# Matrix Exponential Learning Schemes with Low Informational Exchange 

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#### Abstract

We consider a distributed resource allocation problem in networks where each transmitter-receiver pair aims at maximizing its local utility function by adjusting its action matrix, which belongs to a given feasible set. This problem has been addressed recently by applying a matrix exponential learning (MXL) algorithm which has a very appealing convergence rate. In this learning algorithm, however, each transmitter must know an estimate of the gradient matrix of the local utility. The knowledge of the gradient matrix at the transmitters incurs a high signaling overhead especially that the matrix size increases with the dimension of the action matrix. In this paper, we therefore investigate two strategies in order to decrease the informational exchange per iteration of the algorithm. In the first strategy, each receiver sends at each iteration part of the elements of the gradient matrix with respect to a certain probability. In the second strategy, each receiver feeds back "sporadically" the whole gradient matrix. We focus on the analysis of the convergence of the MXL algorithm to optimum under these two strategies. We prove that the algorithm can still converge to optimum almost surely. Upper bounds of the average convergence rate are also derived in both situations with general step-size setting, from which we can clearly see the impact of the incompleteness of the feedback information. The proposed algorithms are applied to solve the energy efficiency maximization problem in a multicarrier multi-user MIMO network. Simulation results further corroborate our claim.


## I. Introduction

Recently, there has been a growing interest in adopting cooperative and non-cooperative game theoretic approaches to model many communications and networking problems, such as power control and resource sharing in wireless/wired and peer-to-peer networks, cognitive radio systems, and distributed routing, flow, and congestion control in communication networks, see, for example [2], [3], [4], [5], [6].

This paper deals with a resource allocation problem in networks where each user tries to maximize its local utility. More specifically, we consider the multi-user, multi-carrier MIMO networks, each user controls its signal covariance matrix and the local utility function is defined as the energy efficiency of each user [7]. This problem has been addressed very recently

[^0]in [8] by applying the matrix exponential learning (MXL) technique, of which the convergence to Nash Equilibrium (NE) has been well demonstrated. The MXL-based algorithm is attractive because of its fast convergence to NE. As most of the gradient based methods [9], [10], the gradient of the utility function should be estimated by receivers and then sent back to transmitters as signaling information. Another possibility is to let the transmitters compute the gradient, which requires a full CSI knowledge of all direct and cross links and hence requires more signaling overhead than feeding back the gradient as we will see later in the paper. Although feeding back the gradient decreases the signaling overhead, it is still of huge burden of the network. In multi-carrier MIMO networks, the feedback is a gradient matrix with its size related to the number of transmission antennas and the number of carriers. When many users are present in the network, such signaling overhead may introduce a huge traffic burden.

For the above reason, our main goal is to investigate some modified MXL-based algorithm which requires less amount of signaling overhead and ensures the convergence to NE. Two possible variants are considered: $i$ ) each receiver feedback at each iteration only part of the elements of the gradient matrix instead of the full gradient matrix, the elements of the action matrix do not update if the associated element of the gradient matrix is not available; ii) each receiver sporadically feedback the whole gradient matrix, instead of feedbacking at each iteration, thus not all of the transmitters are able to update their action at the same time.

As an extension of the work in [8], we have analyzed the two variants of the MXL algorithms and shown that they converge to NE almost surely. In both settings, we also derive the evolution of the average quantum Kullback-Leibler divergence [11], of which the upper bounds are obtained. The theoretical results provide a quantitative description of the impact of incompleteness on the convergence rate. Our analysis is much more complicated compared with that in [8], as we have introduced some additional random terms into the algorithms, i.e., random incompleteness of the feedback information and sporadic update of the action. As a consequence, we have to define some additional stochastic noise to be analyzed by applying Doob's martingale inequality. The sporadic algorithm is more challenging to be analyzed by the fact that the update of action takes place at random time slot for each transmitter, and the step-size is also generated in a random manner. Some concentration inequality, such as Chernoff bound, has to be used to show the convergence of the sporadic algorithm. Another contribution of this paper is that we consider general forms of the step-size $\gamma_{n}$ to derive
the convergence rate, whereas $\gamma_{n}=\alpha / n$ is considered in [8]. Simulations further justify our results.

The rest of the paper is organized as follows. Section $\Pi$ presents the system model and briefly describes the MXL algorithm proposed in [8]. Section III presents the MXL algorithm with incomplete feedback information and presents its convergence. The sporadic MXL is analyzed in Section IV. Section V provides some numerical examples and Section VI concludes this paper.

Throughout this paper, we denote $\mathbf{X}=\operatorname{diag}\left(\mathbf{X}_{k}\right)_{k=1}^{K}$ and $\mathbf{X}_{-k}=\left(\mathbf{X}_{j}\right)_{j \neq k}$. The $m$-th eigenvalue of a matrix $\mathbf{X}$ is denoted by $\operatorname{eig}_{m}(\mathbf{X}),\|\mathbf{X}\|=\operatorname{tr}(\mathbf{X})=\sum_{m=1}^{M}\left|\operatorname{eig}_{m}(\mathbf{X})\right|$ denotes the trace norm of $\mathbf{X}$, and $\|\mathbf{X}\|_{\infty}=\max _{m}\left|\operatorname{eig}_{m}(\mathbf{X})\right|$ represents the singular norm.

## II. System Model and MXL Algorithm

This section presents the basic system model and briefly recalls the MXL algorithm proposed in [8].

## A. Problem formulation

This section introduces the general mathematical system model.

Consider a finite set of transmitter-receiver links $\mathcal{K}=$ $\{1, \ldots, K\}$. Each transmitter $k$ needs to properly control its action matrix $\mathbf{X}_{k}$ in order to maximize its local utility $u_{k}\left(\mathbf{X}_{1}, \ldots, \mathbf{X}_{K}\right)$, which depends on the action of all the transmitters.

Notice that the instant local utility $\widetilde{u}_{k}$ may be affected by some stochastic environment state $\mathbf{S}$, e.g., time-varying channels. In such situation, we consider $u_{k}\left(\mathbf{X}_{1}, \ldots, \mathbf{X}_{K}\right)=$ $\mathbb{E}_{\mathbf{S}}\left[\widetilde{u}_{k}\left(\mathbf{X}_{1}, \ldots, \mathbf{X}_{K} ; \mathbf{S}\right)\right]$. For any link $k$, its local utility $u_{k}$ is assumed to be concave and smooth in $\mathbf{X}_{k}$.

Consider the same setting as in [8], the action matrix $\mathbf{X}_{k}$ has to belong to a pre-defined feasible action set $\mathscr{X}_{k}$, i.e.,

$$
\mathscr{X}_{k}=\left\{\mathbf{X}_{k} \succcurlyeq 0:\left\|\mathbf{X}_{k}\right\| \leq A_{k}\right\}
$$

with $A_{k}$ is a positive constant.
Our problem is then, $\forall k \in \mathcal{K}$,

$$
\begin{cases}\operatorname{maximize} & u_{k}\left(\mathbf{X}_{1}, \ldots, \mathbf{X}_{K}\right)  \tag{1}\\ \text { subject to } & \mathbf{X}_{k} \in \mathscr{X}_{k}\end{cases}
$$

The solution of the problem (1) is the well known Nash equilibrium (NE) $\mathbf{X}^{*}=\left(\mathbf{X}_{1}^{*}, \ldots, \mathbf{X}_{K}^{*}\right) \in \mathcal{X}$, which satisfies

$$
u_{k}\left(\mathbf{X}^{*}\right) \geq u_{k}\left(\mathbf{X}_{k}, \mathbf{X}_{-k}^{*}\right), \forall \mathbf{X}_{k} \in \mathcal{X}_{k}, \forall k \in \mathcal{K}
$$

Note that the existence of NE can be guaranteed as $u_{k}$ is concave with respect to $\mathbf{X}_{k}$ [12]. We will show in Section II-C an example where (1) admits a NE.

## B. MXL algorithm

In order to solve the above problem, a solution has been proposed recently in [8], named Matrix Exponential Learning (MXL). This MXL algorithm is interesting thanks to its robustness against the stochastic environment and its fast convergence to NE. This section briefly describes this algorithm.

Suppose that each transmitter $k$ is able to get a noisy estimation $\widehat{\mathbf{V}}_{k}(n)$ of the individual gradient $\mathbf{V}_{k}(n)=$ $\partial u_{k} / \partial \mathbf{X}_{k}(n)$. According to the MXL algorithm [8], each user $k$ updates its action at each iteration $n \geq 1$

$$
\begin{align*}
& \mathbf{X}_{k}(n)=A_{k} \frac{\exp \left(\mathbf{Y}_{k}(n-1)\right)}{1+\left\|\exp \left(\mathbf{Y}_{k}(n-1)\right)\right\|}  \tag{2}\\
& \mathbf{Y}_{k}(n)=\mathbf{Y}_{k}(n-1)+\gamma_{n} \widehat{\mathbf{V}}_{k}(n) \tag{3}
\end{align*}
$$

in which $\gamma_{n}$ is a pre-defined vanishing step-size and $\mathbf{Y}_{k}(n)$ is an intermediate matrix variable with $\mathbf{Y}_{k}(0)$ an arbitrary Hermitian value. Notice that (3) can be seen as a step of the gradient ascent method and (2) ensures that $\mathbf{X}_{k}(n)$ meets the action constraint. Interested readers may refer to [8] for the detailed properties of the step (2), here $\mathbf{Y}_{k}(n)$ can be seen as the auxiliary dual variable of the primal variable $\mathbf{X}_{k}(n)$. Note that the dimension of the matrices $\mathbf{X}_{k}(n), \mathbf{Y}_{k}(n)$, and $\widehat{\mathbf{V}}_{k}(n)$ is the same.

To guarantee the performance of the MXL algorithm, the following assumptions are made, which are common in the typical stochastic approximation problems [13], [14]:

A1. $\widehat{\mathbf{V}}(n)=\mathbf{V}(n)+\mathbf{Z}(n)$ where $\mathbf{Z}(n)$ is an additive stochastic noise. Introduce $\mathcal{F}_{n}=\{\mathbf{Z}(0), \ldots, \mathbf{Z}(n)\}$, then any element $Z_{i, j}(n)$ satisfies $\mathbb{E}\left[Z_{i, j}(n) \mid \mathcal{F}_{n-1}\right]=\mathbf{0}$ and $\mathbb{E}\left[Z_{i, j}^{2}(n) \mid \mathcal{F}_{n-1}\right] \leq \sigma^{2}$.

A2. The positive vanishing step-size $\gamma_{n}$ satisfies $\sum_{n=1}^{\infty} \gamma_{n}=$ $\infty$ and $\sum_{n=1}^{\infty} \gamma_{n}^{2}<\infty$.

## C. Multicarrier, multi-user MIMO system

This section provides an example in which the assumptions on the concavity of the local utility functions are satisfied and NE exists.

We consider the application of the MXL algorithm in a multicarrier, multi-user MIMO network with $K$ transmitterreceiver links. During the communication, $S$ orthogonal subcarriers are used, e.g., in OFDM systems. Each transmitter is equipped with $N_{t}$ transmission antennas and each receiver has $N_{r}$ reception antennas. For any $k \in \mathcal{K}, j \in \mathcal{K}$, and $s \in \mathcal{S}$, let the matrix $\mathbf{H}_{k j s} \in \mathbb{C}^{N_{r} \times N_{t}}$ describe the channel during the communication between transmitter $k$ any receiver $j$ over subcarrier $s$. The channel matrix over all subcarriers is then $\mathbf{H}_{k j}=\operatorname{diag}\left(\mathbf{H}_{k j s}\right)_{s=1}^{S}$ of size $N_{r} S \times N_{t} S$. In this paper, we assume that $\mathbf{H}_{k j}$ is stochastic, time-varying, ergodic and Gaussian.

