# A New Atomic Norm for DOA Estimation With Gain-Phase Errors

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Abstract—The problem of direction of arrival (DOA) estimation has been studied for decades as an essential technology in enabling radar, wireless communications, and array signal processing related applications. In this paper, the DOA estimation problem in the scenario with gain-phase errors is considered, and a sparse model is formulated by exploiting the signal sparsity in the spatial domain. By proposing a new atomic norm, named as GP-ANM, an optimization method is formulated via deriving a dual norm of GP-ANM. Then, the corresponding semidefinite program (SDP) is given to estimate the DOA efficiently, where the SDP is obtained based on the Schur complement. Moreover, a regularization parameter is obtained theoretically in the convex optimization problem. Simulation results show that the proposed method outperforms the existing methods, including the subspace-based and sparse-based methods in the scenario with gain-phase errors.

*Index Terms*—Atomic norm, DOA estimation, semidefinite program, gain-phase error, sparse signals.

### I. INTRODUCTION

The estimation problem of the direction of arrival (DOA) has been studied for decades in different applications encompassing radar, wireless communications, and array signal processing [1]. Traditionally, the DOA is estimated by the discrete Fourier transform (DFT)-based methods [2]–[4], where the antenna arrays provide spatial samplings. The DFT-based methods realize the DOA estimation via the DFT of received signals spatially sampled by the antenna array, with its inherent sampling resolution characterized by the *Rayleigh criterion* [5].

To overcome the resolution limit of the Rayleigh criterion, different super-resolution methods for DOA estimation have been proposed, and the subspace-based methods have been widely used in the scenarios with multiple measurements to estimate the covariance matrix of received signals in the antenna array. For example, the multiple signal classification

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(MUSIC) method [6] and the estimating signal parameters via rotational invariance techniques (ESPRIT) method [7], where the MUSIC method estimate the DOA with the noise subspace but the ESPRIT uses the signal subspace. Then, the extension algorithms based on the MUSIC and ESPRIT methods are proposed in the present papers, such as Root-MUSIC method [8], space-time MUSIC method [9], G-MUSIC method [10], higher order ESPRIT and virtual ES-PRIT [11], etc. Ref. [12] also develops a frequency estimation method in the continuous domain with sensor calibration and off-grid problems.

Recently, to further improve the DOA estimation performance, the compressed sensing (CS) methods have been proposed by exploiting the signal sparsity in the spatial domain [13]-[16]. Ref. [16,17] propose the CS-based DOA estimation methods in the multiple-input and multiple-output (MIMO) radar systems. A compressed sparse array scheme is proposed in [18]. However, in the CS-based methods, the dictionary matrix is formulated by discretizing the spatial domain. Consequently, the corresponding dictionary matrix is formulated using the discretized spatial angles. When the DOAs are not exactly at the discretized angles, which introduces the off-grid errors, and the off-grid methods have been proposed to solve this problem [19]. For example, the structured dictionary mismatch is considered, and the corresponding sparse reconstruction methods are proposed in [20]. A sparse Bayesian inference is given in [21] with the off-grid consideration. Moreover, an iterative reweighted method [22] estimates the off-grid and sparse signals jointly. In [23], the line spectral estimation is investigated by the Bayesian variational inference using multiple measurement vector (MMV), which outperforms the state-of-the-art MMV methods. Additionally, in [24], a multi-snapshot Newtonized orthogonal matching pursuit (MNOMP) algorithm is given for MMV scenario with relatively low computational complexity. With the prior knowledge of the signal structure, a general SDP method is proposed in [25] to recover the signal using the positive trigonometric polynomials, and the perfect signal reconstruction is achieved with sufficient prior information.

The super-resolution methods based on the sparse theory and avoiding the discretization have been proposed. In [26], *total variation norm* is introduced, and show that the exact locations and amplitudes of the line spectrum can be recovered by solving a convex optimization problem. Therefore, the DOA estimation problem can also be described as a type of *line spectral estimation* problem [27], and a generalized method is proposed in [28] by formulating the sparse signal recovery problems over a continuously indexed dictionary. Then, the *atomic norm* as a specific form of total variation norm is formulated [29]–[32], and an upper bound on the optimization of an atomic norm is given in [33]. Atomic norm minimization (ANM) method [31] with multiple measurement vectors (MMV) is proposed [34], and a Toeplitz covariance matrix reconstruction approach is also given in [35] to formulate a low-rank matrix reconstruction during the DOA estimation. For the general antenna geometries, a method based on total variation minimization is proposed in [36], where the theoretic guarantee for DOA estimation is derived. In [37], a family of nonconvex penalties is used to approximate the rank norm, and an iterative reweighted strategy is also proposed to achieve a better performance than the atomic norm method.

However, the existing ANM methods assume the perfect antenna array during the DOA estimation without considering the inconsistent antennas, where a polynomial with steering vector is formulated to estimate the DOA and will be mismatch in the scenario with inconsistent array [26]. The quantized noisy magnitudes are used to reconstruct the sparse signal in [38], and an approximation is used for the problem of sparse signal reconstruction for the approximate message passing method. The Cramér-Rao lower bound (CRLB) with quantization is given in [39], and the algorithm using atomic norm soft thresholding is shown for the sparse reconstruction. In the practical antenna array, the gain-phase errors among antennas degrade the DOA estimation performance [40,41]. The CS-based method for the DOA estimation is proposed in [42], and ref. [43] describes the localization method for the near-field sources with gain-phase errors. However, the DOA estimation based on the gridless sparse theory in the scenario has not been proposed.

In this paper, the DOA estimation problem in the scenario with gain-phase errors has been investigated. The technical contributions of this paper are summarized below:

- A new atomic norm for DOA estimation with gainphase errors: By introducing additional parameters in MMV, a new atomic norm is formulated, and the corresponding dual norm is theoretically obtained. An optimization problem is formulated for the DOA estimation.
- An semidefinite program (SDP) problem for the new atomic norm: To solve the new atomic norm efficiently, an SDP problem is formulated by the Schur complement.
- Theoretical expressions for the regularization parameter: In the atomic norm-based method, the regularization parameter determines the DOA estimation performance and is theoretically obtained to describe the reconstruction bound.

The remainder of this paper is organized as follows. The DOA estimation model in the uniform linear array (ULA) with gain-phase is formulated in Section II. The atomic normbased DOA estimation method is proposed in Section III. The regularization parameter is theoretically obtained in Section IV. The CRLB of DOA estimation is given in Section VI, and the simulation results are shown in Section VII. Finally, Section VIII concludes the paper. *Notations:* diag{a} denotes a diagonal matrix and the diagonal entries are from the vector a.  $(\cdot)^{T}$  and  $(\cdot)^{H}$  denote the matrix transpose and the Hermitian transpose, respectively.  $\|\cdot\|_{1}, \|\cdot\|_{2}, \|\cdot\|_{F}$  denote the  $\ell_{1}$  norm, the  $\ell_{2}$  norm, and the Frobenius norm, respectively.  $\|\cdot\|^{*}$  denotes the dual norm.  $I_{N}$  denotes an  $N \times N$  identity matrix.  $\otimes$  denotes the Kronecker product. Tr { $\cdot$ } denotes the trace of a matrix.  $\mathcal{R}{a}$  denotes the real part of complex value a. The boldface capital letters denote the matrix, such as A, and the lower-case letters denote the vector, such as a.

