# A Unifying Framework for Adaptive Radar Detection in Homogeneous plus Structured Interference-Part II: Detectors Design 

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#### Abstract

This paper deals with the problem of adaptive multidimensional/multichannel signal detection in homogeneous Gaussian disturbance with unknown covariance matrix and structured (unknown) deterministic interference. The aforementioned problem extends the well-known Generalized Multivariate Analysis of Variance (GMANOVA) tackled in the open literature. In a companion paper, we have obtained the Maximal Invariant Statistic (MIS) for the problem under consideration, as an enabling tool for the design of suitable detectors which possess the Constant False-Alarm Rate (CFAR) property. Herein, we focus on the development of several theoretically-founded detectors for the problem under consideration. First, all the considered detectors are shown to be function of the MIS, thus proving their CFARness property. Secondly, coincidence or statistical equivalence among some of them in such a general signal model is proved. Thirdly, strong connections to well-known simpler scenarios found in adaptive detection literature are established. Finally, simulation results are provided for a comparison of the proposed receivers.


Index Terms-Adaptive Radar Detection, CFAR, Invariance Theory, Maximal Invariants, Double-subspace Model, GMANOVA, Coherent Interference.

## I. Introduction

## A. Motivation and Related Works

THE PROBLEM of adaptive detection has been object of great interest in the last decades. Many works appeared in the open literature, dealing with the design and performance analysis of suitable detectors in several specific settings (see for instance [1] and references therein).

As introduced in a companion paper, herein we focus on a signal model which generalizes that of GMANOVA [2] by considering an additional unknown double-subspace structured deterministic interference. Such model is here denoted as IGMANOVA. The I-GMANOVA model is very general and comprises many adaptive detection setups as special instances, ranging from point-like targets (resp. interference) [3] to extended ones [4], from a single-steering assumption to a vector subspace one [5], [6], and the GMANOVA model itself [2], only to mention a few examples. We recall that attractive modifications of GMANOVA have also appeared in the recent

[^0]literature [7], [8], focusing on the design of computationallyefficient approximate ML estimators when the unknown signal matrix is constrained to be diagonal [7] or block-diagonal [8].

In the case of composite hypothesis testing, the three widely-used design criteria are the Generalized Likelihood Ratio Test (GLRT), the Rao test, and the Wald test [9]. Their use is well-established in the context of adaptive detection literature [4], [10]-[13]; more important they are known to share the same asymptotic performance [9]. However, in the finite-sample case their performance differ and their relative assessment depends on the specific hypothesis testing model being considered. Such statement holds true unless some specific instances occur, such as in [14], where it is proved that they are statistically equivalent in the case of point-like targets and a partially-homogeneous scenario. Other than the aforementioned detectors, in the context of radar adaptive detection it is also customary to consider their two-step variations, with the two-step GLRT (2S-GLRT) being the most common. Those are typically obtained by designing the detector under the assumption of the a known disturbance covariance matrix and replacing it with a sample estimate based on the so-called secondary (or signal-free) data [15].

Furthermore, a few interesting alternative detectors for composite hypothesis testing are the so-called Durbin (naive) test [16] and the Terrell (Gradient) test [17]. These detectors have been shown to be asymptotically efficient as the aforementioned well-known criteria. The same rationale applies to the Lawley-Hotelling (LH) test [18]. Though these detectors are well-known in the statistics field, the development and application of these decision rules is less frequently encountered in radar adaptive detection literature, e.g., [15], [19]. The reason is that an important prerequisite for a wide-spread application of an adaptive detection algorithm consists in showing its CFARness with respect to the nuisance parameters; in this respect, the assessment of such property in radar adaptive detection literature has been somewhat lacking.

Of course, the use of GMANOVA model in the context adaptive radar detection is not new and dates back to the the milestone study in [15], where the development and analysis of the GLRT was first proposed. A similar work was then presented years later in [20], where the focus was on the design of a compression matrix prior to the detection process, aimed at reducing the computational burden and minimizing the performance loss with respect to the standard processing. More recently, GLRT, Rao and Wald tests were developed and compared under the GMANOVA model [21], along with some
other heuristic detectors. Unfortunately, albeit the CFARness of the proposed detectors was proved, no clear connection to the MIS was established. More importantly, no (structured) deterministic interference was considered in all the aforementioned works; the proposed I-GMANOVA model is aimed at filling such gap.

We point out that the closest study to ours (in terms of interference accounting) is the work in [12], where rangespread and vector subspace targets and interference are considered; however the sole GLRT and 2S-GLRT are derived and analyzed. Similarly, a Rao test (and its two-step version) is recently obtained in [22]. It is worth noticing that the model considered by the aforementioned works is included in the IGMANOVA model, and can be readily obtained by assuming a canonical form for the right-subspace matrix of both the signal and the interference.

Summarizing, in our opinion the huge (but scattered) literature on adaptive detection in many case-specific signal models and the presence of several detectors (developed on theoretically-solid assumptions) for generic composite hypothesis testing problems lacks a comprehensive and systematic analysis. First, such analysis may help the generic designer in readily obtaining a plethora of suitable adaptive detectors in some relevant scenarios which can be fitted into the considered I-GMANOVA model. Secondly, the development of detectors closed-form expressions under I-GMANOVA model may allow to easily claim some general statistical equivalence results than those already noticed in some special instances (see e.g., [11], [19], [23]). Thirdly, the available explicit expression for each detector allows for a systematic analysis of its (possible) CFARness (under a quite general signal model). The latter study is greatly simplified when knowledge of the explicit form of the MIS is available for the considered problem; in this respect, the derivation of the MIS and its analysis, object of a companion paper, fulfills this need.

## B. Summary of the contributions and Paper Organization

The main contributions of the second part of this work are thus related to detectors development and CFAR property analysis and can summarized as follows:

- Starting from the canonical form obtained in our companion paper, for the general model under investigation we derive closed-form expressions for the ( $i$ ) GLRT, ( $i i$ ) Rao test, (iii) Wald test, (iv) Gradient test (v) Durbin test, (vi) two-step GLRT (2S-GLRT), and (vii) LH test. As an interesting byproduct of our derivation, we show that Durbin test is statistically equivalent to the Rao test for the considered (adaptive) detection problem, thus extending the findings in [19], obtained for the simpler case of a point-like target without interference. Similarly, we demonstrate the statistical equivalence between Wald test and 2S-GLRT, thus extending the works in [11] and [23], concerning the special instances of point-like targets (no interference) and multidimensional signals, respectively;
- The general expressions of the receivers are exploited to analyze special cases of interest, such as: (a) vector
subspace detection of point-like targets (with possible structured interference) [3], [6], [11], (b) multidimensional signals [10], [23], (c) range-spread (viz. extended) targets [4], [24], [25], and (d) standard GMANOVA (i.e., without structured interference) [15], [21]. In such special instances, possible coincidence or statistical equivalence is investigated among the considered detectors;
- Exploiting the matrix pair form of the MIS obtained in part one, we show that all the considered detector can be expressed as a function of the MIS, thus proving their CFARness with respect to both the covariance of the disturbance and the deterministic (structured) interference;
- Finally, a simulation results section is provided to compare the proposed detectors in terms of the relevant parameters and underline the common trends among them.
The remainder of the paper is organized as follows: in Sec. [I] we describe the hypothesis testing problem under investigation; in Sec. [II] we obtain the general expressions for the detectors considered in this paper and we express them as a function of the MIS; in Sec. IV, we particularize the obtained expressions to the aforementioned special instances of adaptive detection problems; finally, in Sec. V we compare the obtained detectors through simulation results and in Sec. VI we draw some concluding remarks and indicate future research directions. Proofs and derivations are confined to an additional document containing supplemental materia 1 .


## II. Problem Formulation

In a companion paper, we have shown that the considered problem admits an equivalent (but simpler) formulation by exploiting the so-called "canonical form", that is:

$$
\begin{cases}\mathcal{H}_{0}: & \boldsymbol{Z}=\boldsymbol{A}\left[\begin{array}{ll}
\boldsymbol{B}_{t, 0}^{T} & \mathbf{0}_{M \times r}
\end{array}\right]^{T} \boldsymbol{C}+\boldsymbol{N}  \tag{1}\\
\mathcal{H}_{1}: & \boldsymbol{Z}=\boldsymbol{A} \boldsymbol{B}_{s} \boldsymbol{C}+\boldsymbol{N}\end{cases}
$$

where we have assumed that a data matrix $\boldsymbol{Z} \in \mathbb{C}^{N \times K}$ has been collected. Also, we have adopted the following definitions:

[^1]- $\boldsymbol{A} \triangleq\left[\begin{array}{ll}\boldsymbol{E}_{t} & \boldsymbol{E}_{r}\end{array}\right] \in \mathbb{C}^{N \times J}$, where $\boldsymbol{E}_{t} \triangleq\left[\begin{array}{ll}\boldsymbol{I}_{t} & \mathbf{0}_{t \times(N-t)}\end{array}\right]^{T}$ and $\boldsymbol{E}_{r} \triangleq\left[\begin{array}{lll}\mathbf{0}_{r \times t} & \boldsymbol{I}_{r} & \mathbf{0}_{r \times(N-J)}\end{array}\right]^{T}$ are the (known) left subspaces of the interference and useful signal, respectively (we have denoted $J \triangleq r+t$ );
- $\boldsymbol{B}_{s} \triangleq\left[\begin{array}{ll}\boldsymbol{B}_{t, 1}^{T} & \boldsymbol{B}^{T}\end{array}\right]^{T}$, where $\boldsymbol{B}_{t, i} \in \mathbb{C}^{t \times M}$ and $\boldsymbol{B} \in$ $\mathbb{C}^{r \times M}$ are the (unknown) interference (under $\mathcal{H}_{i}$ ) and useful signal matrices, respectively;
- $\boldsymbol{C} \triangleq\left[\begin{array}{ll}\boldsymbol{I}_{M} & \mathbf{0}_{M \times(K-M)}\end{array}\right] \in \mathbb{C}^{M \times K}$ is the (known) right subspace matrix associated to both signal and interference in canonical form;
- $N$ is a disturbance matrix such that $N \sim$ $\mathcal{C} \mathcal{N}_{N \times K}\left(\mathbf{0}_{N \times K}, \boldsymbol{I}_{K}, \boldsymbol{R}\right)$, where $\boldsymbol{R} \in \mathbb{C}^{N \times N}$ is an (unknown) positive definite covariance matrix [15].
We recall that the detection problem in (11) is tantamount to testing the null hypothesis $\boldsymbol{B}=\mathbf{0}_{r \times M}$ (viz. $\|\boldsymbol{B}\|_{F}=0$, denoted with $\mathcal{H}_{0}$ ) against the alternative that $\boldsymbol{B}$ is unrestricted (viz. $\|\boldsymbol{B}\|_{F}>0$, denoted with $\mathcal{H}_{1}$ ), along with the set of nuisance parameters $\boldsymbol{B}_{t, i}$ and $\boldsymbol{R}$.

In the present manuscript we will consider decision rules which declare $\mathcal{H}_{1}\left(\right.$ resp. $\left.\mathcal{H}_{0}\right)$ if $\Phi(\boldsymbol{Z})>\eta$ (resp. $\Phi(\boldsymbol{Z})<$ $\eta$ ), where $\Phi(\cdot) \in \mathbb{C}^{N \times K} \rightarrow \mathbb{R}$ indicates the generic form of a statistic processing the received data $Z$ and $\eta$ denotes the threshold to be set in order to achieve a predetermined probability of false alarm $\left(P_{f a}\right)$.

As a preliminary step towards the derivation of suitable detectors for the problem at hand, we also give the following auxiliary definitions:

- $\boldsymbol{b}_{R} \in \mathbb{R}^{r M \times 1}$ and $\boldsymbol{b}_{I} \in \mathbb{R}^{r M \times 1}$ are obtained as $\boldsymbol{b}_{R} \triangleq$ $\Re\{\boldsymbol{b}\}$ and $\boldsymbol{b}_{I} \triangleq \Im\{\boldsymbol{b}\}$, respectively, where we have defined $\boldsymbol{b} \triangleq \operatorname{vec}(\boldsymbol{B})$;
- $\boldsymbol{\theta}_{r} \triangleq\left[\begin{array}{ll}\boldsymbol{b}_{R}^{T} & \boldsymbol{b}_{I}^{T}\end{array}\right]^{T} \in \mathbb{R}^{2 r M \times 1}$ is the vector collecting the parameters of interest;
- $\boldsymbol{\theta}_{s} \triangleq\left[\begin{array}{ll}\boldsymbol{\theta}_{s, a}^{T} & \boldsymbol{\theta}_{s, b}^{T}\end{array}\right]^{T} \in \mathbb{R}^{\left(2 t M+N^{2}\right) \times 1}$ is the vector of nuisance parameters containing: (a) $\boldsymbol{\theta}_{s, a} \triangleq$ $\left[\begin{array}{ll}\boldsymbol{b}_{t, R}^{T} & \boldsymbol{b}_{t, I}^{T}\end{array}\right]^{T} \in \mathbb{R}^{2 t M \times 1}$ where $\boldsymbol{b}_{t, R}$ and $\boldsymbol{b}_{t, I}$ are the vectors obtained as $\boldsymbol{b}_{t, R} \triangleq \Re\left\{\boldsymbol{b}_{t}\right\}$ and $\boldsymbol{b}_{t, I} \triangleq \Im\left\{\boldsymbol{b}_{t}\right\}$, respectively, where $\boldsymbol{b}_{t} \triangleq \operatorname{vec}\left(\boldsymbol{B}_{t}\right)$ (i.e. $\boldsymbol{B}_{t, i}$ under $\left.\mathcal{H}_{i}\right)$; (b) $\boldsymbol{\theta}_{s, b}$ contains in a given order ${ }^{2}$ the real and imaginary parts of the off-diagonal entries together with the diagonal elements of $\boldsymbol{R}$;
- $\boldsymbol{\theta} \triangleq\left[\begin{array}{ll}\boldsymbol{\theta}_{r}^{T} & \boldsymbol{\theta}_{s}^{T}\end{array}\right]^{T} \in \mathbb{R}^{\left(2 J M+N^{2}\right) \times 1}$ is the overall unknown parameter vector;
- $\widehat{\boldsymbol{\theta}}_{0} \triangleq\left[\begin{array}{cc}\boldsymbol{\theta}_{r, 0}^{T} & \widehat{\boldsymbol{\theta}}_{s, 0}^{T}\end{array}\right]^{T}$, with $\widehat{\boldsymbol{\theta}}_{s, 0}$ denoting the Maximum Likelihood (ML) estimate of $\boldsymbol{\theta}_{s}$ under $\mathcal{H}_{0}$ and $\boldsymbol{\theta}_{r, 0}=$ $\mathbf{0}_{2 r M}$ (that is, the true value of $\boldsymbol{\theta}_{r}$ under $\mathcal{H}_{0}$ );
- $\widehat{\boldsymbol{\theta}}_{1} \triangleq\left[\begin{array}{ll}\widehat{\boldsymbol{\theta}}_{r, 1}^{T} & \widehat{\boldsymbol{\theta}}_{s, 1}^{T}\end{array}\right]^{T}$, with $\widehat{\boldsymbol{\theta}}_{r, 1}$ and $\widehat{\boldsymbol{\theta}}_{s, 1}$ denoting the ML estimates of $\boldsymbol{\theta}_{r}$ and $\boldsymbol{\theta}_{s}$, respectively, under $\mathcal{H}_{1}$.
The probability density function (pdf) of $\boldsymbol{Z}$, when the hypothesis $\mathcal{H}_{1}$ is in force, is denoted with $f_{1}(\cdot)$ and it is given in closed form as:

$$
\begin{gather*}
f_{1}\left(\boldsymbol{Z} ; \boldsymbol{B}_{s}, \boldsymbol{R}\right)=\pi^{-N K} \operatorname{det}(\boldsymbol{R})^{-K} \\
\times \exp \left(-\operatorname{Tr}\left[\boldsymbol{R}^{-1}\left(\boldsymbol{Z}-\boldsymbol{A} \boldsymbol{B}_{s} \boldsymbol{C}\right)\left(\boldsymbol{Z}-\boldsymbol{A} \boldsymbol{B}_{s} \boldsymbol{C}\right)^{\dagger}\right]\right), \tag{2}
\end{gather*}
$$

[^2]while the corresponding pdf under $\mathcal{H}_{0}$, denoted in the following with $f_{0}(\cdot)$, is similarly obtained when replacing $\boldsymbol{A} \boldsymbol{B}_{s} \boldsymbol{C}$ with $\boldsymbol{E}_{t} \boldsymbol{B}_{t, 0} \boldsymbol{C}$ in Eq. (2). In the following, in order for our analysis to apply, we will assume that the condition $(K-M) \geq N$ holds. Such condition is typically satisfied in practical adaptive detection setups [15].

## A. MIS for the considered problem

In what follows, we recall the MIS for the hypothesis testing under investigation, obtained in our companion paper. The mentioned statistic will be exploited in Sec. III to ascertain the CFARness of each considered detector. Before proceeding further, let

$$
\boldsymbol{V}_{c, 1} \triangleq\left[\begin{array}{c}
\boldsymbol{I}_{M}  \tag{3}\\
\mathbf{0}_{(K-M) \times M}
\end{array}\right], \quad \boldsymbol{V}_{c, 2} \triangleq\left[\begin{array}{c}
\mathbf{0}_{M \times(K-M)} \\
\boldsymbol{I}_{K-M}
\end{array}\right]
$$

and observe that $\boldsymbol{P}_{\boldsymbol{C}^{\dagger}}=\left(\boldsymbol{V}_{c, 1} \boldsymbol{V}_{c, 1}^{\dagger}\right)$ and $\boldsymbol{P}_{\boldsymbol{C}^{\dagger}}^{\perp}=\left(\boldsymbol{V}_{c, 2} \boldsymbol{V}_{c, 2}^{\dagger}\right)$. Given these definitions, we denote (as in Part I): (i) $\boldsymbol{Z}_{c} \triangleq$ $\left(\boldsymbol{Z} \boldsymbol{V}_{c, 1}\right) \in \mathbb{C}^{N \times M}$, (ii) $\boldsymbol{Z}_{c, \perp} \triangleq\left(\boldsymbol{Z} \boldsymbol{V}_{c, 2}\right) \in \mathbb{C}^{N \times(K-M)}$ and (iii) $\boldsymbol{S}_{c} \triangleq\left(\boldsymbol{Z}_{c, \perp} \boldsymbol{Z}_{c, \perp}^{\dagger}\right)=\left(\boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}^{\perp} \boldsymbol{Z}^{\dagger}\right) \in \mathbb{C}^{N \times N}$.

It has been shown in our companion paper that the MIS is given by:

$$
\boldsymbol{T}\left(\boldsymbol{Z}_{c}, \boldsymbol{S}_{c}\right)=\left\{\begin{array}{cc}
{\left[\begin{array}{cc}
\boldsymbol{T}_{a} \triangleq\left\{\boldsymbol{Z}_{2.3}^{\dagger} \boldsymbol{S}_{2.3}^{-1} \boldsymbol{Z}_{2.3}\right\} \\
\boldsymbol{T}_{b} \triangleq\left\{\boldsymbol{Z}_{3}^{\dagger} \boldsymbol{S}_{33}^{-1} \boldsymbol{Z}_{3}\right\}
\end{array}\right]} & J<N  \tag{4}\\
\boldsymbol{Z}_{2}^{\dagger} \boldsymbol{S}_{22}^{-1} \boldsymbol{Z}_{2} & J=N
\end{array}\right.
$$

where $\boldsymbol{Z}_{2.3} \triangleq\left(\boldsymbol{Z}_{2}-\boldsymbol{S}_{23} \boldsymbol{S}_{33}^{-1} \boldsymbol{Z}_{3}\right)$ and $\boldsymbol{S}_{2.3} \triangleq\left(\boldsymbol{S}_{22}-\right.$ $\boldsymbol{S}_{23} \boldsymbol{S}_{33}^{-1} \boldsymbol{S}_{32}$ ). Also, we have exploited the following partitioning for matrices $\boldsymbol{Z}_{c}$ and $\boldsymbol{S}_{c}$ :

$$
\boldsymbol{Z}_{c}=\left[\begin{array}{l}
\boldsymbol{Z}_{1}  \tag{5}\\
\boldsymbol{Z}_{2} \\
\boldsymbol{Z}_{3}
\end{array}\right], \quad \boldsymbol{S}_{c}=\left[\begin{array}{lll}
\boldsymbol{S}_{11} & \boldsymbol{S}_{12} & \boldsymbol{S}_{13} \\
\boldsymbol{S}_{21} & \boldsymbol{S}_{22} & \boldsymbol{S}_{23} \\
\boldsymbol{S}_{31} & \boldsymbol{S}_{32} & \boldsymbol{S}_{33}
\end{array}\right]
$$

where $\boldsymbol{Z}_{1} \in \mathbb{C}^{t \times M}, \boldsymbol{Z}_{2} \in \mathbb{C}^{r \times M}$, and $\boldsymbol{Z}_{3} \in \mathbb{C}^{(N-J) \times M}$, respectively. Furthermore, $\boldsymbol{S}_{i j},(i, j) \in\{1,2,3\} \times\{1,2,3\}$, is a sub-matrix whose dimensions can be obtained replacing 1 , 2 and 3 with $t, r$ and $(N-J)$, respectivel $\sqrt{3}$. Additionally, for notational convenience, we also give the following definitions that will be used throughout the manuscript:

$$
\boldsymbol{Z}_{23} \triangleq\left[\begin{array}{l}
\boldsymbol{Z}_{2}  \tag{6}\\
\boldsymbol{Z}_{3}
\end{array}\right], \quad \boldsymbol{S}_{2} \triangleq\left[\begin{array}{ll}
\boldsymbol{S}_{22} & \boldsymbol{S}_{23} \\
\boldsymbol{S}_{32} & \boldsymbol{S}_{33}
\end{array}\right]
$$

Finally we recall that, for the I-GMANOVA model, the induced maximal invariant equals $\boldsymbol{T}_{\mathrm{p}} \triangleq \boldsymbol{B}^{\dagger} \boldsymbol{R}_{2.3}^{-1} \boldsymbol{B} \in \mathbb{C}^{M \times M}$, where $\boldsymbol{R}_{2.3}$ is analogously defined as $\boldsymbol{S}_{2.3}$ when $\boldsymbol{S}_{c}$ is replaced with the true covariance $\boldsymbol{R}$.

## III. Detectors DESIGN

In this section we will consider several decision statistics designed according to well-founded design criteria. Initially, we will concentrate on the derivation of the well-known GLRT (including its two-step version), Rao and Wald tests [9].