Each transmitter $k$ controls its signal covariance matrix $\mathbf{Q}_{k}$ in order to maximize its own energy efficiency (EE) defined as $\mathrm{EE}_{k}=r_{k} /\left(\operatorname{tr}\left(\mathbf{Q}_{k}\right)+P_{\mathrm{c}}\right)$, where $\left.i\right) r_{k}$ denotes the Shannonachievable rate, i.e., $r_{k}=\log \operatorname{det}\left(\mathbf{I}+\sum_{j \in \mathcal{K}} \mathbf{H}_{j k} \mathbf{Q}_{k} \mathbf{H}_{j k}^{\dagger}\right)-$ $\log \operatorname{det}\left(\mathbf{I}+\sum_{j \in \mathcal{K} \backslash\{k\}} \mathbf{H}_{j k} \mathbf{Q}_{k} \mathbf{H}_{j k}^{\dagger}\right)$; ii) $\operatorname{tr}\left(\mathbf{Q}_{k}\right)+P_{\mathrm{c}}$ is the total power consumed by transmitter $k$, including the transmission power $\operatorname{tr}\left(\mathbf{Q}_{k}\right)$ and the total circuit consumption power $P_{\mathrm{c}}$ [15]. Note that $\mathbf{Q}_{k}=\operatorname{diag}\left(\mathbf{Q}_{k s}\right)_{s=1}^{S}$ with $\mathbf{Q}_{k s} \in \mathbb{H}^{N_{t} \times N_{t}}$ the covariance matrix over subcarrier $s$, we use $\mathbb{H}$ to denote the set of Hermittian matrices. Since the covariance matrix is positive semi-definite, we have $\mathbf{Q}_{k s} \succcurlyeq 0$. Introduce $P_{\max }$ the maximum transmission power of any transmitter $k$, then we have $\operatorname{tr}\left(\mathbf{Q}_{k}\right) \leq P_{\text {max }}$. Thus $\mathbf{Q}_{k} \in \mathscr{X}_{k}$ with $A_{k}=P_{\max }$,
$\forall k \in \mathcal{K}$. Recall that our aim is to make each transmitter $k$ maximize its $\mathrm{EE}_{k}$, which depends on the channel state of the network as well as the signal covariance matrices of all the transmitters. Hence, in this situation, our problem (1) turns to, $\forall k \in \mathcal{K}$,

$$
\begin{cases}\operatorname{maximize} & \mathbb{E}\left[\mathrm{EE}_{k}\left(\mathbf{Q}_{1}, \ldots, \mathbf{Q}_{K} ; \mathbf{H}\right)\right]  \tag{4}\\ \text { subject to } & \mathbf{Q}_{k} \in\left\{\operatorname{diag}\left(\mathbf{Q}_{k s}\right)_{s=1}^{S}: \mathbf{Q}_{k s} \in \mathbb{H}^{N_{t} \times N_{t}},\right. \\ & \left.\mathbf{Q}_{k s} \succcurlyeq 0, \text { and } \sum_{s=1}^{S} \operatorname{tr}\left(\mathbf{Q}_{k s}\right) \leq P_{\max }\right\}\end{cases}
$$

At a first look, $\mathrm{EE}_{k}\left(\mathbf{Q}_{1}, \ldots, \mathbf{Q}_{K} ; \mathbf{H}\right)$ is not a concave function of $\mathbf{Q}_{k}$. However, EE can be concave after a variable change: as presented in [7], [8], we apply the following adjusted action matrix $\mathbf{X}_{k}$ instead of $\mathbf{Q}_{k}$, such that $\widetilde{u}_{k}$ is concave with respect to $\mathbf{X}_{k}$, i.e.,

$$
\mathbf{X}_{k}=\frac{P_{\mathrm{c}}+P_{\max }}{P_{\max }} \frac{\mathbf{Q}_{k}}{P_{\mathrm{c}}+\operatorname{tr}\left(\mathbf{Q}_{k}\right)}
$$

Using the transformation, we have [7]

$$
\begin{aligned}
\widetilde{u}_{k}\left(\mathbf{X}_{1}, \ldots, \mathbf{X}_{K} ; \mathbf{H}\right)= & \frac{P_{\mathrm{c}}+P_{\max }\left(1-\operatorname{tr}\left(\mathbf{X}_{k}\right)\right)}{P_{\mathrm{c}}\left(P_{\mathrm{c}}+P_{\max }\right)} \\
& \cdot \log \operatorname{det}\left(\mathbf{I}+\frac{P_{\mathrm{c}} P_{\max } \widetilde{\mathbf{H}}_{k} X_{k} \widetilde{\mathbf{H}}_{k}^{\dagger}}{P_{\mathrm{c}}+P_{\max }\left(1-\operatorname{tr}\left(\mathbf{X}_{k}\right)\right)}\right),
\end{aligned}
$$

where $\widetilde{\mathbf{H}}_{k}=\left(\mathbf{I}+\sum_{j \neq k} \mathbf{H}_{j k} \mathbf{Q}_{k} \mathbf{H}_{j k}^{\dagger}\right)^{-\frac{1}{2}} \mathbf{H}_{k k}$ is the effective channel matrix. By definition, we have $\mathbf{X}_{k} \in \mathbb{H}^{N_{t} S} \times N_{t} S$ and $\mathbf{X}_{k} \succcurlyeq 0$. As $\operatorname{tr}\left(\mathbf{Q}_{k}\right) \leq P_{\text {max }}$, we can deduce that $\operatorname{tr}\left(\mathbf{X}_{k}\right) \leq 1$. Thus $\mathbf{X}_{k} \in \mathscr{X}_{k}$ with $A_{k}=1, \forall k \in \mathcal{K}$. More precisely, our problem (4) turns to, $\forall k \in \mathcal{K}$,

$$
\begin{cases}\operatorname{maximize} & \mathbb{E}\left[\widetilde{u}_{k}\left(\mathbf{X}_{1}, \ldots, \mathbf{X}_{K} ; \mathbf{H}\right)\right]  \tag{5}\\ \text { subject to } & \mathbf{X}_{k} \in\left\{\operatorname{diag}\left(\mathbf{X}_{k s}\right)_{s=1}^{S}: \mathbf{X}_{k s} \in \mathbb{H}^{N_{t} \times N_{t}},\right. \\ & \left.\mathbf{X}_{k s} \succcurlyeq 0, \text { and } \sum_{s=1}^{S} \operatorname{tr}\left(\mathbf{X}_{k s}\right) \leq 1\right\},\end{cases}
$$

which admits a NE, as the concavity assumption is satisfied. The MXL algorithm can be thus easily applied in this system.

Remark 1. As pointed out in [8], a first implementation issue is that $\mathbf{X}_{k}(n)$ should keep Hermitian to satisfy the feasibility constraint. By the basic steps of the MXL algorithm, we find that the estimation of the gradient matrix $\widehat{\mathbf{V}}_{k}(n)$ has to be Hermitian, which cannot be true due to the additive noise $\mathbf{Z}(n)$. A simple solution we propose here is then using $\left(\widehat{\mathbf{V}}_{k}(n)+\widehat{\mathbf{V}}_{k}^{\dagger}(n)\right) / 2$ instead of $\widehat{\mathbf{V}}_{k}(n)$. Another issue is the signaling overhead introduced by the MXL algorithm, we highlight this problem in the next section.

## D. Signaling overhead

In the MXL algorithm, each transmitter $k$ needs to have the full knowledge of the gradient matrix $\widehat{\mathbf{V}}_{k}(n)$. There are two possible options to obtain $\widehat{\mathbf{V}}_{k}(n)$ :

The first option is to make each transmitter compute the gradient. For brevity, we skip the derivation of the gradient which is straightforward [7]. For each transmitter $k$, the computation of $\partial \widetilde{u}_{k} / \partial \mathbf{X}_{k}(n)$ requires the knowledge of the
channel matrices $\mathbf{H}_{j k}$ with all $j \in \mathcal{K}$, including the direct link and all the cross links to receiver $k$. The amount of necessary information is then $K S N_{t} N_{r}$.

The second option is to make each receiver $k$ compute the gradient and then directly feedback the gradient matrix $\widehat{\mathbf{V}}_{k}(n)$. Note that the dimension of $\widehat{\mathbf{V}}_{k}(n)$ is the same as $\mathbf{X}_{k}(n)$, thus the signaling overhead in this situation is of size $S N_{t}^{2}$.

It is obvious that the second option requires less signaling overhead as the number of links is high. For this reason, we focus on the second option in this paper. It is notable that feeding back $S N_{t}^{2}$ elements is still a huge burden of the network. This is the reason why we focus on the variant of the MXL algorithm with less signaling overhead.

## III. MXL with Incomplete Feedback

In this section, we consider a first strategy to reduce the signaling overhead: each receiver $k$ does not send every element of the matrix $\widehat{\mathbf{V}}_{k}(n)$. More precisely:

Sl. Each element $\widehat{V}_{k, i, j}(n)$ of $\widehat{\mathbf{V}}_{k}(n)$ is sent with a constant probability $p_{\mathrm{I}} \in(0,1)$ with $i \geq j$ and $\widehat{V}_{k, j, i}(n)=\widehat{V}_{k, i, j}(n)$. The non-received elements are replaced by 0 .

Notice that the transmitted elements of $\widehat{\mathbf{V}}_{k}(n)$ has symmetric positions in order to ensure that the received gradient matrix is Hermitian.

Introduce a symmetric matrix $\boldsymbol{\Delta}_{k}(n)=\left[\delta_{k, i, j}(n)\right]$ of the same dimension as $\widehat{\mathbf{V}}_{k}(n)$. The elements of $\delta_{k, i, j}(n)$ are i.i.d. Bernoulli random variables with $\mathbb{P}\left[\delta_{k, i, j}(n)=1\right]=p_{\mathrm{I}}$ as $i \geq j$, i.e., $\delta_{k, i, j}(n) \sim \mathcal{B}\left(1, p_{\mathrm{I}}\right)$. Mathematically, the actually transmitted gradient matrix $\widehat{\mathbf{V}}_{k}^{(\mathrm{I})}(n)$ can be seen as the Hadamard (element-wise) product of matrices $\widehat{\mathbf{V}}_{k}(n)$ and $\boldsymbol{\Delta}_{k}(n)$, i.e.,

$$
\widehat{\mathbf{V}}_{k}^{(\mathrm{I})}(n)=\boldsymbol{\Delta}_{k}(n) \circ \widehat{\mathbf{V}}_{k}(n)
$$

In this situation, the MXL algorithm presented in Section $\Pi$ can still be performed, with (2)-(3) replaced by

$$
\begin{align*}
& \mathbf{X}_{k}^{(\mathrm{I})}(n)=A_{k} \frac{\exp \left(\mathbf{Y}_{k}^{(\mathrm{I})}(n-1)\right)}{1+\left\|\exp \left(\mathbf{Y}_{k}^{(\mathrm{I})}(n-1)\right)\right\|}  \tag{6}\\
& \mathbf{Y}_{k}^{(\mathrm{I})}(n)=\mathbf{Y}_{k}^{(\mathrm{I})}(n-1)+\gamma_{n} \boldsymbol{\Delta}_{k}(n) \circ \widehat{\mathbf{V}}_{k}(n) \tag{7}
\end{align*}
$$

In the rest of this section, we investigate the almost surely convergence of MXL-I (the MXL performed with incomplete feedback), as well as the convergence rate of MXL-I. Note that we only present our analysis with $A_{k}=1$ to lighten the equations. The general case can be analyzed in the same way with rescaling.

## A. Convergence of MXL-I

The analysis in this section is performed under the following assumption:

A3. $\mathbf{X}^{*} \in \mathcal{X}$ is globally stable, which means that

$$
\begin{equation*}
\operatorname{tr}\left(\left(\mathbf{X}-\mathbf{X}^{*}\right) \mathbf{V}(\mathbf{X})\right) \leq 0, \forall \mathbf{X} \in \mathcal{X} \tag{8}
\end{equation*}
$$

Notice that the global stability implies the uniqueness of NE $\mathrm{X}^{*}$.

We consider the generalized quantum Kullback-Leibler divergence [11]

$$
\begin{align*}
\mathrm{d}_{\mathrm{KL}}\left(\mathbf{X}^{*}, \mathbf{X}\right)= & \operatorname{tr}\left(\mathbf{X}^{*}\left(\log \mathbf{X}^{*}-\log \mathbf{X}\right)\right) \\
& +\left(1-\operatorname{tr}\left(\mathbf{X}^{*}\right)\right) \log \frac{1-\operatorname{tr}\left(\mathbf{X}^{*}\right)}{1-\operatorname{tr}(\mathbf{X})} \tag{9}
\end{align*}
$$

to have a measure of the distance between the NE $\mathbf{X}^{*}$ and an arbitrary action $\mathbf{X} \in \mathcal{X}$. Note that $\mathrm{d}_{\mathrm{KL}}\left(\mathbf{X}^{*}, \mathbf{X}\right)$ is the Bregman divergence [16] associated with the strictly convex generating function $h(\mathbf{X})$, with

$$
\begin{gather*}
h(\mathbf{X})=\operatorname{tr}(\mathbf{X} \log \mathbf{X})+(1-\operatorname{tr}(\mathbf{X}))(\log 1-\operatorname{tr}(\mathbf{X}))  \tag{10}\\
\mathrm{d}_{\mathrm{KL}}\left(\mathbf{X}^{*}, \mathbf{X}\right)=h\left(\mathbf{X}^{*}\right)-h(\mathbf{X})-\left\langle\nabla h(\mathbf{X}), \mathbf{X}^{*}-\mathbf{X}\right\rangle . \tag{11}
\end{gather*}
$$

In fact, $h(\mathbf{X})$ is a modified von Neumann entropy [11]. According to the general property of Bregman divergence, we know that $d_{\text {KL }} \geq 0$ is a measure of difference, increasing with the difference between $\mathbf{X}^{*}$ and $\mathbf{X}$. Besides, $\mathrm{d}_{\mathrm{KL}}=0$ if and only if $\mathbf{X}^{*}=\mathbf{X}$.