## II. SYSTEM MODEL WITH GAIN-PHASE ERRORS

In an ULA system, the DOA is estimated from the received signal by the antenna array, where a steering vector is used to describe the gain and phase among the perfect antennas. However, the gain-phase errors could cause the model mismatch in characterizing the steering vector, which eventually degrades the DOA estimation performance. Suppose the DOA estimation problem for K signals in the ULA with unknown gain-phase errors, and the received signal with P snapshots (multiple measurements) can be expressed as

$$Y = GAS + N, \tag{1}$$

where  $\mathbf{Y} \in \mathbb{C}^{N \times P}$  and N denotes the number of antennas, and the spacing between neighboring antennas is d. The signals are denoted by a matrix  $\mathbf{S} \triangleq [\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_{K-1}]^T$ , where the k-th signal is defined as  $\mathbf{s}_k \triangleq [\mathbf{s}_{k,0}, \mathbf{s}_{k,1}, \dots, \mathbf{s}_{k,P-1}]^T$ . The steering matrix is denoted as  $\mathbf{A} \triangleq [\mathbf{a}(\theta_0), \mathbf{a}(\theta_1), \dots, \mathbf{a}(\theta_{K-1})]$ , where  $\theta_k$  is the DOA of the k-th signal. The steering vector is defined as

$$\boldsymbol{a}(\theta) \triangleq \left[1, e^{j\xi\sin\theta}, \dots, e^{j(N-1)\xi\sin\theta}\right]^{\mathrm{T}}, \qquad (2)$$

where  $\xi \triangleq \frac{2\pi d}{\lambda}$  and  $\lambda$  denotes the wavelength. In the imperfect ULA systems, the received signals are effected by the antenna inconsistency, and we use a diagonal matrix G in (1) to describe the gain-phase errors. The diagonal matrix  $G \in \mathbb{C}^{N \times N}$  can be expressed as

$$\boldsymbol{G} \triangleq (\boldsymbol{I}_N + \operatorname{diag}\{\boldsymbol{g}\}) \operatorname{diag}\{e^{j\boldsymbol{\phi}}\},\tag{3}$$

where we define  $\boldsymbol{g} \triangleq \begin{bmatrix} g_0, g_1, \dots, g_{N-1} \end{bmatrix}^{\mathrm{T}}$  as the gain-error vector  $(g_n \in \mathbb{R})$  and  $\boldsymbol{\phi} \triangleq \begin{bmatrix} \phi_0, \phi_1, \dots, \phi_{N-1} \end{bmatrix}^{\mathrm{T}}$  as the phaseerror vector  $(\phi_n \in [0, 2\pi))$ .

In this paper, by exploiting the signal sparsity in the spatial domain, we will estimate the DOA parameters  $\boldsymbol{\theta} = [\theta_0, \theta_1, \dots, \theta_{K-1}]^T$  from the received signal  $\boldsymbol{Y}$  with the unknown antenna inconsistency including the gain errors  $\boldsymbol{g}$  and the phase errors  $\boldsymbol{\phi}$ . To avoid the discretized grids in the spatial domain, we will propose a new atomic norm and formulate the DOA estimation problem as an optimization problem with new atomic norm.

#### III. ATOMIC NORM-BASED GRIDLESS DOA ESTIMATION

#### A. Preliminary Atomic Norm

To improve the DOA estimation performance by exploiting the signal sparsity, the ANM-based methods have been proposed. Different from the exiting sparse-based methods using a dictionary matrix formulated by the discretized angles, such as the  $\ell_1$  norm method [44]–[46], the mixed  $\ell_{2,0}$  norm approximation [47], the ANM methods reconstruct the sparse signals without discretizing the spatial domain, so the ANM methods are the *gridless* sparse methods [48].

Usually, for the DOA estimation with perfect antennas, the system mode is formulated as

$$Y = AS + N, \tag{4}$$

so the *atomic set* is defined as [49]–[51]

$$\mathcal{A} \triangleq \left\{ \boldsymbol{a}(\theta) \boldsymbol{b}^{\mathrm{T}} : \theta \in [0, 2\pi), \|\boldsymbol{b}\|_{2} = 1 \right\}.$$
 (5)

Then, the DOA estimation in (4) is transferred into the following optimization problem (ANM)

$$\min_{\boldsymbol{X}} \frac{1}{2} \|\boldsymbol{Y} - \boldsymbol{X}\|_F^2 + \tau \|\boldsymbol{X}\|_{\mathcal{A}},$$
(6)

where the atomic norm is defined as

$$\|\boldsymbol{X}\|_{\mathcal{A}} = \inf \left\{ t > 0 : \boldsymbol{X} \in t \operatorname{conv}(\mathcal{A}) \right\}$$
(7)  
= 
$$\inf \left\{ \sum_{k} c_{k} : \boldsymbol{X} = \sum_{k} c_{k} \boldsymbol{a}(\theta) \boldsymbol{b}^{\mathrm{T}}, c_{k} \ge 0 \right\}.$$

Then, the ANM problem can be solved by SDP [19,27,52].

## B. New Atomic Norm Method for DOA Estimation

1) The Definition of New Atomic Norm: Without the noise, the received signal can be also expressed as

$$\boldsymbol{X} = \boldsymbol{G}\boldsymbol{A}\boldsymbol{S} = \sum_{k=0}^{K-1} (\boldsymbol{I}_N + \text{diag}\{\boldsymbol{e}\})\boldsymbol{a}(\theta_k)\boldsymbol{s}_k^{\mathrm{T}}.$$
 (8)

From the gain-phase model in (3), the antenna inconsistency e in (8) is  $e = (\mathbf{1}_N + g) \operatorname{diag} \{e^{j\phi}\} - \mathbf{1}_N$ , where  $\mathbf{1}_N$  is a  $N \times 1$  vector with all the entries being 1.

Then, we propose a new atomic decomposition to describe X in the scenario with gain-phase errors and to improve the robustness, and it is defined as

$$\|\boldsymbol{X}\|_{\tilde{\mathcal{A}},0} \triangleq \left\{ K : \boldsymbol{X} = \sum_{k=0}^{K-1} b_k (\operatorname{diag}\{\boldsymbol{e}\} + \boldsymbol{I}_N) \boldsymbol{a}(\theta_k) \boldsymbol{d}_k^{\mathrm{T}}, \\ \|\boldsymbol{e}\|_2 \le C_{\mathrm{e}}, \|\boldsymbol{d}_k\|_2 \le 1, b_k \ge 0 \right\},$$
(9)

where  $C_e$  is used to control the gain and phase errors. Note that, the  $\ell_2$  norm for the gain-phase error  $||e||_2 \leq C_e$  can be easily extended to the sparse norm  $||e||_1 \leq C_e$ . When we have  $||e||_1 \leq C_e$ , we can obtain  $||e||_2 \leq C_e$  with  $||e||_1 \geq ||e||_2$ . Therefore, the proposed atomic norm can be used in the scenario with the sparse gain-phase errors, where only a few antennas are inconsistent.

However, it is not computationally feasible to find the minimum K in (9) by the atomic decomposition of X. A

new atomic norm  $\ell_{\tilde{\mathcal{A}}}$  is proposed by a convex relaxation of  $\ell_{\tilde{\mathcal{A}},0}$ , and is defined as

$$\|\boldsymbol{X}\|_{\tilde{\mathcal{A}}} \triangleq \inf \left\{ \sum_{k} b_{k} \middle| \boldsymbol{X} = \sum_{k} b_{k} (\operatorname{diag}\{\boldsymbol{e}\} + \boldsymbol{I}_{N}) \boldsymbol{a}(\theta_{k}) \boldsymbol{d}_{k}^{\mathrm{T}}, \\ \|\boldsymbol{e}\|_{2} \leq C_{\mathrm{e}}, \|\boldsymbol{d}_{k}\|_{2} \leq 1, b_{k} \geq 0 \right\}$$
(10)

$$= \inf \left\{ \|\boldsymbol{b}\|_1 \middle| \boldsymbol{X} = \sum_k b_k (\operatorname{diag}\{\boldsymbol{e}\} + \boldsymbol{I}_N) \boldsymbol{a}(\theta_k) \boldsymbol{d}_k^{\mathrm{T}}, \\ \|\boldsymbol{e}\|_2 \le C_{\mathrm{e}}, \|\boldsymbol{d}_k\|_2 \le 1, b_k \ge 0 \right\},$$

where **b** is defined as  $\mathbf{b} \triangleq \begin{bmatrix} b_0, b_1, \dots, b_{K-1} \end{bmatrix}^T$ . This optimization is named as Gain-Phase ANM (GP-ANM) to be different from the existing ANM methods. From GP-ANM, the novel DOA estimation method will be proposed, and the corresponding algorithm will be given in the following sections.