[^3]Then, we will devise the explicit form of recently used detection statistics, such as the Gradient (Terrell) test [17], the Durbin (naive) test [16], which have been shown to be asymptotically distributed as the three aforementioned detectors (under very mild conditions). Finally, for the sake of completeness, we will obtain the LH test for the problem at hand, following the lead of [15].

## A. GLR

The generic form of the GLR in terms of the complexvalued unknowns is given by [9]:

$$
\begin{equation*}
\frac{\max _{\left\{\boldsymbol{B}_{s}, \boldsymbol{R}\right\}} f_{1}\left(\boldsymbol{Z} ; \boldsymbol{B}_{s}, \boldsymbol{R}\right)}{\max _{\left\{\boldsymbol{B}_{t, 0}, \boldsymbol{R}\right\}} f_{0}\left(\boldsymbol{Z} ; \boldsymbol{B}_{t, 0}, \boldsymbol{R}\right)} \tag{7}
\end{equation*}
$$

First, it can be readily shown that the ML estimate of $\boldsymbol{R}$ under $\mathcal{H}_{1}$ (resp. under $\mathcal{H}_{0}$ ), parametrized by $\boldsymbol{B}_{s}$ (resp. $\boldsymbol{B}_{t, 0}$ ) is:

$$
\begin{gather*}
\hat{\boldsymbol{R}}_{1}\left(\boldsymbol{B}_{s}\right) \triangleq K^{-1}\left(\boldsymbol{Z}-\boldsymbol{A} \boldsymbol{B}_{s} \boldsymbol{C}\right)\left(\boldsymbol{Z}-\boldsymbol{A} \boldsymbol{B}_{s} \boldsymbol{C}\right)^{\dagger}  \tag{8}\\
\hat{\boldsymbol{R}}_{0}\left(\boldsymbol{B}_{t, 0}\right) \triangleq K^{-1}\left(\boldsymbol{Z}-\boldsymbol{E}_{t} \boldsymbol{B}_{t, 0} \boldsymbol{C}\right)\left(\boldsymbol{Z}-\boldsymbol{E}_{t} \boldsymbol{B}_{t, 0} \boldsymbol{C}\right)^{\dagger} \tag{9}
\end{gather*}
$$

After substitution of Eqs. (8) and (9) in $f_{1}(\cdot)$ and $f_{0}(\cdot)$, respectively, the concentrated likelihoods are expressed as:

$$
\begin{gather*}
f_{1}\left(\boldsymbol{Z} ; \boldsymbol{B}_{s}, \hat{\boldsymbol{R}}_{1}\left(\boldsymbol{B}_{s}\right)\right)=(K /(\pi e))^{K N} \\
\times \operatorname{det}\left[\left(\boldsymbol{Z}-\boldsymbol{A} \boldsymbol{B}_{s} \boldsymbol{C}\right)\left(\boldsymbol{Z}-\boldsymbol{A} \boldsymbol{B}_{s} \boldsymbol{C}\right)^{\dagger}\right]^{-K}  \tag{10}\\
f_{0}\left(\boldsymbol{Z} ; \boldsymbol{B}_{t, 0}, \hat{\boldsymbol{R}}_{0}\left(\boldsymbol{B}_{t, 0}\right)\right)=(K /(\pi e))^{K N} \\
\times \operatorname{det}\left[\left(\boldsymbol{Z}-\boldsymbol{E}_{t} \boldsymbol{B}_{t, 0} \boldsymbol{C}\right)\left(\boldsymbol{Z}-\boldsymbol{E}_{t} \boldsymbol{B}_{t, 0} \boldsymbol{C}\right)^{\dagger}\right]^{-K} \tag{11}
\end{gather*}
$$

Then the ML estimates of $\boldsymbol{B}_{s}$ and $\boldsymbol{B}_{t, 0}$ under $\mathcal{H}_{1}$ and $\mathcal{H}_{0}$, respectively, are obtained as the solutions to the following optimization problems:

$$
\begin{align*}
& \widehat{\boldsymbol{B}}_{s} \triangleq \arg \min _{\boldsymbol{B}_{s}} \operatorname{det}\left[\left(\boldsymbol{Z}-\boldsymbol{A} \boldsymbol{B}_{s} \boldsymbol{C}\right)\left(\boldsymbol{Z}-\boldsymbol{A} \boldsymbol{B}_{s} \boldsymbol{C}\right)^{\dagger}\right]  \tag{12}\\
& \widehat{\boldsymbol{B}}_{t, 0} \triangleq \arg \min _{\boldsymbol{B}_{t, 0}} \operatorname{det}\left[\left(\boldsymbol{Z}-\boldsymbol{E}_{t} \boldsymbol{B}_{t, 0} \boldsymbol{C}\right)\left(\boldsymbol{Z}-\boldsymbol{E}_{t} \boldsymbol{B}_{t, 0} \boldsymbol{C}\right)^{\dagger}\right] \tag{13}
\end{align*}
$$

It has been shown in [15] that the optimizers have the closed form:

$$
\begin{align*}
\widehat{\boldsymbol{B}}_{s} & =\left(\boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{Z} \boldsymbol{C}^{\dagger}\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{-1}  \tag{14}\\
\widehat{\boldsymbol{B}}_{t, 0} & =\left(\boldsymbol{E}_{t}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{E}_{t}\right)^{-1} \boldsymbol{E}_{t}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{Z} \boldsymbol{C}^{\dagger}\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{-1} \tag{15}
\end{align*}
$$

Substituting Eqs. (14) and (15) into (10) and (11), respectively, provides (after lengthy manipulations):

$$
\begin{align*}
& f_{1}\left(\boldsymbol{Z} ; \widehat{\boldsymbol{B}}_{s}, \hat{\boldsymbol{R}}_{1}\right)=(K /(\pi e))^{K N} \operatorname{det}\left[\boldsymbol{S}_{c}\right]^{-K} \\
& \times \operatorname{det}\left[\boldsymbol{I}_{M}+\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)\right]^{-K}  \tag{16}\\
& f_{0}\left(\boldsymbol{Z} ; \widehat{\boldsymbol{B}}_{t, 0}, \hat{\boldsymbol{R}}_{0}\right)=(K /(\pi e))^{K N} \operatorname{det}\left[\boldsymbol{S}_{c}\right]^{-K} \\
& \times \operatorname{det}\left[\boldsymbol{I}_{M}+\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)\right]^{-K} \tag{17}
\end{align*}
$$

where we have defined $\boldsymbol{A}_{1} \triangleq\left(\boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{A}\right), \boldsymbol{A}_{0} \triangleq\left(\boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{E}_{t}\right)$ and $\boldsymbol{Z}_{W 1} \triangleq\left(\boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{Z}\right)$, respectively. Finally, substituting Eqs. (16) and (17) into Eq. (7) and after taking the $k$-th root, the following explicit statistic is obtained:

$$
\begin{align*}
t_{\mathrm{glr}} & \triangleq \frac{\operatorname{det}\left[\boldsymbol{I}_{M}+\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)\right]}{\operatorname{det}\left[\boldsymbol{I}_{M}+\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)\right]}  \tag{18}\\
& =\frac{\operatorname{det}\left[\boldsymbol{I}_{K}+\boldsymbol{Z}_{W 1}^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{Z}_{W 1} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]}{\operatorname{det}\left[\boldsymbol{I}_{K}+\boldsymbol{Z}_{W 1}^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp} \boldsymbol{Z}_{W 1} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]} \tag{19}
\end{align*}
$$

where the last expression follows from Sylvester's determinant theorem [26]. Furthermore, we observe that Eq. (18] can be also re-arranged in the following useful equivalent forms (again obtained via Sylvester's determinant theorem):

$$
\begin{align*}
& t_{\mathrm{glr}}=\operatorname{det}\left[\boldsymbol{I}_{M}-\boldsymbol{D}_{0}^{-1 / 2}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \mathcal{P}_{\Delta}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right) \boldsymbol{D}_{0}^{-1 / 2}\right]^{-1}  \tag{20}\\
& \quad=\operatorname{det}\left[\boldsymbol{I}_{M}+\boldsymbol{D}_{1}^{-1 / 2}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\Delta}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right) \boldsymbol{D}_{1}^{-1 / 2}\right] \tag{21}
\end{align*}
$$

where $\boldsymbol{P}_{\Delta} \triangleq\left(\boldsymbol{P}_{\boldsymbol{A}_{1}}-\boldsymbol{P}_{\boldsymbol{A}_{0}}\right)$ and $\boldsymbol{D}_{i} \triangleq\left[\boldsymbol{I}_{M}+\right.$ $\left.\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{i}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)\right]$, respectively. Finally, it is worth noticing that Eq. (21) is in the well-known Wilks’ Lambda statistic form [27]. Moreover, the latter expression generalizes the GLR in [15] to the interference scenario (i.e., $t \neq 0$ ).

For the sake of completeness, we also report the closed form ML estimates of $\boldsymbol{R}$ obtained under $\mathcal{H}_{0}$ and $\mathcal{H}_{1}$ (after backsubstitution of (14) and (15) in Eqs. (8) and (9), respectively):

$$
\begin{align*}
\hat{\boldsymbol{R}}_{1}\left(\widehat{\boldsymbol{B}}_{s}\right)= & K^{-1}\left[\boldsymbol{S}_{c}+\left(\boldsymbol{Z}-\boldsymbol{S}_{c}^{1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{1}} \boldsymbol{Z}_{W 1}\right) \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right. \\
& \left.\times\left(\boldsymbol{Z}-\boldsymbol{S}_{c}^{1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{1}} \boldsymbol{Z}_{W 1}\right)^{\dagger}\right]  \tag{22}\\
\hat{\boldsymbol{R}}_{0}\left(\widehat{\boldsymbol{B}}_{t, 0}\right)= & K^{-1}\left[\boldsymbol{S}_{c}+\left(\boldsymbol{Z}-\boldsymbol{S}_{c}^{1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}} \boldsymbol{Z}_{W 1}\right) \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right. \\
& \left.\times\left(\boldsymbol{Z}-\boldsymbol{S}_{c}^{1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}} \boldsymbol{Z}_{W 1}\right)^{\dagger}\right] \tag{23}
\end{align*}
$$

and underline that we will use the short-hand notation $\hat{\boldsymbol{R}}_{i}$ in what follows. Finally, before proceeding further, we state some useful properties of ML covariance estimates (later exploited in this paper) in the form of the following lemma.
Lemma 1. The ML estimates of $\boldsymbol{R}$ under $\mathcal{H}_{1}$ and $\mathcal{H}_{0}$ satisfy the following equalities:

$$
\begin{align*}
\hat{\boldsymbol{R}}_{1}^{-1} \boldsymbol{A} & =K \boldsymbol{S}_{c}^{-1} \boldsymbol{A}=K\left[\begin{array}{ll}
\boldsymbol{S}_{c}^{-1} \boldsymbol{E}_{t} & \boldsymbol{S}_{c}^{-1} \boldsymbol{E}_{r}
\end{array}\right]  \tag{24}\\
\hat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{E}_{t} & =K \boldsymbol{S}_{c}^{-1} \boldsymbol{E}_{t} \tag{25}
\end{align*}
$$

Proof: Provided as supplementary material.
CFARness of GLRT: Using the expression in Eq. 18), we here verify that $t_{\text {glr }}$ can be expressed in terms of the MIS (cf. Eq. (4)), thus proving its CFARness. Indeed, it can be shown that

$$
\begin{align*}
& \left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)=\boldsymbol{Z}_{2.3}^{\dagger} \boldsymbol{S}_{2.3}^{-1} \boldsymbol{Z}_{2.3}+\boldsymbol{Z}_{3}^{\dagger} \boldsymbol{S}_{33}^{-1} \boldsymbol{Z}_{3}  \tag{26}\\
& \left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)=\boldsymbol{Z}_{3}^{\dagger} \boldsymbol{S}_{33}^{-1} \boldsymbol{Z}_{3} \tag{27}
\end{align*}
$$

from which it follows

$$
\begin{equation*}
t_{\mathrm{glr}}=\frac{\operatorname{det}\left[\boldsymbol{I}_{M}+\boldsymbol{T}_{a}+\boldsymbol{T}_{b}\right]}{\operatorname{det}\left[\boldsymbol{I}_{M}+\boldsymbol{T}_{b}\right]} \tag{28}
\end{equation*}
$$

which demonstrates invariance of the GLR(T) with respect to the nuisance parameters and thus ensures CFAR property.

## B. Rao statistic

The generic form for the Rao statistic is given by [9]:

$$
\begin{equation*}
\left.\left.\frac{\partial \ln f_{1}(\boldsymbol{Z} ; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}_{r}^{T}}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}_{0}}\left[\boldsymbol{I}^{-1}\left(\widehat{\boldsymbol{\theta}}_{0}\right)\right]_{\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}} \frac{\partial \ln f_{1}(\boldsymbol{Z} ; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}_{r}}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}_{0}} \tag{29}
\end{equation*}
$$

[^4]where
\[

$$
\begin{equation*}
\boldsymbol{I}(\boldsymbol{\theta}) \triangleq \mathbb{E}\left\{\frac{\partial \ln f_{1}(\boldsymbol{Z} ; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{\partial \ln f_{1}(\boldsymbol{Z} ; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}^{T}}\right\}, \tag{30}
\end{equation*}
$$

\]

denotes the Fisher Information Matrix (FIM) and $\left[\boldsymbol{I}^{-1}(\boldsymbol{\theta})\right]_{\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}}$ indicates the sub-matrix obtained by selecting from the FIM inverse only the elements corresponding to the vector $\boldsymbol{\theta}_{r}$. It is shown (the proof is provided as supplementary material) that the aforementioned statistic is given in closed form as:

$$
\begin{equation*}
t_{\text {rao }} \triangleq \operatorname{Tr}\left[\boldsymbol{Z}_{d, 0}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{E}_{r} \widehat{\boldsymbol{\Gamma}}_{22}^{\circ} \boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{Z}_{d, 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right] \tag{31}
\end{equation*}
$$

where we have partitioned $\widehat{\boldsymbol{\Gamma}}^{\circ} \triangleq\left(\boldsymbol{A}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{A}\right)^{-1}$ as:

$$
\widehat{\boldsymbol{\Gamma}}^{\circ}=\left[\begin{array}{ll}
\widehat{\boldsymbol{\Gamma}}_{11}^{\circ} & \widehat{\boldsymbol{\Gamma}}_{12}^{\circ}  \tag{32}\\
\widehat{\boldsymbol{\Gamma}}_{21}^{\circ} & \widehat{\boldsymbol{\Gamma}}_{22}^{\circ}
\end{array}\right]
$$

and, $\widehat{\boldsymbol{\Gamma}}_{i j}^{\circ},(i, j) \in\{1,2\} \times\{1,2\}$, is a sub-matrix whose dimensions can be obtained replacing 1 and 2 with $t$ and $r$, respectively. Additionally, we have defined:

$$
\begin{equation*}
\boldsymbol{Z}_{d, 0} \triangleq\left(\boldsymbol{Z}-\boldsymbol{S}_{c}^{1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right) \tag{33}
\end{equation*}
$$

Eq. (31) can be rewritten in a more familiar (and convenient) way. Indeed, it is proved ${ }^{5}$ that:

$$
\begin{equation*}
\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{Z}_{d, 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}=\boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}^{\perp} \boldsymbol{Z}_{W 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}} \tag{34}
\end{equation*}
$$

where $\overline{\boldsymbol{A}}_{0} \triangleq\left(\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{E}_{t}\right)$ and $\boldsymbol{Z}_{W 0} \triangleq\left(\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{Z}\right)$, respectively, and also

$$
\begin{equation*}
\boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}^{\perp} \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{E}_{r} \boldsymbol{\Gamma}_{22}^{\circ} \boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}^{\perp}=\left(\boldsymbol{P}_{\overline{\boldsymbol{A}}_{1}}-\boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}\right), \tag{35}
\end{equation*}
$$

where $\overline{\boldsymbol{A}}_{1} \triangleq \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{A}$ and $\overline{\boldsymbol{A}}_{0} \triangleq \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{E}_{t}$, respectively. Therefore, an alternative form of $t_{\text {rao }}$ is obtained substituting Eq. (34) into Eq. (31) and exploiting (35), thus leading to the compact expression:

$$
\begin{equation*}
t_{\mathrm{rao}}=\operatorname{Tr}\left[\boldsymbol{Z}_{W 0}^{\dagger}\left(\boldsymbol{P}_{\overline{\boldsymbol{A}}_{1}}-\boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}\right) \boldsymbol{Z}_{W 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right] . \tag{36}
\end{equation*}
$$

CFARness of Rao Test: We now express the Rao statistic as a function of the MIS, aiming at showing its CFARness. To this end, we first notice that Eq. (36) can be rewritten as:

$$
\begin{equation*}
t_{\text {rao }}=\operatorname{Tr}\left[\left(\boldsymbol{Z}_{W 0} \boldsymbol{V}_{c, 1}\right)^{\dagger}\left(\boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}^{\perp}-\boldsymbol{P}_{\overline{\boldsymbol{A}}_{1}}^{\perp}\right)\left(\boldsymbol{Z}_{W 0} \boldsymbol{V}_{c, 1}\right)\right] . \tag{37}
\end{equation*}
$$

Moreover, exploiting the equalitie 6

$$
\begin{gather*}
\left(\boldsymbol{Z}_{W 0} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}^{\perp}\left(\boldsymbol{Z}_{W 0} \boldsymbol{V}_{c, 1}\right)=K\left\{\boldsymbol{Z}_{23}^{\dagger} \boldsymbol{S}_{2}^{-1} \boldsymbol{Z}_{23}\right. \\
\left.-\boldsymbol{Z}_{23}^{\dagger} \boldsymbol{S}_{2}^{-1} \boldsymbol{Z}_{23}\left(\boldsymbol{I}_{M}+\boldsymbol{Z}_{23}^{\dagger} \boldsymbol{S}_{2}^{-1} \boldsymbol{Z}_{23}\right)^{-1} \boldsymbol{Z}_{23}^{\dagger} \boldsymbol{S}_{2}^{-1} \boldsymbol{Z}_{23}\right\}  \tag{38}\\
\left(\boldsymbol{Z}_{W 0} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\overline{\boldsymbol{A}}_{1}}^{\perp}\left(\boldsymbol{Z}_{W 0} \boldsymbol{V}_{c, 1}\right)=K\left\{\boldsymbol{Z}_{3}^{\dagger} \boldsymbol{S}_{33}^{-1} \boldsymbol{Z}_{3}\right. \\
\left.-\boldsymbol{Z}_{3}^{\dagger} \boldsymbol{S}_{33}^{-1} \boldsymbol{Z}_{3}\left(\boldsymbol{I}_{M}+\boldsymbol{Z}_{3}^{\dagger} \boldsymbol{S}_{33}^{-1} \boldsymbol{Z}_{3}\right)^{-1} \boldsymbol{Z}_{3}^{\dagger} \boldsymbol{S}_{33}^{-1} \boldsymbol{Z}_{3}\right\}, \tag{39}
\end{gather*}
$$

Eq. (36) can be rewritten as:

$$
\begin{align*}
t_{\mathrm{rao}}= & K \operatorname{Tr}\left[\boldsymbol{T}_{a}-\left(\boldsymbol{T}_{a}+\boldsymbol{T}_{b}\right)\right. \\
& \times\left(\boldsymbol{I}_{M}+\boldsymbol{T}_{a}+\boldsymbol{T}_{b}\right)^{-1}\left(\boldsymbol{T}_{a}+\boldsymbol{T}_{b}\right) \\
& \left.+\boldsymbol{T}_{b}\left(\boldsymbol{I}_{M}+\boldsymbol{T}_{b}\right)^{-1} \boldsymbol{T}_{b}\right] \tag{40}
\end{align*}
$$

which is only function of the MIS, thus proving its CFARness.

[^5]
## C. Wald statistic

The generic form for the Wald statistic is given by [9]:

$$
\begin{equation*}
\left(\hat{\boldsymbol{\theta}}_{r, 1}-\boldsymbol{\theta}_{r, 0}\right)^{T}\left\{\left[\boldsymbol{I}^{-1}\left(\widehat{\boldsymbol{\theta}}_{1}\right)\right]_{\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}}\right\}^{-1}\left(\hat{\boldsymbol{\theta}}_{r, 1}-\boldsymbol{\theta}_{r, 0}\right) \tag{41}
\end{equation*}
$$

It is shown (the proof is provided as supplementary material) that the aforementioned statistic is given in closed form as:
$t_{\text {wald }} \triangleq \operatorname{Tr}\left[\boldsymbol{Z}_{W 1}^{\dagger} K \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{E}_{r} \widehat{\boldsymbol{\Gamma}}_{22}^{1} \boldsymbol{E}_{r}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{Z}_{W 1} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]$
where $\widehat{\boldsymbol{\Gamma}}_{i j}^{1}$ indicates the $(i, j)$-th sub-matrix of $\widehat{\boldsymbol{\Gamma}}^{1} \triangleq$ $\left(\boldsymbol{A}^{\dagger} \widehat{\boldsymbol{R}}_{1}^{-1} \boldsymbol{A}\right)^{-1}$, obtained using the same partitioning as in Eq. 32) for matrix $\widehat{\Gamma}^{\circ}$. The above expression can be rewritten in a more compact way, as shown in what follows. Indeed, the inner matrix in Eq. (42) is rewritten a:7:

$$
\begin{equation*}
K \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{E}_{r} \widehat{\boldsymbol{\Gamma}}_{22}^{1} \boldsymbol{E}_{r}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp}=\mathcal{P}_{\Delta} \tag{43}
\end{equation*}
$$

which then gives:

$$
\begin{equation*}
t_{\mathrm{wald}}=\operatorname{Tr}\left[\boldsymbol{Z}_{W 1}^{\dagger} \boldsymbol{\mathcal { P }}_{\Delta} \boldsymbol{Z}_{W 1} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right] \tag{44}
\end{equation*}
$$

CFARness of Wald Test: Finally we prove CFARness of Wald statistic. First, it is apparent that Eq. (44) can be rewritten as:

$$
\begin{equation*}
t_{\text {wald }}=\operatorname{Tr}\left[\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger}\left(\boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp}-\boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp}\right) \boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right] \tag{45}
\end{equation*}
$$

Secondly, exploiting Eqs. (26) and 27) (as in the GLR case), Eq. (45) is rewritten as:

$$
\begin{equation*}
t_{\mathrm{wald}}=\operatorname{Tr}\left[\boldsymbol{Z}_{2.3}^{\dagger} \boldsymbol{S}_{2.3}^{-1} \boldsymbol{Z}_{2.3}\right]=\operatorname{Tr}\left[\boldsymbol{T}_{a}\right] \tag{46}
\end{equation*}
$$

Therefore $t_{\text {wald }}$ depends on the data matrix uniquely through the MIS (actually, only through the first component).