We are interested in the evolution of the divergence $\mathrm{d}_{\mathrm{KL}}\left(\mathbf{X}^{*}, \mathbf{X}^{(\mathrm{I})}(n)\right)$ to analyze the convergence of algorithm. To begin with our analysis, we recall a preliminary property of the MXL algorithm that has been presented in [8].

Lemma 1. In the MXL algorithm, we have

$$
\begin{align*}
& \mathrm{d}_{K L}\left(\mathbf{X}^{*}, \mathbf{X}(n+1)\right) \leq \mathrm{d}_{K L}\left(\mathbf{X}^{*}, \mathbf{X}(n)\right) \\
& \quad+\gamma_{n} \operatorname{tr}\left(\left(\mathbf{X}(n)-\mathbf{X}^{*}\right) \widehat{\mathbf{V}}(\mathbf{X})\right)+\gamma_{n}^{2}\|\widehat{\mathbf{V}}(\mathbf{X})\|_{\infty}^{2} \tag{12}
\end{align*}
$$

Proof: See Appendix A
Let us now establish our first theoretical result in this paper, which is the almost sure convergence of our modified MXL algorithm.

Theorem 1. Suppose that Assumptions Al-A3 are satisfied, then MXL-I converges to NE almost surely, i.e., $d_{n}^{(\mathrm{I})} \rightarrow 0$ a.s..

Proof: See Appendix B
Due to the presence of the additional stochastic term $\boldsymbol{\Delta}_{k}(n)$, the analysis of MXL-I is more complicated than that of MXL presented in [8]. As we can see in the proof in Appendix B , there is an additional term of stochastic noise that has to be analyzed and cannot be done by applying the tool used in [8]. In fact, we develop a new analysis by using Doob's martingale inequality to prove that such novel noise has the required similar property compared with the classical noise term $\mathbf{Z}=\widehat{\mathbf{V}}(\mathbf{X})-\mathbf{V}(\mathbf{X})$. The demonstration detail is provided in Appendix C.

## B. Convergence rate of MXL-I

In this section, we are interested in the evolution of the expected value of the divergence over all the stochastic items, i.e., $D_{n}^{(\mathrm{I})}=\mathbb{E}\left[d_{n}^{(\mathrm{I})}\right]=\mathbb{E}\left[\mathrm{d}_{\mathrm{KL}}\left(\mathbf{X}^{*}, \mathbf{X}^{(\mathrm{I})}(n)\right)\right]$. To simplify the analysis, we further assume that the NE $\mathbf{X}^{*}$ is strongly stable

A4. Given a constant $B>0$, the $\mathrm{NE} \mathbf{X}^{*}$ is $B$-strongly stable, i.e.,

$$
\begin{equation*}
\operatorname{tr}\left(\left(\mathbf{X}-\mathbf{X}^{*}\right) \mathbf{V}(\mathbf{X})\right) \leq-B \mathrm{~d}_{\mathrm{KL}}\left(\mathbf{X}^{*}, \mathbf{X}\right), \forall \mathbf{X} \in \mathcal{X} \tag{13}
\end{equation*}
$$

We aim to derive an upper bound of $D_{n}^{(\mathrm{I})}$ in order to find the convergence rate of MXL-I. We can also observe the influence of the incompleteness of the feedback information on the convergence rate.

Despite of the fact that we have an additional stochastic term $\boldsymbol{\Delta}(n)$, it is also worth mentioning that we consider a general setting of the step-size, while $\gamma_{n}=n^{-1}$ in [8]. Our result is presented in what follows.

Theorem 2. Assume that the assumptions A1, A2, and A4 hold, the MXL-I algorithm is performed such that each element of the gradient matrix is sent with a constant probability $p_{\mathrm{I}}$. Define

$$
\varepsilon=\max \frac{\gamma_{n}-\gamma_{n+1}}{\gamma_{n}^{2}}
$$

then if

$$
\begin{equation*}
\varepsilon<p_{\mathrm{I}} B<\frac{1}{\gamma_{1}} \tag{14}
\end{equation*}
$$

we have

$$
\begin{equation*}
D_{n}^{(\mathrm{I})} \leq \lambda \gamma_{n}, \text { with } \lambda=\max \left\{\frac{D_{1}^{(\mathrm{I})}}{\gamma_{1}}, \frac{p_{\mathrm{I}} C}{p_{\mathrm{I}} B-\varepsilon}\right\} \tag{15}
\end{equation*}
$$

Proof: See Appendix D.
Thanks to the fact that $\gamma_{n}$ is vanishing and the constant $\lambda$ is bounded, we find that the upper bound of $D_{n}^{(\mathrm{I})}$ is vanishing by Theorem 2, We may consider another definition of convergence.

Definition 1. We say that the MXL algorithm converges in mean square to NE if $\mathbb{E}\left[\left\|\mathbf{X}^{*}-\mathbf{X}\right\|^{2}\right] \rightarrow 0$ as $n \rightarrow \infty$.

Since $h(\mathbf{X})$ is $1 / 2$-strongly convex, we have $\mathrm{d}_{\mathrm{KL}}\left(\mathbf{X}^{*}, \mathbf{X}\right) \geq$ $\frac{1}{4}\left\|\mathbf{X}^{*}-\mathbf{X}\right\|^{2}$ by the property of Bregman divergence [16], which means that $D_{n}^{(\mathrm{I})} \rightarrow 0$ implies $\mathbb{E}\left[\left\|\mathbf{X}^{*}-\mathbf{X}\right\|^{2}\right] \rightarrow 0$. Then we can easily conclude the following result.

Corollary 1. As long as the condition (14) holds, we have $D_{n}^{(\mathrm{I})} \rightarrow 0$ as $n \rightarrow \infty$, the MXL algorithm converges in mean square to $N E$.

The decreasing order of the average divergence is the same as that of the step-size $\gamma_{n}$, which does not depend on $p_{\mathrm{I}}$. In fact, the original MXL with full gradient knowledge is a special case with $p_{\mathrm{I}}=1$ and $D_{n}^{(\mathrm{O})}$ can also be bounded by (15) 1 It is obvious that the incompleteness of the feedback only affects the constant term $\lambda$ of the upper bound of $D_{n}^{(\mathrm{I})}$, while the decreasing order only depends on the step-size $\gamma_{n}$.

Now we consider an example of $\gamma_{n}$ and investigate $\varepsilon$ to show that the condition (14) can be easily satisfied.

Example 1. In most work related to stochastic approximation, a common setting of $\gamma_{n}$ is

$$
\begin{equation*}
\gamma_{n}=\alpha n^{-\nu} \tag{16}
\end{equation*}
$$

with $\nu \in(0.5,1]$ and $\alpha \in \mathbb{R}^{+}$. Here the constraint on $\nu$ is to make $\gamma_{n}$ satisfy the assumption $A 2$.

[^1]We check first whether $\varepsilon$ is bounded when $\gamma_{n}$ follows (16). Due to the fact that the denominator $\gamma_{n}^{2} \rightarrow 0$ as $n \rightarrow \infty$, we mainly need to check

$$
\begin{align*}
\lim _{n \rightarrow \infty} \frac{\gamma_{n}-\gamma_{n+1}}{\gamma_{n}^{2}} & =\lim _{n \rightarrow \infty} \frac{1-\left(1+\frac{1}{n}\right)^{-\nu}}{\alpha n^{-\nu}}=\lim _{x \rightarrow 0} \frac{1-(1+x)^{-\nu}}{\alpha x^{\nu}} \\
& =\lim _{x \rightarrow 0} \frac{(1+x)^{-\nu-1}}{\alpha x^{\nu-1}}=\lim _{x \rightarrow 0} \frac{1}{\alpha} x^{1-\nu} \\
& = \begin{cases}\infty, & \text { if } \nu>1 \\
1 / \alpha, & \text { if } \nu=1 \\
0, & \text { else. }\end{cases} \tag{17}
\end{align*}
$$

Hence $\frac{\gamma_{n}-\gamma_{n+1}}{\gamma_{n}^{2}}$ is bounded in our case as $\nu \leq 1$, we can see clearly the importance of the assumption $A 2$.

Then we aim to find an upper bound of $\varepsilon$. In fact

$$
\varepsilon=\max _{n \geq 1} \frac{1-\left(1+\frac{1}{n}\right)^{-\nu}}{\alpha n} \leq \max _{0<x \leq 1} \frac{g(x)}{\alpha}
$$

with

$$
g(x)=x^{-\nu}\left(1-(1+x)^{-\nu}\right)
$$

the inequality mainly comes from the fact that $\frac{1}{n}$ takes discrete values from the set $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$, while $x$ takes any real values from the interval $(0,1]$. The lemma in what follows describes an upper bound of $\varepsilon$.

Lemma 2. For any $x \in(0,1]$ and $\nu \in(0.5,1]$, we have

$$
g(x) \leq \bar{\varepsilon}(\nu)= \begin{cases}\nu\left(\frac{1-\nu}{2 \nu}\right)^{1-\nu}, & \text { if } \nu \in\left(\log _{2} 1.5,1\right]  \tag{18}\\ 1-2^{-\nu}, & \text { if } \nu \in\left(0.5, \log _{2} 1.5\right]\end{cases}
$$

## Proof: See Appendix E

By applying Lemma 2, the condition (14) holds if $\frac{1}{\alpha} \bar{\varepsilon}(\nu)<$ $p_{\mathrm{I}} B<\frac{1}{\alpha}$. As $\nu \in(0.5,1]$, we can verify that $\bar{\varepsilon}(\nu) \leq 1$ with the equality if and only if $\nu=1$. We can conclude that for any fixed $\nu, \alpha$ should be chosen from the interval $\left(\frac{\bar{\varepsilon}(\nu)}{p_{1} B}, \frac{1}{p_{1} B}\right)$.

For other more complicated forms of $\gamma_{n}$, the conditions (14) can also be verified numerically.

## IV. Sporadic MXL

In this section, we introduce and analyze the sporadic version of the MXL algorithm.

Different from the situation that all the receivers send incomplete feedback information, now we consider the scenario that a part of transmitters send complete feedback information at each time instant. Particularly, we consider the following setting:
$S 2$. At each iteration $n$ of the algorithm, each receiver $k$ has a probability $p_{\mathrm{S}}$ to send the feedback $\widehat{\mathbf{V}}_{k}(n)$.

The transmitters update their action only when they have received the feedback. Let a Bernoulli random variable $\eta_{k}(n) \in$ $\{0,1\}$ indicate whether receiver $k$ feeds back the gradient matrix. We have $\eta_{k}(n) \sim \mathcal{B}\left(1, p_{\mathrm{S}}\right)$ according to the setting S2.

In the sporadic MXL (MXL-S) algorithm, at each iteration $n \geq 1$, each transmitter $k$ update their action using

$$
\begin{align*}
& \mathbf{X}_{k}^{(\mathrm{S})}(n)=A_{k} \frac{\exp \left(\mathbf{Y}_{k}^{(\mathrm{S})}(n-1)\right)}{1+\left\|\exp \left(\mathbf{Y}_{k}^{(\mathrm{S})}(n-1)\right)\right\|}  \tag{19}\\
& \mathbf{Y}_{k}^{(\mathbf{S})}(n)=\mathbf{Y}_{k}^{(\mathbf{S})}(n-1)+\gamma_{n_{k}} \eta_{k}(n) \widehat{\mathbf{V}}_{k}(n) \tag{20}
\end{align*}
$$

where $n_{k}$ is maintained by each transmitter $k$ to indicate the index of the step-size $\gamma$. to be applied during algorithm. In this paper, we consider $n_{k}=\widetilde{n}_{k}+\eta_{k}(t)$, with $\widetilde{n}_{k}$ a purely Binomial random variable, $\widetilde{n}_{k} \sim \mathcal{B}\left(n-1, p_{\mathrm{S}}\right)$.

The analysis of MXL-S is more complex than that of MXLI. The main difference is that: in MXL-S, transmitters update their action at random time slot and the step-sizes used by different transmitters are completely independent; while in MXL-I, transmitters update their action at each time slot and their step-size is always identical.