2) DOA Estimation Using GP-ANM: With the received signals Y and the additive noise N, the DOA estimation problem can be described by following optimization problem

$$\min_{\boldsymbol{X}} \frac{1}{2} \|\boldsymbol{Y} - \boldsymbol{X}\|_F^2 + \tau \|\boldsymbol{X}\|_{\tilde{\mathcal{A}}},$$
(11)

where the first term is used to control the reconstruction performance and the second one is for the sparsity of X. The regularization parameter  $\tau$  is adopted to control the balance between the reconstruction performance and the sparsity. We will show how to get the regularization parameter  $\tau$  in the following sections. The optimization problem in (11) is a special case in [53], where a generalization of SDP over infinite dictionary is investigated.

Before solving the optimization problem (11), we first introduce the dual norm [51] for the proposed atomic norm. We define the dual norm of atomic norm as

$$\|\boldsymbol{U}\|_{\tilde{\mathcal{A}}}^* \triangleq \sup_{\|\boldsymbol{X}\|_{\tilde{\mathcal{A}}} \leq 1} \langle \boldsymbol{X}, \boldsymbol{U} \rangle, \qquad (12)$$

where atomic norm is given in (10).

Based on the dual norm, the dual problem of (11) can be obtained from the following proposition

**Proposition 1.** For an optimization problem  $\min_{\mathbf{X}} \frac{1}{2} \| \mathbf{Y} - \mathbf{X} \|_F^2 + \tau \| \mathbf{X} \|_{\tilde{\mathcal{A}}}$ , where  $\mathbf{Y} \in \mathbb{C}^{N \times P}$ ,  $\mathbf{X} \in \mathbb{C}^{N \times P}$  and  $\tau \ge 0$ , the dual problem is

$$\min_{\boldsymbol{U}} \quad \|\boldsymbol{Y} - \boldsymbol{U}\|_{F}^{2}$$
(13)  
s.t. 
$$\|\boldsymbol{U}\|_{\tilde{A}}^{*} \leq \tau,$$

where  $\|\boldsymbol{U}\|_{\tilde{A}}^*$  denotes the dual norm of  $\|\boldsymbol{U}\|_{\tilde{A}}$ .

*Proof.* Using a Lagrange multiplier U, we first formulate the following Lagrange function of the optimization problem (11) as

$$L(\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{U}) \triangleq \frac{1}{2} \|\boldsymbol{Y} - \boldsymbol{Z}\|_{F}^{2} + \tau \|\boldsymbol{X}\|_{\tilde{\mathcal{A}}} + \langle \boldsymbol{Z} - \boldsymbol{X}, \boldsymbol{U} \rangle,$$
(14)

where the inner product between matrices is defined as  $\langle \boldsymbol{X}, \boldsymbol{Y} \rangle \triangleq \mathcal{R}\{\operatorname{Tr}(\boldsymbol{Y}^{\mathrm{H}}\boldsymbol{X})\}$ . Using the Lagrange function, the dual problem of (11) is given as [54]

$$\max_{\boldsymbol{U}} \min_{\boldsymbol{X},\boldsymbol{Z}} L(\boldsymbol{X},\boldsymbol{Z},\boldsymbol{U}) = \max_{\boldsymbol{U}} \left\{ L_1(\boldsymbol{U}) - L_2(\boldsymbol{U}) \right\}, \quad (15)$$

where  $L_1(\boldsymbol{U}) \triangleq \min_{\boldsymbol{Z}} \frac{1}{2} \|\boldsymbol{Y} - \boldsymbol{Z}\|_F^2 + \langle \boldsymbol{Z}, \boldsymbol{U} \rangle$ , and  $L_2(\boldsymbol{U}) \triangleq \max_{\boldsymbol{X}} \{ \langle \boldsymbol{X}, \boldsymbol{U} \rangle - \tau \| \boldsymbol{X} \|_{\tilde{\mathcal{A}}} \}.$ 

Then, with the definition of dual norm,  $L_2(U)$  can be simplified as

$$L_{2}(\boldsymbol{U}) = \tau \max_{\boldsymbol{X}} \left\{ \left\langle \boldsymbol{X}, \frac{1}{\tau} \boldsymbol{U} \right\rangle - \|\boldsymbol{X}\|_{\tilde{\mathcal{A}}} \right\}$$
$$= \tau I \left( \|\boldsymbol{U}\|_{\tilde{\mathcal{A}}}^{*} \leq \tau \right), \qquad (16)$$

where the indicate function is defined as

$$I\left(\|\boldsymbol{U}\|_{\tilde{\mathcal{A}}}^* \leq \tau\right) = \begin{cases} 0, & \|\boldsymbol{U}\|_{\tilde{\mathcal{A}}}^* \leq \tau\\ \infty, & \text{otherwise} \end{cases}.$$
 (17)

Additionally,  $L_1(U)$  can be obtained as

$$L_1(\boldsymbol{U}) = -\frac{1}{2} \|\boldsymbol{Y} - \boldsymbol{U}\|_F^2 + \frac{1}{2} \|\boldsymbol{Y}\|_F^2.$$
(18)

Therefore, the dual problem in (15) can be simplified as (13).  $\Box$ 

With the dual problem, we obtain a SDP problem to solve (15) efficiently. The SDP problem is given as the following proposition.

**Proposition 2.** With the gain-phase error vector e, the feasible set of the dual problem (13) for DOA estimation with multiple measurements is included by the following SDP problem, so the dual problem (13) can be relaxed and simplified as

$$\min_{\substack{\boldsymbol{U} \in \mathbb{C}^{N \times P} \\ \boldsymbol{Q} \in \mathbb{C}^{N \times N}}} \|\boldsymbol{Y} - \boldsymbol{U}\|_{F} \tag{19}$$
s.t.
$$\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{U} \\ \boldsymbol{U}^{H} & \tau^{2} \boldsymbol{I}_{P} \end{bmatrix} \succeq 0$$

$$\sum_{n} Q_{n,n+k} = 0 \quad (k \neq 0)$$

$$\operatorname{Tr}(\boldsymbol{Q}) + (C_{e} + 2\sqrt{N})C_{e} \|\boldsymbol{Q}\|_{2} - 1 \leq 0$$

$$\boldsymbol{Q} \text{ is Hermitian,}$$

where the  $\ell_2$  norm of a matrix  $\|Q\|_2$  is the largest singular value of Q.

To show that the optimization problem (19) is a type of SDP problem, we give the proof in Appendix A. The proof for Proposition 2 is given in Appendix B. Note that the constraints in the SDP problem (19) are sufficient to the dual norm constraint in (13), so the denoised result U in (19) is only a sufficient approximation of the optimal results in (13).

To estimate the DOA from the solution of (19), we can formulate a quadratic  $\|\hat{U}a(\theta)\|_2^2$ , which is also a polynomial of  $a(\theta)$ . Inspired by [31], we can estimate the DOA by searching the peak of  $\|\hat{U}a(\theta)\|_2^2$ , so we get a straightforward corollary.

Corollary 3. The DOA polynomial is formulated as

$$\left\|\hat{\boldsymbol{U}}\boldsymbol{a}(\theta)\right\|_{2}^{2} \leq \tau^{2} \left(1 - (C_{e} + 2\sqrt{N})C_{e}\|\hat{\boldsymbol{Q}}\|_{2}\right), \qquad (20)$$

where  $\hat{U}$  and  $\hat{Q}$  are the solutions of the SDP problem in (19). Additionally, the quantity  $1 - (C_e + 2\sqrt{N})C_e \|\hat{Q}\|_2$  is positive.