## D. Gradient statistic

The Gradient (Terrell) test requires the evaluation of the following statistic [17], [28]:

$$
\begin{equation*}
\left.\frac{\partial \ln f_{1}(\boldsymbol{Z} ; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}_{r}^{T}}\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_{0}}\left(\hat{\boldsymbol{\theta}}_{r, 1}-\boldsymbol{\theta}_{r, 0}\right) \tag{47}
\end{equation*}
$$

The appeal of Eq. 47) arises from the fact that it does not require neither to invert the FIM nor to evaluate a compressed likelihood function under both hypotheses (as opposed to GLR, Wald, and Rao statistics). As a consequence, this formal simplicity can make the Gradient statistic easy to compute. Moreover, under some mild technical conditions, such test is asymptotically equivalent to the GLR, Rao and Wald statistics [17].

It is shown (the proof is provided as supplementary material) that the Gradient statistic is given in closed form as:
$t_{\mathrm{grad}} \triangleq \Re\left\{\operatorname{Tr}\left[\boldsymbol{Z}_{W 1}^{\dagger} K \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{E}_{r} \widehat{\boldsymbol{\Gamma}}_{22}^{1} \boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{Z}_{d, 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]\right\}$

[^6]where $\boldsymbol{Z}_{d, 0}$ is given in Eq. (33). The expression in Eq. (48) can be cast in a more compact form, as shown below. First, we notice that the following equality holds $\sqrt{8}$ :
\[

$$
\begin{equation*}
\widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{Z}_{d, 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}=\boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp}\left(\boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1}\right) \boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}} \tag{49}
\end{equation*}
$$

\]

which, after substitution in Eq. (48) and exploitation of Eq. (43), gives the final form

$$
\begin{equation*}
t_{\text {grad }}=\Re\left\{\operatorname{Tr}\left[\boldsymbol{Z}_{W 1}^{\dagger} \mathcal{P}_{\Delta}\left(\boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}\right) \boldsymbol{Z}_{W 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]\right\} \tag{50}
\end{equation*}
$$

where identical steps as for Wald test have been exploited.
CFARness of Gradient Test: First, Eq. (50) can be readily rewritten as:

$$
\begin{align*}
t_{\text {grad }}= & \Re\left\{\operatorname { T r } \left[\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger}\left(\boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp}-\boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp}\right)\right.\right. \\
& \left.\left.\times\left(\boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}\right) \boldsymbol{Z}_{W 0} \boldsymbol{V}_{c, 1}\right]\right\} . \tag{51}
\end{align*}
$$

Moreover, exploiting the following equalitie: 9

$$
\begin{gather*}
\left(\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{Z}_{W 0} \boldsymbol{V}_{c, 1}\right) \\
=K\left(\boldsymbol{T}_{a}+\boldsymbol{T}_{b}\right)\left[\boldsymbol{I}_{M}-\left(\boldsymbol{I}_{M}+\boldsymbol{T}_{a}+\boldsymbol{T}_{b}\right)^{-1}\left(\boldsymbol{T}_{a}+\boldsymbol{T}_{b}\right)\right]  \tag{52}\\
\left(\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp} \boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{Z}_{W 0} \boldsymbol{V}_{c, 1}\right) \\
=K \boldsymbol{T}_{b}\left[\boldsymbol{I}_{M}-\left(\boldsymbol{I}_{M}+\boldsymbol{T}_{a}+\boldsymbol{T}_{b}\right)^{-1}\left(\boldsymbol{T}_{a}+\boldsymbol{T}_{b}\right)\right] \tag{53}
\end{gather*}
$$

it thus follows that:

$$
\begin{align*}
t_{\mathrm{grad}}=\Re\{\operatorname{Tr}[ & K \boldsymbol{T}_{a}\left(\boldsymbol{I}_{M}-\left(\boldsymbol{I}_{M}+\boldsymbol{T}_{a}+\boldsymbol{T}_{b}\right)^{-1}\right. \\
& \left.\left.\left.\times\left(\boldsymbol{T}_{a}+\boldsymbol{T}_{b}\right)\right)\right]\right\} \tag{54}
\end{align*}
$$

which shows that also the Gradient test satisfies the CFAR property.

## E. Durbin statistic

The Durbin test (also referred to as "Naive test") consists in the evaluation of the following decision statistic [16]:

$$
\begin{gather*}
\left(\hat{\boldsymbol{\theta}}_{r, 01}-\boldsymbol{\theta}_{r, 0}\right)^{T}\left\{\left[\boldsymbol{I}\left(\widehat{\boldsymbol{\theta}}_{0}\right)\right]_{\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}}\left[\boldsymbol{I}^{-1}\left(\widehat{\boldsymbol{\theta}}_{0}\right)\right]_{\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}} \times\right. \\
\left.\left[\boldsymbol{I}\left(\widehat{\boldsymbol{\theta}}_{0}\right)\right]_{\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}}\right\}\left(\hat{\boldsymbol{\theta}}_{0,1}-\boldsymbol{\theta}_{r, 0}\right), \tag{55}
\end{gather*}
$$

where the estimate $\hat{\boldsymbol{\theta}}_{r, 01}$ is defined as:

$$
\begin{equation*}
\hat{\boldsymbol{\theta}}_{r, 01} \triangleq \arg \max _{\boldsymbol{\theta}_{r}} f_{1}\left(\boldsymbol{Z} ; \boldsymbol{\theta}_{r}, \hat{\boldsymbol{\theta}}_{s, 0}\right) . \tag{56}
\end{equation*}
$$

In general, the Durbin statistic is asymptotically equivalent to GLR, Rao and Wald statistics, as shown in [16]. However, for the considered problem, a stronger result holds with respect to the Rao statistic, as stated by the following theorem.

Theorem 2. The Durbin statistic for the hypothesis testing model considered in Eq. (1) is statistically equivalent to the Rao statistic. Therefore, the test is also CFAR.

Proof: Provided as supplementary material.
It is worth noticing that the present result generalizes the statistical equivalence observed between Rao and Durbin statistics for the simpler scenario of point-like targets and single-steering assumption in [19]. On the other hand, Thm. 2 proves that such result holds for the (very general) hypothesis testing problem considered in this work.

[^7]
## F. Two-step GLR (2S-GLR)

It is also worth considering a two-step GLR (2S-GLR), which first consists in evaluating the GLR statistic under the assumption that $\boldsymbol{R}$ is known and then plugging-in a reasonable estimate of $\boldsymbol{R}$. The GLR statistic for known $\boldsymbol{R}$ can be expressed in implicit form as [15]:

$$
\begin{equation*}
\frac{\max _{\boldsymbol{B}_{s}} f_{1}\left(\boldsymbol{Z} ; \boldsymbol{B}_{s}, \boldsymbol{R}\right)}{\max _{\boldsymbol{B}_{t, 0}} f_{0}\left(\boldsymbol{Z} ; \boldsymbol{B}_{t, 0}, \boldsymbol{R}\right)} \tag{57}
\end{equation*}
$$

The ML estimates of $\boldsymbol{B}_{s}$ and $\boldsymbol{B}_{t, 0}$ are more easily obtained from optimizing the logarithm of $f_{1}(\cdot)$ and $f_{0}(\cdot)$, respectively, that is:

$$
\begin{align*}
& -K \ln \left(\pi^{N} \operatorname{det}[\boldsymbol{R}]\right) \\
& -\operatorname{Tr}\left[\boldsymbol{R}^{-1}\left(\boldsymbol{Z}-\boldsymbol{A} \boldsymbol{B}_{s} \boldsymbol{C}\right)\left(\boldsymbol{Z}-\boldsymbol{A} \boldsymbol{B}_{s} \boldsymbol{C}\right)^{\dagger}\right]  \tag{58}\\
& -K \ln \left(\pi^{N} \operatorname{det}[\boldsymbol{R}]\right) \\
& -\operatorname{Tr}\left[\boldsymbol{R}^{-1}\left(\boldsymbol{Z}-\boldsymbol{E}_{t} \boldsymbol{B}_{t, 0} \boldsymbol{C}\right)\left(\boldsymbol{Z}-\boldsymbol{E}_{t} \boldsymbol{B}_{t, 0} \boldsymbol{C}\right)^{\dagger}\right] \tag{59}
\end{align*}
$$

Maximization of Eqs. (58) and (59) with respect to $\boldsymbol{B}_{s}$ and $\boldsymbol{B}_{t, 0}$, respectively, can be obtained following the same steps employed in [15] and thus it is omitted for brevity. Therefore, after optimization, the following statistic is obtained (as the logarithm of Eq. (57)):

$$
\begin{equation*}
\operatorname{Tr}\left[\boldsymbol{Z}^{\dagger} \boldsymbol{R}^{-1 / 2}\left(\boldsymbol{P}_{\breve{\boldsymbol{A}}_{1}}-\boldsymbol{P}_{\breve{\boldsymbol{A}}_{0}}\right) \boldsymbol{R}^{-1 / 2} \boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right] \tag{60}
\end{equation*}
$$

where we have defined $\breve{\boldsymbol{A}}_{1} \triangleq\left(\boldsymbol{R}^{-1 / 2} \boldsymbol{A}\right)$ and $\breve{\boldsymbol{A}}_{0} \triangleq$ $\left(\boldsymbol{R}^{-1 / 2} \boldsymbol{E}_{t}\right)$, respectively. We recall that the expression in Eq. 60) depends on $\boldsymbol{R}$. We now turn our attention on finding an estimate for the covariance $\boldsymbol{R}$. Clearly, in order to obtain a meaningful estimate to be be plugged in both the numerator and denominator of Eq. (57), such estimate should be based only on signal-free data (also commonly denoted as "secondary data").
It is not difficult to show that the covariance estimate based only on secondary data is given by 10 :

$$
\begin{equation*}
\widehat{\boldsymbol{R}}_{s d}=(K-M)^{-1} \boldsymbol{S}_{c} . \tag{61}
\end{equation*}
$$

Thus, substitution of Eq. (61) into Eq. (60) leads to the final form of 2S-GLR:

$$
\begin{gather*}
\operatorname{Tr}\left[\boldsymbol{Z}^{\dagger} \sqrt{K-M} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{\mathcal { P }}_{\Delta} \boldsymbol{S}_{c}^{-1 / 2} \sqrt{K-M} \boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right] \propto \\
t_{2 \mathrm{~s}-\mathrm{glr}} \triangleq \operatorname{Tr}\left[\boldsymbol{Z}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{\mathcal { P }}_{\Delta} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right] \tag{62}
\end{gather*}
$$

From direct comparison of Eqs. (44) and 62), a general equivalence result is obtained, stated in the form of the following lemma.

Lemma 3. The $2 S-G L R$ statistic is statistically equivalent to the Wald statistic. Therefore, the test is also CFAR.

The aforementioned lemma extends the statistical equivalence observed between 2S-GLR and Wald statistics in the simpler cases of point-like targets [11], range-spread targets [24] and multidimensional signals [23].

[^8]
## G. Lawley-Hotelling (LH) statistic

Finally, for the sake of a complete comparison, we also consider (and generalize) the simpler statistic proposed in [15, pag. 37] as a reasonable approximation to GLR. Indeed, Wilks' Lambda form of GLR in Eq. (21) can be rewritten as:

$$
\begin{align*}
t_{\mathrm{glr}}= & \operatorname{det}\left[\boldsymbol{I}_{M}+\boldsymbol{D}_{1}^{-1 / 2}\left\{\widehat{\boldsymbol{B}}_{s}^{\dagger}\left(\boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{A}\right) \widehat{\boldsymbol{B}}_{s}\right.\right. \\
& \left.\left.-\widehat{\boldsymbol{B}}_{t, 0}^{\dagger}\left(\boldsymbol{E}_{t}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{E}_{t}\right) \widehat{\boldsymbol{B}}_{t, 0}\right\} \boldsymbol{D}_{1}^{-1 / 2}\right] \tag{63}
\end{align*}
$$

where we exploited $\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{-1}=\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{-1 / 2}$ and closedform estimates for $\widehat{\boldsymbol{B}}_{s}$ and $\widehat{\boldsymbol{B}}_{t, 0}^{\dagger}$ (cf. Eqs. (14) and (15), respectively). As the number of samples $K$ grows large, we can invoke approximation $\boldsymbol{S}_{c} \approx(K-M) \boldsymbol{R}$, that is, the sample covariance based on secondary data will accurately approximate the true covariance matrix. Accordingly, we can safely approximate

$$
\begin{align*}
\boldsymbol{D}_{1} & \approx \boldsymbol{I}_{M} \\
\widehat{\boldsymbol{B}}_{s} & \approx\left(\boldsymbol{A}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{C}^{\dagger}\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{-1} \\
\widehat{\boldsymbol{B}}_{t, 0} & \approx\left(\boldsymbol{E}_{t}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{E}_{t}\right)^{-1} \boldsymbol{E}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{C}^{\dagger}\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{-1} \tag{64}
\end{align*}
$$

since $\left(\boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{A}\right) \approx(K-M)^{-1}\left(\boldsymbol{A}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{A}\right)$ and $\left(\boldsymbol{E}_{t}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{E}_{t}\right) \approx(K-M)^{-1}\left(\boldsymbol{E}_{t}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{E}_{t}\right)$, respectively. Therefore, based on these approximations, it is apparent that the second contribution within the determinant in Eq. (63) will be a vanishing term as the number of observations increases. Hence, GLR statistic will be given by the determinant of an identity matrix plus a small perturbing term.

Additionally, we remark that when $\mathbf{\Upsilon} \in \mathbb{H}^{M \times M}$ is a small perturbing matrix, $\operatorname{det}\left[\boldsymbol{I}_{M}+\mathbf{\Upsilon}\right]$ can be (accurately) approximated at first order as $\prod_{i=1}^{M}\left(1+v_{i}\right) \approx 1+\sum_{i=1}^{M} v_{i}=$ $1+\operatorname{Tr}[\Upsilon]$, where $v_{i}$ denotes the $i$-th eigenvalue of $\Upsilon$. Based on those reasons, we formulate the LH statistic as:

$$
\begin{align*}
t_{\mathrm{lh}} & \triangleq \operatorname{Tr}\left[\boldsymbol{D}_{1}^{-1 / 2}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \mathcal{P}_{\Delta}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right) \boldsymbol{D}_{1}^{-1 / 2}\right]  \tag{65}\\
& =\operatorname{Tr}\left[\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \mathcal{P}_{\Delta}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right) \boldsymbol{D}_{1}^{-1}\right]
\end{align*}
$$

CFARness of LH statistic: The CFARness is proved by using Eqs. (26) and (27) from Sec. III-A within Eq. (65), thus obtaining:

$$
\begin{equation*}
t_{\mathrm{lh}}=\operatorname{Tr}\left[\boldsymbol{T}_{a}\left(\boldsymbol{I}_{M}+\boldsymbol{T}_{b}\right)^{-1}\right] \tag{66}
\end{equation*}
$$

Finally, in Tab. $\Pi$ it is shown a recap table, summarizing all the considered detectors and their respective expressions in terms of the MIS in Eq. (4).

## IV. Detectors in Special Cases

## A. Adaptive (Vector Subspace) Detection of a Point-like Target

In the present case we start from general formulation in Eq. (1) and assume that: $(i) t=0$ (i.e., there is no interference, thus $J=r$ and $\left.\boldsymbol{A}=\left[\begin{array}{ll}\boldsymbol{I}_{r} & \mathbf{0}_{r \times(N-r)}\end{array}\right]^{T} \in \mathbb{C}^{N \times r}\right) ;(i i) M=1$, i.e., the matrix $\boldsymbol{B}$ collapses to a vector $\boldsymbol{b} \in \mathbb{C}^{J \times 1}$ and (iii) $\boldsymbol{c} \triangleq\left[\begin{array}{llll}1 & 0 & \cdots & 0\end{array}\right] \in \mathbb{C}^{1 \times K}$ (i.e., a row vector). Such case has been extensively dealt in adaptive detection literature [3], [5], [15], [29], [30]. The hypothesis testing in canonical form is then:

$$
\begin{cases}\mathcal{H}_{0}: & \boldsymbol{Z}=\boldsymbol{N}  \tag{67}\\ \mathcal{H}_{1}: & \boldsymbol{Z}=\boldsymbol{A} \boldsymbol{b} \boldsymbol{c}+\boldsymbol{N}\end{cases}
$$

Clearly, since in this case $M=1$ holds, $(K-1)$ vector components are assumed signal-free, that is, $\boldsymbol{Z}$ admits the partitioning $\boldsymbol{Z}=\left[\begin{array}{ll}\boldsymbol{z}_{p} & \boldsymbol{Z}_{s}\end{array}\right]=\left[\begin{array}{ll}\boldsymbol{z}_{c} & \boldsymbol{Z}_{c, \perp}\end{array}\right]$, where $\boldsymbol{z}_{p}$ denotes the signal vector related to the cell under test and the columns of $Z_{s}$ represent the secondary (training) data. Also, $\boldsymbol{P}_{\boldsymbol{A}_{0}}=\mathbf{0}_{N \times N}$ (resp. $\boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp}=\boldsymbol{I}_{N}$ ) holds, because of the absence of the structured interference. In the latter case, it can be shown that the simplified projector form holds:

$$
\boldsymbol{P}_{\boldsymbol{C}^{\dagger}}=\left[\begin{array}{cc}
1 & \mathbf{0}_{K-1}^{T}  \tag{68}\\
\mathbf{0}_{K-1} & \mathbf{0}_{(K-1) \times(K-1)}
\end{array}\right]
$$

Given the results in Eq. 68, it can be shown that $\boldsymbol{S}_{c}=$ $\boldsymbol{Z}_{s} \boldsymbol{Z}_{s}^{\dagger}$ and $\hat{\boldsymbol{R}}_{0}=\frac{1}{K} \boldsymbol{S}_{0}$, where $\boldsymbol{S}_{0} \triangleq\left(\boldsymbol{z}_{p} \boldsymbol{z}_{p}^{\dagger}+\boldsymbol{Z}_{s} \boldsymbol{Z}_{s}^{\dagger}\right)$ hold, respectively. In some cases we will also use the ShermanWoodbury formula [26] applied to $\boldsymbol{S}_{0}^{-1}$, that is:

$$
\begin{equation*}
\boldsymbol{S}_{0}^{-1}=\boldsymbol{S}_{c}^{-1}-\frac{\boldsymbol{S}_{c}^{-1} \boldsymbol{z}_{p} \boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1}}{1+\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{z}_{p}} \tag{69}
\end{equation*}
$$

$G L R$ : In the specific case of $M=1$, the following form of the GLR is obtained from Eq. (20):

$$
\begin{equation*}
t_{\mathrm{glr}}=\frac{1}{1-\eta}, \quad \eta \triangleq \frac{\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{1}} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{z}_{p}}{1+\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{z}_{p}} \tag{70}
\end{equation*}
$$

since we have exploited $\boldsymbol{D}_{0} \rightarrow d_{0}=\left(1+\boldsymbol{z}_{p, 1}^{\dagger} \boldsymbol{z}_{p, 1}\right)$ and $\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{\mathcal { P }}_{\Delta}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right) \rightarrow\left(\boldsymbol{z}_{p, 1}^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}} \boldsymbol{z}_{p, 1}\right)$, where $\boldsymbol{z}_{p, 1} \triangleq$ $\left(\boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{z}_{p}\right)$. Clearly, $t_{\mathrm{glr}}$ is an increasing function of $\eta$, the latter thus being an equivalent form of the statistic and coinciding with the so-called multi-rank signal model GLR described in [3], [15].
Rao/Durbin statistic: For the present scenario, Eq. (36) specializes into:

$$
\begin{align*}
& t_{\mathrm{rao}}=\operatorname{Tr} {\left[\boldsymbol{Z}_{W 0}^{\dagger} \boldsymbol{P}_{\overline{\boldsymbol{A}}_{1}} \boldsymbol{Z}_{W 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]=\operatorname{Tr}\left[\boldsymbol{z}_{p}^{\dagger}\left(\hat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{P}_{\overline{\boldsymbol{A}}_{1}} \hat{\boldsymbol{R}}_{0}^{-1 / 2}\right) \boldsymbol{z}_{p}\right] } \\
& \propto \boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{0}^{-1} \boldsymbol{A}\left(\boldsymbol{A}^{\dagger} \boldsymbol{S}_{0}^{-1} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^{\dagger} \boldsymbol{S}_{0}^{-1} \boldsymbol{z}_{p} \triangleq \eta_{\mathrm{rao}} \tag{71}
\end{align*}
$$

Eq. (71) can be further simplified by exploiting the Woodbury identity in 69) (and similar steps as in [13]), thus obtaining the following simplified form of the Rao statistic:

$$
\begin{equation*}
\eta_{\mathrm{rao}}=\frac{1}{1+\boldsymbol{z}_{p, 1}^{\dagger} \boldsymbol{z}_{p, 1}}\left[\frac{\boldsymbol{z}_{p, 1}^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}} \boldsymbol{z}_{p, 1}}{1+\boldsymbol{z}_{p, 1} \boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp} \boldsymbol{z}_{p, 1}}\right] \tag{72}
\end{equation*}
$$

Finally, for $r=1$ (i.e., a single-steering case) $(\boldsymbol{A} \rightarrow \boldsymbol{a} \in$ $\mathbb{C}^{N \times 1}$, Eq. 72 reduces to:

$$
\begin{equation*}
\eta_{\mathrm{rao}}=\frac{\left|\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right|^{2} /\left(\boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)}{\left[1+\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{z}_{p}\right]\left[1+\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{z}_{p}-\frac{\left|\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right|^{2}}{\left(\boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)}\right]} \tag{73}
\end{equation*}
$$

which coincides with the well-known Rao statistic for the single-steering case developed in [13].

