \mathbf{S}_{3}^{\dagger} \end{cases}$$
(118)

which provides the "completing" solutions for matrix G:

$$G_1 = S_{.1}^{1/2} \bar{S}_{.1}^{-1/2}$$
 (119)

$$\boldsymbol{G}_{2}^{\dagger} = \boldsymbol{\bar{S}}_{2}^{-1} \left(\boldsymbol{G}_{3}^{-1} \boldsymbol{S}_{3}^{\dagger} - \boldsymbol{\bar{S}}_{3}^{\dagger} \boldsymbol{G}_{1}^{\dagger} \right)$$
(120)

Up to now, we have shown how matrices G_i can be constructed. Finally, it can be easily shown that matrix F_1 should be chosen as $F_1 = Z_1 - \sum_{i=1}^{3} G_{1,i} \overline{Z}_i$. This concludes proof for (b).

APPENDIX C MIS obtained by enforcing additional invariance in range-spread case

In this appendix we show that the statistic

$$\begin{cases} \boldsymbol{x}_1 \triangleq \operatorname{eig}(\boldsymbol{T}_b) \\ \boldsymbol{x}_2 \triangleq \operatorname{eig}(\boldsymbol{T}_b + \boldsymbol{T}_a) \end{cases}$$
(121)

where $T_a = (aa^{\dagger})$ $(a \in \mathbb{C}^{M \times 1})$ and $T_b \in \mathbb{H}^{M \times M}$, is a MIS for the elementary action

$$\ell_{2,b}^{\star}(\boldsymbol{T}_{a},\boldsymbol{T}_{b}) = \left(\boldsymbol{U}_{d}^{\dagger}\,\boldsymbol{T}_{a}\,\boldsymbol{U}_{d},\,\boldsymbol{U}_{d}^{\dagger}\,\boldsymbol{T}_{b}\,\boldsymbol{U}_{d}\right)\,,\qquad(122)$$

where $U_d \in \mathcal{U}(M)$. First, we observe that Eq. (121) is in one-to-one mapping with:

$$\begin{cases} \boldsymbol{x}_1 = \operatorname{eig}(\boldsymbol{T}_b) \\ \bar{\boldsymbol{x}}_2 \triangleq |\boldsymbol{k}| \end{cases}$$
(123)

where $\mathbf{k} \triangleq (\mathbf{U}_b^{\dagger} \mathbf{a})$ and the modulus $|\cdot|$ in Eq. (123) should be intended element-wise. Also, \mathbf{U}_b denotes the eigenvector matrix of \mathbf{T}_b , that is $\mathbf{T}_b = \mathbf{U}_b \mathbf{\Lambda}_b \mathbf{U}_b^{\dagger}$. The existence of the aforementioned mapping can be proved as follows. We start by observing that $eig(T_b + aa^{\dagger})$ can be obtained as the zeros (with respect to the variable *s*) of the rational function [23]:

$$w(s) = \left(1 + \boldsymbol{k}^{\dagger} \left(\boldsymbol{\Lambda}_{b} - s \, \boldsymbol{I}_{M}\right)^{-1} \boldsymbol{k}\right) \,. \tag{124}$$

Also, since $(\Lambda_b - sI_M)^{-1}$ is a diagonal matrix, w(s) depends only on $|\mathbf{k}|$. Therefore $\operatorname{eig}(\mathbf{T}_b + \mathbf{a}\mathbf{a}^{\dagger})$ can be obtained starting from Λ_b (viz. $\operatorname{eig}(\mathbf{T}_b)$) and $|\mathbf{k}|$. Vice versa, the vector $|\mathbf{x}|$ is obtained from $\operatorname{eig}(\mathbf{T}_b + \mathbf{a}\mathbf{a}^{\dagger})$ and $\operatorname{eig}(\mathbf{T}_b)$ by inverting Eq. (124), that is:

$$\boldsymbol{k}^{\dagger} \left(\boldsymbol{\Lambda}_{b} - \boldsymbol{x}_{2,i} \, \boldsymbol{I}_{M} \right)^{-1} \boldsymbol{k} = -1, \tag{125}$$

$$\sum_{n=1}^{M} \frac{|k_n|^2}{(\lambda_{b,n} - x_{2,i})} = -1,$$
(126)

$$\boldsymbol{\alpha}_i^T \boldsymbol{\epsilon} = -1, \quad i \in \{1, \dots, M\}$$
 (127)

where $\lambda_{b,n}$ is the *n*-th diagonal element of Λ_b and

$$\boldsymbol{\epsilon} \triangleq \begin{bmatrix} |k_1|^2 & \cdots & |k_M|^2 \end{bmatrix}^T, \tag{128}$$

$$\boldsymbol{\alpha}_{i} \triangleq \begin{bmatrix} (\lambda_{b,1} - x_{2,i})^{-1} & \cdots & (\lambda_{b,N} - x_{2,i})^{-1} \end{bmatrix}^{T}.$$
 (129)

It is shown hereinafter that the linear system in Eq. (127) (with respect to the unknown vector ϵ) admits a unique solution.

Indeed, the generic α_i represents a scaled version of $(E_p v_{a+b,i})$, where $E_p \triangleq \text{diag}\{\epsilon\}$ and $v_{a+b,i}$ denotes the *i*-th eigenvector of $T_b + aa^{\dagger}$ [23, Eq. (2.1)]. However, since we assume that the eigenvalues are distinct with probability one, the eigenvectors $v_{b,i}$ will be linearly independent. Therefore, it follows that also the set $\{\alpha_i\}_{i=1}^M$ constitutes a linearly independent basis. Such conclusion clearly implies that the system is invertible and admits a unique solution; therefore there exists a one-to-one correspondence between the statistics in Eq. (121) and (123).