*Proof.* From (64), we can find that for any  $\theta$ , we have

$$\|\hat{\boldsymbol{U}}\boldsymbol{a}(\theta)\|_{2}^{2} \leq \tau^{2}\boldsymbol{a}^{\mathrm{H}}(\theta)\hat{\boldsymbol{Q}}\boldsymbol{a}(\theta).$$
(21)

Therefore, from the construction of  $\hat{Q}$  in (67), we obtain

$$\|\hat{\boldsymbol{U}}\boldsymbol{a}(\theta)\|_{2}^{2} \leq \tau^{2} \left(1 - (C_{e} + 2\sqrt{N})C_{e}\|\hat{\boldsymbol{Q}}\|_{2}\right).$$
(22)

Then, the DOA can be obtained by searching the peak values of  $\|\hat{U}\boldsymbol{a}(\theta)\|_2^2$ , which is closed to  $\tau^2 \left(1 - (C_e + 2\sqrt{N})C_e \|\hat{\boldsymbol{Q}}\|_2\right)$ . The estimated DOAs are denoted as  $\hat{\theta}_k$   $(k = 0, 1, \dots, K - 1)$ .

To estimate the other unknown parameters including e,  $b_k$ ,  $d_k$ , we formulate the following optimization problem

$$\min_{\boldsymbol{e},\boldsymbol{b}_{k},\boldsymbol{d}_{k}} \|\boldsymbol{Y} - \sum_{k} b_{k} (\operatorname{diag}\{\boldsymbol{e}\} + \boldsymbol{I}) \boldsymbol{a}(\hat{\theta}_{k}) \boldsymbol{d}_{k}^{\mathrm{T}} \|_{F}$$
(23)  
s.t.  $\|\boldsymbol{e}\|_{2} = C_{\mathrm{e}}, \|\boldsymbol{d}_{k}\|_{2} \leq 1, b_{k} \geq 0.$ 

Since the upper bound of the gain-phase errors e is used in the proposed GP-ANM method to estimate the DOA, the constraint  $||e||_2 = C_e$  is used for the estimation of unknown parameters. For the gain-phase errors e, we can formulate the following optimization problem

$$\min_{\boldsymbol{e}} \quad f(\boldsymbol{e}) \triangleq \|\boldsymbol{Y} - \sum_{k} b_{k} (\operatorname{diag}\{\boldsymbol{e}\} + \boldsymbol{I}) \boldsymbol{a}(\hat{\theta}_{k}) \boldsymbol{d}_{k}^{\mathrm{T}}\|_{F}^{2} \quad (24)$$
s.t.  $\|\boldsymbol{e}\|_{2} = C_{\mathrm{e}}.$ 

f(e) can be rewritten as

$$f(\boldsymbol{e}) = \|\boldsymbol{Y} - (\operatorname{diag}\{\boldsymbol{e}\} + \boldsymbol{I}) \underbrace{\sum_{k} b_{k} \boldsymbol{a}(\hat{\theta}_{k}) \boldsymbol{d}_{k}^{\mathrm{T}}}_{\boldsymbol{H}} \|_{F}^{2} \qquad (25)$$
$$= \sum_{n=0}^{N-1} \|\bar{\boldsymbol{y}}_{n} - (e_{n}+1)\bar{\boldsymbol{h}}_{n}\|_{2}^{2},$$

where  $\bar{\boldsymbol{y}}_n$  denotes the *n*-th row of  $\boldsymbol{Y}$  and  $\bar{\boldsymbol{h}}_n$  is the *n*-th row of  $\boldsymbol{H}$ . Therefore, the Lagrange function for  $\boldsymbol{e}$  with the Lagrange parameter  $\lambda_{\rm e} \geq 0$  can be obtained as

$$L(e) = f(e) + \lambda_{e}(||e||_{2}^{2} - C_{e}^{2}).$$
(26)

With  $\frac{\partial L(e)}{\partial e^*} = 0$ , we can obtain the estimated gain-phase as

$$\hat{e}_n = \frac{\boldsymbol{h}_n^{\mathrm{H}}(\bar{\boldsymbol{y}}_n - \bar{\boldsymbol{h}}_n)}{\lambda_{\mathrm{e}} + \bar{\boldsymbol{h}}_n^{\mathrm{H}} \bar{\boldsymbol{h}}_n},$$
(27)

where we choose  $\lambda_e$  to ensure that  $\|e\|_2 = C_e$ .

For the unknown parameter  $b_k$  and  $d_k$ , we can formulate  $d'_k = d_k b_k$ . With the estimated  $\hat{\theta}_k$  and  $\hat{e}$ , we have

$$\min_{\boldsymbol{d}'_{k}} \quad f(\boldsymbol{d}'_{k}) \triangleq \|\boldsymbol{Y} - \sum_{k} (\operatorname{diag}\{\hat{\boldsymbol{e}}\} + \boldsymbol{I}) \boldsymbol{a}(\hat{\theta}_{k}) \boldsymbol{d}'^{\mathrm{T}}_{k}\|_{F}^{2}.$$
(28)

Then, we define  $D' = [d'_0, d'_1, \dots, d'_{K-1}]$  and  $\hat{A} = [a(\hat{\theta}_0), a(\hat{\theta}_1), \dots, a(\hat{\theta}_{K-1})]$ , and D' can be estimated as

$$\hat{\boldsymbol{D}}' = \left[\hat{\boldsymbol{A}}^{\dagger} (\operatorname{diag}\{\hat{\boldsymbol{e}}\} + \boldsymbol{I})^{-1} \boldsymbol{Y}\right]^{\mathrm{T}}, \qquad (29)$$

where  $\dagger$  denotes the pseudo-inverse operation. Then,  $d_k$  can be estimated from the normalized  $d'_k$  and  $b_k$  is the normalization coefficient. By alternatively estimating the unknown parameterse,  $b_k$ ,  $d_k$ . We can finally estimated all the unknown parameters.

The details of the proposed method for the DOA estimation is given in Algorithm 1. The computational complexity of the proposed method is almost the same with the traditional atomic norm minimization (ANM) method. Only a  $\ell_2$  norm for the matrix Q is added in the SDP problem, so the computational complexity is  $\mathcal{O}(N^3)$  more than the ANM method at each iteration.

# Algorithm 1 DOA Estimation Using GP-ANM

- 1: *Input:* received signal Y, noise variance  $\sigma_n^2$ , the number of antennas N, and the number of measurements (snapshots) P.
- 2: Initialization:  $\tau = \eta \sigma_n \sqrt{4NP \ln(N)}$ .
- 3: Formulate the SDP problem as (19), and obtain the matrix  $\hat{U}$ .
- 4: Get the polynomial  $f(\hat{U}) = \left\| \hat{U} \boldsymbol{a}(\theta) \right\|_{2}^{2}$ .
- 5: Use the peak searching of  $f(\hat{U})$ , and get the estimated DOA  $\hat{\theta}$ .
- 6: The other unknown parameters can be obtained by the alternative estimations in the problem (23).
- 7: *Output:* the estimated DOA  $\theta$ .

#### IV. The regularization parameter au

In (11), the regularization parameter is important and has a great effect on reconstructing the sparse signal, so we will obtain the regularization parameter in this section.

Usually, the regularization parameter  $\tau$  can be chosen as [55]

$$\tau \approx \eta \mathcal{E}\left\{\|\boldsymbol{N}\|_{\tilde{\mathcal{A}}}^*\right\} \quad (\eta \ge 1).$$
(30)

To get  $\mathcal{E}\left\{\|N\|_{\tilde{A}}^*\right\}$ , we can obtain the following proposition to determine the regularization

**Proposition 4.** The entries in  $\mathbf{N} \in \mathbb{C}^{N \times P}$  follow the zeromean Gaussian distribution with the variance being  $\sigma_N^2$  and the entries are independent. With the probability more than  $1 - 2e^{-t^2/2}$ , the upper bound of  $\{\|\mathbf{N}\|_{\tilde{A}}^*\}$  can be obtained as

$$\mathcal{E}\left\{\|\boldsymbol{N}\|_{\mathcal{A}}^{*}\right\} \leq \min\left\{bd_{1}, bd_{2}\right\} C_{e} + \sigma_{N}\sqrt{4NP\ln N}, \quad (31)$$

where the definition of dual atomic norm is defined in (57),  $bd_1 \triangleq \sqrt{2}\sigma_N \frac{\Gamma((NP+1)/2)}{\Gamma(NP/2)}$ , and  $bd_2 \triangleq \left(\sqrt{N} + \sqrt{P} + t\right)\sigma_N$ .