The desirable property of $\gamma_{n_{k}} \eta_{k}(n)$ is stated in the following lemma, which is essential to our main result.
Lemma 3. Denote $\bar{\gamma}_{n}=\mathbb{E}\left[\gamma_{n_{k}} \eta_{k}(n)\right]$ and $\dot{\gamma}_{n}=$ $\sqrt{\mathbb{E}\left[\left(\gamma_{n_{k}} \eta_{k}(n)\right)^{2}\right]}$, then we have

$$
\begin{align*}
& \bar{\gamma}_{n}=\sum_{\ell=1}^{n} \gamma_{\ell} p_{\mathrm{S}}^{\ell}\left(1-p_{\mathrm{S}}\right)^{n-\ell}\binom{n-1}{\ell-1}  \tag{21}\\
& \dot{\gamma}_{n}^{2}=\sum_{\ell=1}^{n} \gamma_{\ell}^{2} p_{\mathrm{S}}^{\ell}\left(1-p_{\mathrm{S}}\right)^{n-\ell}\binom{n-1}{\ell-1} \tag{22}
\end{align*}
$$

More importantly,

$$
\begin{gather*}
\sum_{n=1}^{\infty} \bar{\gamma}_{n}=\sum_{n=1}^{\infty} \gamma_{n} \rightarrow \infty  \tag{23}\\
\sum_{n=1}^{\infty} \bar{\gamma}_{n}^{2} \leq \sum_{n=1}^{\infty} \dot{\gamma}_{n}^{2}=\sum_{n=1}^{\infty} \gamma_{n}^{2}<\infty \tag{24}
\end{gather*}
$$

Proof: See Appendix F.
In the rest of this section, the convergence of MXL-S is discussed.

## A. Convergence of $M X L-S$

As in Section III-A, we assume that the $\mathrm{NE} \mathbf{X}^{*} \in \mathcal{X}$ is globally stable. We investigate the a.s. convergence of MXLS, i.e., to check whether $d_{n}^{(\mathrm{S})}=\mathrm{d}_{\mathrm{KL}}\left(\mathbf{X}^{*}, \mathbf{X}^{(\mathrm{S})}(n)\right) \rightarrow 0$ a.s..

The main challenge is that the update performed by each transmitter is at random time slot and the step-size is also generated in a random manner. Our result is stated in what follows.

Theorem 3. Suppose that Assumptions A1-A3 are satisfied, then MXL-S converges to NE a.s., i.e., $d_{n}^{(S)} \rightarrow 0$ a.s..

Proof: See Appendix G
The main idea of the proof is to compare the MXL-S with an original MXL using the average step-size $\bar{\gamma}_{n}$ defined in (21), which converges to NE a.s. according to the property of $\bar{\gamma}_{n}$ as stated in Lemma 3. Their difference can be seen as another type of stochastic noise to be learned with the help of

Lemma3 i.e., the property of $\dot{\gamma}_{n}$. Note that the tool used in the proof can be applied to analyze the MXL with asynchronous update of the transmitter, even if the purpose here is beyond this issue.

## B. Convergence rate of $M X L-S$

For the MXL-S algorithm, we use $D_{n}^{(\mathrm{S})}$ to denote the average quantum KL divergence. We re-consider Assumption A4 to get the result as presented in Theorem 4

Theorem 4. Assume that the assumptions A1, A2, and A4 hold, the MXL-S algorithm is performed such that each receiver feeds back with probability $p_{\mathrm{S}}$. Let

$$
\begin{equation*}
\epsilon=\max \frac{1}{\grave{\gamma}_{n}^{2}}\left(\frac{\stackrel{\grave{\gamma}}{n}_{2}^{\bar{\gamma}_{n}}}{\left.-\frac{\check{\gamma}_{n+1}^{2}}{\bar{\gamma}_{n+1}}\right), ~, ~}\right. \tag{25}
\end{equation*}
$$

then if $\epsilon<B<\frac{1}{\gamma_{1}}$, we have

$$
\begin{equation*}
D_{n}^{(\mathrm{S})} \leq \mu \frac{\dot{\gamma}_{n}^{2}}{\bar{\gamma}_{n}} \text { with } \mu=\max \left\{\frac{D_{1}^{(\mathrm{S})}}{\gamma_{1}}, \frac{C}{B-\epsilon}\right\} \tag{26}
\end{equation*}
$$

Proof: See Appendix I.
Compared with $D_{n}^{(\mathrm{I})}$, the decreasing order of $D_{n}^{(\mathrm{S})}$ is more complicated, as it depends on the equivalent step-size $\dot{\gamma}_{n}^{2} / \bar{\gamma}_{n}$ which is affected by $p_{\mathrm{S}}$. We have obtained an upper bound of $\dot{\gamma}_{n}^{2} / \bar{\gamma}_{n}$ described in the following lemma, which can be used to prove the convergence of the sporadic MXL to NE from a theoretical point of view.

Lemma 4. If $\gamma_{n}$ is convex over $n$, then for any constant $\xi \in$ $(0,1)$ we have

$$
\begin{equation*}
\frac{\stackrel{\circ}{\gamma}_{n+1}^{2}}{\bar{\gamma}_{n+1}} \leq \frac{\exp \left(-\frac{1}{2} \xi^{2} p_{\mathrm{S}} n\right) \gamma_{1}^{2}+\gamma_{\left\lfloor(1-\xi) p_{\mathrm{S}} n\right\rfloor+2}^{2}}{\gamma_{\left\lfloor p_{\mathrm{S}} n\right\rfloor+2}} \tag{27}
\end{equation*}
$$

## Proof: See Appendix J.

Note that the convexity of $\gamma_{n}$ is not a big assumption as $\gamma_{n}$ satisfies A2. In fact, $\gamma_{n}=\alpha n^{-\nu}$ as introduced in Example 1 is convex.

Corollary 2. If $\gamma_{n}=\alpha n^{-\nu}$ with $\alpha \in \mathbb{R}^{+}$and $\nu \in(0.5,1]$, then $D_{n}^{(S)} \rightarrow 0$ as $n \rightarrow \infty$, the MXL algorithm converges in mean square to $N E$.

$$
\begin{align*}
& \text { Proof: From Lemma 4, we can evaluate } \\
& \lim _{n \rightarrow \infty} \frac{\dot{\gamma}_{n+1}^{2}}{\bar{\gamma}_{n+1}} \\
& \leq \lim _{n \rightarrow \infty} \alpha \frac{\exp \left(-\frac{1}{2} \xi^{2} p_{\mathrm{S}} n\right)+\left(\left\lfloor(1-\xi) p_{\mathrm{S}} n\right\rfloor+2\right)^{-2 \nu}}{\left(\left\lfloor p_{\mathrm{S}} n\right\rfloor+2\right)^{-\nu}} \\
& =\lim _{n \rightarrow \infty} \alpha\left(\frac{\left\lfloor p_{\mathrm{S}} n\right\rfloor+2}{\left(\left\lfloor(1-\xi) p_{\mathrm{S}} n\right\rfloor+2\right)^{2}}\right)^{\nu} \\
& =\lim _{n \rightarrow \infty} \alpha\left((1-\xi)^{2} p_{\mathrm{S}}\right)^{-\nu} n^{-\nu}=0 \tag{28}
\end{align*}
$$

therefore $\dot{\gamma}_{n}^{2} / \bar{\gamma}_{n}$ is vanishing when $\gamma_{n}=\alpha n^{-\nu}$. In fact, we can see that $\dot{\gamma}_{n}^{2} / \bar{\gamma}_{n} \propto n^{-\nu}$ as $n$ is large enough from 28. Besides, the constant term $\mu$ is bounded, we can then conclude that $D_{n}^{(\mathrm{S})} \rightarrow 0$ as $n \rightarrow \infty$, which implies the convergence of the MXL algorithm to NE as $\gamma_{n}=\alpha n^{-\nu}$.


Fig. 1. For a static channel, evolution of energy efficiency of link 8, average results from 100 simulations by performing: (i) MXL-I with $p_{\mathrm{I}} \in\{0.2,0.5\}$ and $p_{\mathrm{S}}=1$; (ii) MXL-S with $p_{\mathrm{S}} \in\{0.2,0.5\}$ and $p_{\mathrm{I}}=1$; (iii) original MXL with $p_{\mathrm{I}}=p_{\mathrm{S}}=1$.

## V. Numerical Example

In our simulation, we consider $K=9$ pairs of transmitterreceiver links. For each link, we set $P_{\mathrm{c}}=20 \mathrm{dBm}, P_{\max }=$ $30 \mathrm{dBm}, N_{t}=4, N_{r}=8$ and $S=3$. The additive noise $\mathbf{Z}$ is generated as Gaussian random variable with zero-mean and variance 1. For each different setting, 100 independent simulations are preformed to obtain the average results.

In the MXL algorithm, we set the initial transmit power as $P_{\max } / 2$ and the step-size is $\gamma_{n}=0.2 n^{-0.7}$. For short, we use MXL-I and MXL-S to name the MXL algorithm with incomplete feedback and the sporadic MXL, respectively.

We start with an easy situation: the channel matrix keeps static with its initial value randomly generated. The evolution of the EE of different links has similar shape. Consider an arbitrary link 8, Figure 1 compares the average evolution of $u_{8}(\mathrm{EE})$ obtained by performing MXL-I with $p_{\mathrm{I}} \in\{0.2,0.5\}$ and $p_{\mathrm{S}}=1, \mathrm{MXL}-\mathrm{S}$ with $p_{\mathrm{S}} \in\{0.2,0.5\}$ and $p_{\mathrm{I}}=1$, as well as the original MXL with $p_{\mathrm{I}}=p_{\mathrm{S}}=1$. Furthermore, Figure 2 shows the average divergence $D_{n}^{(\mathrm{I})}$ and $D_{n}^{(\mathrm{S})}$ in all these cases. We can see that the MXL algorithm tends to converge to NE in all the cases. For the same level of traffic, for example, MXL-I with $p_{\mathrm{I}}=0.5$ and MXL-S with $p_{\mathrm{S}}=0.5$, we find that MXL-S converges faster than MXL-I. In fact, MXL-S converges slightly slower compared with the original MXL, even if half of the signaling information is reduced. Another interesting result is that the performance of MXL-S is less sensitive to $p_{\mathrm{S}}$, while MXL-I is more sensitive to $p_{\mathrm{I}}$. As we can see from the results, the difference between the curves related to MXL-S with $p_{\mathrm{S}}=0.5$ and $p_{\mathrm{S}}=0.2$ are much smaller than the difference presented in MXL-I with $p_{\mathrm{I}}=0.5$ and $p_{\mathrm{I}}=0.2$.

Then we consider a much more challenging situation where channel matrix is stochastic and its elements are randomly and independently generated at each iteration. Results are presented in Figures 3 and 4, which are similar compared with Figures 1 and 2. We can see that average EE is quite sensitive to the stochastic channel, while the evolution $D_{n}$ is smooth by the average of 100 simulations. Our claim is thus further justified by the Figures 3 and 4


Fig. 2. For a static channel, evolution of the average quantum KullbackLeibler divergence to NE, average results from 100 simulations by performing: (i) MXL-I with $p_{\mathrm{I}} \in\{0.2,0.5\}$ and $p_{\mathrm{S}}=1$; (ii) MXL-S with $p_{\mathrm{S}} \in\{0.2,0.5\}$ and $p_{\mathrm{I}}=1$; (iii) original MXL with $p_{\mathrm{I}}=p_{\mathrm{S}}=1$.


Fig. 3. For a stochastic channel, evolution of energy efficiency of link 8 , average results from 100 simulations by performing: (i) MXL-I with $p_{\mathrm{I}} \in$ $\{0.2,0.5\}$ and $p_{\mathrm{S}}=1$; (ii) MXL-S with $p_{\mathrm{S}} \in\{0.2,0.5\}$ and $p_{\mathrm{I}}=1$; (iii) original MXL with $p_{\mathrm{I}}=p_{\mathrm{S}}=1$.


Fig. 4. For a stochastic channel, evolution of the average quantum KullbackLeibler divergence to NE, average results from 100 simulations by performing: (i) MXL-I with $p_{\mathrm{I}} \in\{0.2,0.5\}$ and $p_{\mathrm{S}}=1$; (ii) MXL-S with $p_{\mathrm{S}} \in\{0.2,0.5\}$ and $p_{\mathrm{I}}=1$; (iii) original MXL with $p_{\mathrm{I}}=p_{\mathrm{S}}=1$.

## VI. CONCLUSION

In this paper, we investigate the performance of the MXL algorithm under different feedback strategies. We have proposed two variants of the MXL algorithm in order to reduce the signaling overhead: one is by making receivers feedback only part the elements of the gradient matrix per iteration; the other is by making receivers sporadically feedback the whole gradient matrix. For both strategies, we have proved the convergence of the MXL algorithm to NE and evaluated the upper bounds of the average convergence rate as well. From the theoretical results we can clearly see that the incompleteness of the feedback information does not seriously affect the convergence rate of the MXL algorithm. In the simulations, we consider a distributed energy efficiency maximization problem in a multi-user, multicarrier MIMO network. The results are provided to justify our claim. In some scenario, such as the simulation that we considered, the second proposed strategy performs better in terms of the convergence rate.