Wald/2S-GLR statistic: Starting from Eq. (44), we particularize the Wald statistic as follows:

$$
\begin{gather*}
t_{\text {wald }}=\operatorname{Tr}\left[\boldsymbol{Z}_{W 1}^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}} \boldsymbol{Z}_{W 1} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]=\operatorname{Tr}\left[\boldsymbol{z}_{p, 1}^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}} \boldsymbol{z}_{p, 1}\right]  \tag{74}\\
=\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{A}\left(\boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{z}_{p}
\end{gather*}
$$

Table I
DETECTORS COMPARISON AND THEIR FUNCTIONAL DEPENDENCE OF THE MIS (VIZ. CFARNESS). AUXILIARY DEFINITIONS: $\boldsymbol{T}_{a+b} \triangleq\left(\boldsymbol{T}_{a}+\boldsymbol{T}_{b}\right)$ AND $\boldsymbol{D}_{i} \triangleq\left[\boldsymbol{I}_{M}+\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{i}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)\right]$.

| Detector | Standard Expression | MIS function |
| :---: | :---: | :---: |
| GLR | $\operatorname{det}\left[\boldsymbol{D}_{0}\right] / \operatorname{det}\left[\boldsymbol{D}_{1}\right]$ | $\operatorname{det}\left[\boldsymbol{I}_{M}+\boldsymbol{T}_{a+b}\right] / \operatorname{det}\left[\boldsymbol{I}_{M}+\boldsymbol{T}_{b}\right]$ |
| Rao/Durbin | $\operatorname{Tr}\left[\boldsymbol{Z}_{W 0}^{\dagger}\left(\boldsymbol{P}_{\overline{\boldsymbol{A}}_{1}}-\boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}\right) \boldsymbol{Z}_{W 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]$ | $K \operatorname{Tr}\left[\boldsymbol{T}_{a}-\boldsymbol{T}_{a+b}\left(\boldsymbol{I}_{M}+\boldsymbol{T}_{a+b}\right)^{-1} \boldsymbol{T}_{a+b}+\boldsymbol{T}_{b}\left(\boldsymbol{I}_{M}+\boldsymbol{T}_{b}\right)^{-1} \boldsymbol{T}_{b}\right]$ |
| Wald/2S-GLR | $\operatorname{Tr}\left[\boldsymbol{Z}_{W 1}^{\dagger}\left(\boldsymbol{P}_{\boldsymbol{A}_{1}}-\boldsymbol{P}_{\boldsymbol{A}_{0}}\right) \boldsymbol{Z}_{W 1} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]$ | $\operatorname{Tr}\left[\boldsymbol{T}_{a}\right]$ |
| Gradient | $\Re\left\{\operatorname{Tr}\left[\boldsymbol{Z}_{W 1}^{\dagger}\left(\boldsymbol{P}_{\boldsymbol{A}_{1}}-\boldsymbol{P}_{\boldsymbol{A}_{0}}\right)\left(\boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}\right) \boldsymbol{Z}_{W 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]\right\}$ | $\Re\left\{\operatorname{Tr}\left[K \boldsymbol{T}_{a}\left(\boldsymbol{I}_{M}-\left(\boldsymbol{I}_{M}+\boldsymbol{T}_{a+b}\right)^{-1} \boldsymbol{T}_{a+b}\right)\right]\right\}$ |
| LH | $\operatorname{Tr}\left[\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger}\left(\boldsymbol{P}_{\boldsymbol{A}_{1}}-\boldsymbol{P}_{\boldsymbol{A}_{0}}\right)\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right) \boldsymbol{D}_{1}^{-1}\right]$ | $\operatorname{Tr}\left[\boldsymbol{T}_{a}\left(\boldsymbol{I}_{M}+\boldsymbol{T}_{b}\right)^{-1}\right]$. |

In the special case $r=1$ (i.e., a single-steering case), Eq. (74) becomes:

$$
\begin{equation*}
t_{\mathrm{wald}}=\frac{\left|\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right|^{2}}{\boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}} \tag{75}
\end{equation*}
$$

which is recognized as the well-known Adaptive Matched Filter (AMF) [11], [31].

Gradient statistic: In this case the gradient statistic in Eq. (50) specializes into

$$
\begin{align*}
t_{\mathrm{grad}} & =\Re\left\{\operatorname{Tr}\left[\boldsymbol{Z}_{W 1}^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}\left(\boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}\right) \boldsymbol{Z}_{W_{0}} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]\right\}  \tag{76}\\
& \left.=\Re\left\{\boldsymbol{z}_{p, 1}^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}} \boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}\right) \boldsymbol{z}_{p, 0}\right\} \\
& =K \Re\left\{\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{A}\left(\boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^{\dagger} \boldsymbol{S}_{0}^{-1} \boldsymbol{z}_{p}\right\}
\end{align*}
$$

where $\boldsymbol{z}_{p, 0} \triangleq\left(\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{z}_{p}\right)$. It is interesting to note that, exploiting Eq. (69), the gradient statistic is rewritten as:

$$
\begin{align*}
t_{\mathrm{grad}} & =K \Re\left\{\frac{\left(\boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{z}_{p}\right)^{\dagger}\left(\boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{A}\right)^{-1}\left(\boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{z}_{p}\right)}{1+\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{z}_{p}}\right\} \\
& =K \frac{\left(\boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{z}_{p}\right)^{\dagger}\left(\boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{A}\right)^{-1}\left(\boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{z}_{p}\right)}{1+\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{z}_{p}} \tag{77}
\end{align*}
$$

where in last line we have omitted $\Re\{\cdot\}$ since Eq. (77) is formed by Hermitian quadratic forms (at both numerator and denominator); thus it is always real-valued. Therefore Eq. (77) is statistically equivalent to Kelly's GLR in Eq. (70).

LH statistic: We recall that for point-like targets, the condition $M=1$ holds. Therefore the LH test is statistically equivalent to the GLRT since the operators $\operatorname{Tr}[\cdot]$ and $\operatorname{det}[\cdot]$ are non-influential when applied to a scalar value. This follows since the expressions in Eqs. (21) and (65) are thus related by a monotone transformation.

## B. Adaptive Vector Subspace Detection with Structured Interference

In the present case we start from general formulation in Eq. (1) and assume that: (i) $M=1$, i.e., the matrices $\boldsymbol{B}$ and $\boldsymbol{B}_{t, i}$ collapse to the vectors $\boldsymbol{b} \in \mathbb{C}^{r \times 1}$ and $\boldsymbol{b}_{t, i} \in \mathbb{C}^{t \times 1}$, respectively; (ii) $\boldsymbol{c} \triangleq\left[\begin{array}{llll}1 & 0 & \cdots & 0\end{array}\right] \in \mathbb{C}^{1 \times K}$ (i.e., a row vector). Such case has been dealt in [6]. Given the aforementioned assumptions, the problem in canonical form
is given as:

$$
\begin{cases}\mathcal{H}_{0}: & \boldsymbol{Z}=\boldsymbol{A}\left[\begin{array}{ll}
\boldsymbol{b}_{t, 0}^{T} & \mathbf{0}_{r}^{T}
\end{array}\right]^{T} \boldsymbol{c}+\boldsymbol{N}  \tag{78}\\
\mathcal{H}_{1}: & \boldsymbol{Z}=\boldsymbol{A}\left[\begin{array}{ll}
\boldsymbol{b}_{t, 1}^{T} & \boldsymbol{b}^{T}
\end{array}\right]^{T} \boldsymbol{c}+\boldsymbol{N}\end{cases}
$$

Clearly, since in this case $M=1$ holds, $(K-1)$ vector components are assumed signal-free, that is, $\boldsymbol{Z}$ admits the partitioning $\boldsymbol{Z}=\left[\begin{array}{ll}\boldsymbol{z}_{p} & \boldsymbol{Z}_{s}\end{array}\right]=\left[\begin{array}{ll}\boldsymbol{z}_{c} & \boldsymbol{Z}_{c, \perp}\end{array}\right]$, where $\boldsymbol{z}_{p}$ denotes the signal vector related to the cell of interest and the columns of $\boldsymbol{Z}_{s}$ represent the secondary (or training) data. In the latter case, it can be shown that the same simplified projector form in Eq. (68) holds. Given the results in Eq. (68) , it can be shown that $\boldsymbol{S}_{c}=\boldsymbol{Z}_{s} \boldsymbol{Z}_{s}^{\dagger}$ and $\hat{\boldsymbol{R}}_{0}=\frac{1}{K} \boldsymbol{S}_{0}$, where $\boldsymbol{S}_{0} \triangleq\left(\boldsymbol{S}_{c}+\boldsymbol{S}_{c}^{1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{z}_{p} \boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{1 / 2}\right)$ hold, respectively. In some cases we will also use the ShermanWoodbury formula [26] applied to $\boldsymbol{S}_{0}^{-1}$ and consider the product $\boldsymbol{S}_{0}^{-1} \boldsymbol{z}_{p}$, which provides:

$$
\begin{align*}
\boldsymbol{S}_{0}^{-1} \boldsymbol{z}_{p}= & \boldsymbol{S}_{c}^{-1} \boldsymbol{z}_{p}-\left(\boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2}\right) \boldsymbol{z}_{p}  \tag{79}\\
& \times \frac{\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{z}_{p}}{1+\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{z}_{p}}
\end{align*}
$$

$G L R$ : In the specific case of $M=1$, the following form of the GLRT is obtained from Eq. (20):

$$
\begin{equation*}
t_{\mathrm{glr}}=\frac{1}{1-\eta}, \quad \eta \triangleq \frac{\boldsymbol{z}_{p, 1}^{\dagger}\left(\boldsymbol{P}_{\boldsymbol{A}_{1}}-\boldsymbol{P}_{\boldsymbol{A}_{0}}\right) \boldsymbol{z}_{p, 1}}{1+\boldsymbol{z}_{p, 1}^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{z}_{p, 1}} \tag{80}
\end{equation*}
$$

since we have exploited $\boldsymbol{D}_{0} \quad \rightarrow \quad d_{0} \quad=\quad(1+$ $\left.\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{z}_{p}\right)$ and $\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\Delta}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right) \rightarrow$ $\boldsymbol{z}_{p, 1}^{\dagger}\left(\boldsymbol{P}_{\boldsymbol{A}_{1}}-\boldsymbol{P}_{\boldsymbol{A}_{0}}\right) \boldsymbol{z}_{p, 1}$, where $\boldsymbol{z}_{p, 1} \triangleq\left(\boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{z}_{p}\right)$. Clearly, Eq. (80) is an increasing function of $\eta$, which can be thus seen as an equivalent form of the GLR.
Rao/Durbin statistic: For the present scenario, Eq. (36) specializes into:

$$
\begin{equation*}
t_{\mathrm{rao}}=\boldsymbol{z}_{p, 0}^{\dagger}\left(\boldsymbol{P}_{\overline{\boldsymbol{A}}_{1}}-\boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}\right) \boldsymbol{z}_{p, 0} \tag{81}
\end{equation*}
$$

where $\boldsymbol{z}_{p, 0} \triangleq\left(\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{z}_{p}\right)$.
Wald/2S-GLR statistic: Starting from Eq. (44), we particularize the Wald statistic as follows:

$$
\begin{equation*}
t_{\text {wald }}=\boldsymbol{z}_{p, 1}^{\dagger}\left(\boldsymbol{P}_{\boldsymbol{A}_{1}}-\boldsymbol{P}_{\boldsymbol{A}_{0}}\right) \boldsymbol{z}_{p, 1} \tag{82}
\end{equation*}
$$

Gradient statistic: In this case the gradient statistic in Eq. (50) specializes into:

$$
\begin{equation*}
t_{\text {grad }}=\Re\left\{\boldsymbol{z}_{p, 1}^{\dagger}\left(\boldsymbol{P}_{\boldsymbol{A}_{1}}-\boldsymbol{P}_{\boldsymbol{A}_{0}}\right)\left(\boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}\right) \boldsymbol{z}_{p, 0}\right\} \tag{83}
\end{equation*}
$$

We now rewrite Eq. 83) as:

$$
\begin{equation*}
t_{\mathrm{grad}}=K \Re\left\{\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2}\left(\boldsymbol{P}_{\boldsymbol{A}_{1}}-\boldsymbol{P}_{\boldsymbol{A}_{0}}\right)\left(\boldsymbol{S}_{c}^{1 / 2} \boldsymbol{S}_{0}^{-1}\right) \boldsymbol{z}_{p}\right\} \tag{84}
\end{equation*}
$$

Exploiting Eq. (79) and observing that $\left(\boldsymbol{P}_{\boldsymbol{A}_{1}}-\boldsymbol{P}_{\boldsymbol{A}_{0}}\right) \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp}=$ $\boldsymbol{P}_{\boldsymbol{A}_{1}}-\boldsymbol{P}_{\boldsymbol{A}_{0}}$ holds, Eq. [84) is expressed as:

$$
\begin{align*}
t_{\mathrm{grad}} & =K \Re\left\{\frac{\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2}\left(\boldsymbol{P}_{\boldsymbol{A}_{1}}-\boldsymbol{P}_{\boldsymbol{A}_{0}}\right) \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{z}_{p}}{1+\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{z}_{p}}\right\} \\
& =K \frac{\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2}\left(\boldsymbol{P}_{\boldsymbol{A}_{1}}-\boldsymbol{P}_{\boldsymbol{A}_{0}}\right) \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{z}_{p}}{1+\boldsymbol{z}_{p}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{z}_{p}} \tag{85}
\end{align*}
$$

where in last line we have omitted $\Re\{\cdot\}$ since Eq. 85) is formed by Hermitian quadratic forms (at both numerator and denominator); thus it is always real-valued. Therefore Eq. (85) is statistically equivalent to GLR in Eq. 80).

LH statistic: As in the case of no-interference in Sec. IV-A, the condition $M=1$ holds. Therefore the LH statistic is statistically equivalent to the GLR.

## C. Multidimensional Signals

In the present case we start from formulation in Eq. (1) and assume that: (i) $t=0$ (i.e. there is no interference, meaning $J=r$ ), (ii) $\boldsymbol{A}=\boldsymbol{E}_{r}=\boldsymbol{I}_{N}$ (thus $J=r=N$ ) and (iii) $\boldsymbol{C} \triangleq\left[\begin{array}{ll}\boldsymbol{I}_{M} & \mathbf{0}_{M \times(K-M)}\end{array}\right]$. Such case has been dealt in [10], [23]. Thus, the hypothesis testing in canonical form is given by:

$$
\begin{cases}\mathcal{H}_{0}: & \boldsymbol{Z}=\boldsymbol{N}  \tag{86}\\ \mathcal{H}_{1}: & \boldsymbol{Z}=\boldsymbol{B} C+\boldsymbol{N}\end{cases}
$$

Clearly, since in this case $J=N$ holds, $(K-M)$ vector components are assumed signal-free, that is, $\boldsymbol{Z}$ admits the partitioning $\boldsymbol{Z}=\left[\begin{array}{ll}\boldsymbol{Z}_{M} & \boldsymbol{Z}_{s}\end{array}\right]=\left[\begin{array}{ll}\boldsymbol{Z}_{c} & \boldsymbol{Z}_{c, \perp}\end{array}\right]$, where $\boldsymbol{Z}_{M}$ denotes the signal matrix collecting the cells containing the useful signals and the columns of $\boldsymbol{Z}_{s}$ are the training data. In the latter case, it can be shown that the simplified projector form holds:

$$
\boldsymbol{P}_{\boldsymbol{C}^{\dagger}}=\left[\begin{array}{cc}
\boldsymbol{I}_{M} & \mathbf{0}_{M \times(K-M)}  \tag{87}\\
\mathbf{0}_{(K-M) \times M} & \mathbf{0}_{(K-M) \times(K-M)}
\end{array}\right]
$$

Given the results in Eq. 68), it can be shown that $\boldsymbol{S}_{c}=\boldsymbol{Z}_{s} \boldsymbol{Z}_{s}^{\dagger}$ and $\hat{\boldsymbol{R}}_{0}=\frac{1}{K} \boldsymbol{S}_{0}$, where $\boldsymbol{S}_{0} \triangleq\left(\boldsymbol{Z}_{M} \boldsymbol{Z}_{M}^{\dagger}+\boldsymbol{Z}_{s} \boldsymbol{Z}_{s}^{\dagger}\right)$ holds, respectively. Also, it is not difficult to show that $\boldsymbol{P}_{\boldsymbol{A}_{1}}=\boldsymbol{P}_{\overline{\boldsymbol{A}}_{1}}=\boldsymbol{I}_{N}$ and $\boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp}=\boldsymbol{P}_{\overline{\boldsymbol{A}}_{1}}^{\perp}=\mathbf{0}_{N \times N}$, respectively.

GLR: In order to specialize GLR expression we start from Eq. (18). Indeed, it can be easily shown that:

$$
\begin{align*}
t_{\mathrm{glr}} & =\frac{\operatorname{det}\left[\boldsymbol{I}_{M}+\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)\right]}{\operatorname{det}\left[\boldsymbol{I}_{M}+\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)\right]} \\
& =\operatorname{det}\left[\boldsymbol{I}_{M}+\boldsymbol{Z}_{M}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{Z}_{M}\right] \\
& =\operatorname{det}\left[\boldsymbol{I}_{M}+\boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{Z}_{M} \boldsymbol{Z}_{M}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2}\right] \\
& =\operatorname{det}\left[\boldsymbol{S}_{c}+\boldsymbol{Z}_{M} \boldsymbol{Z}_{M}^{\dagger}\right] / \operatorname{det}\left[\boldsymbol{S}_{c}\right] \tag{88}
\end{align*}
$$

where we have exploited $\boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp}=\mathbf{0}_{N \times N}$ and Sylvester's determinant theorem in third and fourth lines, respectively. It is apparent that the latter expressions coincide with those in [10, Eqs. (18) and (20)], respectively.

Rao/Durbin statistic: For the present setup Eq. (36) specializes into:

$$
\begin{align*}
t_{\text {rao }}=\operatorname{Tr}\left[\boldsymbol{Z}_{W 0}^{\dagger} \boldsymbol{P}_{\overline{\boldsymbol{A}}_{1}} \boldsymbol{Z}_{W 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right] & =\operatorname{Tr}\left[\boldsymbol{Z}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right] \\
=K \operatorname{Tr}\left[\left(\boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right)^{\dagger} \boldsymbol{S}_{0}^{-1}\left(\boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right)\right] & =K \operatorname{Tr}\left[\boldsymbol{Z}_{M}^{\dagger} \boldsymbol{S}_{0}^{-1} \boldsymbol{Z}_{M}\right] \tag{89}
\end{align*}
$$

which coincides with the specific result obtained in [23], which was originally derived as a modified two-step GLRT procedure in [10].

Wald/2S-GLR statistic: Starting from Eq. (44), we particularize the Wald statistic as follows:

$$
\begin{gather*}
t_{\text {wald }}=\operatorname{Tr}\left[\boldsymbol{Z}_{W 1}^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}} \boldsymbol{Z}_{W 1} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]=\operatorname{Tr}\left[\boldsymbol{Z}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right] \\
=\operatorname{Tr}\left[\left(\boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right)^{\dagger} \boldsymbol{S}_{c}^{-1}\left(\boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right)\right]=\operatorname{Tr}\left[\boldsymbol{Z}_{M}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{Z}_{M}\right] \tag{90}
\end{gather*}
$$

which coincides with the specific result obtained in [23].
Gradient statistic: In this case the gradient statistic in Eq. (50) specializes into:

$$
\begin{align*}
& t_{\mathrm{grad}}=\Re\left\{\operatorname{Tr}\left[\boldsymbol{Z}_{W 1}^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}\left(\boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}\right) \boldsymbol{Z}_{W 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]\right\} \\
&=K \Re\left\{\operatorname{Tr}\left[\boldsymbol{Z}^{\dagger} \boldsymbol{S}_{0}^{-1} \boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]\right\}=K \operatorname{Tr}\left[\boldsymbol{Z}_{M}^{\dagger} \boldsymbol{S}_{0}^{-1} \boldsymbol{Z}_{M}\right] \tag{91}
\end{align*}
$$

It is interesting to observe that in this specific scenario, Gradient statistic coincides with Rao statistic in Eq. 89).

LH statistic: In this specific instance, LH statistic in Eq. (65) specializes into:

$$
\begin{gather*}
t_{\mathrm{lh}}=\operatorname{Tr}\left[\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right) \times\right.  \tag{92}\\
\left.\left(\boldsymbol{I}_{M}+\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)\right)^{-1}\right]=\operatorname{Tr}\left[\boldsymbol{Z}_{M}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{Z}_{M}\right]
\end{gather*}
$$

since $\boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp}=\mathbf{0}_{N \times N}\left(\right.$ resp. $\left.\boldsymbol{P}_{\boldsymbol{A}_{1}}=\boldsymbol{I}_{N}\right)$ for multidimensional signal setup. From inspection of the last line, it is apparent that LH statistic coincides with Wald/2S-GLR statistic in Eq. 90) for this specific scenario.