The proof for Proposition 4 is given in Appendix C. Then, the regularization parameter  $\tau$  is

$$\tau \approx \eta \left( \min \left\{ \mathsf{bd}_1, \mathsf{bd}_2 \right\} C_{\mathsf{e}} + \sigma_{\mathsf{N}} \sqrt{4NP \ln N} \right).$$
 (32)

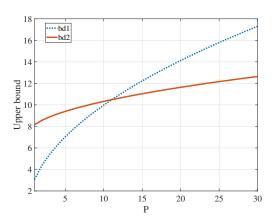


Fig. 1. The upper bound of  $\mathcal{E} \{ \| \mathbf{N} \|_2 \}$ .

We will show that with the regularization parameter  $\tau$ , the probability of  $\|N\|_{\tilde{A}}^* \geq \frac{\tau}{\eta}$  can be obtained as

$$\mathbb{P}\left(\|\boldsymbol{N}\|_{\tilde{\mathcal{A}}}^{*} \geq \frac{\tau}{\eta}\right) \tag{33}$$

$$= \mathbb{P}\left(\sup_{\substack{\|\boldsymbol{e}\|_{2} \leq C_{e} \\ \theta \in [0, 2\pi)}} \|\boldsymbol{N}^{\mathsf{H}}[\boldsymbol{e} + \boldsymbol{a}(\theta)]\|_{2} \geq \alpha + \beta\right)$$

$$\leq \mathbb{P}\left(\sup_{\|\boldsymbol{e}\|_{2} \leq C_{e}} \|\boldsymbol{N}^{\mathsf{H}}\boldsymbol{e}\|_{2} + \sup_{\theta \in [0, 2\pi)} \|\boldsymbol{N}^{\mathsf{H}}\boldsymbol{a}(\theta)\|_{2} \geq \alpha + \beta\right)$$

$$\leq \mathbb{P}\left(C_{e} \|\boldsymbol{N}\|_{F} + \sup_{\theta \in [0, 2\pi)} \|\boldsymbol{N}^{\mathsf{H}}\boldsymbol{a}(\theta)\|_{2} \geq \alpha + \beta\right)$$

$$\leq \frac{1}{N^{2}} + \left(1 - \frac{1}{N^{2}}\right) z(N, P),$$

where we define  $\alpha \triangleq \min \{ bd_1, bd_2 \} C_e$ ,  $\beta \triangleq \sigma_N \sqrt{4NP \ln N}$ and

$$z(N,P) = \begin{cases} 2e^{-t^{2}/2}, \sqrt{N} + \sqrt{P} + t \le \sqrt{2}\frac{\Gamma((NP+1)/2)}{\Gamma(NP/2)}\\ \frac{\Gamma(NP/2, (\sqrt{N} + \sqrt{P} + t)^{2}/2)}{\Gamma(NP/2)}, \text{ otherwise} \end{cases}$$
(34)

The incomplete Gamma function is defined as  $\Gamma(s, x) \triangleq \int_x^\infty t^{s-1} e^{-t} dt$ . When we can choose t = 4, the probability of  $\mathcal{E}\{\|\mathbf{N}\|_2\} \leq$ 

When we can choose t = 4, the probability of  $\mathcal{E} \{ \|N\|_2 \} \le \sqrt{N} + \sqrt{P} + 4$  is more than 0.9993. In Fig. 1, we show the two types of upper bounds, where the antenna number N is 10, and the number of measurements P is from 1 to 30. When  $P \le 11$ , we have  $bd_1 \le bd_2$ , and  $bd_1 > bd_2$  with P > 11. Hence, for larger P, we choose  $bd_2$  as the tighter upper bound, and for smaller P, we choose  $bd_1$ .

Therefore, according to Theorem III.6 in [49], with probability  $1 - \frac{1}{N^2} - (1 - \frac{1}{N^2}) z(N, P)$ , the reconstruction error is limited by

$$\left\|\hat{\boldsymbol{X}} - \boldsymbol{X}_*\right\|_F^2 \le \tau^2,\tag{35}$$

where  $\hat{X}$  denotes the estimated X by minimizing the atomic norm, and  $X_*$  denotes the ground-truth X.

# V. SPARSE GAIN-PHASE ERRORS

In the scenario with only a few gain-phase errors, the gainphase errors are sparse. The proposed type of atomic norm can be rewritten as

$$\|\boldsymbol{X}\|_{\tilde{\mathcal{A}}} = \inf \left\{ \|\boldsymbol{b}\|_1 \middle| \boldsymbol{X} = \sum_k b_k (\operatorname{diag}\{\boldsymbol{e}\} + \boldsymbol{I}_N) \boldsymbol{a}(\theta_k) \boldsymbol{d}_k^{\mathrm{T}} \right\}$$
$$\|\boldsymbol{e}\|_1 \leq C_{\mathrm{e}}, \|\boldsymbol{d}_k\|_2 \leq 1, b_k \geq 0 \right\},$$

where the  $\ell_1$  norm  $\|\boldsymbol{e}\|_1$  is used to describe the sparse gainphase errors. We formulate  $\boldsymbol{e}' \triangleq \operatorname{diag}(\boldsymbol{e})\boldsymbol{a}(\theta) \in \mathbb{C}^{N \times 1}$ , then, the  $\ell_1$  norm of  $\boldsymbol{e}'$  can be simplified as

$$\|e'\|_1 = \|\operatorname{diag}(e)a(\theta)\|_1 = \|e\|_1 \le C_{e}.$$

Therefore, the corresponding dual norm can be obtained as

$$\|\boldsymbol{U}\|_{\tilde{\mathcal{A}}}^{*} = \sup_{\|\boldsymbol{X}\|_{\tilde{\mathcal{A}}} \leq 1} \langle \boldsymbol{X}, \boldsymbol{U} \rangle \qquad (36)$$
$$= \sup_{\substack{\|\boldsymbol{e}\|_{1} \leq C_{e} \\ \theta \in [0, 2\pi)}} \|\boldsymbol{U}^{\mathrm{H}}[\boldsymbol{e} + \boldsymbol{a}(\theta)]\|_{2},$$

where we reuse the notation e instead of e' to avoid introducing additional symbol e'.

If  $\|\boldsymbol{e}\|_2 \leq \frac{1}{\sqrt{N}}C_{\mathrm{e}}$ , we have  $\|\boldsymbol{e}\|_1 \leq C_{\mathrm{e}}$ . Therefore, we can obtain

$$\|\boldsymbol{U}\|_{\tilde{\mathcal{A}}}^{*} \leq \sup_{\substack{\|\boldsymbol{e}\|_{2} \leq \frac{C_{e}}{\sqrt{N}}\\\theta \in [0, 2\pi)}} \|\boldsymbol{U}^{H}[\boldsymbol{e} + \boldsymbol{a}(\theta)]\|_{2}$$
(37)  
$$\leq \sup_{\theta \in [0, 2\pi)} \tau^{2} \left[ \left(2 + \frac{1}{\sqrt{N}}\right) C_{e} \|\boldsymbol{Q}\|_{2} + \boldsymbol{a}^{H}(\theta) \boldsymbol{Q} \boldsymbol{a}(\theta) \right].$$

Similarly, the SDP problem with the  $\ell_1$  norm in atomic norm can be obtained as

$$\min_{\substack{\boldsymbol{U} \in \mathbb{C}^{N \times P} \\ \boldsymbol{Q} \in \mathbb{C}^{N \times N}}} \|\boldsymbol{Y} - \boldsymbol{U}\|_{F} \tag{38}$$
s.t.
$$\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{U} \\ \boldsymbol{U}^{\mathrm{H}} & \tau^{2} \boldsymbol{I}_{P} \end{bmatrix} \succeq 0$$

$$\sum_{n} Q_{n,n+k} = 0 \quad (k \neq 0)$$

$$\operatorname{Tr}(\boldsymbol{Q}) + \left(2 + \frac{1}{\sqrt{N}}\right) C_{\mathrm{e}} \|\boldsymbol{Q}\|_{2} - 1 \leq 0$$

$$\boldsymbol{Q} \text{ is Hermitian,}$$

which can be solved efficiently.