## Appendix

## A. Proof sketch of Lemma 1

We present a brief proof that has been presented in [8]. Consider

$$
\begin{equation*}
h^{*}(\mathbf{Y})=\max _{\mathbf{X} \in \mathbb{H}_{+}^{M}: \operatorname{tr}(\mathbf{X}) \leq 1}[\operatorname{tr}(\mathbf{Y X})-h(\mathbf{X})], \tag{29}
\end{equation*}
$$

where $h^{*}$ denotes the convex conjugate of $h$ over a spectrahedron, with $\mathbf{Y} \in \mathbb{H}^{M}$. As stated in Proposition A. 1 of [8], the closed expression of $h^{*}(\mathbf{Y})$ can be derived, i.e., $h^{*}(\mathbf{Y})=\log (1+\operatorname{tr}(\exp (\mathbf{Y})))$. Introduce

$$
\begin{equation*}
G(\mathbf{Y})=\frac{\exp (\mathbf{Y})}{1+\operatorname{tr}(\exp (\mathbf{Y}))} \tag{30}
\end{equation*}
$$

then it is straightforward to deduce that $G(\mathbf{Y})$ is the gradient matrix of $h^{*}(\mathbf{Y})$, i.e., $G(\mathbf{Y})=\nabla h^{*}(\mathbf{Y})$.

With the above definition, for the NE $\mathbf{X}^{*}$ and any $\mathbf{Y} \in \mathbb{H}^{M}$, we consider the Fenchel coupling

$$
\begin{equation*}
F\left(\mathbf{X}^{*}, \mathbf{Y}\right)=h\left(\mathbf{X}^{*}\right)+h^{*}(\mathbf{Y})-\operatorname{tr}\left(\mathbf{Y} \mathbf{X}^{*}\right) \tag{31}
\end{equation*}
$$

We have $F\left(\mathbf{X}^{*}, \mathbf{Y}\right) \geq 0$ by Fenchel-Young inequality, with equality iif $\mathbf{X}^{*}=\nabla h^{*}(\mathbf{Y})=G(\mathbf{Y})$. According to the equivalence between Fenchel coupling and KL divergence presented in [17], we have $\mathrm{d}_{\mathrm{KL}}\left(\mathbf{X}^{*}, G(\mathbf{Y})\right)=F\left(\mathbf{X}^{*}, \mathbf{Y}\right)$.

In the MXL algorithm, we find that 2 can be written as $\mathbf{X}(n)=G(\mathbf{Y}(n))$. Hence,

$$
\begin{align*}
& \mathrm{d}_{\mathrm{KL}}\left(\mathbf{X}^{*}, \mathbf{X}(n+1)\right)=\mathrm{d}_{\mathrm{KL}}\left(\mathbf{X}^{*}, G(\mathbf{Y}(n+1))\right) \\
& =F\left(\mathbf{X}^{*}, \mathbf{Y}(n+1)\right) \stackrel{(a)}{=} F\left(\mathbf{X}^{*}, \mathbf{Y}(n)+\mathbf{U}(n)\right) \\
& \stackrel{(b)}{\leq} F\left(\mathbf{X}^{*}, \mathbf{Y}(n)\right)+\operatorname{tr}\left(\left(G(\mathbf{Y}(n))-\mathbf{X}^{*}\right) \mathbf{U}(n)\right)+\|\mathbf{U}(n)\|_{\infty}^{2} \\
& =\mathrm{d}_{\mathrm{KL}}\left(\mathbf{X}^{*}, \mathbf{X}(n)\right)+\operatorname{tr}\left(\left(\mathbf{X}(n)-\mathbf{X}^{*}\right) \mathbf{U}(n)\right)+\|\mathbf{U}(n)\|_{\infty}^{2}, \tag{32}
\end{align*}
$$

where we introduce $\mathbf{U}(n)=\mathbf{Y}(n+1)-\mathbf{Y}(n)$ in $(a)$, recall that $\mathbf{U}(n)=\gamma_{n} \widehat{\mathbf{V}}(n)$ in the MXL algorithm, according to (3); $(b)$ is by Proposition A. 2 in [8], which can be proved by the strong smoothness of $h^{*}$.

## B. Proof of Theorem 1

Define

$$
d_{n}^{(\mathrm{I})}=\mathrm{d}_{\mathrm{KL}}\left(\mathbf{X}^{*}, \mathbf{X}^{(\mathrm{I})}(n)\right)
$$

According to Lemma 1, we have

$$
\begin{align*}
d_{n+1}^{(\mathrm{I})} \leq d_{n}^{(\mathrm{I})}+ & \gamma_{n} \operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right) \boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)\right) \\
+ & \gamma_{n}^{2}\|\boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)\|_{\infty}^{2} \tag{33}
\end{align*}
$$

by the fact that $\mathbf{U}(n)=\gamma_{n} \boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)$ in MXL-I according to (7). Note that the presence of $\boldsymbol{\Delta}(n)$ makes the proof more complicated, compared with that in [8]. Introduce

$$
\begin{align*}
e_{\boldsymbol{\Delta}}(n) & =\operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right) \boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)\right) \\
& -\mathbb{E}_{\boldsymbol{\Delta}}\left[\operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right) \boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)\right)\right] . \tag{34}
\end{align*}
$$

Since the elements of $\boldsymbol{\Delta}(n)$ follow i.i.d. Bernoulli distribution, it is easy to obtain $\mathbb{E}_{\boldsymbol{\Delta}}(\boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n))=p_{\mathrm{I}} \widehat{\mathbf{V}}(n)$, hence

$$
\begin{align*}
& \mathbb{E}_{\boldsymbol{\Delta}}\left[\operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right) \boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)\right)\right] \\
& =p_{\mathrm{I}} \operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right) \widehat{\mathbf{V}}(n)\right) \tag{35}
\end{align*}
$$

Similar to (34, we define another stochastic noise arisen by the the approximation of the gradient matrix,

$$
\begin{align*}
e_{\mathbf{Z}}^{(\mathrm{I})}(n) & =p_{\mathrm{I}} \operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right) \widehat{\mathbf{V}}(n)\right) \\
& -p_{\mathrm{I}} \mathbb{E}_{\mathbf{Z}}\left[\operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right)(\mathbf{V}(n)+\mathbf{Z}(n))\right)\right] \\
& =p_{\mathrm{I}} \operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right)(\widehat{\mathbf{V}}(n)-\mathbf{V}(n))\right)  \tag{36}\\
& =p_{\mathrm{I}} \operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right) \mathbf{Z}(n)\right), \tag{37}
\end{align*}
$$

which comes from the assumption that the additive noise $\mathbf{Z}$ has zero mean.

Using equations (34)-(37), we evaluate

$$
\begin{align*}
& \operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right) \boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)\right) \\
& =\mathbb{E}_{\boldsymbol{\Delta}}\left[\operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right) \boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)\right)\right]+e_{\boldsymbol{\Delta}}(n) \\
& =p_{\mathrm{I}} \operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right) \widehat{\mathbf{V}}(n)\right)+e_{\boldsymbol{\Delta}}(n) \\
& =p_{\mathrm{I}} \operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right) \mathbf{V}(n)\right)+e_{\boldsymbol{\Delta}}(n)+e_{\mathbf{Z}}^{(\mathrm{I})}(n) \tag{38}
\end{align*}
$$

Combine (33) and (38), we have

$$
\begin{align*}
& d_{n+1}^{(\mathrm{I})} \leq d_{n}^{(\mathrm{I})}+p_{\mathrm{I}} \gamma_{n} \operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right) \mathbf{V}(n)\right) \\
& \quad+\gamma_{n} e_{\boldsymbol{\Delta}}(n)+\gamma_{n} e_{\mathbf{Z}}^{(\mathrm{I})}(n)+\gamma_{n}^{2}\|\boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)\|_{\infty}^{2} . \tag{39}
\end{align*}
$$

Perform the sum of (39), we get
$d_{N+1}^{(\mathrm{I})} \leq d_{1}^{(\mathrm{I})}+p_{\mathrm{I}} \sum_{n=1}^{N} \gamma_{n} \operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right) \mathbf{V}(n)\right)+E_{N}^{(\mathrm{I})}$,
in which
$E_{N}^{(\mathrm{I})}=\sum_{n=1}^{N} \gamma_{n}\left(e_{\boldsymbol{\Delta}}(n)+e_{\mathbf{Z}}^{(\mathrm{I})}(n)+\gamma_{n}\|\boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)\|_{\infty}^{2}\right)$.
We introduce an important lemma with the proof presented in Appendix C.

Lemma 5. As long as Assumptions A1-A2 hold, we have $\left|E_{N}^{(\mathrm{I})}\right|<\infty$ a.s..

The rest part of the proof is straightforward and similar to that in [8]. Recall that $\operatorname{tr}\left(\left(\mathbf{X}(n)-\mathbf{X}^{*}\right) \mathbf{V}(n)\right) \leq 0$ by Assumption A3. The basic idea is to suppose that there exists a small positive constant $c$ and sufficient large $N_{0}$ such that

$$
\begin{equation*}
\operatorname{tr}\left(\left(\mathbf{X}(n)-\mathbf{X}^{*}\right) \mathbf{V}(n)\right) \leq-c, \quad \forall n \geq N_{0} \tag{42}
\end{equation*}
$$

which leads to

$$
\begin{align*}
\lim _{N \rightarrow \infty} \sum_{n=N_{0}}^{N} \gamma_{n} \operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right) \mathbf{V}(n)\right) & \leq-c \sum_{n=N_{0}}^{\infty} \gamma_{n} \\
& <-\infty \tag{43}
\end{align*}
$$

Meanwhile, with $\left|E_{N}^{(\mathrm{I})}\right|<\infty$ a.s., we finally get that $\lim _{N \rightarrow \infty} d_{N+1}^{(\mathrm{I})}<-\infty$, which obviously violates the fact that $d_{N+1}^{(\mathrm{I})} \geq 0$. Therefore, the hypothesis (42) does not hold. Therefore, we can say that $\mathbf{X}_{k}(n)$ converges to $\mathbf{X}_{k}^{*}, \forall k$.

## C. Proof of Lemma 5

In order to prove Lemma 5, it is sufficient to show the following:

$$
\begin{align*}
\lim _{N \rightarrow \infty} \sum_{n=1}^{N} \gamma_{n}^{2}\|\boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)\|_{\infty}^{2} & <\infty  \tag{44}\\
\lim _{N \rightarrow \infty}\left|\sum_{n=1}^{N} \gamma_{n} e_{\boldsymbol{\Delta}}(n)\right| & <\infty, \quad \text { a.s. }  \tag{45}\\
\lim _{N \rightarrow \infty}\left|\sum_{n=1}^{N} \gamma_{n} e_{\mathbf{Z}}^{(\mathrm{I})}(n)\right| & <\infty, \quad \text { a.s. } \tag{46}
\end{align*}
$$

We show them separately in this appendix.

1) Proof of (44): All the elements of the gradient matrix $\widehat{\mathbf{V}}_{k}$ should have bounded value, since they are approximated by each receiver and then have to be transmitted within feedback packets. We can say that there exist a constant $C<\infty$ such that

$$
\begin{equation*}
\sum_{i, j}\left|\widehat{V}_{i, j}(n)\right|^{2} \leq C \tag{47}
\end{equation*}
$$