Once established the correspondence between Eqs. (121) and (123), it suffices to show that Eq. (123) is a MIS for the group of trasformations specified in Eq. (122). In order to accomplish this task, we first prove *invariance* of statistic in Eq. (123). Indeed, given the transformations:

$$\left\{ \widetilde{\boldsymbol{T}}_{a} = (\boldsymbol{U}_{d}^{\dagger} \, \boldsymbol{a} \boldsymbol{a}^{\dagger} \, \boldsymbol{U}_{d}), \quad \widetilde{\boldsymbol{T}}_{b} = (\boldsymbol{U}_{d}^{\dagger} \, \boldsymbol{T}_{b} \, \boldsymbol{U}_{d}), \quad (130)\right.$$

It is readily shown that $eig(\tilde{T}_b)$ can obtained as the zeros of:

$$\det(s\boldsymbol{I} - \boldsymbol{U}_d^{\dagger} \boldsymbol{T}_b \boldsymbol{U}_d) = 0 \iff \det(s\boldsymbol{I} - \boldsymbol{T}_b) = 0 \quad (131)$$

thus coinciding with $eig(T_b)$. Also, it holds

$$\begin{split} |\tilde{U}_{b}^{\dagger} \tilde{a}| &= |U_{b}^{\dagger} U_{d} U_{d}^{\dagger} a| \\ &= |U_{b}^{\dagger} a| \end{split} \tag{132}$$

Therefore the statistic in Eq. (123) is invariant. We then prove *maximality*. Under the assumption

$$\begin{cases} \operatorname{eig}(\boldsymbol{T}_{b}) = \operatorname{eig}(\widetilde{\boldsymbol{T}}_{b}) \\ |\boldsymbol{U}_{b}^{\dagger} \boldsymbol{a}| = |\widetilde{\boldsymbol{U}}_{b}^{\dagger} \widetilde{\boldsymbol{a}}| \end{cases}$$
(133)

it can be readily shown that there exists a unitary matrix V that ensures the equality $(V^{\dagger}T_bV) = \tilde{T}_b$, namely $V = (U_b D_b \tilde{U}_b^{\dagger})$, where D_b is a diagonal matrix of arbitrary phasors. Similarly we have employed the eigendecomposition $\tilde{T}_b = (\tilde{U}_b \tilde{\Lambda}_b \tilde{U}_b^{\dagger})$. Additionally, in order to complete

maximality proof, we need to prove that the aforementioned transformation, when applied to $T_a = aa^{\dagger}$, can be adjusted to satisfy:

$$\boldsymbol{V}^{\dagger}(\boldsymbol{a}\boldsymbol{a}^{\dagger})\boldsymbol{V} = \tilde{\boldsymbol{a}}\tilde{\boldsymbol{a}}^{\dagger} \tag{134}$$

After substitution, such condition can be rewritten as:

$$\widetilde{U}_b D_b^{\dagger} U_b^{\dagger} (a a^{\dagger}) U_b D_b \widetilde{U}_b^{\dagger} = \widetilde{a} \widetilde{a}^{\dagger}$$
(135)

$$\boldsymbol{D}_{b}^{\dagger}\boldsymbol{U}_{b}^{\dagger}(\boldsymbol{a}\boldsymbol{a}^{\dagger})\boldsymbol{U}_{b}\boldsymbol{D}_{b} = \boldsymbol{U}_{b}^{\dagger}\tilde{\boldsymbol{a}}\tilde{\boldsymbol{a}}^{\dagger}\boldsymbol{U}_{b}$$
(136)

$$\left[\boldsymbol{D}_{b}^{\dagger}\left(\boldsymbol{U}_{b}^{\dagger}\boldsymbol{a}\right)\right]\left[\boldsymbol{D}_{b}^{\dagger}\left(\boldsymbol{U}_{b}^{\dagger}\boldsymbol{a}\right)\right]^{\prime}=\left(\widetilde{\boldsymbol{U}}_{b}^{\dagger}\tilde{\boldsymbol{a}}\right)\left(\widetilde{\boldsymbol{U}}_{b}^{\dagger}\tilde{\boldsymbol{a}}\right)^{\prime}$$
(137)

The above rank-one matrix equality can be achieved by enforcing the vector equality

$$\left[\boldsymbol{D}_{b}^{\dagger}\left(\boldsymbol{U}_{b}^{\dagger}\,\boldsymbol{a}\right)\right] = \left(\widetilde{\boldsymbol{U}}_{b}^{\dagger}\tilde{\boldsymbol{a}}\right) \tag{138}$$

by choosing each element of the diagonal matrix D_b^{\dagger} in order to rotate each phase term of $(U_b^{\dagger} a)$ aiming at imposing $\angle (U_b^{\dagger} a) = \angle (\widetilde{U}_b^{\dagger} \widetilde{a})$, since $|U_b^{\dagger} a| = |\widetilde{U}_b^{\dagger} \widetilde{a}|$ by definition (cf. Eq. (133)). Therefore Eq. (123) (resp. Eq. (121)) is a MIS for the aforementioned group of transformations.

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