# VI. CRLB FOR DOA ESTIMATION WITH GAIN-PHASE Errors

For the DOA estimation problem Y = GAS + N, we use  $A \triangleq [a(\theta_0), a(\theta_1), \dots, a(\theta_{K-1})]$  to denote the steering matrix, and we assume  $n = \text{vec}\{N\} \sim C\mathcal{N}(\mathbf{0}, \sigma_n^2 I)$ . We consider K unknown signals  $S = [s_0, s_1, \dots, s_{K-1}]^T$ , and we assume s follows the zero mean Gaussian distribution with  $\mathcal{E}(ss^H) = B$  and  $s \sim C\mathcal{N}(\mathbf{0}, B)$ , where  $s \triangleq \text{vec}\{S\}$ . Then, in this section, the CRLB will be derived theoretically to indicate the DOA estimation performance of the proposed method.

The received signal can be written in a vector form as

$$\boldsymbol{y} \triangleq \operatorname{vec}\{\boldsymbol{Y}\} = (\boldsymbol{I} \otimes \boldsymbol{G}\boldsymbol{A})\boldsymbol{s} + \boldsymbol{n},$$
 (39)

where  $G \triangleq \text{diag}\{g\}$ . Therefore, with the DOA parameter  $\boldsymbol{\theta} \triangleq [\theta_0, \theta_1, \dots, \theta_{K-1}]^{\text{T}}$  and the gain-phase error  $\boldsymbol{g} \triangleq [g_0, g_1, \dots, g_{N-1}]^{\text{T}}$ , and the received signal follows the Gaussian distribution

$$\boldsymbol{y} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{C}),$$
 (40)

where  $C \triangleq (I \otimes GA)B[I \otimes (GA)^{H}] + \sigma_{n}^{2}I$ . The probability density function of Gaussian distribution  $y \sim C\mathcal{N}(0, C)$  can be expressed as

$$f(\boldsymbol{x}) = \frac{1}{\pi^N \det\{\boldsymbol{C}\}} e^{-\boldsymbol{y}^{\mathsf{H}} \boldsymbol{C}^{-1} \boldsymbol{y}}.$$
 (41)

The Fisher information matrix F can be written as

$$\boldsymbol{F} \triangleq \begin{bmatrix} \boldsymbol{F}_{1,1} & \boldsymbol{F}_{1,2} \\ \boldsymbol{F}_{2,1} & \boldsymbol{F}_{2,2} \end{bmatrix},$$
(42)

where we have

$$\boldsymbol{F}_{1,1} = -\mathcal{E}\left\{ \left. \frac{\partial \ln f(\boldsymbol{y}; \boldsymbol{\theta}, \boldsymbol{g})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}} \right| \boldsymbol{\theta}, \boldsymbol{g} \right\}, \tag{43}$$

$$\boldsymbol{F}_{1,2} = -\mathcal{E}\left\{ \left. \frac{\partial \ln f(\boldsymbol{y}; \boldsymbol{\theta}, \boldsymbol{g})}{\partial \boldsymbol{\theta} \partial \boldsymbol{g}} \right| \boldsymbol{\theta}, \boldsymbol{g} \right\},\tag{44}$$

$$\boldsymbol{F}_{2,1} = -\mathcal{E}\left\{ \left. \frac{\partial \ln f(\boldsymbol{y}; \boldsymbol{\theta}, \boldsymbol{g})}{\partial \boldsymbol{g} \partial \boldsymbol{\theta}} \right| \boldsymbol{\theta}, \boldsymbol{g} \right\}, \tag{45}$$

$$\boldsymbol{F}_{2,2} = -\mathcal{E}\left\{ \left. \frac{\partial \ln f(\boldsymbol{y}; \boldsymbol{\theta}, \boldsymbol{g})}{\partial \boldsymbol{g} \partial \boldsymbol{g}} \right| \boldsymbol{\theta}, \boldsymbol{g} \right\}.$$
 (46)

The entries of Fisher information matrix F are given in Appendix D.

The CRLB of DOA estimation can be expressed as

$$\operatorname{var}\{\boldsymbol{\theta}\} \ge \sum_{k=0}^{K-1} \left[\boldsymbol{F}^{-1}\right]_{k,k}.$$
(47)

However, in the parameter estimation problems, when the dimension of the unknown parameter is high, the FIM will be singular or very nearly so, especially in the case with sparse reconstruction, where the number of samples is much less than that in the oversampling scenario. The derivation of CRLB using  $F^{-1}$  can be only obtained by assuming that the FIM is positive defined [56]. In our problem of sparse estimation with unknown gain-phase errors, the Fisher information matrix is singular, and it is inconvenient to obtain the inverse of the Fisher information matrix, so we use a lower bound of FIM to describe the estimation performance as [57]

v

$$ar\{\boldsymbol{\theta}\} \ge \sum_{k=0}^{K-1} F_{k,k}^{-1}.$$
(48)

TABLE I SIMULATION PARAMETERS

Parameter	Val1ue	
The signal-to-noise ratio (SNR) of received signal	20 dB	
The number of pulses $P$	5	
The number of antennas $N$	10	
The number of signals $K$	3	
The space between antennas $d$	0.5 wavelength	
The detection DOA range	$[-70^{\circ}, 70^{\circ}]$	
The standard deviation of gain error $\sigma_A$	0.15	
The standard deviation of phase error $\sigma_{\rm P}$	10 in degree	

TABLE II DOA ESTIMATION

Methods	Signal 1	Signal 2	Signal 3	RMSE (deg)
Ground-truth	$-56.8889^\circ$	$-7.6806^\circ$	$5.9595^{\circ}$	-
ANM	$-56.4480^\circ$	$5.6840^{\circ}$	$28.7000^\circ$	232
MUSIC	$-56.4900^\circ$	$-7.6020^{\circ}$	$6.0900^{\circ}$	0.06076
SOMP	$-56.3640^\circ$	$-8.2460^\circ$	$5.7400^{\circ}$	0.2144
SBL	$-56.0000^\circ$	$-7.0000^{\circ}$	$5.6000^{\circ}$	0.4608
Proposed method	$-56.6860^\circ$	$-7.6300^\circ$	$5.9640^{\circ}$	0.01458

#### VII. SIMULATION RESULTS

The simulation parameters are given in Table I. The number of Monte Carlo simulations is  $10^3$ . We consider the DOA estimation in the scenario with much few measurements (snapshots) P = 5. The minimum separation between signals in degree is  $\Delta \ge 10^\circ$ . All the simulation results are obtained on a PC with Matlab R2018b with a 2.9 GHz Intel Core i5 and 8 GB of RAM. The code of proposed algorithm will be available online after that the paper is accepted.