Hence, we have

$$
\begin{align*}
& \lim _{N \rightarrow \infty} \sum_{n=1}^{N} \gamma_{n}^{2}\|\boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)\|_{\infty}^{2} \\
& \leq \lim _{N \rightarrow \infty} \sum_{n=1}^{N} \gamma_{n}^{2} \sum_{i, j}\left|\delta_{i, j}(n) \widehat{V}_{i, j}(n)\right|^{2} \\
& \leq \lim _{N \rightarrow \infty} \sum_{n=1}^{N} \gamma_{n}^{2} \sum_{i, j}\left|\widehat{V}_{i, j}(n)\right|^{2} \\
& \leq \lim _{N \rightarrow \infty} C \sum_{n=1}^{N} \gamma_{n}^{2}<\infty \tag{48}
\end{align*}
$$

recall that $\delta_{i, j}(n) \in\{0,1\}$.
2) Proof of (45) and (46): By definition (34), it is obvious that: i). $\mathbb{E}_{\boldsymbol{\Delta}}\left[e_{\boldsymbol{\Delta}}(n)\right]=0$; ii). $e_{\boldsymbol{\Delta}}\left(n_{1}\right)$ is independent of $e_{\boldsymbol{\Delta}}\left(n_{2}\right)$ due to the independence of $\boldsymbol{\Delta}\left(n_{1}\right)$ and $\boldsymbol{\Delta}\left(n_{2}\right)$. Therefore, $\sum_{n=1}^{N} \gamma_{n} e_{\boldsymbol{\Delta}}(n)$ is martingale, so that we can use Doob's inequality, to have, for any $\rho>0$,

$$
\begin{align*}
& \mathbb{P}\left[\sup _{N}\left|\sum_{n=1}^{N} \gamma_{n} e_{\boldsymbol{\Delta}}(n)\right| \geq \rho\right] \\
& \leq \rho^{-2} \mathbb{E}_{\boldsymbol{\Delta}}\left[\left|\sum_{n=1}^{N} \gamma_{n} e_{\boldsymbol{\Delta}}(n)\right|^{2}\right] \\
& \stackrel{(a)}{=} \rho^{-2} \sum_{n=1}^{N} \gamma_{n}^{2} \mathbb{E}_{\boldsymbol{\Delta}}\left[e_{\boldsymbol{\Delta}}^{2}(n)\right] \\
& \leq \rho^{-2} \sum_{n=1}^{N} \gamma_{n}^{2} \mathbb{E}_{\boldsymbol{\Delta}}\left[\left(\operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right) \boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)\right)\right)^{2}\right] \\
& \stackrel{(b)}{\leq} \rho^{-2} c_{1} \sum_{n=1}^{N} \gamma_{n}^{2}, \tag{49}
\end{align*}
$$

in which: $(a)$ is by the fact that $\mathbb{E}_{\boldsymbol{\Delta}}\left[e_{\boldsymbol{\Delta}}\left(n_{1}\right) e_{\boldsymbol{\Delta}}\left(n_{2}\right)\right]=0$ for any $n_{1} \neq n_{2}$; we have $\left|\operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right)\right)\right|<\infty$ and $|\operatorname{tr}(\boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n))|<\infty$ as $\widehat{V}_{i, j}(n)$ takes bounded value, there should exists $c_{1}<\infty$ such that

$$
\begin{equation*}
\mathbb{E}_{\boldsymbol{\Delta}}\left[\left(\operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{I})}(n)-\mathbf{X}^{*}\right) \boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)\right)\right)^{2}\right]<c_{1}, \tag{50}
\end{equation*}
$$

as stated in (b). We can say that (45) is true, as (49) implies that the probability that $\left|\sum_{n=1}^{N} \gamma_{n} e_{\Delta}(n)\right| \geq \rho$ decreases with $\rho$.

In a similar way, 46 can be proved. Since the additive noise $\mathbf{Z}(n)$ is i.i.d., $\sum_{n=1}^{N} \gamma_{n} e_{\mathbf{Z}}^{(\mathrm{I})}(n)$ is also martingale. Use the same step as in 49), we mainly need to verify whether $\mathbb{E}_{\mathbf{Z}}\left[\left(e_{\mathbf{Z}}^{(\mathrm{I})}(n)\right)^{2}\right]$ is bounded, which is straightforward to prove by the assumption that $\mathbf{Z}(n)$ has bounded variance.

## D. Proof of Theorem 2

By definition $D_{n}^{(\mathrm{I})}=\mathbb{E}\left[d_{n}^{(\mathrm{I})}\right]$, we perform the expectation of (33) to get

$$
\begin{align*}
D_{n+1}^{(\mathrm{I})} \leq D_{n}^{(\mathrm{I})}+ & \gamma_{n} \mathbb{E}\left[\operatorname{tr}\left(\left(\mathbf{X}(n)-\mathbf{X}^{*}\right) \boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)\right)\right] \\
& +\gamma_{n}^{2} \mathbb{E}\left[\|\boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)\|_{\infty}^{2}\right] \tag{51}
\end{align*}
$$

Since the elements of $\Delta_{k}(n)$ follow Bernoulli distribution, it is easy to evaluate

$$
\begin{align*}
& \mathbb{E}\left[\operatorname{tr}\left(\left(\mathbf{X}(n)-\mathbf{X}^{*}\right) \boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)\right)\right] \\
& =p \mathbb{E}\left[\operatorname{tr}\left(\left(\mathbf{X}(n)-\mathbf{X}^{*}\right)(\mathbf{V}(n)+\mathbf{Z}(n))\right)\right] \\
& \stackrel{(a)}{=} p \mathbb{E}\left[\operatorname{tr}\left(\left(\mathbf{X}(n)-\mathbf{X}^{*}\right) \mathbf{V}(n)\right)\right] \\
& \stackrel{(b)}{\leq}-p_{\mathrm{I}} B D_{n}^{(\mathrm{I})}, \tag{52}
\end{align*}
$$

in which $(a)$ is due to the fact that $\mathbf{Z}(n)$ has zero-mean elements and (b) is by (13).

Meanwhile, we evaluate

$$
\begin{align*}
\mathbb{E}\left[\|\widehat{\mathbf{V}}(n)\|_{\infty}^{2}\right] & \leq \mathbb{E}\left[\sum_{m}\left|\operatorname{eig}_{m}(\widehat{\mathbf{V}}(n))\right|^{2}\right] \\
& =\mathbb{E}\left[\sum_{i, j}\left|\widehat{V}_{i, j}(n)\right|^{2}\right] \leq C \tag{53}
\end{align*}
$$

which comes from the bound in (47). Similar to (53), we have

$$
\begin{align*}
\mathbb{E}\left[\|\boldsymbol{\Delta}(n) \circ \widehat{\mathbf{V}}(n)\|_{\infty}^{2}\right] & \leq \mathbb{E}\left[\sum_{i, j}\left|\delta_{i, j}(n) \widehat{V}_{i, j}(n)\right|^{2}\right] \\
& =p_{\mathrm{I}} \mathbb{E}\left[\sum_{i, j}\left|\widehat{V}_{i, j}(n)\right|^{2}\right] \\
& \leq p_{\mathrm{I}} C . \tag{54}
\end{align*}
$$

Combining (51), 52), and (54), we obtain

$$
\begin{equation*}
D_{n+1}^{(\mathrm{I})} \leq\left(1-p_{\mathrm{I}} B \gamma_{n}\right) D_{n}^{(\mathrm{I})}+p_{\mathrm{I}} C \gamma_{n}^{2} \tag{55}
\end{equation*}
$$

Our aim is to show the existence of a bounded constant $\lambda$ such that $D_{n}^{(\mathrm{I})} \leq \lambda \gamma_{n}$. The proof is by induction.

Obviously, we have $D_{1}^{(\mathrm{I})} \leq \lambda \gamma_{1}$ if $\lambda \geq D_{1}^{(\mathrm{I})} / \gamma_{1}$. Then under the condition that $D_{n}^{(\mathrm{I})} \leq \lambda \gamma_{n}$, we need to verify whether $D_{n+1}^{(\mathrm{I})} \leq \lambda \gamma_{n+1}$. As $\gamma_{n} \leq \gamma_{1}<\frac{1}{p_{\mathrm{I}} B} \forall n$, from 55), we need to have

$$
\begin{align*}
D_{n+1}^{(\mathrm{I})} & \leq\left(1-p_{\mathrm{I}} B \gamma_{n}\right) \lambda \gamma_{n}+p_{\mathrm{I}} C \gamma_{n}^{2} \\
& \leq \lambda \gamma_{n+1} \tag{56}
\end{align*}
$$

meaning that $\lambda$ should be such that

$$
\begin{equation*}
\lambda \geq \frac{p_{\mathrm{I}} C}{p_{\mathrm{I}} B-\max _{n}\left(\frac{\gamma_{n}-\gamma_{n+1}}{\gamma_{n}^{2}}\right)}=\frac{p_{\mathrm{I}} C}{p_{\mathrm{I}} B-\varepsilon} \tag{57}
\end{equation*}
$$

under the condition $p_{\mathrm{I}} B>\varepsilon$. In this way, we conclude that $D_{n}^{(\mathrm{I})} \leq \lambda \gamma_{n}$ if $\lambda \geq \max \left\{\frac{D_{1}^{(\mathrm{I})}}{\gamma_{1}}, \frac{p_{\mathrm{I}} C}{p_{\mathrm{I}} B-\varepsilon}\right\}$, which concludes the proof.

## E. Proof of Lemma 2

Consider the first situation where $\nu=1,18$ holds as

$$
\begin{equation*}
g(x)=x^{-1}\left(1-\frac{1}{1+x}\right)=\frac{1}{1+x} \leq 1 \tag{58}
\end{equation*}
$$

In the other situation, i.e., $\nu \in(0.5,1)$, we evaluate the derivative of $g(x)$, i.e., $g^{\prime}(x)=\nu x^{-\nu-1} h(x)$ with

$$
\begin{equation*}
h(x)=(2 x+1)(1+x)^{-\nu-1}-1 . \tag{59}
\end{equation*}
$$

Thus the monotonicity of $g(x)$ depends on whether $h(x)$ is positive or negative. We further evaluate

$$
\begin{equation*}
h^{\prime}(x)=2 \nu(1+x)^{-\nu-2}\left(\frac{1-\nu}{2 \nu}-x\right) \tag{60}
\end{equation*}
$$

from which we deduce that $h(x)$ is an increasing function as $x \in\left(0, \frac{1-\nu}{2 \nu}\right]$ and $h(x)$ decreases while $x \in\left[\frac{1-\nu}{2 \nu}, 1\right]$. Note that $\frac{1-\nu}{2 \nu} \in(0,1)$. It is easy to get $\lim _{x \rightarrow 0} h(x)=0$ and $h\left(\frac{1-\nu}{2 \nu}\right)>0$. Besides, $h(1)=3 \cdot 2^{-\nu-1}-1$ depends on the value of $\nu$.

If $\nu \in\left(0.5, \log _{2} 1.5\right]$, then $h(1) \geq 0$, which implies that $g(x)$ is an increasing function over $(0,1]$, thus $g(x) \leq g(1)=$ $1-2^{-\nu}$.

If $\nu \in\left(\log _{2} 1.5,1\right)$, then $h(1)<0$. Based on the monotonicity of $h(x)$, we concludes that there exists a single point $x_{0} \in\left(\frac{1-\nu}{2 \nu}, 1\right)$ such that $h\left(x_{0}\right)=0$ and $g(x) \leq g\left(x_{0}\right)$. From $h\left(x_{0}\right)=0$, we get $\left(1+x_{0}\right)^{-\nu}=\frac{1+x_{0}}{1+2 x_{0}}$, thus

$$
\begin{align*}
g(x) & \leq g\left(x_{0}\right)=x_{0}^{-\nu}\left(1-\frac{1+x_{0}}{1+2 x_{0}}\right) \\
& =\frac{x_{0}^{1-\nu}}{1+2 x_{0}} \leq \nu\left(\frac{1-\nu}{2 \nu}\right)^{1-\nu} \tag{61}
\end{align*}
$$

which takes the equality as $x_{0}=\frac{1-\nu}{2 \nu}$.

## F. Proof of Lemma 3

For any $k$ and $n$, we evaluate

$$
\begin{align*}
& \mathbb{E}\left[\gamma_{n_{k}} \eta_{k}(n)\right] \\
& =\mathbb{P}\left[\eta_{k}(n)=1\right] \mathbb{E}\left[\gamma_{\tilde{n}_{k}+\eta_{k}(n)} \eta_{k}(n) \mid \eta_{k}(n)=1\right] \\
& =p_{\mathrm{S}} \sum_{\ell=1}^{n} \gamma_{\ell} \mathbb{P}\left[\widetilde{n}_{k}=\ell-1\right] \\
& =\sum_{\ell=1}^{n} \gamma_{\ell} p_{\mathrm{S}}^{\ell}\left(1-p_{\mathrm{S}}\right)^{n-\ell}\binom{n-1}{\ell-1} \tag{62}
\end{align*}
$$

recall that $\eta_{k}(n) \sim \mathcal{B}\left(1, p_{\mathrm{S}}\right)$ and $\widetilde{n}_{k} \sim \mathcal{B}\left(n-1, p_{\mathrm{S}}\right)$. We find that $\bar{\gamma}_{n}$ defined in (21) represents $\mathbb{E}\left[\gamma_{n_{k}} \eta_{k}(n)\right]$. Similarly, we have

$$
\begin{align*}
& \mathbb{E}\left[\left(\gamma_{n_{k}} \eta_{k}(n)\right)^{2}\right] \\
& =\mathbb{P}\left[\eta_{k}(n)=1\right] \mathbb{E}\left[\left(\gamma_{\tilde{n}_{k}+\eta_{k}(n)} \eta_{k}(n)\right)^{2} \mid \eta_{k}(n)=1\right] \\
& =p_{\mathrm{S}} \sum_{\ell=1}^{n} \gamma_{\ell}^{2} \mathbb{P}\left[\widetilde{n}_{k}=\ell-1\right] \\
& =\sum_{\ell=1}^{n} \gamma_{\ell}^{2} p_{\mathrm{S}}^{\ell}\left(1-p_{\mathrm{S}}\right)^{n-\ell}\binom{n-1}{\ell-1} \tag{63}
\end{align*}
$$

then $\dot{\gamma}_{n}=\sqrt{\mathbb{E}\left[\left(\gamma_{n_{k}} \eta_{k}(n)\right)^{2}\right]}$ are also obtained.
In order to prove 23 and 24, we first present a useful lemma in what follows with its proof presented in the end of this appendix.