## D. Range-spread Targets

In the present case we start from general formulation in Eq. (1) and assume that: $(i) t=0$ (i.e., there is no interference, thus $J=r$ ); (ii) $r=1$, thus the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ collapse to $\boldsymbol{a} \triangleq\left[\begin{array}{cccc}1 & 0 & \cdots & 0\end{array}\right]^{T} \in \mathbb{C}^{N \times 1}$ and $\boldsymbol{b} \in \mathbb{C}^{1 \times M}$ (i.e., a row vector), respectively; (iii) $\boldsymbol{C} \triangleq\left[\begin{array}{ll}\boldsymbol{I}_{M} & \mathbf{0}_{M \times K-M}\end{array}\right]$. Such case has been dealt in [4], [24], [25]. Therefore, the hypothesis testing in canonical form is given by:

$$
\left\{\begin{array}{ll}
\mathcal{H}_{0}: & Z=\boldsymbol{N}  \tag{93}\\
\mathcal{H}_{1}: & Z=a b C+N
\end{array} .\right.
$$

Additionally, $(K-M)$ vector components are assumed signalfree, that is, $\boldsymbol{Z}$ admits the partitioning $\boldsymbol{Z}=\left[\begin{array}{ll}\boldsymbol{Z}_{e} & \boldsymbol{Z}_{s}\end{array}\right]$ where $\boldsymbol{Z}_{e} \in \mathbb{C}^{N \times M}$ comprises the cells containing the extended target and $\boldsymbol{Z}_{s} \in \mathbb{C}^{N \times(K-M)}$ collects the secondary (training) data. In the latter case, the following simplified projector form holds:

$$
\boldsymbol{P}_{\boldsymbol{C}^{\dagger}}=\left[\begin{array}{cc}
\boldsymbol{I}_{M} & \mathbf{0}_{M \times(K-M)}  \tag{94}\\
\mathbf{0}_{(K-M) \times M} & \mathbf{0}_{(K-M) \times M}
\end{array}\right]
$$

Based on the structure of Eq. [94), it follows that $\boldsymbol{S}_{c}=\boldsymbol{Z}_{s} \boldsymbol{Z}_{s}^{\dagger}$ and $\widehat{\boldsymbol{R}}_{0}=\frac{1}{K} \boldsymbol{S}_{0}$, where $\boldsymbol{S}_{0} \triangleq\left(\boldsymbol{Z}_{e} \boldsymbol{Z}_{e}+\boldsymbol{Z}_{s} \boldsymbol{Z}_{s}^{\dagger}\right)$. Moreover, it can be shown that $\boldsymbol{P}_{\boldsymbol{a}_{1}}$ and $\boldsymbol{P}_{\overline{\boldsymbol{a}}_{1}}$ (where we have analogously defined $\boldsymbol{a}_{1} \triangleq\left(\boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{a}\right)$ and $\left.\overline{\boldsymbol{a}}_{1} \triangleq\left(\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{a}\right)\right)$ assumes the following simplified expression:

$$
\begin{equation*}
\boldsymbol{P}_{\boldsymbol{a}_{1}}=\frac{\boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{a} \boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2}}{\boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}} ; \quad \boldsymbol{P}_{\overline{\boldsymbol{a}}_{1}}=\frac{\boldsymbol{S}_{0}^{-1 / 2} \boldsymbol{a} \boldsymbol{a}^{\dagger} \boldsymbol{S}_{0}^{-1 / 2}}{\boldsymbol{a}^{\dagger} \boldsymbol{S}_{0}^{-1} \boldsymbol{a}} \tag{95}
\end{equation*}
$$

In some cases, we will use the Woodbury identity applied to $\boldsymbol{S}_{0}^{-1}$, that is:

$$
\begin{equation*}
\boldsymbol{S}_{0}^{-1}=\boldsymbol{S}_{c}^{-1}-\boldsymbol{S}_{c}^{-1} \boldsymbol{Z}_{e}\left(\boldsymbol{I}_{M}+\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{Z}_{e}\right)^{-1} \boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \tag{96}
\end{equation*}
$$

$G L R$ : Aiming at particularizing the expression of the GLR for the present case, we follow the same derivation as in [15] to show that Eq. 20) can be specialized exploiting the equalities

$$
\begin{align*}
\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{a}_{1}}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right) & =\frac{\left(\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)\left(\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)^{\dagger}}{\left(\boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)}  \tag{97}\\
\boldsymbol{D}_{0} & =\boldsymbol{I}_{M}+\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{Z}_{e} \tag{98}
\end{align*}
$$

thus obtaining $t_{\mathrm{glr}}=\left[1 /\left(1-\eta^{\prime}\right)\right]$, where:

$$
\begin{equation*}
\eta^{\prime} \triangleq \frac{\left(\boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{Z}_{e}\right)\left[\boldsymbol{I}_{M}+\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{Z}_{e}\right]^{-1}\left(\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)}{\left(\boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)} \tag{99}
\end{equation*}
$$

The result in Eq. (99) is obtained after substitution of Eqs. (97) and (98) into Eq. (20) and exploiting Sylvester's determinant theorem. Such GLR form ${ }^{11}$ (as $t_{\mathrm{glr}}$ is a monotone function of $\eta^{\prime}$ ) corresponds to that found in [15].

Rao/Durbin statistic: In the present case Eq. (36) specializes into:

$$
\begin{gather*}
t_{\mathrm{rao}}=\operatorname{Tr}\left[\boldsymbol{Z}_{W 0}^{\dagger} \boldsymbol{P}_{\overline{\boldsymbol{a}}_{1}} \boldsymbol{Z}_{W 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right] \\
=\operatorname{Tr}\left[\left(\boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right)^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{P}_{\overline{\boldsymbol{a}}_{1}} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}\left(\boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right)\right] \\
=\operatorname{Tr}\left[\boldsymbol{Z}_{e}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{P}_{\overline{\boldsymbol{a}}_{1}} \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{Z}_{e}\right] \\
=\frac{K \operatorname{Tr}\left[\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{0}^{-1} \boldsymbol{a} \boldsymbol{a}^{\dagger} \boldsymbol{S}_{0}^{-1} \boldsymbol{Z}_{e}\right]}{\left(\boldsymbol{a}^{\dagger} \boldsymbol{S}_{0}^{-1} \boldsymbol{a}\right)}=K \frac{\left\|\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{0}^{-1} \boldsymbol{a}\right\|^{2}}{\left(\boldsymbol{a}^{\dagger} \boldsymbol{S}_{0}^{-1} \boldsymbol{a}\right)} \tag{100}
\end{gather*}
$$

Eq. (100) is recognized as the result found in [24].
Wald/2S-GLR statistic: Starting from Eq. (44), we particularize the test as follows:

$$
\begin{gather*}
t_{\mathrm{wald}}=\operatorname{Tr}\left[\boldsymbol{Z}_{W 1}^{\dagger} \boldsymbol{P}_{\boldsymbol{a}_{1}} \boldsymbol{Z}_{W 1} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right] \\
=\operatorname{Tr}\left[\left(\boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right)^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{a}_{1}} \boldsymbol{S}_{c}^{-1 / 2}\left(\boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right)\right] \\
=\operatorname{Tr}\left[\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{a}_{1}} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{Z}_{e}\right] \\
=\frac{\operatorname{Tr}\left[\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a} \boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{Z}_{e}\right]}{\left(\boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)}=\frac{\left\|\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right\|^{2}}{\left(\boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)} \tag{101}
\end{gather*}
$$

which agrees with the result in [24] and can be shown to coincide with the generalized AMF proposed in [4], thus extending the theoretical findings in [11].

[^9]Gradient statistic: In this case Eq. (50) reduces to:

$$
\begin{align*}
t_{\mathrm{grad}}= & \Re\left\{\operatorname{Tr}\left[\boldsymbol{Z}_{W 1}^{\dagger} \boldsymbol{P}_{\boldsymbol{a}_{1}}\left(\boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}\right) \boldsymbol{Z}_{W 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]\right\} \\
& =K \frac{\Re\left\{\operatorname{Tr}\left[\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a} \boldsymbol{a}^{\dagger} \boldsymbol{S}_{0}^{-1} \boldsymbol{Z}_{e}\right]\right\}}{\left(\boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)} \\
& =K \frac{\Re\left\{\left[\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right]^{\dagger}\left[\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{0}^{-1} \boldsymbol{a}\right]\right\}}{\left(\boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)} \tag{102}
\end{align*}
$$

where we have used $\Re\left\{\operatorname{Tr}\left[\boldsymbol{m} \boldsymbol{n}^{\dagger}\right]\right\}=\Re\left[\boldsymbol{m}^{\dagger} \boldsymbol{n}\right]$, with $\boldsymbol{m}$ and $\boldsymbol{n}$ being two column vectors of proper size. Furthermore, by exploiting Eq. 96, the following equality holds

$$
\begin{equation*}
\left(\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{0}^{-1} \boldsymbol{a}\right)=\left[\boldsymbol{I}_{M}+\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{Z}_{e}\right]^{-1}\left(\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right) \tag{103}
\end{equation*}
$$

which, substituted into Eq. (102), gives:

$$
\begin{equation*}
t_{\mathrm{grad}}=K \frac{\left[\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right]^{\dagger}\left[\boldsymbol{I}_{M}+\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{Z}_{e}\right]^{-1}\left[\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right]}{\left(\boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)} \tag{104}
\end{equation*}
$$

where we have omitted $\Re\{\cdot\}$ since Eq. (104) is an Hermitian quadratic form (i.e., it is always real-valued). Therefore, the Gradient statistic is statistically equivalent to the GLR in Eq. (99).

LH statistic: In this case the general LH statistic form in Eq. (65) specializes into:

$$
\begin{align*}
t_{\mathrm{lh}} & =\operatorname{Tr}\left[\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{a}_{1}}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right) \boldsymbol{D}_{1}^{-1}\right] \\
& =\operatorname{Tr}\left[\frac{\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a} \boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{Z}_{e}}{\left(\boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)} \boldsymbol{D}_{1}^{-1}\right] \tag{105}
\end{align*}
$$

where $\boldsymbol{D}_{1}=\boldsymbol{I}_{M}+\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{a}_{1}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)$ in this specific case. Matrix $D_{1}$ can be further rewritten as:

$$
\begin{equation*}
\boldsymbol{D}_{1}=\left(\boldsymbol{I}_{M}+\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{Z}_{e}\right)-\frac{\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a} \boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{Z}_{e}}{\left(\boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)} \tag{106}
\end{equation*}
$$

Applying the Woodbury identity on $\boldsymbol{D}_{1}^{-1}$, we obtain:

$$
\begin{align*}
\boldsymbol{D}_{1}^{-1}= & \left\{\boldsymbol{D}_{0}^{-1}+\right. \\
& \left.\frac{\boldsymbol{D}_{0}^{-1}\left(\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)\left(\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)^{\dagger} \boldsymbol{D}_{0}^{-1}}{\left(\boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)\left[1-\frac{\left(\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)^{\dagger} \boldsymbol{D}_{0}^{-1}\left(\boldsymbol{Z}_{e}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}\right)}{\boldsymbol{a}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{a}}\right]}\right\} \tag{107}
\end{align*}
$$

where we exploited the definition of $\boldsymbol{D}_{0}$ in Eq. (98). Thus, after substitution into Eq. (105), we obtain

$$
\begin{equation*}
t_{\mathrm{lh}}=\eta^{\prime}+\frac{\left(\eta^{\prime}\right)^{2}}{1-\eta^{\prime}}=\frac{\eta^{\prime}}{1-\eta^{\prime}} \propto \eta^{\prime} \tag{108}
\end{equation*}
$$

with $\eta^{\prime}$ given by Eq. (99). Thus LH statistic is statistically equivalent to the $G L R$ for range-spread targets.

## E. Standard GMANOVA

In the present case no interference is present $(t=0$, thus $J=r$ ). This reduces to the standard adaptive detection problem via the GMANOVA model considered in [15], [20], [21] and whose canonical form is:

$$
\begin{cases}\mathcal{H}_{0}: & \boldsymbol{Z}=\boldsymbol{N}  \tag{109}\\ \mathcal{H}_{1}: & \boldsymbol{Z}=\boldsymbol{A B C}+\boldsymbol{N}\end{cases}
$$

Clearly, under the above assumptions, it holds $\boldsymbol{P}_{\boldsymbol{A}_{0}}=\boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}=$ $\mathbf{0}_{N \times N}$. Therefore, the ML covariance estimate under $\mathcal{H}_{0}$ simplifies into $\widehat{\boldsymbol{R}}_{0}=K^{-1} \boldsymbol{S}_{0}$, where $\boldsymbol{S}_{0} \triangleq \boldsymbol{Z} \boldsymbol{Z}^{\dagger}$ (cf. Eq. (23)).
$G L R$ : Direct specialization of Eq. 18) gives the explicit statistic:

$$
\begin{equation*}
t_{\mathrm{glr}}=\frac{\operatorname{det}\left[\boldsymbol{I}_{M}+\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)\right]}{\operatorname{det}\left[\boldsymbol{I}_{M}+\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)\right]} \tag{110}
\end{equation*}
$$

which coincides with the classical expression ${ }^{12}$ of GLR obtained in [15].

Rao/Durbin statistic: Direct particularization of Eq. 36) gives:

$$
\begin{equation*}
t_{\mathrm{rao}}=\operatorname{Tr}\left[\boldsymbol{Z}_{W 0}^{\dagger} \boldsymbol{P}_{\overline{\boldsymbol{A}}_{1}} \boldsymbol{Z}_{W 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right] \tag{111}
\end{equation*}
$$

which provides the result obtained in [21].
Wald/2S-GLR statistic: Direct specialization of Eq. (44) gives leads to:

$$
\begin{equation*}
t_{\mathrm{wald}}=\operatorname{Tr}\left[\boldsymbol{Z}_{W 1}^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}} \boldsymbol{Z}_{W 1} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right] \tag{112}
\end{equation*}
$$

which is the same result obtained in [21].
Gradient statistic: Direct application of Eq. 50) provides:

$$
\begin{equation*}
t_{\mathrm{grad}}=\Re\left\{\operatorname{Tr}\left[\boldsymbol{Z}_{W 1}^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}\left(\boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}\right) \boldsymbol{Z}_{W 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]\right\} \tag{113}
\end{equation*}
$$

LH statistic: In this case the LH statistic specializes into:

$$
\begin{equation*}
t_{\mathrm{lh}}=\operatorname{Tr}\left[\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right) \boldsymbol{D}_{1}^{-1}\right] \tag{114}
\end{equation*}
$$

where $D_{1}$ is defined as in Sec. III-A Eq. (114) clearly coincides with the statistic obtained in [15, pag. 37].

## V. Simulation results

In Fig. 1 we report $P_{d}$ vs. the SINR for all the considered detectors, defined as $\rho \triangleq \operatorname{Tr}\left[\boldsymbol{B}^{\dagger} \boldsymbol{R}_{2.3}^{-1} \boldsymbol{B}\right]$, that is, the trace of the induced maximal invariant (cf. Sec. 【II-A). We underline that such term is also proportional to the non-centrality parameter $\lambda \triangleq\left(\boldsymbol{\theta}_{r, 1}-\boldsymbol{\theta}_{r, 0}\right)^{T}\left\{\left[\boldsymbol{I}^{-1}\left(\boldsymbol{\theta}_{0}\right)\right]_{\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}}\right\}^{-1}\left(\boldsymbol{\theta}_{r, 1}-\boldsymbol{\theta}_{r, 0}\right)$, representing the synthetic parameter on which the asymptotic performances of all the considered test depend [9]. The curves have been obtained via standard Monte Carlo counting techniques. More specifically, the thresholds necessary to ensure a preassigned value of $P_{f a}$ have been evaluated exploiting $100 / P_{f a}$ independent trials, while the $P_{d}$ values are estimated over $5 \cdot 10^{3}$ independent trials. As to the disturbance, it is modeled as an exponentially-correlated Gaussian vector with covariance matrix (in canonical form) $\boldsymbol{R}=\sigma_{n}^{2} \boldsymbol{I}_{N}+\sigma_{c}^{2} \boldsymbol{M}_{c}$, where $\sigma_{n}^{2}>0$ is the thermal noise power, $\sigma_{c}^{2}>0$ is the clutter power, and the $(i, j)$-th element of $\boldsymbol{R}_{c}$ is given by $0.95^{|i-j|}$. The clutter-to-noise ratio $\sigma_{c}^{2} / \sigma_{n}^{2}$ is set here to 30 dB , with $\sigma_{n}^{2}=1$. We point out that the specific value of the deterministic interference $\boldsymbol{B}_{t}$ does not need to be specified at each trial considered (for both $P_{f a}$ and $P_{d}$ evaluation); the reason is that the performance of each detector depends on the unknown parameters solely through the induced maximal

[^10]invariant, which is independent on $\boldsymbol{B}_{t}$ (cf. Sec. II-A). Finally, all the numerical examples assume $P_{f a}=10^{-4}$.

In order to average the performance of $P_{d}$ with respect to $\boldsymbol{B}$, for each independent trial we generate the signal matrix as $\boldsymbol{B}=\alpha_{B} \boldsymbol{B}_{g}$, where $\boldsymbol{B}_{g} \sim \mathcal{C N}\left(\mathbf{0}_{r \times M}, \boldsymbol{I}_{M}, \boldsymbol{I}_{r}\right)$ and $\alpha_{B} \in \mathbb{R}$. The latter coefficient is a scaling factor used to achieve the desired SINR value, that is, $\alpha_{b} \triangleq \sqrt{\rho / \operatorname{Tr}\left[\boldsymbol{B}_{g}^{\dagger} \boldsymbol{R}_{2.3}^{-1} \boldsymbol{B}_{g}\right]}$.

For our simulation $\sqrt{13}$ we assume $M=3$ (i.e., an extended target), $N=8$, and two different scenarios of signal and interference lying in a vector subspace, that is (i) $r=2$ and $t=4$ (sub-plots $(a)$ and $(b))$ and $(i i) r=4$ and $t=2$ (sub-plots $(c)$ and $(d)$ ). Additionally, for each of these setups, the cases corresponding to $K=12$ and $K=19$ columns for $\boldsymbol{Z}$ have been considered, representing two extreme casestudies. Indeed, the first case clearly corresponds to a so-called sample-starved scenario (i.e. the number of signal-free data required to achieve a consistent (invertible) estimate of $\boldsymbol{R}$ is just satisfied, that is, $(K-M)=9)$ while the second case to a setup where an adequate number of samples needed to obtain an accurate estimate for $\boldsymbol{R}$ is provided (i.e., in this case $(K-M)=2 N=16$, with a consequent loss of 3 dB in estimating $\boldsymbol{R}$ with the sample covariance approach, with respect to the known covariance case, as dictated from [33]).

The following observations can be made from inspection of the results. First, as $K$ grows large, all the considered detectors converge to the same performance, corresponding to the non-adaptive case. Differently, in the sample-starved case (viz. the difference $K-M$ is close to $N$ ) a significant difference in detection performance can be observed among them. First of all, the GLR has the best performance in the medium-high SNR range. Differently, the Rao and Gradient tests perform significantly better than Wald and LH tests for a moderate number of $K$ (i.e., $K=12$ ) in the case $r>t$ (cf. sub-plot $(c)$, corresponding to $r=4$ and $t=2$ ). On the other hand, for the same case $K=12$, but $r=2$ and $t=4$, Wald and LH tests outperform Rao and Gradient tests when $\rho$ is higher than $\approx 18 \mathrm{~dB}$. This is easily explained since both Wald (viz. 2S-GLR) and LH test both rely on an accurate estimate of true covariance $\boldsymbol{R}$ based on the sole signal-free data (cf. Secs. III-C III-F and III-G respectively). Differently, both Gradient and Rao tests employ a covariance estimate under the hypothesis $\mathcal{H}_{0}$ (that is, $\widehat{\boldsymbol{R}}_{0}$ ). The latter covariance estimate also relies on the use of the additional contributions of $\boldsymbol{Z}$ corrupted by the signal $\boldsymbol{B}$. Although using them to evaluate $\widehat{\boldsymbol{R}}_{0}$ may be detrimental when the number of signal-free samples is adequate or the SINR is high (cf. sub-plots $(a)$ and (b)), when the SINR is low (i.e., the energy spread among the different columns is not so high) and the number of signal-free samples is not sufficient to guarantee

[^11]

Figure 1. $\quad P_{d}$ vs. $\rho$ for all the considered detectors; common parameters: $M=3$ and $N=8$. Case ( $a$ ) (top-left) $r=2$, $t=4$ and $K=12$; Case (b) (top-right) $r=2, t=4$ and $K=19$; Case ( $c$ ) (bottom-left) $r=4, t=2$ and $K=12$; Case ( $d$ ) (bottom-right) $r=4, t=2$ and $K=19$.
a reliable estimate of $\boldsymbol{R}$, the degradation of using signalcorrupted terms is overcome by the (beneficial) availability of additional samples for covariance estimation.

## VI. Conclusions

In this second part of this work, we have derived several detectors for adaptive detection in a GMANOVA signal model with structured interference (viz. I-GMANOVA). We derived the GLR, Rao, Wald, 2S-GLR, Durbin, Gradient, and LH statistics. All the aforementioned statistics have been shown to be CFAR with respect to the nuisance parameters, by proving that all can be written in terms of the MIS (obtained in the first part of this work). For the considered general model, we also established statistical equivalence between: (i) Wald and 2S-GLR statistics and (ii) Durbin and Rao statistics.

Furthermore, the following statistical-equivalence results have been proved in the following special setups:

- For point-like targets (with possible point-like interference), we have shown that Gradient and LH tests are statistically equivalent to Kelly's GLRT;
- For multidimensional signals, we have shown that: (a) Rao test is statistically equivalent to Gradient test and (b) Wald test (2S-GLRT) is statistically equivalent to LH test;
- For range-spread targets and rank-one subspace $(r=1)$, we have shown that Gradient and LH tests are statistically equivalent to the GLRT.
Finally, simulation results were provided to compare the performance of the aforementioned detectors.


## VII. Supplementary Material Organization

The following additional sections contain supplemental material for part II of this work. More specifically, Sec. VIII contains the proof of Lem. 1 in the paper, while Secs. IX, X and XI provide the derivation of Rao, Wald, and Gradient (Terrell) tests, respectively. Furthermore, Sec. XIIprovides the statistical equivalence between Rao and Durbin tests (Thm. 2). Additionally, Sec. XIII provides a series of useful equalities for showing the CFARness of all the considered detectors. Finally, Sec. XIV provides some numerical results aimedat confirming the special equivalence results obtained in the manuscript.

## VIII. Proof of Lemma 1

We only provide the proof for Eq. (24), as the equality for $\widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{E}_{t}$ in Eq. (25) can be obtained following similar steps. We first rewrite Eq. (22) as:

$$
\begin{equation*}
\widehat{\boldsymbol{R}}_{1}=K^{-1}\left[\boldsymbol{S}_{c}+\boldsymbol{S}_{c}^{1 / 2}\left(\boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp} \boldsymbol{Z}_{W 1}\right) \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\left(\boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp} \boldsymbol{Z}_{W 1}\right)^{\dagger} \boldsymbol{S}_{c}^{1 / 2}\right] \tag{115}
\end{equation*}
$$

Taking the inverse and exploiting Woodbury identity [26] gives:

$$
\begin{align*}
& \widehat{\boldsymbol{R}}_{1}^{-1}=K {\left[\boldsymbol{S}_{c}^{-1}-\boldsymbol{S}_{c}^{-1 / 2}\left(\boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp} \boldsymbol{Z}_{W 1}\right) \boldsymbol{V}_{c, 1}\right.} \\
& \times\left\{\boldsymbol{I}_{M}+\right.\left.\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)\right\}^{-1} \\
&\left.\times \boldsymbol{V}_{c, 1}^{\dagger} \boldsymbol{Z}_{W 1}^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2}\right] \tag{116}
\end{align*}
$$

It is apparent that the second term in Eq. (116) is null when post-multiplied by $\boldsymbol{A}$, since $\left(\boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{A}\right)=\boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp} \boldsymbol{A}_{1}=$ $\mathbf{0}_{N \times J}$, thus leading to the claimed result.