The gain errors among antennas are generated by a Gaussian distribution

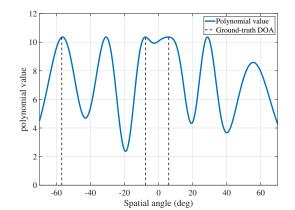
$$g_n \sim \mathcal{N}(0, \sigma_{\rm A}^2), \quad n = 0, 1, \dots, N - 1,$$
 (49)

where  $\sigma_A^2$  denotes the variance of gain errors. The phase errors in degree also follow a Gaussian distribution

$$\phi_n \sim \mathcal{N}(0, \sigma_{\mathbf{P}}^2), \quad n = 0, 1, \dots, N - 1,$$
 (50)

where  $\sigma_{\rm P}^2$  denotes the variance of phase errors. Then, the normalized gain for the *n*-th antenna with gain-phase error is  $(1 + g_n)e^{j\phi_n}$ . Hence, the parameter  $C_{\rm e}$  can be chosen as the one with  $C_e^2 \ge N(\sigma_{\rm A}^2 + \sigma_{\rm P}^2) = 0.514$ .

First, we try to estimate 3 signals from the received signals, and the ground-truth DOAs are  $-56.8889^{\circ}$ ,  $-7.6806^{\circ}$ , and  $5.9595^{\circ}$ . When the ANM method is adopted, the DOAs are estimated by the polynomial of the ANM method. As shown in Fig. 2, the polynomial of ANM method is given. Since the antennas in the array have gain-phase errors, the polynomial has multiple peak values, and the DOAs cannot be estimated well. The estimated DOAs are  $-56.4480^{\circ}$ ,  $5.6840^{\circ}$ , and  $28.7000^{\circ}$ , so the estimation error is much large. However, when the proposed method with GP-ANM is adopted, we can obtain the polynomial in Fig. 3. The peak values are well distinguished, and the estimated DOAs are  $-56.6860^{\circ}$ ,  $-7.6300^{\circ}$ , and  $5.9640^{\circ}$ . Therefore, the proposed method outperforms the



\_ Fig. 2. The polynomial in ANM method.

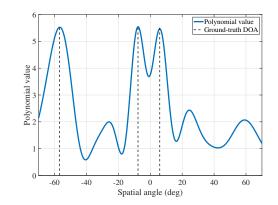


Fig. 3. The polynomial in the proposed method.

traditional ANM method in the DOA estimation with gainphase errors.

With the simulation parameters in Table I, the probability of signal reconstruction is shown in Fig. 4 with the different number of antennas, and this figure is different from the direct performance of DOA estimation, such as the root mean square error (RMSE) of DOA estimation. When the number of antennas increases, the probability that the sparse reconstruction signal can approach the ground-truth signal is also improved. Therefore, a high probability that the sparse signal can be reconstructed with limited error can be achieved by selecting an appropriate regularization parameter  $\tau$ .

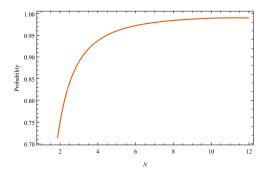


Fig. 4. The probability for signal reconstruction in (35).

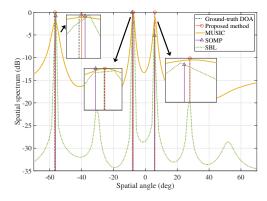


Fig. 5. The spatial spectrum for DOA estimation.

Additional, we compare the DOA estimation performance of the proposed method with existing methods, including MU-SIC, simultaneous orthogonal matching pursuit (SOMP) [58], and sparse Bayesian learning (SBL) [45] methods. MUSIC method is a subspace-based method and has been widely used in the DOA estimation with better performance and robustness. SOMP method is the sparse-based method and has been widely used in the sparse reconstruction problem. SBL method is a sparse method and has great reconstruction performance but has high computational complexity. The DOA estimation performance is measured by the RMSE. RMSE is defined as

$$\mathbf{RMSE} \triangleq \sqrt{\frac{1}{KN_{\mathrm{mc}}} \sum_{n_{\mathrm{mc}}=0}^{N_{\mathrm{mc}}-1} \sum_{k=0}^{K-1} \left(\theta_{n_{\mathrm{mc},k}} - \hat{\theta}_{n_{\mathrm{mc},k}}\right)^2}, \quad (51)$$

where  $N_{\rm mc}$  denotes the number of Monte Carlo simulations, and K denotes the number of signals in one simulation.  $\theta_{n_{\text{mc},k}}$  is the ground-truth DOA of the k-th signal during the  $n_{\rm mc}$ -th simulation, and  $\hat{\theta}_{n_{\rm mc,k}}$  is the corresponding estimated DOA. In this paper, we assume that the number of signals can be estimated precisely using the traditional methods, such as Akaike information theoretic criteria (AIC) and minimum description length (MDL) [59]–[61]. The RMSEs of ANM, MUSIC, SOMP, SBL and the proposed method are shown in Table. I. The RMSE of proposed method is 0.01458 in deg, and 76% better than MUSIC. Additionally, since the multiple peak values in the polynomial of ANM method, the DOA cannot be estimated well and the RMSE of ANM method is much larger than other methods. In the SBL method, the spatial angle is discretized into grids with the grid size being  $0.5^{\circ}$  to have a comparable computational time with the proposed method. The spatial spectrums of these 4 methods are shown in Fig. 5, where we can see that the spectrum of SBL is much better than that of MUSIC method. SOMP and proposed methods are the sparse-based method, so we show the reconstruction results in the figure of spatial spectrum. The spatial spectrum of proposed method is much close to the ground-truth DOA.

Then, to show the DOA estimation performance with different variances of grain-phase errors, we give the DOA estimation performance with different variances in Fig. 6 and Fig. 7, where Fig. 6 uses the traditional ANM method and Fig. 7 uses the proposed method. When the variance of the

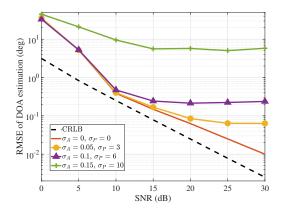


Fig. 6. The DOA estimation with different gain-phase errors using the proposed method (ANM).

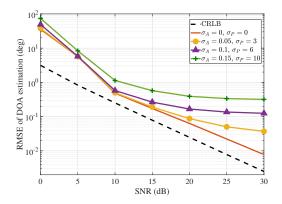


Fig. 7. The DOA estimation with different gain-phase errors using the proposed method (Proposed method).

gain-phase error is small, both ANM and proposed methods can approach the CRLB in DOA estimation. However, when the variance of the gain-phase error is large, the ANM method degrades the RMSE significantly. The proposed method can also keep the estimation performance well. Therefore, with the GP-ANM, the effect of gain-phase error can be reduced effectively.

For different SNRs, the DOA estimation performance is

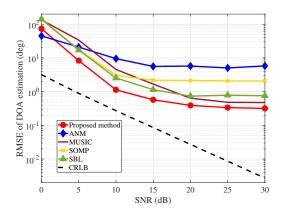


Fig. 8. The DOA estimation with different SNRs.

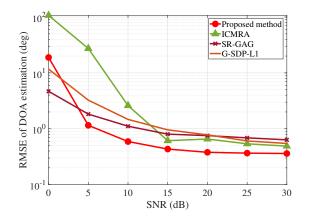
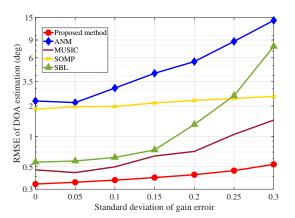


Fig. 9. The DOA estimation performance compared with ICMRA, SR-GAG Fig. 10. The DOA estimation with different gain errors. and G-SDP-L1 methods.