Lemma 6. Consider an arbitrary sequence $\left\{a_{n}\right\}$ and define

$$
\begin{equation*}
\bar{a}_{n}=\sum_{\ell=1}^{n} a_{\ell}(1-p)^{\ell} p^{n-\ell}\binom{n-1}{\ell-1} \tag{64}
\end{equation*}
$$

with $p \in[0,1]$, we always have

$$
\begin{equation*}
\sum_{n=1}^{\infty} \bar{a}_{n}=\sum_{n=1}^{\infty} a_{n} \tag{65}
\end{equation*}
$$

Replace $a_{n}$ by $\gamma_{n}$ and $p$ by $1-p_{\mathrm{s}}$, we get that $\sum_{n=1}^{\infty} \bar{\gamma}_{n}=$ $\sum_{n=1}^{\infty} \gamma_{n}$, then 23) can be proved as $\sum_{n=1}^{\infty} \gamma_{n} \rightarrow \infty$.

Replace $a_{n}$ by $\gamma_{n}^{2}$ and $p$ by $1-p_{\mathrm{s}}$, we have $\sum_{n=1}^{\infty} \dot{\gamma}_{n}^{2}=$ $\sum_{n=1}^{\infty} \gamma_{n}^{2}$. Due to the fact that

$$
\begin{equation*}
\dot{\gamma}_{n}^{2}=\mathbb{E}\left[\left(\gamma_{n_{k}} \eta_{k}(n)\right)^{2}\right] \geq\left(\mathbb{E}\left[\gamma_{n_{k}} \eta_{k}(n)\right]\right)^{2}=\bar{\gamma}_{n}^{2} \tag{66}
\end{equation*}
$$

we can finally justify (24), with the assumption that $\sum_{n=1}^{\infty} \gamma_{n}^{2}<\infty$.

The proof of Lemma 6 is presented in the following.
Proof: We evaluate

$$
\begin{align*}
\sum_{n=1}^{\infty} \bar{a}_{n} & =\sum_{n=0}^{\infty} \bar{a}_{n+1} \\
& =(1-p) \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} a_{\ell+1}(1-p)^{\ell} p^{n-\ell}\binom{n}{\ell} \\
& \stackrel{(a)}{=}(1-p) \sum_{\ell=0}^{\infty} a_{\ell+1} \sum_{n=\ell}^{\infty}(1-p)^{\ell} p^{n-\ell}\binom{n}{\ell} \\
& \stackrel{(b)}{=}(1-p) \sum_{\ell=0}^{\infty} a_{\ell+1} w_{\ell} \tag{67}
\end{align*}
$$

where $(a)$ is obtained by changing the order of summation and $(b)$ is by introducing $w_{\ell}=\sum_{n=\ell}^{\infty}(1-p)^{\ell} p^{n-\ell}\binom{n}{\ell}$, which is independent of $n$. We should then focus on the expression of $w_{\ell}$. We have

$$
\begin{align*}
w_{\ell} & \stackrel{(a)}{=} \sum_{m=0}^{\infty}(1-p)^{\ell} p^{m}\binom{m+\ell}{\ell} \\
& =\frac{(1-p)^{\ell}}{\ell!} \sum_{m=0}^{\infty}(m+\ell)(m+\ell-1) \cdots(m+1) p^{m} \\
& \stackrel{(b)}{=} \frac{(1-p)^{\ell}}{\ell!} \sum_{m=0}^{\infty}\left(p^{m+\ell}\right)^{(\ell)}=\frac{(1-p)^{\ell}}{\ell!}\left(\sum_{m=0}^{\infty} p^{m+\ell}\right)^{(\ell)} \\
& =\frac{(1-p)^{\ell}}{\ell!}\left(p^{\ell}(1-p)^{-1}\right)^{(\ell)} \tag{68}
\end{align*}
$$

in which $(a)$ is by change of variable $m=n-\ell$; in $(b)$, we denote $\left(p^{m+\ell}\right)^{(\ell)}$ as the $\ell$-th order derivative of the function $p^{m+\ell}$.

By applying the general Leibniz rule to evaluate the $\ell$ th order derivative of the function $p^{\ell}(1-p)^{-1}$, 68) can be written as

$$
\begin{align*}
w_{\ell} & =\frac{(1-p)^{\ell}}{\ell!} \sum_{k=0}^{\ell}\binom{\ell}{k}\left((1-p)^{-1}\right)^{(k)}\left(p^{\ell}\right)^{(\ell-k)} \\
& \stackrel{(a)}{=} \frac{(1-p)^{\ell}}{\ell!} \sum_{k=0}^{\ell}\binom{\ell}{k} k!(1-p)^{-k-1} \frac{\ell!}{k!} p^{k} \\
& =\sum_{k=0}^{\ell}\binom{\ell}{k}(1-p)^{\ell-k-1} p^{k} \\
& =(1-p)^{-1} \tag{69}
\end{align*}
$$

where $(a)$ is obtained by $\left((1-p)^{-1}\right)^{(k)}=k!(1-p)^{-k-1}$ and $\left(p^{\ell}\right)^{(\ell-k)}=\frac{\ell!}{k!} p^{k}$. We deduce that $w_{\ell}$ is in fact a constant for any $\ell$.

Combine 67) and 69, we get that

$$
\begin{align*}
\sum_{n=1}^{\infty} \bar{a}_{n} & =(1-p) \sum_{\ell=0}^{\infty} a_{\ell+1}(1-p)^{-1} \\
& =\sum_{\ell=0}^{\infty} a_{\ell+1} \tag{70}
\end{align*}
$$

which concludes the proof.

## G. Proof of Theorem 3

Introduce a all-ones matrix $\mathbf{1}_{k}$ of the same shape as $\mathbf{V}_{k}$ and denote $\boldsymbol{\Gamma}(n)=\operatorname{diag}\left(\gamma_{n_{k}} \eta_{k}(n) \mathbf{1}_{k}\right)_{k=1}^{K}$. From (19), we have $\mathbf{Y}^{(\mathrm{S})}(n)=\mathbf{Y}^{(\mathrm{S})}(n-1)+\boldsymbol{\Gamma}(n) \circ \widehat{\mathbf{V}}(n)$.

Similar to (33), we have

$$
\begin{align*}
& d_{n+1}^{(\mathbf{S})} \leq d_{n}^{(\mathbf{S})}+ \operatorname{tr} \\
&\left.+\left(\mathbf{X}^{(\mathbf{S})}(n)-\mathbf{X}^{*}\right) \boldsymbol{\Gamma}(n) \circ \widehat{\mathbf{V}}(n)\right)  \tag{71}\\
&+\|\boldsymbol{\Gamma}(n) \circ \widehat{\mathbf{V}}(n)\|_{\infty}^{2}
\end{align*}
$$

As $e_{\boldsymbol{\Delta}}(n)$ and $e_{\mathbf{Z}}^{(\mathrm{I})}(n)$ defined in Appendix B , we define

$$
\begin{align*}
e_{\boldsymbol{\Gamma}}(n)=\operatorname{tr} & \left(\left(\mathbf{X}^{(\mathrm{S})}(n)-\mathbf{X}^{*}\right) \boldsymbol{\Gamma}(n) \circ \widehat{\mathbf{V}}(n)\right) \\
& -\mathbb{E}_{\boldsymbol{\Gamma}}\left[\operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{S})}(n)-\mathbf{X}^{*}\right) \boldsymbol{\Gamma}(n) \circ \widehat{\mathbf{V}}(n)\right)\right] \\
=\operatorname{tr} & \left(\left(\mathbf{X}^{(\mathrm{S})}(n)-\mathbf{X}^{*}\right) \boldsymbol{\Gamma}(n) \circ \widehat{\mathbf{V}}(n)\right) \\
& -\bar{\gamma}_{n} \operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{S})}(n)-\mathbf{X}^{*}\right) \widehat{\mathbf{V}}(n)\right) \tag{72}
\end{align*}
$$

and

$$
\begin{equation*}
e_{\mathbf{Z}}^{(\mathrm{S})}(n)=\bar{\gamma}_{n} \operatorname{tr}\left(\left(\mathbf{X}^{(\mathbf{S})}(n)-\mathbf{X}^{*}\right) \mathbf{Z}(n)\right) \tag{73}
\end{equation*}
$$

in which $\bar{\gamma}_{n}=\mathbb{E}\left[\gamma_{n_{k}} \eta_{k}(n)\right]$ as defined in 21. By introducing (72)-(73) into (71), we get

$$
\begin{align*}
d_{n+1}^{(\mathrm{S})} & \leq d_{n}^{(\mathrm{S})}+\bar{\gamma}_{n} \operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{S})}(n)-\mathbf{X}^{*}\right) \mathbf{V}(n)\right) \\
& +e_{\boldsymbol{\Gamma}}(n)+e_{\mathbf{Z}}^{(\mathrm{S})}(n)+\|\boldsymbol{\Gamma}(n) \circ \widehat{\mathbf{V}}(n)\|_{\infty}^{2} \tag{74}
\end{align*}
$$

from which we deduce that

$$
\begin{equation*}
d_{N+1}^{(\mathrm{S})} \leq d_{1}^{(\mathrm{S})}+\sum_{n=1}^{N} \bar{\gamma}_{n} \operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{S})}(n)-\mathbf{X}^{*}\right) \mathbf{V}(n)\right)+E_{N}^{(\mathrm{S})} \tag{75}
\end{equation*}
$$

with

$$
\begin{equation*}
E_{N}^{(\mathrm{S})}=\sum_{n=1}^{N}\left(e_{\boldsymbol{\Gamma}}(n)+e_{\mathbf{Z}}^{(\mathrm{S})}(n)+\|\boldsymbol{\Gamma}(n) \circ \widehat{\mathbf{V}}(n)\|_{\infty}^{2}\right) \tag{76}
\end{equation*}
$$

Similar to Lemma 5, it is important to investigate the property of $E_{N}^{(\mathrm{S})}$.

Lemma 7. As long as Assumptions A1-A2 hold, we have $\left|E_{N}^{(\mathrm{S})}\right|<\infty$ a.s..

Proof: See Appendix H

With the result presented in Lemma 7, as well as the important property stated in Lemma 3, i.e., $\sum_{n=1}^{\infty} \bar{\gamma}_{n} \rightarrow \infty$, the rest part of the proof is the same as that in Appendix B since (75) has the same form compared with 40.

## H. Proof of Lemma 7

The proof steps presented in this appendix is similar to that in Appendix C. We need to prove

$$
\begin{align*}
\lim _{N \rightarrow \infty} \sum_{n=1}^{N}\|\boldsymbol{\Gamma}(n) \circ \widehat{\mathbf{V}}(n)\|_{\infty}^{2} & <\infty  \tag{77}\\
& \lim _{N \rightarrow \infty}\left|\sum_{n=1}^{N} e_{\mathbf{Z}}^{(\mathrm{S})}(n)\right|<\infty, \quad \text { a.s. }  \tag{78}\\
& \lim _{N \rightarrow \infty}\left|\sum_{n=1}^{N} e_{\boldsymbol{\Gamma}}(n)\right|<\infty, \quad \text { a.s. } \tag{79}
\end{align*}
$$

The proof of (77) is quite similar to that of (44). We focus on the demonstration of (78) and (79).