## IX. Derivation of Rao statistic

In this appendix we report the derivation for Rao statistic in Eq. (31). Before proceeding, we define the auxiliary notation $\boldsymbol{b}_{s, R} \triangleq\left[\begin{array}{ll}\boldsymbol{b}_{R}^{T} & \boldsymbol{b}_{t, R}^{T}\end{array}\right]^{T}$ and $\boldsymbol{b}_{s, I} \triangleq\left[\begin{array}{ll}\boldsymbol{b}_{I}^{T} & \boldsymbol{b}_{t, I}^{T}\end{array}\right]^{T}$. First, it can be shown that:

$$
\begin{align*}
& \frac{\partial \ln f_{1}\left(\boldsymbol{Z} ; \boldsymbol{B}_{s}, \boldsymbol{R}\right)}{\partial \boldsymbol{b}_{s, R}}=\left[\begin{array}{l}
2 \Re\left\{\boldsymbol{g}_{A}\right\} \\
2 \Re\left\{\boldsymbol{g}_{B}\right\}
\end{array}\right] \in \mathbb{R}^{J M \times 1}  \tag{117}\\
& \frac{\partial \ln f_{1}\left(\boldsymbol{Z} ; \boldsymbol{B}_{s}, \boldsymbol{R}\right)}{\partial \boldsymbol{b}_{s, I}}=\left[\begin{array}{l}
2 \Im\left\{\boldsymbol{g}_{A}\right\} \\
2 \Im\left\{\boldsymbol{g}_{B}\right\}
\end{array}\right] \in \mathbb{R}^{J M \times 1} \tag{118}
\end{align*}
$$

where:

$$
\begin{align*}
& \boldsymbol{g}_{A} \triangleq \operatorname{vec}\left(\boldsymbol{E}_{r}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{Z}_{d} \boldsymbol{C}^{\dagger}\right) \in \mathbb{C}^{r M \times 1}  \tag{119}\\
& \boldsymbol{g}_{B} \triangleq \operatorname{vec}\left(\boldsymbol{E}_{t}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{Z}_{d} \boldsymbol{C}^{\dagger}\right) \in \mathbb{C}^{t M \times 1} \tag{120}
\end{align*}
$$

In Eqs. 119 and 120, we have adopted the simplified notation $\boldsymbol{Z}_{d} \triangleq\left(\boldsymbol{Z}-\boldsymbol{A} \boldsymbol{B}_{s} \boldsymbol{C}\right)$. The results in Eqs. 117) and (118) are obtained by exploiting the following steps:

1) Evaluate the complex derivatives $\frac{\partial \ln f_{1}\left(\boldsymbol{Z} ; \boldsymbol{B}_{s}, \boldsymbol{R}\right)}{\partial \boldsymbol{b}}$ and $\frac{\partial \ln f_{1}\left(\boldsymbol{Z} ; \boldsymbol{B}_{s}, \boldsymbol{R}\right)}{\partial \boldsymbol{b}_{t}}$ (as well as $\frac{\partial \ln f_{1}\left(\boldsymbol{Z} ; \boldsymbol{B}_{s}, \boldsymbol{R}\right)}{\partial \boldsymbol{b}^{*}}$ and $\left.\frac{\partial \ln f_{1}\left(\boldsymbol{Z}_{;} ; \boldsymbol{B}_{s}, \boldsymbol{R}\right)}{\partial \boldsymbol{b}_{t}^{*}}\right)$ by standard complex ${ }^{\partial \boldsymbol{b}^{*}}$ differentiation rules [34];
2) Exploit that for any $f(\boldsymbol{x}): \mathbb{C}^{p \times 1} \rightarrow \mathbb{R}$, it holds $\frac{\partial f(\boldsymbol{x})}{\partial \Re\{\boldsymbol{x}\}}=2 \Re\left\{\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}^{*}}\right\}$ and $\frac{\partial f(\boldsymbol{x})}{\partial \Im\{\boldsymbol{x}\}}=2 \Im\left\{\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}^{*}}\right\}$ (see e.g., [35]);
3) Obtain $\frac{\partial \ln f_{1}\left(\boldsymbol{Z} ; \boldsymbol{B}_{s}, \boldsymbol{R}\right)}{\partial \boldsymbol{b}_{s, R}}$ and $\frac{\partial \ln f_{1}\left(\boldsymbol{Z} ; \boldsymbol{B}_{s}, \boldsymbol{R}\right)}{\partial \boldsymbol{b}_{s, I}}$ as composition of the gradients obtained at step 2).
By identical steps, it can be also proved that:

$$
\frac{\partial \ln f_{1}\left(\boldsymbol{Z} ; \boldsymbol{B}_{s}, \boldsymbol{R}\right)}{\partial \boldsymbol{\theta}_{r}}=\frac{\partial \ln f_{1}(\boldsymbol{Z} ; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}_{r}}=\left[\begin{array}{l}
2 \Re\left\{\boldsymbol{g}_{A}\right\}  \tag{121}\\
2 \Im\left\{\boldsymbol{g}_{A}\right\}
\end{array}\right],
$$

which gives the explicit expression for the gradient of the loglikelihood required for evaluation of Rao statistic (cf. Eq. (29)). On the other hand, the block $\left(\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}\right)$ of the inverse of the FIM is evaluated as follows. First, we notice that [35]:

$$
\begin{equation*}
\left[\boldsymbol{I}^{-1}(\boldsymbol{\theta})\right]_{\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}}=\left[\boldsymbol{I}_{a}^{-1}(\boldsymbol{\theta})\right]_{\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}} \tag{122}
\end{equation*}
$$

where $\left[\boldsymbol{I}_{a}(\boldsymbol{\theta})\right]$ is here used to denote the block of the FIM comprising only the contributions related to $\left(\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{s, a}\right)$. The aforementioned property follows from the cross-terms $\left(\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{s, b}\right)$ and $\left(\boldsymbol{\theta}_{s, a}, \boldsymbol{\theta}_{s, b}\right)$ being null in the (overall) FIM $\boldsymbol{I}(\boldsymbol{\theta})$. Additionally, the following equality holds (recalling that $\left[\begin{array}{ll}\boldsymbol{\theta}_{r}^{T} & \boldsymbol{\theta}_{s, a}^{T}\end{array}\right]=\left[\begin{array}{llll}\boldsymbol{b}_{R}^{T} & \boldsymbol{b}_{I}^{T} & \boldsymbol{b}_{t, R}^{T} & \boldsymbol{b}_{t, I}^{T}\end{array}\right]$ and exploiting Eqs. (117) and (118)):

$$
\frac{\partial \ln f_{1}\left(\boldsymbol{Z} ; \boldsymbol{B}_{s}, \boldsymbol{R}\right)}{\partial\left[\begin{array}{c}
\boldsymbol{\theta}_{r}  \tag{123}\\
\boldsymbol{\theta}_{s, a}
\end{array}\right]}=\boldsymbol{P}\left[\begin{array}{l}
\frac{\partial \ln f_{1}\left(\boldsymbol{Z} ; \boldsymbol{B}_{s}, \boldsymbol{R}\right)}{\partial \boldsymbol{b}_{s}} \\
\frac{\partial \ln f_{1}\left(\boldsymbol{Z} ; \boldsymbol{B}_{s}, \boldsymbol{R}\right)}{} \\
\partial \boldsymbol{b}_{s, I}
\end{array}\right]
$$

where $\boldsymbol{P} \in \mathbb{R}^{2 J M \times 2 J M}$ is a suitable permutation matrix ${ }^{14}$, defined as

$$
\boldsymbol{P} \triangleq\left[\begin{array}{cccc}
\boldsymbol{I}_{r M} & \mathbf{0}_{r M \times t M} & \mathbf{0}_{r M \times r M} & \mathbf{0}_{r M \times t M}  \tag{124}\\
\mathbf{0}_{r M \times r M} & \mathbf{0}_{r M \times t M} & \boldsymbol{I}_{r M} & \mathbf{0}_{r M \times t M} \\
\mathbf{0}_{t M \times r M} & \boldsymbol{I}_{t M} & \mathbf{0}_{t M \times r M} & \mathbf{0}_{t M \times t M} \\
\mathbf{0}_{t M \times r M} & \mathbf{0}_{t M \times t M} & \mathbf{0}_{t M \times r M} & \boldsymbol{I}_{t M}
\end{array}\right]
$$

Before proceeding, we define the matrix $\boldsymbol{\Omega} \triangleq\left(\boldsymbol{A}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{A}\right)$ and the partitioning:

$$
\boldsymbol{\Omega}=\left[\begin{array}{ll}
\boldsymbol{\Omega}_{11} & \boldsymbol{\Omega}_{12}  \tag{125}\\
\boldsymbol{\Omega}_{21} & \boldsymbol{\Omega}_{22}
\end{array}\right]=\left[\begin{array}{ll}
\boldsymbol{E}_{t}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{E}_{t} & \boldsymbol{E}_{t}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{E}_{r} \\
\boldsymbol{E}_{r}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{E}_{t} & \boldsymbol{E}_{r}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{E}_{r}
\end{array}\right]
$$

where, $\boldsymbol{\Omega}_{i j},(i, j) \in\{1,2\} \times\{1,2\}$, is a sub-matrix whose dimensions can be obtained replacing 1 and 2 with $t$ and $r$, respectively. Then, the sub-FIM $\boldsymbol{I}_{a}(\boldsymbol{\theta})$ is obtained starting from Eq. (123) as $\boldsymbol{I}_{a}(\boldsymbol{\theta})=\left(\boldsymbol{P} \boldsymbol{\Psi} \boldsymbol{P}^{T}\right)$, where $\boldsymbol{\Psi}$ has the following special structure:

$$
\boldsymbol{\Psi} \triangleq\left[\begin{array}{cc}
2 \Re\{\boldsymbol{K}\} & -2 \Im\{\boldsymbol{K}\}  \tag{126}\\
2 \Im\{\boldsymbol{K}\} & 2 \Re\{\boldsymbol{K}\}
\end{array}\right]
$$

and the matrix $K \in \mathbb{C}^{J M \times J M}$ is defined as:

$$
\boldsymbol{K} \triangleq\left[\begin{array}{ll}
\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{T} \otimes \boldsymbol{\Omega}_{22} & \left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{T} \otimes \boldsymbol{\Omega}_{21}  \tag{127}\\
\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{T} \otimes \boldsymbol{\Omega}_{12} & \left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{T} \otimes \boldsymbol{\Omega}_{11}
\end{array}\right]
$$

Finally, the inverse $\boldsymbol{I}_{a}^{-1}(\boldsymbol{\theta})$ is obtained as $\boldsymbol{I}_{a}^{-1}(\boldsymbol{\theta})=$ $\left(\boldsymbol{P} \boldsymbol{\Psi}^{-1} \boldsymbol{P}^{T}\right)$ (as $\boldsymbol{P}$ is orthogonal). In the latter case, the inverse matrix $\Psi^{-1}$ has the same structure as $\Psi$ (cf. Eq. 126) except for $\boldsymbol{K}$ and the factor 2 replaced by $\boldsymbol{K}^{-1}$ and $\frac{1}{2}$, respectively. By exploiting the following block structure of $\boldsymbol{K}^{-1}$

$$
\boldsymbol{K}^{-1}=\left[\begin{array}{ll}
\boldsymbol{K}^{11} & \boldsymbol{K}^{12}  \tag{128}\\
\boldsymbol{K}^{21} & \boldsymbol{K}^{22}
\end{array}\right]
$$

with $\boldsymbol{K}^{11} \in \mathbb{C}^{r M \times r M}, \boldsymbol{K}^{12} \in \mathbb{C}^{r M \times t M}, \boldsymbol{K}^{21} \in \mathbb{C}^{t M \times r M}$ and $\boldsymbol{K}^{22} \in \mathbb{C}^{t M \times t M}$, respectively, and the structure of $\boldsymbol{P}$, it can be shown that:

$$
\left[\boldsymbol{I}^{-1}(\boldsymbol{\theta})\right]_{\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}}=\left[\begin{array}{cc}
\frac{1}{2} \Re\left\{\boldsymbol{K}^{11}\right\} & -\frac{1}{2} \Im\left\{\boldsymbol{K}^{11}\right\}  \tag{129}\\
\frac{1}{2} \Im\left\{\boldsymbol{K}^{11}\right\} & \frac{1}{2} \Re\left\{\boldsymbol{K}^{11}\right\}
\end{array}\right]
$$

Similarly, it is not difficult to show that $\boldsymbol{K}^{11}$ is given in closedform as:

$$
\begin{align*}
\boldsymbol{K}^{11}= & \left\{\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{T} \otimes \boldsymbol{\Omega}_{22}-\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{T} \otimes \boldsymbol{\Omega}_{21}\right. \\
& \left.\times\left[\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{T} \otimes \boldsymbol{\Omega}_{11}\right]^{-1}\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{T} \otimes \boldsymbol{\Omega}_{12}\right\}^{-1}  \tag{130}\\
= & \left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{-T} \otimes \boldsymbol{\Gamma}_{22} \tag{131}
\end{align*}
$$

where $\boldsymbol{\Gamma}_{i j}$ is a sub-matrix obtained from $\boldsymbol{\Gamma} \triangleq \boldsymbol{\Omega}^{-1}$ exploiting identical partitioning (in terms of size) as done in Eq. (125) for $\boldsymbol{\Omega}$. The compact expression in Eq. (131) is obtained from the use of mixed product and associative properties of Kronecker

[^12]operato ${ }^{16}$. Then, combining Eqs. (121), (129) and (131) leads to:
\[

$$
\begin{gather*}
\left\{\frac{\partial \ln f_{1}(\boldsymbol{Z} ; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}_{r}^{T}}\left[\boldsymbol{I}^{-1}(\boldsymbol{\theta})\right]_{\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}} \frac{\partial \ln f_{1}(\boldsymbol{Z} ; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}_{r}}\right\} \propto \\
\operatorname{vec}\left(\boldsymbol{E}_{r}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{Z}_{d} \boldsymbol{C}^{\dagger}\right)^{\dagger}\left[\left(\boldsymbol{C C}^{\dagger}\right)^{-T} \otimes \boldsymbol{\Gamma}_{22}\right] \\
\times \operatorname{vec}\left(\boldsymbol{E}_{r}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{Z}_{d} \boldsymbol{C}^{\dagger}\right)  \tag{132}\\
=\operatorname{Tr}\left[\boldsymbol{Z}_{d}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{E}_{r} \boldsymbol{\Gamma}_{22} \boldsymbol{E}_{r}^{\dagger} \boldsymbol{R}^{-1} \boldsymbol{Z}_{d} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right] \tag{133}
\end{gather*}
$$
\]

where we have exploited the well-known equivalence between a real-valued Hermitian quadratic form and its real symmetric quadratic counterpart in Eq. (132) and some standard properties of $\operatorname{vec}(\cdot)$ operato $\sqrt{17}$ in obtaining Eq. (133). Finally, the substitution $\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}_{0}$ provides:

$$
\begin{equation*}
\operatorname{Tr}\left[\boldsymbol{Z}_{d, 0}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{E}_{r} \widehat{\boldsymbol{\Gamma}}_{22}^{\circ} \boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{Z}_{d, 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right] \tag{134}
\end{equation*}
$$

where $\widehat{\boldsymbol{R}}_{0}$ and $\boldsymbol{Z}_{d, 0}$ are defined in Eqs. (23) and (33), respectively. Similarly, $\widehat{\boldsymbol{\Gamma}}_{i j}^{\circ}$ denotes a sub-matrix obtained from $\widehat{\boldsymbol{\Gamma}}_{i j}^{\circ}=\left(\boldsymbol{A}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{A}\right)^{-1}$ by exploiting identical partitioning (in terms of size) as done in Eq. 125). This provides the explicit expression for the Rao statistic.

## Proof of Eq. (34)

The mentioned result is proved as:

$$
\begin{gather*}
\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{Z}_{d, 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}} \\
=\widehat{\boldsymbol{R}}_{0}^{-1 / 2}\left(\boldsymbol{Z}-\boldsymbol{S}_{c}^{1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right) \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}  \tag{135}\\
=\widehat{\boldsymbol{R}}_{0}^{-1 / 2}\left(\boldsymbol{I}_{N}-\boldsymbol{S}_{c}^{1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}} \boldsymbol{S}_{c}^{-1 / 2}\right) \boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}} \tag{136}
\end{gather*}
$$

where we have exploited Eq. (33) and $\boldsymbol{P}_{\boldsymbol{C}^{\dagger}} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}=\boldsymbol{P}_{\boldsymbol{C}^{\dagger}}$, respectively. The above expression can be further rewritten by exploiting Eq. (25) of Lem. 1 as:

$$
\begin{gather*}
\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{Z}_{d, 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}} \\
\left.=\left(\widehat{\boldsymbol{R}}_{0}^{-1 / 2}-\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{E}_{t} \boldsymbol{E}_{t}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{E}_{t}\right)^{-1} \boldsymbol{E}_{t}^{\dagger} \boldsymbol{S}_{c}^{-1}\right) \boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}(1  \tag{137}\\
=\left(\widehat{\boldsymbol{R}}_{0}^{-1 / 2}-\boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}\right) \boldsymbol{Z} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}=\boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}^{\perp} \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{Z} \boldsymbol{P}_{C^{\dagger}} \tag{138}
\end{gather*}
$$

where we have denoted $\overline{\boldsymbol{A}}_{0} \triangleq \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{E}_{t}$, which finally provides Eq. (34) (after having defined $\boldsymbol{Z}_{W 0} \triangleq \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{Z}$ ).

## Proof of Eq. (35)

First, it can be shown that:

$$
\begin{equation*}
\widehat{\boldsymbol{\Gamma}}_{22}^{\circ} \boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{\boldsymbol { A }}_{0}}^{\perp}=\left(\widehat{\boldsymbol{\Gamma}}_{21}^{\circ} \boldsymbol{E}_{t}^{\dagger}+\widehat{\boldsymbol{\Gamma}}_{22}^{\circ} \boldsymbol{E}_{r}^{\dagger}\right) \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \tag{139}
\end{equation*}
$$

[^13]Therefore, in view of Eq. 139), it holds:

$$
\begin{gather*}
\boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{E}_{r} \widehat{\boldsymbol{\Gamma}}_{22}^{\circ} \boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}^{\perp}  \tag{140}\\
=\left[\left(\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{E}_{t}\right)\left(\widehat{\boldsymbol{\Gamma}}_{21}^{\circ}\right)^{\dagger}+\left(\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{E}_{r}\right)\left(\widehat{\boldsymbol{\Gamma}}_{22}^{\circ}\right)^{\dagger}\right]\left(\widehat{\boldsymbol{\Gamma}}_{22}^{\circ}\right)^{-1} \\
\times\left[\widehat{\boldsymbol{\Gamma}}_{21}^{\circ}\left(\boldsymbol{E}_{t}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}\right)+\widehat{\boldsymbol{\Gamma}}_{22}^{\circ}\left(\boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}\right)\right]  \tag{141}\\
=\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{A}\left(\boldsymbol{A}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \\
-\left(\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{E}_{t}\right)\left(\boldsymbol{E}_{t}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{E}_{t}\right)^{-1}\left(\boldsymbol{E}_{t}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}\right)  \tag{142}\\
=\left(\boldsymbol{P}_{\overline{\boldsymbol{A}}_{1}}-\boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}\right) \tag{143}
\end{gather*}
$$

where we have exploited the equality $\left(\boldsymbol{E}_{t}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{E}_{t}\right)^{-1}=\widehat{\boldsymbol{\Gamma}}_{11}^{\circ}-$ $\widehat{\boldsymbol{\Gamma}}_{12}^{\circ}\left(\widehat{\boldsymbol{\Gamma}}_{22}^{\circ}\right)^{-1} \widehat{\boldsymbol{\Gamma}}_{21}^{\circ}$ (which can be deduced from Eq. (125) after substitution $\boldsymbol{R}=\widehat{\boldsymbol{R}}_{0}$ and from $\widehat{\boldsymbol{\Gamma}}^{\circ}$ definition). Finally, in Eq. (143) we have further defined $\overline{\boldsymbol{A}}_{1} \triangleq\left(\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{A}\right)$.

## X. Derivation of Wald statistic

In this section we report the derivation for Wald statistic in Eq. (42). In order to prove the aforementioned result, we build upon the explicit expression of $\left[\boldsymbol{I}^{-1}(\boldsymbol{\theta})\right]_{\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}}$ obtained in Eq. 129). Such result allows to readily evaluate $\left\{\left[\boldsymbol{I}^{-1}\left(\widehat{\boldsymbol{\theta}}_{1}\right)\right]_{\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}}\right\}^{-1}$ in Eq. (41) by (i) substitution $\boldsymbol{R}=\widehat{\boldsymbol{R}}_{1}$ and (ii) matrix inversior 18 as:

$$
\left\{\left[\boldsymbol{I}^{-1}\left(\widehat{\boldsymbol{\theta}}_{1}\right)\right]_{\left.\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}\right\}}\right\}^{-1}=\left[\begin{array}{cc}
2 \Re\left\{\left(\boldsymbol{K}_{1}^{11}\right)^{-1}\right\} & -2 \Im\left\{\left(\boldsymbol{K}_{1}^{11}\right)^{-1}\right\}  \tag{144}\\
2 \Im\left\{\left(\boldsymbol{K}_{1}^{11}\right)^{-1}\right\} & 2 \Re\left\{\left(\boldsymbol{K}_{1}^{11}\right)^{-1}\right\}
\end{array}\right\},
$$

where $\left(\boldsymbol{K}_{1}^{11}\right)^{-1} \triangleq\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{T} \otimes\left(\widehat{\boldsymbol{\Gamma}}_{22}^{1}\right)^{-1}$, and $\widehat{\boldsymbol{\Gamma}}_{i j}^{1}$ is a submatrix obtained from $\widehat{\boldsymbol{\Gamma}}^{1} \triangleq\left(\boldsymbol{A}^{\dagger} \hat{\boldsymbol{R}}_{1}^{-1} \boldsymbol{A}\right)^{-1}$ exploiting identical partitioning (in terms of size) as done in Eq. (125) for $\boldsymbol{\Omega}$.
We have now to evaluate $\boldsymbol{\theta}_{r, 0}$ and $\hat{\boldsymbol{\theta}}_{r, 1}$, respectively. First, we recall that $\boldsymbol{\theta}_{r, 0}=\mathbf{0}_{2 r M}$, while $\hat{\boldsymbol{\theta}}_{r, 1}=$ $\left[\Re\{\operatorname{vec}(\widehat{\boldsymbol{B}})\}^{T} \quad \Im\{\operatorname{vec}(\widehat{\boldsymbol{B}})\}^{T}\right]^{T}$, with $\widehat{\boldsymbol{B}}$ representing the ML estimate of the complex-valued signal matrix under $\mathcal{H}_{1}$. This estimate can be shown to be equal to:

$$
\begin{equation*}
\widehat{\boldsymbol{B}}=K \widehat{\boldsymbol{\Gamma}}_{22}^{1} \boldsymbol{E}_{r}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{Z} \boldsymbol{C}^{\dagger}\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{-1} \tag{145}
\end{equation*}
$$

The above result is obtained starting from Eq. (14) and observing that $\widehat{\boldsymbol{B}}_{s}=\left[\begin{array}{cc}\widehat{\boldsymbol{B}}_{t, 1}^{T} & \widehat{\boldsymbol{B}}^{T}\end{array}\right]^{T}$. Therefore, collecting the above results, Wald statistic is obtained as

$$
\begin{gather*}
\left(\hat{\boldsymbol{\theta}}_{r, 1}-\boldsymbol{\theta}_{r, 0}\right)^{T}\left\{\left[\boldsymbol{I}^{-1}\left(\widehat{\boldsymbol{\theta}}_{1}\right)\right]_{\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}}\right\}^{-1}\left(\hat{\boldsymbol{\theta}}_{r, 1}-\boldsymbol{\theta}_{r, 0}\right) \\
=2 \operatorname{vec}(\widehat{\boldsymbol{B}})^{\dagger}\left(\boldsymbol{K}_{1}^{11}\right)^{-1} \operatorname{vec}(\widehat{\boldsymbol{B}})  \tag{146}\\
\propto \operatorname{vec}\left(\widehat{\boldsymbol{\Gamma}}_{22}^{1} \boldsymbol{E}_{r}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{Z} \boldsymbol{C}^{\dagger}\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{-1}\right)^{\dagger} \\
\times K\left\{\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{T} \otimes\left(\widehat{\boldsymbol{\Gamma}}_{22}^{1}\right)^{-1}\right\} \\
\times \operatorname{vec}\left(\widehat{\boldsymbol{\Gamma}}_{22}^{1} \boldsymbol{E}_{r}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{Z} \boldsymbol{C}^{\dagger}\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{-1}\right)  \tag{147}\\
=\operatorname{Tr}\left[\boldsymbol{Z}_{W 1}^{\dagger}\left(K \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{E}_{r} \widehat{\boldsymbol{\Gamma}}_{22}^{1} \boldsymbol{E}_{r}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp}\right) \boldsymbol{Z}_{W 1} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right] \tag{148}
\end{gather*}
$$

where we have again exploited (as in the derivation of Rao statistic) the well-known equivalence between a real-valued

[^14]Hermitian quadratic form and its real symmetric quadratic counterpart in Eq. (146) and some standard properties of $\operatorname{vec}(\cdot)$ operator in obtaining Eq. 148. This provides the closed form expression for Wald statistic.