1) Proof of 788: Similar to $\sum_{n=1}^{N} e_{\mathbf{Z}}^{(\mathrm{I})}(N)$, $\sum_{n=1}^{N} e_{\mathbf{Z}}^{(\mathrm{S})}(N)$ is also martingale. We apply Doob's inequality again to have, for any $\rho>0$,

$$
\begin{align*}
& \mathbb{P}\left[\sup _{N}\left|\sum_{n=1}^{N} e_{\mathbf{Z}}^{(\mathrm{S})}(n)\right| \geq \rho\right] \\
& \leq \rho^{-2} \mathbb{E}_{\mathbf{Z}}\left[\left(\sum_{n=1}^{N} e_{\mathbf{Z}}^{(\mathrm{S})}(n)\right)^{2}\right]=\rho^{-2} \sum_{n=1}^{N} \mathbb{E}_{\mathbf{Z}}\left[\left(e_{\mathbf{Z}}^{(\mathrm{S})}(n)\right)^{2}\right] \\
& =\rho^{-2} \sum_{n=1}^{N} \bar{\gamma}_{n}^{2} \mathbb{E}_{\mathbf{Z}}\left[\left(\operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{S})}(n)-\mathbf{X}^{*}\right) \mathbf{Z}(n)\right)\right)^{2}\right] \\
& \leq \rho^{-2} c_{2} \sum_{n=1}^{N} \bar{\gamma}_{n}^{2} \tag{80}
\end{align*}
$$

where we have, similar to 50,

$$
\begin{equation*}
\mathbb{E}_{\mathbf{Z}}\left[\left(\operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{S})}(n)-\mathbf{X}_{k}^{*}\right) \mathbf{Z}_{k}(n)\right)\right)^{2}\right] \leq c_{2}<\infty \tag{81}
\end{equation*}
$$

According to Lemma 3, $\sum_{n=1}^{\infty} \bar{\gamma}_{n}^{2}$ is also bounded, therefore we conclude that $\left|\sum_{n=1}^{N} e_{\mathbf{Z}}^{(\mathrm{S})}(n)\right|<\infty$ almost surely as $N \rightarrow$ $\infty$.
2) Proof of (79): Due to the fact that the step-size of each user is generated independently and randomly, we can easily show that $\sum_{n=1}^{N} e_{\boldsymbol{\Gamma}}(n)$ is also martingale. We have, by Doob's
inequality, for any $\rho>0$,

$$
\begin{align*}
& \mathbb{P}\left[\sup _{N}\left|\sum_{n=1}^{N} e_{\boldsymbol{\Gamma}}(n)\right| \geq \rho\right] \\
& \leq \rho^{-2} \mathbb{E}_{\boldsymbol{\Gamma}}\left[\left(\sum_{n=1}^{N} e_{\boldsymbol{\Gamma}}(n)\right)^{2}\right]=\rho^{-2} \sum_{n=1}^{N} \mathbb{E}_{\boldsymbol{\Gamma}}\left[e_{\boldsymbol{\Gamma}}^{2}(n)\right] \\
& \leq \rho^{-2} \sum_{n=1}^{N} \mathbb{E}_{\boldsymbol{\Gamma}}\left[\left(\operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{S})}(n)-\mathbf{X}^{*}\right) \boldsymbol{\Gamma}(n) \circ \widehat{\mathbf{V}}(n)\right)\right)^{2}\right] \\
& \stackrel{(a)}{=} \rho^{-2} \sum_{n=1}^{N} \mathbb{E}_{\boldsymbol{\Gamma}}\left[\left(\sum_{k=1}^{K} \gamma_{n_{k}} \eta_{k}(n)\right.\right. \\
& \left.\left.\cdot \operatorname{tr}\left(\left(\mathbf{X}_{k}^{(\mathrm{S})}(n)-\mathbf{X}_{k}^{*}\right) \widehat{\mathbf{V}}_{k}(n)\right)\right)^{2}\right] \\
& \stackrel{(b)}{=} \rho^{-2} \sum_{n=1}^{N} \sum_{k=1}^{K} \mathbb{E}_{\boldsymbol{\Gamma}}\left[\left(\gamma_{n_{k}} \eta_{k}(n)\right)^{2}\right] \\
& =\rho^{-2} \sum_{n=1}^{N} \dot{\gamma}_{n}^{2} \sum_{k=1}^{K}\left(\operatorname{tr}\left(\left(\mathbf{X}_{k}^{(\mathrm{S})}(n)-\mathbf{X}_{k}^{*}\right) \widehat{\mathbf{V}}_{k}((n))\right)^{2}\right. \\
& \text { (c) } \rho^{-2} c_{3} \sum_{n=1}^{N} \dot{\gamma}_{n}^{2},
\end{align*}
$$

where $(a)$ is comes from the fact that $\mathbf{X}, \boldsymbol{\Gamma}$, and $\widehat{\mathbf{V}}$ are all block-diagonal matrices with the same shape; (b) is by $\mathbb{E}_{\boldsymbol{\Gamma}}\left[\eta_{k_{1}}(n) \eta_{k_{2}}(n)\right]=0$ for any $k_{1} \neq k_{2}$; in $(c)$, there exists $c_{3}<\infty$ such that

$$
\begin{equation*}
\sum_{k=1}^{K}\left(\operatorname{tr}\left(\left(\mathbf{X}_{k}^{(\mathrm{S})}(n)-\mathbf{X}_{k}^{*}\right) \widehat{\mathbf{V}}_{k}(n)\right)\right)^{2} \leq c_{3} \tag{83}
\end{equation*}
$$

as the entries of $\widehat{\mathbf{V}}$ and $\mathbf{X}$ have bounded values. In the end, (79) can be justified as $\sum_{n=1}^{N} \stackrel{\circ}{\gamma}_{n}^{2}<\infty$ by Lemma 3 .

## I. Proof of Theorem 4

Perform the expectation of 71, we have

$$
\begin{align*}
D_{n+1}^{(\mathrm{S})} \leq D_{n}^{(\mathrm{S})}+ & \mathbb{E}\left[\operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{S})}(n)-\mathbf{X}^{*}\right) \boldsymbol{\Gamma}(n) \circ \widehat{\mathbf{V}}(n)\right)\right] \\
& +\mathbb{E}\left[\|\boldsymbol{\Gamma}(n) \circ \widehat{\mathbf{V}}(n)\|_{\infty}^{2}\right] \tag{84}
\end{align*}
$$

Recall that $\bar{\gamma}_{n}$ defined in represents $\mathbb{E}\left[\gamma_{n_{k}} \eta_{k}(n)\right]$.

$$
\begin{align*}
& \mathbb{E}\left[\operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{S})}(n)-\mathbf{X}^{*}\right) \boldsymbol{\Gamma}(n) \circ \widehat{\mathbf{V}}(n)\right)\right] \\
& =\mathbb{E}\left[\operatorname{tr}\left(\left(\mathbf{X}^{(\mathrm{S})}(n)-\mathbf{X}^{*}\right) \operatorname{diag}\left(\gamma_{n_{k}} \eta_{k}(n) \widehat{\mathbf{V}}_{k}(n)\right)_{k=1}^{K}\right)\right] \\
& =\bar{\gamma}_{n} \mathbb{E}\left[\operatorname{tr}\left(\left(\left(\mathbf{X}^{(\mathrm{S})} \mathbf{X}\right)(n)-\mathbf{X}^{*}\right) \widehat{\mathbf{V}}(n)\right)\right] \\
& \leq-B \bar{\gamma}_{n} D_{n}^{(\mathrm{S})} \tag{85}
\end{align*}
$$

and

$$
\begin{align*}
\mathbb{E}\left[\|\boldsymbol{\Gamma}(n) \circ \widehat{\mathbf{V}}(n)\|_{\infty}^{2}\right] & \leq \mathbb{E}\left[\sum_{i, j, k}\left(\gamma_{n_{k}} \eta_{k}(n)\right)^{2}\left|\widehat{V}_{k, i, j}(n)\right|^{2}\right] \\
& =\stackrel{\gamma}{n}_{n}^{2} \mathbb{E}\left[\sum_{i, j}\left|\widehat{V}_{i, j}(n)\right|^{2}\right] \leq C \stackrel{\gamma}{n}_{n}^{2} \tag{86}
\end{align*}
$$

Thus (84) leads to

$$
\begin{equation*}
D_{n+1}^{(\mathrm{S})} \leq\left(1-B \bar{\gamma}_{n}\right) D_{n}^{(\mathrm{S})}+C \dot{\gamma}_{n}^{2} \tag{87}
\end{equation*}
$$

The rest of the proof is still by induction, the steps are similar to that in Section I We mainly need to find $\nu$ such that $D_{n}^{(\mathrm{S})} \leq$ $\mu \stackrel{\gamma}{\gamma}_{n}^{2} / \bar{\gamma}_{n}$. Since $B \bar{\gamma}_{n} \leq B \gamma_{1} \leq 1$, we should have

$$
\begin{equation*}
D_{n+1}^{(\mathrm{S})} \leq\left(1-B \bar{\gamma}_{n}\right) \mu \frac{\dot{\gamma}_{n}^{2}}{\bar{\gamma}_{n}}+C \dot{\gamma}_{n}^{2} \leq \mu \frac{\stackrel{\circ}{n+1}_{2}^{\gamma_{n+1}}}{\bar{\gamma}_{n+1}} \tag{88}
\end{equation*}
$$

leading to

$$
\begin{equation*}
\mu \geq \frac{C}{B-\max _{n}\left(\frac{1}{\hat{\gamma}_{n}^{2}}\left(\frac{\dot{\gamma}_{n}^{2}}{\bar{\gamma}_{n}}-\frac{\dot{\gamma}_{n+1}^{2}}{\gamma_{n+1}}\right)\right)}=\frac{C}{B-\epsilon} \tag{89}
\end{equation*}
$$

under the condition that $B>\epsilon$, which concludes the proof.

## J. Proof of Lemma 4

For any $n=1,2, \ldots$, consider a random sequence $W_{n}$ with $W_{n} \sim \mathcal{B}\left(n, p_{\mathrm{S}}\right)$. By definition, we have

$$
\bar{\gamma}_{n+1}=p_{\mathrm{S}} \mathbb{E}\left[\gamma_{W_{n}+1}\right], \quad \dot{\gamma}_{n+1}^{2}=p_{\mathrm{S}} \mathbb{E}\left[\gamma_{W_{n}+1}^{2}\right]
$$

Due to the fact that $\gamma_{\ell}$ is convex over $\ell$, we can obtain a lower bound of $\bar{\gamma}_{n+1}$ using Jensen's inequality, i.e.,

$$
\begin{equation*}
\bar{\gamma}_{n+1} \geq p_{\mathrm{S}} \gamma_{\left\lfloor\mathbb{E}\left[W_{n}\right]\right\rfloor+1+1}=p_{\mathrm{S}} \gamma_{\left\lfloor p_{\mathrm{s} n\rfloor+2}\right.} \tag{90}
\end{equation*}
$$

Now we need to find an upper bound of $\dot{\gamma}_{n+1}^{2}$. Consider an arbitrary $0<\xi<1$, we have

$$
\begin{align*}
& \stackrel{\circ}{\gamma}_{n+1}^{2}= p_{\mathrm{S}} \sum_{\ell=0}^{n} \mathbb{P}\left[W_{n}=\ell\right] \gamma_{\ell+1}^{2} \\
& \stackrel{(a)}{\leq} p_{\mathrm{S}}\left(\sum_{\ell=0}^{\left\lfloor(1-\xi) p_{\mathrm{S}} n\right\rfloor} \mathbb{P}\left[W_{n}=\ell\right] \gamma_{1}^{2}\right. \\
&\left.+\sum_{\ell=\left\lfloor(1-\xi) p_{\mathrm{S}} n\right\rfloor+1}^{n} \mathbb{P}\left[W_{n}=\ell\right] \gamma_{\left\lfloor(1-\xi) p_{\mathrm{S}} n\right\rfloor+2}^{2}\right) \\
&= p_{\mathrm{S}}\left(\gamma_{1}^{2} \mathbb{P}\left[W_{n} \leq\left\lfloor(1-\xi) p_{\mathrm{S}} n\right\rfloor\right]+\right. \\
&\left.+\gamma_{\left\lfloor(1-\xi) p_{\mathrm{S}} n\right\rfloor+2}^{2} \mathbb{P}\left[W_{n}>\left\lfloor(1-\xi) p_{\mathrm{S}} n\right\rfloor\right]\right) \\
& \quad(b)  \tag{91}\\
&< p_{\mathrm{S}}\left(\exp \left(-\frac{1}{2} \xi^{2} p_{\mathrm{S}} n\right) \gamma_{1}^{2}+\gamma_{\left\lfloor(1-\xi) p_{\mathrm{S}} n\right\rfloor+2}^{2}\right)
\end{align*}
$$

where $(a)$ is by the monotonicity of $\gamma_{\ell}$, i.e., $\gamma_{\ell+1} \leq 1$ for any $\ell \geq 0$ and $\gamma_{\ell+1} \leq \gamma_{\left\lfloor(1-\xi) n p_{\mathrm{S}}\right\rfloor+2}$ for any $\ell \geq\left\lfloor(1-\xi) p_{\mathrm{S}} n\right\rfloor+1$; in $(b)$, we consider $\mathbb{P}\left[W_{n}>\left\lfloor(1-\xi) p_{\mathrm{S}} n\right\rfloor\right]<1$ and we apply Chernoff Bound, i.e.,

$$
\begin{equation*}
\mathbb{P}\left[W_{n} \leq(1-\xi) \mathbb{E}\left[W_{n}\right]\right] \leq \exp \left(-\frac{1}{2} \xi^{2} \mathbb{E}\left[W_{n}\right]\right) \tag{92}
\end{equation*}
$$

recall that $\mathbb{E}\left[W_{n}\right]=p_{\mathrm{S}} n$. Combining (90) and 91, we can finally obtain 27.

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[^1]:    ${ }^{1}$ Note that the upper bound of $D_{n}^{(\mathrm{O})}$ presented in [8] is $D_{n} \leq \lambda^{\prime} / n$ with $\lambda^{\prime}$ some constant, which is derived by considering a special example $\gamma_{n}=\alpha / n$. Whereas, we consider general setting of the step size $\gamma_{n}$. In fact, 15 turns to be $D_{n} \leq \lambda \alpha / n$ as $\gamma_{n}=\alpha / n$ and. $p_{\mathrm{I}}=1$