## Proof of Eq. 43

We first notice that the following equality holds:

$$
\begin{equation*}
\widehat{\boldsymbol{\Gamma}}_{22}^{1} \boldsymbol{E}_{r}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp}=\left(\widehat{\boldsymbol{\Gamma}}_{21}^{1} \boldsymbol{E}_{t}^{\dagger}+\widehat{\boldsymbol{\Gamma}}_{22}^{1} \boldsymbol{E}_{r}^{\dagger}\right) \boldsymbol{S}_{c}^{-1 / 2} \tag{149}
\end{equation*}
$$

The above result is an almost evident consequence of the application of matrix inversion formula for a $2 \times 2$ block matrix. Therefore, in view of the above equality, we observe that:

$$
\begin{gather*}
K \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{E}_{r} \widehat{\boldsymbol{\Gamma}}_{22}^{1} \boldsymbol{E}_{r}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \\
=K\left[\left(\boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{E}_{t}\right)\left(\widehat{\boldsymbol{\Gamma}}_{21}^{1}\right)^{\dagger}+\left(\boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{E}_{r}\right)\left(\widehat{\boldsymbol{\Gamma}}_{22}^{1}\right)^{\dagger}\right]\left(\widehat{\boldsymbol{\Gamma}}_{22}^{1}\right)^{-1} \\
\times\left[\widehat{\boldsymbol{\Gamma}}_{21}^{1}\left(\boldsymbol{E}_{t}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2}\right)+\widehat{\boldsymbol{\Gamma}}_{22}^{1}\left(\boldsymbol{E}_{r}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2}\right)\right]  \tag{150}\\
=\left(\boldsymbol{P}_{\boldsymbol{A}_{1}}-\boldsymbol{P}_{\boldsymbol{A}_{0}}\right)=\mathcal{P}_{\Delta} \tag{151}
\end{gather*}
$$

which thus provides Eq. (43).

## XI. Derivation of Gradient statistic

The derivation of Gradient statistic is readily obtained from intermediate results obtained in derivation of Rao and Wald statistics, in Secs. [X] and X] respectively. Indeed, exploiting Eqs. (121) and 145) provides:

$$
\begin{align*}
& \left.\frac{\partial \ln f_{1}(\boldsymbol{Z} ; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}_{r}^{T}}\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_{0}}\left(\hat{\boldsymbol{\theta}}_{r, 1}-\boldsymbol{\theta}_{r, 0}\right)=2 \Re\left\{\operatorname{vec}(\widehat{\boldsymbol{B}})^{\dagger} \boldsymbol{g}_{A}^{\circ}\right\}  \tag{152}\\
& \propto K \Re\left\{\operatorname { T r } \left[\left(\widehat{\boldsymbol{\Gamma}}_{22}^{1} \boldsymbol{E}_{r}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{Z} \boldsymbol{C}^{\dagger}\left(\boldsymbol{C C}^{\dagger}\right)^{-1}\right)^{\dagger}\right.\right. \\
& \left.\left.\times \boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{Z}_{d, 0} \boldsymbol{C}^{\dagger}\right]\right\}  \tag{153}\\
& =\Re\left\{\operatorname{Tr}\left[\boldsymbol{Z}_{W_{1}}^{\dagger} K \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{E}_{r} \widehat{\boldsymbol{\Gamma}}_{22}^{1} \boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{Z}_{d, 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]\right\} \tag{154}
\end{align*}
$$

In Eq. (152) we have exploited the equivalence between the real part of an inner product in the complex domain and its real-valued equivalent counterpar 19 , along with the definition $\boldsymbol{g}_{A}^{\circ} \triangleq \operatorname{vec}\left(\boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{Z}_{d, 0} \boldsymbol{C}^{\dagger}\right)$. Furthermore, in obtaining Eq. (153), we have exploited standard properties of $\operatorname{vec}(\cdot)$ operator. Finally, we recall that $\boldsymbol{Z}_{d, 0}$ is given in Eq. (33). This concludes the derivation of Gradient statistic.

## Proof of Eq. 49

We start by observing that

$$
\begin{equation*}
\widehat{\boldsymbol{R}}_{0}^{-1} Z_{d, 0} P_{C^{+}}=\widehat{\boldsymbol{R}}_{0}^{-1 / 2} P_{\overrightarrow{\boldsymbol{A}}_{0}}^{\perp} \hat{\boldsymbol{R}}_{0}^{-1 / 2} Z P_{C^{+}}, \tag{155}
\end{equation*}
$$

[^15]which readily follows from application of Eq. (34). Then, we rewrite the matrix $\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}^{\perp} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}$ as:
\[

$$
\begin{gather*}
\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}^{\perp} \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \\
=\boldsymbol{S}_{c}^{-1 / 2}\left[\boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1}-\boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}\right]  \tag{156}\\
=\boldsymbol{S}_{c}^{-1 / 2}\left[\boldsymbol{I}_{N}-\boldsymbol{S}_{c}^{1 / 2}\left(\widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{E}_{t}\right)\right. \\
\left.\times\left(\boldsymbol{E}_{t}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{E}_{t}\right)^{-1} \boldsymbol{E}_{t}^{\dagger} \boldsymbol{S}_{c}^{-1 / 2}\right] \boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1}  \tag{157}\\
=\boldsymbol{S}_{c}^{-1 / 2} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1} \tag{158}
\end{gather*}
$$
\]

where in Eq. (158) we have exploited Lem. (1) Eq. (25). Finally, from straightforward combination of the results in Eqs. (155) and (158), the final result follows.

## XII. EQuivalence between Rao and Durbin STATISTICS (THEOREM 2)

In this section we prove statistical equivalence between Rao and Durbin statistics by explicitly deriving the closed form expression of Durbin statistic (implicitly expressed in Eq. (55)). To this end, analogously as for the Rao and Wald statistics, it can be shown that:

$$
\begin{align*}
{\left[\boldsymbol{I}^{-1}\left(\widehat{\boldsymbol{\theta}}_{0}\right)\right]_{\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}} } & =\left[\begin{array}{cc}
\frac{1}{2} \Re\left\{\boldsymbol{T}_{0}\right\} & -\frac{1}{2} \Im\left\{\boldsymbol{T}_{0}\right\} \\
\frac{1}{2} \Im\left\{\boldsymbol{T}_{0}\right\} & \frac{1}{2} \Re\left\{\boldsymbol{T}_{0}\right\}
\end{array}\right]  \tag{159}\\
{\left[\boldsymbol{I}\left(\widehat{\boldsymbol{\theta}}_{0}\right)\right]_{\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}} } & =\left[\begin{array}{cc}
2 \Re\left\{\overline{\boldsymbol{T}}_{0}\right\} & -2 \Im\left\{\overline{\boldsymbol{T}}_{0}\right\} \\
2 \Im\left\{\overline{\boldsymbol{T}}_{0}\right\} & 2 \Re\left\{\overline{\boldsymbol{T}}_{0}\right\}
\end{array}\right] \tag{160}
\end{align*}
$$

where:

$$
\begin{align*}
& \boldsymbol{T}_{0} \triangleq\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{-T} \otimes \widehat{\boldsymbol{\Gamma}}_{22}^{\circ}  \tag{161}\\
& \overline{\boldsymbol{T}}_{0} \triangleq\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{T} \otimes\left(\boldsymbol{E}_{r}^{\dagger} \hat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{E}_{r}\right) \tag{162}
\end{align*}
$$

The result in Eq. (159) is obtained starting from the explicit expression of $\left[\boldsymbol{I}^{-1}(\boldsymbol{\theta})\right]_{\boldsymbol{\theta}_{r}, \boldsymbol{\theta}_{r}}$ in Eq. 129 and plugging back $\boldsymbol{R}=\widehat{\boldsymbol{R}}_{0}$. Differently, the estimate $\hat{\boldsymbol{\theta}}_{r, 01}$ can be obtained following the steps described next. Without loss of generality, we consider maximization of $\ln (\cdot)$ of the objective in Eq. (56) and evaluate the following estimate:

$$
\begin{gather*}
\widehat{\boldsymbol{B}}_{0}=  \tag{163}\\
\arg \min _{\boldsymbol{B}} \operatorname{Tr}\left[\left(\boldsymbol{Z}-\boldsymbol{A}\left[\begin{array}{c}
\widehat{\boldsymbol{B}}_{t, 0} \\
\boldsymbol{B}
\end{array}\right] \boldsymbol{C}\right)^{\dagger} \hat{\boldsymbol{R}}_{0}^{-1}\left(\boldsymbol{Z}-\boldsymbol{A}\left[\begin{array}{c}
\widehat{\boldsymbol{B}}_{t, 0} \\
\boldsymbol{B}
\end{array}\right] \boldsymbol{C}\right)\right]
\end{gather*}
$$

Once we have obtained $\widehat{\boldsymbol{B}}_{0}, \hat{\boldsymbol{\theta}}_{r, 01}$ is evaluated through the simple operation $\hat{\boldsymbol{\theta}}_{r, 01}=\left[\Re\left\{\operatorname{vec}\left(\widehat{\boldsymbol{B}}_{0}\right)\right\}^{T} \quad \Im\left\{\operatorname{vec}\left(\widehat{\boldsymbol{B}}_{0}\right)\right\}^{T}\right]^{T}$. Thus, it is not difficult to show that the solution to the optimization problem in Eq. (163) is given in closed-form as:

$$
\begin{equation*}
\widehat{\boldsymbol{B}}_{0}=\left(\boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{E}_{r}\right)^{-1} \boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{Z}_{d, 0} \boldsymbol{C}^{\dagger}\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{-1} \tag{164}
\end{equation*}
$$

Substituting Eqs. (159), (160) and (164) into Eq. (55), provides 20 :

$$
\begin{gather*}
2 \operatorname{vec}\left(\widehat{\boldsymbol{B}}_{0}\right)^{\dagger}\left(\overline{\boldsymbol{T}}_{0} \boldsymbol{T}_{0} \overline{\boldsymbol{T}}_{0}\right) \operatorname{vec}\left(\widehat{\boldsymbol{B}}_{0}\right) \\
\propto \operatorname{vec}\left[\left(\boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{E}_{r}\right)^{-1} \boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{Z}_{d, 0} \boldsymbol{C}^{\dagger}\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{-1}\right]^{\dagger} \\
\times\left\{\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{T} \otimes\left[\left(\boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{E}_{r}\right) \widehat{\boldsymbol{\Gamma}}_{22}^{\circ}\left(\boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{E}_{r}\right)\right]\right\} \\
\times \operatorname{vec}\left[\left(\boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{E}_{r}\right)^{-1} \boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{Z}_{d, 0} \boldsymbol{C}^{\dagger}\left(\boldsymbol{C} \boldsymbol{C}^{\dagger}\right)^{-1}\right]  \tag{165}\\
=\operatorname{Tr}\left[\boldsymbol{Z}_{d, 0}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{E}_{r} \widehat{\boldsymbol{\Gamma}}_{22}^{\circ} \boldsymbol{E}_{r}^{\dagger} \widehat{\boldsymbol{R}}_{0}^{-1} \boldsymbol{Z}_{d, 0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right] \tag{166}
\end{gather*}
$$

where we have exploited standard properties of $\operatorname{vec}(\cdot)$ in Eq. (166).

Clearly, the last result coincides with the Rao Test for the I-GMANOVA model, as apparent from comparison with Rao statistic reported in Eq. 31) in the manuscript.

## XIII. Proofs of useful equalities

## Proof of Eqs. (26) and (27)

Hereinafter we provide the proof of Eqs. (26) and 27, which are fundamental for proving CFARness of GLR, Wald and LH statistics. We start by observing that:

$$
\begin{gather*}
\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right) \\
=\boldsymbol{Z}_{c}^{\dagger}\left\{\boldsymbol{S}_{c}^{-1}-\left(\boldsymbol{S}_{c}^{-1} \boldsymbol{E}_{t}\right)\left(\boldsymbol{E}_{t}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{E}_{t}\right)^{-1} \boldsymbol{E}_{t}^{\dagger} \boldsymbol{S}_{c}^{-1}\right\} \boldsymbol{Z}_{c} \tag{167}
\end{gather*}
$$

Analogously, we write:

$$
\begin{gather*}
\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right) \\
=\boldsymbol{Z}_{c}^{\dagger}\left\{\boldsymbol{S}_{c}^{-1}-\left(\boldsymbol{S}_{c}^{-1} \boldsymbol{A}\right)\left(\boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1}\right\} \boldsymbol{Z}_{c} \tag{168}
\end{gather*}
$$

Before proceeding further, we define the following partitioning for matrix $S_{c}^{-1}$ as:

$$
\boldsymbol{S}_{c}^{-1}=\left[\begin{array}{lll}
\boldsymbol{S}^{11} & \boldsymbol{S}^{12} & \boldsymbol{S}^{13}  \tag{169}\\
\boldsymbol{S}^{21} & \boldsymbol{S}^{22} & \boldsymbol{S}^{23} \\
\boldsymbol{S}^{31} & \boldsymbol{S}^{32} & \boldsymbol{S}^{33}
\end{array}\right]
$$

Furthermore, $\boldsymbol{S}^{i j},(i, j) \in\{1,2,3\} \times\{1,2,3\}$, is a sub-matrix whose dimensions can be obtained replacing 1,2 and 3 with $t, r$ and $(N-J)$, respectively. First, exploiting $\boldsymbol{E}_{t}$ structure leads to:

$$
\left(\boldsymbol{E}_{t}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{E}_{t}\right)=\boldsymbol{S}^{11}, \quad\left(\boldsymbol{S}_{c}^{-1} \boldsymbol{E}_{t}\right)=\left[\begin{array}{c}
\boldsymbol{S}^{11}  \tag{170}\\
\boldsymbol{S}^{21} \\
\boldsymbol{S}^{31}
\end{array}\right]
$$

Accordingly, the matrix within curly brackets in Eq. 167) can be rewritten as:

$$
\begin{gather*}
\left\{\boldsymbol{S}_{c}^{-1}-\left(\boldsymbol{S}_{c}^{-1} \boldsymbol{E}_{t}\right)\left(\boldsymbol{E}_{t}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{E}_{t}\right)^{-1} \boldsymbol{E}_{t}^{\dagger} \boldsymbol{S}_{c}^{-1}\right\}= \\
{\left[\begin{array}{cc}
\mathbf{0}_{t \times t} & \mathbf{0}_{t \times(N-t)} \\
\mathbf{0}_{(N-t) \times t} & \boldsymbol{S}_{2}^{-1}
\end{array}\right],} \tag{171}
\end{gather*}
$$

[^16]where we have denoted:
\[

\boldsymbol{S}_{2} \triangleq\left[$$
\begin{array}{ll}
\boldsymbol{S}_{22} & \boldsymbol{S}_{23}  \tag{172}\\
\boldsymbol{S}_{32} & \boldsymbol{S}_{33}
\end{array}
$$\right]
\]

Secondly, after substitution in Eq. 167), we obtain:

$$
\begin{equation*}
\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)=\boldsymbol{Z}_{23}^{\dagger} \boldsymbol{S}_{2}^{-1} \boldsymbol{Z}_{23} \tag{173}
\end{equation*}
$$

where $\boldsymbol{Z}_{23} \triangleq\left[\begin{array}{ll}\boldsymbol{Z}_{2}^{T} & \boldsymbol{Z}_{3}^{T}\end{array}\right]^{T}$. Finally, by exploiting the block inverse expression of $S_{2}$ in Eq. (173), it follows that

$$
\begin{equation*}
\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)=\boldsymbol{Z}_{2.3}^{\dagger} \boldsymbol{S}_{2.3}^{-1} \boldsymbol{Z}_{2.3}+\boldsymbol{Z}_{3}^{\dagger} \boldsymbol{S}_{33}^{-1} \boldsymbol{Z}_{3}, \tag{174}
\end{equation*}
$$

which proves Eq. (26). Similarly, exploiting $\boldsymbol{A}$ structure, it can be shown that:

$$
\boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{A}=\left[\begin{array}{ll}
\boldsymbol{S}^{11} & \boldsymbol{S}^{12}  \tag{175}\\
\boldsymbol{S}^{21} & \boldsymbol{S}^{22}
\end{array}\right] ; \quad \boldsymbol{S}_{c}^{-1} \boldsymbol{A}=\left[\begin{array}{ll}
\boldsymbol{S}^{11} & \boldsymbol{S}^{12} \\
\boldsymbol{S}^{21} & \boldsymbol{S}^{22} \\
\boldsymbol{S}^{31} & \boldsymbol{S}^{32}
\end{array}\right]
$$

Accordingly, we can rewrite the matrix within the curly brackets in Eq. 168) as: :

$$
\begin{align*}
\left\{\boldsymbol{S}_{c}^{-1}\right. & \left.-\left(\boldsymbol{S}_{c}^{-1} \boldsymbol{A}\right)\left(\boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^{\dagger} \boldsymbol{S}_{c}^{-1}\right\} \\
& =\left[\begin{array}{cc}
\mathbf{0}_{J \times J} & \mathbf{0}_{J \times(N-J)} \\
\mathbf{0}_{(N-J) \times J} & \boldsymbol{S}_{33}^{-1}
\end{array}\right] . \tag{176}
\end{align*}
$$

Finally, gathering the above results leads to:

$$
\begin{equation*}
\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)=\boldsymbol{Z}_{3}^{\dagger} \boldsymbol{S}_{33}^{-1} \boldsymbol{Z}_{3} \tag{177}
\end{equation*}
$$

which proves Eq. 27.

## Proof of Eqs. (38) and (39)

Hereinafter we provide a proof of Eqs. (38) and (39), which are fundamental for proving CFARness of Rao (Durbin) statistic. Firstly, it can be shown that:

$$
\begin{align*}
\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}^{\perp} \widehat{\boldsymbol{R}}_{0}^{-1 / 2} & =\left[\begin{array}{cc}
\mathbf{0}_{t \times t} & \mathbf{0}_{t \times(N-t)} \\
\mathbf{0}_{(N-t) \times t} & \widehat{\boldsymbol{R}}_{0,2}^{-1}
\end{array}\right]  \tag{178}\\
\widehat{\boldsymbol{R}}_{0}^{-1 / 2} \boldsymbol{P}_{\overline{\boldsymbol{A}}_{1}}^{\perp} \widehat{\boldsymbol{R}}_{0}^{-1 / 2} & =\left[\begin{array}{cc}
\mathbf{0}_{J \times J} & \mathbf{0}_{J \times(N-J)} \\
\mathbf{0}_{(N-J) \times J} & \widehat{\boldsymbol{R}}_{0,33}^{-1}
\end{array}\right] ; \tag{179}
\end{align*}
$$

where we have defined the following partitioning:
$\widehat{\boldsymbol{R}}_{0}=\left[\begin{array}{lll}\widehat{\boldsymbol{R}}_{0,11} & \widehat{\boldsymbol{R}}_{0,12} & \widehat{\boldsymbol{R}}_{0,13} \\ \widehat{\boldsymbol{R}}_{0,21} & \widehat{\boldsymbol{R}}_{0,22} & \widehat{\boldsymbol{R}}_{0,23} \\ \widehat{\boldsymbol{R}}_{0,31} & \widehat{\boldsymbol{R}}_{0,32} & \widehat{\boldsymbol{R}}_{0,33}\end{array}\right], \quad \widehat{\boldsymbol{R}}_{0,2} \triangleq\left[\begin{array}{ll}\widehat{\boldsymbol{R}}_{0,22} & \widehat{\boldsymbol{R}}_{0,23} \\ \widehat{\boldsymbol{R}}_{0,32} & \widehat{\boldsymbol{R}}_{0,33}\end{array}\right]$,
where $\widehat{\boldsymbol{R}}_{0, i j},(i, j) \in\{1,2,3\} \times\{1,2,3\}$, is a sub-matrix whose dimensions can be obtained replacing 1,2 and 3 with $t$, $r$ and $(N-J)$, respectively. Then, exploiting the above results we obtain:

$$
\begin{align*}
\left(\boldsymbol{Z}_{W 0} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}^{\perp}\left(\boldsymbol{Z}_{W 0} \boldsymbol{V}_{c, 1}\right) & =\boldsymbol{Z}_{23}^{\dagger} \widehat{\boldsymbol{R}}_{0,2}^{-1} \boldsymbol{Z}_{23}  \tag{181}\\
\left(\boldsymbol{Z}_{W 0} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\overline{\boldsymbol{A}}_{1}}^{\perp}\left(\boldsymbol{Z}_{W 0} \boldsymbol{V}_{c, 1}\right) & =\boldsymbol{Z}_{3}^{\dagger} \widehat{\boldsymbol{R}}_{0,33}^{-1} \boldsymbol{Z}_{3} \tag{182}
\end{align*}
$$

Furthermore, it is not difficult to show, starting from Eq. (23), that

$$
\begin{align*}
\widehat{\boldsymbol{R}}_{0,2} & =K^{-1}\left[\boldsymbol{S}_{2}+\boldsymbol{Z}_{23} \boldsymbol{Z}_{23}^{\dagger}\right]  \tag{183}\\
\widehat{\boldsymbol{R}}_{0,33} & =K^{-1}\left[\boldsymbol{S}_{33}+\boldsymbol{Z}_{3} \boldsymbol{Z}_{3}^{\dagger}\right] \tag{184}
\end{align*}
$$

Finally, after substitution into Eqs. (181) and (182) and application of Woodbury identity, the claimed result is obtained.

## Proof of Eqs. (52) and (53)

In what follows, we provide a proof of Eqs. (52) and 53), which are exploited in Sec. III-D for proving CFARness of Gradient statistic. We first observe that the following equalities hold:

$$
\begin{gather*}
\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}\left(\boldsymbol{Z}_{W 0} \boldsymbol{V}_{c, 1}\right) \\
=\left(\boldsymbol{Z}_{W 0} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\overline{\boldsymbol{A}}_{0}}^{\perp}\left(\boldsymbol{Z}_{W 0} \boldsymbol{V}_{c, 1}\right)  \tag{185}\\
\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp} \boldsymbol{S}_{c}^{1 / 2} \widehat{\boldsymbol{R}}_{0}^{-1 / 2}\left(\boldsymbol{Z}_{W 0} \boldsymbol{V}_{c, 1}\right) \\
=K\left[\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)\right. \\
-\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp} \boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\left(\boldsymbol{I}_{M}+\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{-1} \\
\left.\times\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp} \boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right] \tag{186}
\end{gather*}
$$

where we have used Eq. (25) of Lem. 1 in deriving the righthand side of Eq. (185). Differently, Eq. (186) is obtained exploiting the following steps: $(i)$ use of $\widehat{\boldsymbol{R}}_{0}$ explicit expression given by Eq. (23), (ii) application of Woodbury Identity to $\widehat{\boldsymbol{R}}_{0}^{-1}$ and (iii) simplification through the use of the equality $\boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp} \boldsymbol{P}_{\boldsymbol{A}_{0}}^{\perp}=\boldsymbol{P}_{\boldsymbol{A}_{1}}^{\perp}$. Finally, by exploiting Eq. (38) into Eq. (185) and Eqs. (26) and (27) into Eq. (186), we demonstrate the considered equalities.

## XIV. Simulation results showing specific COINCIDENCE RESULTS

In this section we confirm, through numerical results, the statistical equivalence results obtained among the considered detectors for specific adaptive detection scenarios. We remark that the simulation parameters (i.e., the the structure of the covariance $\boldsymbol{R}$ and the generation process for unknown signal matrix $\boldsymbol{B}$ ) are the same as those used in the manuscript (as well as the Monte Carlo setup) and thus are not reported for the sake of brevity.

First, in Fig. 2 we show $P_{d}$ vs. $\rho$ (given $P_{f a}=10^{-4}$ ) for a setup with point-like signal and interference $(M=1)$ where the signal belongs to a two-dimensional vector suspace ( $r=2$ ) while the interference to a four-dimensional vector subspace $(t=4)$. We assume that each column of $Z$ is a vector of $N=8$ elements and $K=13$ samples are assumed. It is apparent the statistical equivalence among GLR, Gradient and LH tests.

Similarly, in Fig. 3 we show $P_{d}$ vs. $\rho$ (given $P_{f a}=10^{-4}$ ) for a setup with multidimensional signals $(N=r=8)$ where $M=8$ and $K=24$. It is apparent the statistical equivalence between Rao and Gradient tests and between Wald and LH tests.

Finally, in Fig. 4 we report $P_{d}$ vs. $\rho$ (given $P_{f a}=10^{-4}$ ) for a range-spread target $(M=8)$ with a rank-one signal subspace $(r=1)$ and no interference $(t=0)$, where $K=24$ and $N=8$. It is apparent the statistical equivalence among GLR, Gradient and LH tests.

## REFERENCES

[1] F. Gini, A. Farina, and M. S. Greco, "Selected list of references on radar signal processing," IEEE Trans. Aerosp. Electron. Syst., vol. 37, no. 1, pp. 329-359, Jan. 2001.
[2] A. Dogandzic and A. Nehorai, "Generalized multivariate analysis of variance-a unified framework for signal processing in correlated noise," IEEE Signal Process. Mag., vol. 20, no. 5, pp. 39-54, 2003.


Figure 2. $\quad P_{d}$ vs. $\rho$ for all the considered detectors; vector subspace detection with point-like $(M=1)$ signal $(r=2)$ and interference $(t=4)$. Parameters $K=13$ and $N=8$.


Figure 3. $\quad P_{d}$ vs. $\rho$ for all the considered detectors; Multidimensional signals ( $r=8, t=0$ and $N=8$ ). Parameters $K=24$ and $M=8$.
[3] E. J. Kelly, "An adaptive detection algorithm," IEEE Trans. Aerosp. Electron. Syst., no. 2, pp. 115-127, 1986.
[4] E. Conte, A. De Maio, and G. Ricci, "GLRT-based adaptive detection algorithms for range-spread targets," IEEE Trans. Signal Process., vol. 49, no. 7, pp. 1336-1348, 2001.
[5] R. S. Raghavan, N. Pulsone, and D. J. McLaughlin, "Performance of the GLRT for adaptive vector subspace detection," IEEE Trans. Aerosp. Electron. Syst., vol. 32, no. 4, pp. 1473-1487, Oct. 1996.
[6] A. De Maio and D. Orlando, "Adaptive detection of a subspace signal embedded in subspace structured plus Gaussian interference via invariance," IEEE Trans. Signal Process. submitted, 2014.
[7] L. Xu, P. Stoica, and J. Li, "A diagonal growth curve model and some signal-processing applications," IEEE Trans. Signal Process., vol. 54, no. 9, pp. 3363-3371, 2006.
[8] -, "A block-diagonal growth curve model," Digital Signal Processing, vol. 16, no. 6, pp. 902-912, 2006.
[9] S. M. Kay, Fundamentals of Statistical Signal Processing, Volume 2: Detection Theory. Prentice Hall PTR, Jan. 1998.
[10] E. Conte, A. De Maio, and C. Galdi, "CFAR detection of multidimensional signals: an invariant approach," IEEE Trans. Signal Process., vol. 51, no. 1, pp. 142-151, 2003.
[11] A. De Maio, "A new derivation of the adaptive matched filter," IEEE Signal Process. Lett., vol. 11, no. 10, pp. 792-793, Oct. 2004.


Figure 4. $\quad P_{d}$ vs. $\rho$ for all the considered detectors; Range-spread targets $(M=8)$ with rank-one subspace $(r=1)$ and no-interference $(t=0)$. Parameters $K=24$ and $N=8$.
[12] F. Bandiera, A. De Maio, A. S. Greco, and G. Ricci, "Adaptive radar detection of distributed targets in homogeneous and partially homogeneous noise plus subspace interference," IEEE Trans. Signal Process., vol. 55, no. 4, pp. 1223-1237, 2007.
[13] A. De Maio, "Rao test for adaptive detection in Gaussian interference with unknown covariance matrix," IEEE Trans. Signal Process., vol. 55, no. 7, pp. 3577-3584, 2007.
[14] A. De Maio and S. Iommelli, "Coincidence of the Rao test, Wald test, and GLRT in partially homogeneous environment," IEEE Signal Process. Lett., vol. 15, pp. 385-388, 2008.
[15] E. J. Kelly and K. M. Forsythe, "Adaptive detection and parameter estimation for multidimensional signal models," Massachussets Inst. of Tech. Lexington Lincoln Lab, Tech. Rep. No. TR-848., 1989.
[16] J. Durbin, "Testing for serial correlation in least-squares regression when some of the regressors are lagged dependent variables," Econometrica: Journal of the Econometric Society, pp. 410-421, 1970.
[17] G. R. Terrell, "The gradient statistic," Computing Science and Statistics, vol. 34, pp. 206-215, 2002.
[18] D. N. Lawley, "A generalization of Fisher's z test," Biometrika, vol. 30, no. 1-2, pp. 180-187, 1938.
[19] A. De Maio and E. Conte, "Adaptive detection in Gaussian interference with unknown covariance after reduction by invariance," IEEE Trans. Signal Process., vol. 58, no. 6, pp. 2925-2934, 2010.
[20] K. A. Burgess and B. D. V. Veen, "Subspace-based adaptive generalized likelihood ratio detection," IEEE Trans. Signal Process., vol. 44, no. 4, pp. 912-927, Apr. 1996.
[21] W. Liu, W. Xie, J. Liu, and Y. Wang, "Adaptive double subspace signal detection in Gaussian background, Part I: Homogeneous environments," IEEE Trans. Signal Process., vol. 62, no. 9, pp. 2345-2357, May 2014.
[22] W. Liu, J. Liu, L. Huang, D. Zou, and Y. Wang, "Rao tests for distributed target detection in interference and noise," Signal Processing, 2015.
[23] W. Liu, W. Xie, and Y. Wang, "Rao and Wald tests for distributed targets detection with unknown signal steering," IEEE Signal Process. Lett., vol. 20, no. 11, pp. 1086-1089, Nov. 2013.
[24] X. Shuai, L. Kong, and J. Yang, "Adaptive detection for distributed targets in Gaussian noise with Rao and Wald tests," Science China Information Sciences, vol. 55, no. 6, pp. 1290-1300, 2012.
[25] R. S. Raghavan, "Maximal invariants and performance of some invariant hypothesis tests for an adaptive detection problem," IEEE Trans. Signal Process., vol. 61, no. 14, pp. 3607-3619, Jul. 2013.
[26] D. S. Bernstein, Matrix mathematics: theory, facts, and formulas. Princeton University Press, 2009.
[27] S. S. Wilks, "Certain generalizations in the analysis of variance," Biometrika, pp. 471-494, 1932.
[28] A. J. Lemonte and S. L. P. Ferrari, "The local power of the gradient test," Annals of the Institute of Statistical Mathematics, vol. 64, no. 2, pp. 373-381, 2012.
[29] S. Bose and A. Steinhardt, "A maximal invariant framework for adaptive
detection with structured and unstructured covariance matrices," IEEE Trans. Signal Process., vol. 43, no. 9, pp. 2164-2175, Sep. 1995.
[30] L. L. Scharf and B. Friedlander, "Matched subspace detectors," IEEE Trans. Signal Process., vol. 42, no. 8, pp. 2146-2157, Aug. 1994.
[31] D. R. Fuhrmann, E. J. Kelly, and R. Nitzberg, "A CFAR adaptive matched filter detector," IEEE Trans. Aerosp. Electron. Syst., vol. 28, no. 1, pp. 208-216, Jan. 1992.
[32] H. Wang and L. Cai, "On adaptive multiband signal detection with GLR algorithm," IEEE Trans. Aerosp. Electron. Syst., vol. 27, pp. 225-233, 1991.
[33] I. S. Reed, J. D. Mallett, and L. E. Brennan, "Rapid convergence rate in adaptive arrays," IEEE Trans. Aerosp. Electron. Syst., no. 6, pp. 853863, Nov. 1974.
[34] A. Hjørungnes, Complex-valued matrix derivatives: with applications in signal processing and communications. Cambridge University Press, 2011.
[35] S. M. Kay, Fundamentals of Statistical Signal Processing: Estimation Theory. Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1993.


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[^1]:    ${ }^{1}$ Notation - Lower-case (resp. Upper-case) bold letters denote vectors (resp. matrices), with $a_{n}$ (resp. $A_{n, m}$ ) representing the $n$-th (resp. the ( $n, m$ )-th) element of the vector $\boldsymbol{a}$ (resp. matrix $\boldsymbol{A}) ; \mathbb{R}^{N}, \mathbb{C}^{N}$, and $\mathbb{H}^{N \times N}$ are the sets of $N$-dimensional vectors of real numbers, of complex numbers, and of $N \times N$ Hermitian matrices, respectively; upper-case calligraphic letters and braces denote finite sets; $\mathbb{E}\{\cdot\},(\cdot)^{T},(\cdot)^{\dagger}, \operatorname{Tr}[\cdot],\|\cdot\|, \Re\{\cdot\}$ and $\Im\{\cdot\}$, denote expectation, transpose, Hermitian, matrix trace, Euclidean norm, real part, and imaginary part operators, respectively; $\mathbf{0}_{N \times M}$ (resp. $\boldsymbol{I}_{N}$ ) denotes the $N \times M$ null (resp. identity) matrix; $\mathbf{0}_{N}\left(\right.$ resp. $\left.\mathbf{1}_{N}\right)$ denotes the null (resp. ones) column vector of length $N$; $\operatorname{vec}(\boldsymbol{M})$ stacks the first to the last column of the matrix $\boldsymbol{M}$ one under another to form a long vector; $\operatorname{det}(\boldsymbol{A})$ and $\|\boldsymbol{A}\|_{F}$ denote the determinant and Frobenius norm of matrix $\boldsymbol{A} ; \boldsymbol{A} \otimes \boldsymbol{B}$ indicates the Kronecker product between $\boldsymbol{A}$ and $\boldsymbol{B}$ matrices; $\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}}$ denotes the gradient of scalar valued function $f(\boldsymbol{x})$ w.r.t. vector $\boldsymbol{x}$ arranged in a column vector, while $\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}^{T}}$ its transpose (i.e. a row vector); the symbol " $\sim$ " means "distributed as"; $\boldsymbol{x} \stackrel{\boldsymbol{x}}{\sim} \mathcal{C} \mathcal{N}_{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes a complex (proper) Gaussian-distributed vector $\boldsymbol{x}$ with mean vector $\boldsymbol{\mu} \in \mathbb{C}^{N \times 1}$ and covariance matrix $\boldsymbol{\Sigma} \in \mathbb{C}^{N \times N} ; \boldsymbol{X} \sim$ $\mathcal{C} \mathcal{N}_{N \times M}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ denotes a complex (proper) Gaussian-distributed matrix $\boldsymbol{X}$ with mean $\boldsymbol{A} \in \mathbb{C}^{N \times M}$ and $\operatorname{Cov}[\operatorname{vec}(\boldsymbol{X})]=\boldsymbol{B} \otimes \boldsymbol{C} ; \boldsymbol{P}_{A}$ denotes the orthogonal projection of the full-column-rank matrix $\boldsymbol{A}$, that is $\boldsymbol{P}_{\boldsymbol{A}} \triangleq$ $\left[\boldsymbol{A}\left(\boldsymbol{A}^{\dagger} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^{\dagger}\right]$, while $\boldsymbol{P}_{A}^{\perp}$ its complement, that is $\boldsymbol{P}_{A}^{\perp} \triangleq\left(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{A}}\right)$.

[^2]:    ${ }^{2}$ More specifically, $\boldsymbol{\theta}_{s, b} \triangleq \boldsymbol{\Xi}(\boldsymbol{R})$, where $\boldsymbol{\Xi}(\cdot)$ denotes the one-to-one mapping providing $\boldsymbol{\theta}_{s, b}$ from $\boldsymbol{R}$.

[^3]:    ${ }^{3}$ Hereinafter, in the case $J=N$, the "3-components" are no longer present in the partitioning.

[^4]:    ${ }^{4}$ The proof of the aforementioned equalities is non-trivial and thus provided as supplementary material.

[^5]:    ${ }^{5}$ The proof is provided as supplementary material.
    ${ }^{6}$ Their proof is provided as supplementary material for this manuscript.

[^6]:    ${ }^{7}$ The proof is provided as supplementary material.

[^7]:    ${ }^{8}$ The proof is deferred to supplementary material.
    ${ }^{9}$ The proof is provided as supplementary material.

[^8]:    ${ }^{10}$ It should be noted that the same result would be obtained by considering $\widehat{\boldsymbol{R}}_{1}^{-1}$ (i.e., the ML estimate under $\mathcal{H}_{1}$ ) as the signal-free covariance estimate. Indeed, since Eq. 60 depends only on $\boldsymbol{R}$ through the quantities $\left(\boldsymbol{R}^{-1} \boldsymbol{A}\right)$ and $\left(\boldsymbol{R}^{-1} \boldsymbol{E}_{t}\right)$ an thus Lem. 1 could be exploited to obtain the same final statistic.

[^9]:    ${ }^{11}$ It is worth pointing out that an alternative (equivalent) form of GLR was obtained in [4], [32] for the range-spread case. The aforementioned expression can be simply obtained starting from general formula in Eq. [18, straightforward application of Sylvester's determinant theorem and exploitation of the simplified assumptions of range-spread scenario.

[^10]:    ${ }^{12}$ We point out that Eq. 110 can be also re-arranged in a similar form as Eq. 21) (i.e., a Wilks' Lambda statistic form). Such expression, being equal to $t_{\mathrm{glr}}=\operatorname{det}\left[\boldsymbol{I}_{M}+\boldsymbol{D}_{1}^{-1 / 2}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right)^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_{1}}\left(\boldsymbol{Z}_{W 1} \boldsymbol{V}_{c, 1}\right) \boldsymbol{D}_{1}^{-1 / 2}\right]$, represents the alternative GLR form obtained in [15].

[^11]:    ${ }^{13}$ Of course, due to the high number of setup parameters involved in the detection problem (i.e., $N, K, M, r, t$ ), we do not claim the following conclusions to be general for any type of setup. Nonetheless, we illustrate a generic setup in order to show some common trends observed among the detectors. A general numerical comparison is omitted due to the lack of space and since performance comparison in some specific setups (such as those considered in Sec. IV can be found in the related literature. Nonetheless, the supplementary material attached to this paper contains some additional numerical results aimed at confirming the statistical equivalence results obtained for the considered special scenarios.

[^12]:    ${ }^{14}$ Recall that every permutation matrix is a special orthogonal matrix, that is, $\boldsymbol{P}^{-1}=\boldsymbol{P}^{T}$.
    ${ }^{15}$ Such result is obtained by exploiting the equality $\boldsymbol{\Psi}^{-1} \boldsymbol{\Psi}=\boldsymbol{I}$ and the real-imaginary parts decompositions of $\boldsymbol{\Psi}=\boldsymbol{\Psi}_{R}+j \boldsymbol{\Psi}_{I}$ and $\boldsymbol{\Psi}^{-1}=\overline{\boldsymbol{\Psi}}_{R}+$ $j \overline{\mathbf{\Psi}}_{I}$, from which it follows the set of equations (i) $\left(\boldsymbol{\Psi}_{R} \overline{\mathbf{\Psi}}_{R}-\boldsymbol{\Psi}_{I} \overline{\mathbf{\Psi}}_{I}\right)=\boldsymbol{I}$ and (ii) $\left(\boldsymbol{\Psi}_{I} \overline{\mathbf{\Psi}}_{R}+\mathbf{\Psi}_{R} \overline{\mathbf{\Psi}}_{I}\right)=\mathbf{0}$.

[^13]:    ${ }^{16}$ The mixed product property states that $\left(\boldsymbol{V}_{1} \otimes \boldsymbol{V}_{2}\right)\left(\boldsymbol{V}_{3} \otimes \boldsymbol{V}_{4}\right)=\left(\boldsymbol{V}_{1} \boldsymbol{V}_{3}\right) \otimes$ $\left(\boldsymbol{V}_{2} \boldsymbol{V}_{4}\right)$, where $\boldsymbol{V}_{i}$ are generic matrices of compatible sizes. Differently, the associative property states that $\boldsymbol{V}_{1} \otimes\left(\boldsymbol{V}_{2}+\boldsymbol{V}_{3}\right)=\boldsymbol{V}_{1} \otimes \boldsymbol{V}_{2}+\boldsymbol{V}_{1} \otimes \boldsymbol{V}_{3}$.
    ${ }^{17}$ More specifically, we have exploited $\operatorname{vec}\left(\boldsymbol{V}_{1} \boldsymbol{V}_{2} \boldsymbol{V}_{3}\right)=\left(\boldsymbol{V}_{2}^{T} \otimes\right.$ $\left.\boldsymbol{V}_{1}\right) \operatorname{vec}\left(\boldsymbol{V}_{3}\right)$ and $\operatorname{vec}\left(\boldsymbol{V}_{1}\right)^{\dagger} \operatorname{vec}\left(\boldsymbol{V}_{2}\right)=\operatorname{Tr}\left[\boldsymbol{V}_{1}^{\dagger} \boldsymbol{V}_{2}\right]$, with $\boldsymbol{V}_{i}$ being generic matrices.

[^14]:    ${ }^{18}$ We again use the property that the inverse of a block-symmetric matrix in the form of Eq. 129 gives rise to a similar structure for its inverse, as exploited for the derivation of Rao statistic.

[^15]:    ${ }^{19}$ More specifically, given two complex vectors with real/imaginary parts decomposition $\boldsymbol{v}_{1}=\boldsymbol{v}_{1, R}+j \boldsymbol{v}_{1, I}$ an $\boldsymbol{v}_{2}=\boldsymbol{v}_{2, R}+j \boldsymbol{v}_{2, I}$, it holds $\Re\left\{\boldsymbol{v}_{1}^{\dagger} \boldsymbol{v}_{2}\right\}=\Re\left\{\boldsymbol{v}_{2}^{\dagger} \boldsymbol{v}_{1}\right\}=\boldsymbol{v}_{1, E}^{T} \boldsymbol{v}_{2, E}$, where $\boldsymbol{v}_{i, E} \triangleq\left[\begin{array}{ll}\boldsymbol{v}_{i, R}^{T} & \boldsymbol{v}_{i, I}^{T}\end{array}\right]^{T}$.

[^16]:    ${ }^{20}$ We have exploited the fact that $\overline{\boldsymbol{T}}_{0} \boldsymbol{T}_{0} \overline{\boldsymbol{T}}_{0}$ is Hermitian and that the product of block-symmetric real counterparts of Hermitian matrices can be expressed as an equivalent block-symmetric real counterpart with the component matrix being given by the product of the aforementioned matrices. Finally, we have used the equivalence between an Hermitian quadratic form and its real block-symmetric counterpart.