shown in Fig. 8, where the SNR is from 0 dB to 30 dB. When the SNR is higher than 15 dB, the estimation performance is almost the same. MUSIC method can achieve better estimation performance than the ANM, SOMP, and SBL methods in the scenario with gain-phase errors. The proposed method achieves the best estimation performance among these methods when the SNR is higher than 5 dB. Since the CRLB does not consider the gain-phase error, the CRLB can be further improved with higher SNR, but the estimation performance has platform effect and cannot be improved when SNR is higher than 20 dB. Moreover, as shown in Fig. 8, when the SNR of the received signal is 20 dB, the RMSEs of the DOA estimation using the ANM method, the SOMP method, the SBL method, the MUSIC method and the proposed method are 5.741°, 2.144°, 0.735° 0.639° 0.391°, respectively. With the gain-phase errors, the ANM method cannot estimate the DOA accurately, but the MUSIC method as a robust method can achieve a higher DOA resolution than the ANM method. Compared with the ANM method, the proposed method can improve the DOA resolution about  $5.35^{\circ}$  in the scenario with the standard derivation of gain error being  $\sigma_A = 0.15$  and the that of phase error being  $\sigma_{\rm P} = 10$  in degree. Additionally, the DOA estimation performance of the proposed method is also compared with the improved covariance matrix reconstruction approach (ICMRA) [37], soft recovery approach for general antenna geometries (SR-GAG) [36], and a generalization of SDP formulation of  $\ell_1$  norm optimization problem (G-SDP-L1) [53]. The ICMRA method is based on the low-rank reconstruction, where a covariance matrix of the received signals is used for the DOA estimation. In the simulation section, the number of antennas is 10 and the snapshots are 5, so the covariance matrix cannot be accurately estimated. The SR-GAG method is proposed for a general antenna geometry. When this method is applied to the system model considered in this paper, the method will be the same with the ANM method, since the antenna geometry is ULA. The G-SDP-L1 method is a general case of gauge function and atomic norm, but this extension cannot describe the gain-phase errors well. Moreover, these methods have not considered the gain-phase errors in the system model. Therefore, better performance can



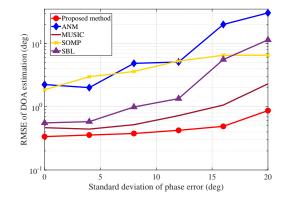


Fig. 11. The DOA estimation with different phase errors.

be achieved by the proposed method.

When we keep the standard deviation of phase errors  $\sigma_{\rm P}$  as 10 in degree, and change that of gain errors, the corresponding RMSE of DOA estimation is shown in Fig. 10.  $\sigma_A$  changes from 0 to 0.3, and the estimation error is only improved from 0.33 to 0.32 using the proposed method. However, the existing methods degrade the DOA estimation performance significantly with larger gain errors. Moreover, keeping  $\sigma_A = 0.15$ , we change  $\sigma_{\rm P}$  from 0 to 20 in degree, and the DOA estimation

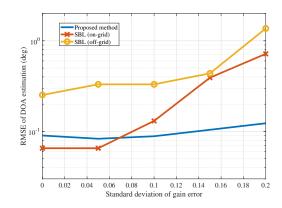


Fig. 12. The DOA estimation performance compared with the SBL method.

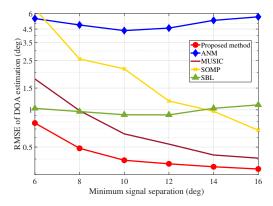


Fig. 13. The DOA estimation with different minimum signal separations.

performance is shown in Fig. 11. As shown in this figure, the proposed method achieves the best estimation performance among these methods. Therefore, in the scenario with gainphase errors, the proposed method can work well. Moreover, with only the gain errors, the DOA estimation performance of the proposed method is also compared with that of the SBL method, as shown in Fig. 12. "SBL (on-grid)" is the DOA estimation performance with the signal angles being at the discretized angles exactly in the spatial domain, and "SBL (off-grid)" means that the signals can be not precisely at the discretized angles. As shown in Fig. 12, when the signals are on-grid, the SBL method outperforms the proposed method in the scenario with small gain-phase errors. However, in the scenario with large gain-phase errors or the off-grid signals, the estimation performance of the SBL method is worse than that of the proposed method.

In the super-resolution methods, the minimum separation between signals is important and shows the ability of superresolution, so we show the DOA estimation performance with different minimum separations in Fig. 13. With larger separation, the correlation between the received signal can be reduced so that the better estimation performance can be achieved. Additionally, the number of measurements is vital for the complexity consideration, and the corresponding estimation performance is shown in Fig. 14. The proposed can outperform the existing methods when the number of measurements is more than 3.

With different numbers of antennas and signals, the DOA estimation performance is shown in Fig. 15 and Fig. 16, respectively. As shown in these two figures, when the number of antennas is more than 10 or the number of signals is less than 4, the proposed method can achieve better estimation performance. Furthermore, the computational time is shown in Table III, the proposed method has relative higher computational complexity. As shown in the results of this section, the proposed method can generally achieve better DOA estimation performance in the scenario with gain-phase errors.

## VIII. CONCLUSIONS

The DOA estimation problem has been considered in the scenario with gain-phased errors, and the GP-ANM has been proposed to formulate the optimization problem. Then, the

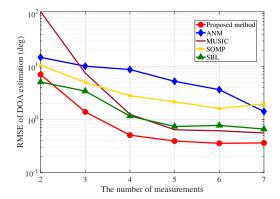


Fig. 14. The DOA estimation with different numbers of measurements.

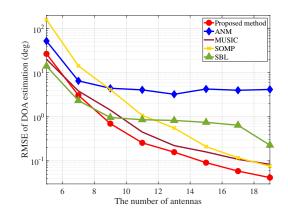


Fig. 15. The DOA estimation with different numbers of antennas.

TABLE III Computational Time

	ANM	MUSIC	SOMP	SBL	Proposed method
Time (s)	2.3047	0.0319	0.0501	0.0780	1.0840

SDP formulation has been derived to solve the DOA estimation problem efficiently, and the corresponding regularization parameter has been obtained theoretically. Simulation results show that the proposed DOA estimation method outperforms

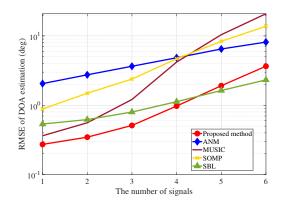


Fig. 16. The DOA estimation with different numbers of signals.

the existing methods in the scenario with gain-phase errors. Future work will focus on the generalized atomic norm in the applications with imperfect antennas.

## APPENDIX A

THE SDP PROOF FOR THE OPTIMIZATION PROBLEM (19)

First, in the constraint  $\operatorname{Tr}(\boldsymbol{Q}) + (C_e + 2\sqrt{N})C_e \|\boldsymbol{Q}\|_2 - 1 \leq 0$ , both the norm operation  $\|\boldsymbol{Q}\|_2$  and the trace operation  $\operatorname{Tr}(\boldsymbol{Q})$ are convex functions, so this constraint is a convex constraint. The optimization problem (19) is a convex optimization problem.

Then, to show that the constraint  $\text{Tr}(\mathbf{Q}) + (C_e + 2\sqrt{N})C_e \|\mathbf{Q}\|_2 - 1 \le 0$  is a SDP constraint, we can formulate a semidefinite matrix

$$\begin{bmatrix} (1 - \operatorname{Tr}(\boldsymbol{Q}))\boldsymbol{I} & (C_{\mathsf{e}} + 2\sqrt{N})C_{\mathsf{e}}\boldsymbol{Q}^{\mathsf{H}} \\ (C_{\mathsf{e}} + 2\sqrt{N})C_{\mathsf{e}}\boldsymbol{Q} & (1 - \operatorname{Tr}(\boldsymbol{Q}))\boldsymbol{I} \end{bmatrix} \succeq 0, \quad (52)$$

with the Schur complement theory, if and only if we have

$$(1 - \operatorname{Tr}(\boldsymbol{Q}))\boldsymbol{I} \succeq 0 \tag{53}$$

$$(1 - \operatorname{Tr}(\boldsymbol{Q}))^2 C_{\mathrm{e}}^2 \boldsymbol{I} - (C_{\mathrm{e}} + 2\sqrt{N})^2 \boldsymbol{Q}^{\mathrm{H}} \boldsymbol{Q} \succeq 0.$$
 (54)

From (54), for arbitrary vector  $\boldsymbol{t}$ , we can obtain a function  $f(\boldsymbol{t}) \triangleq \boldsymbol{t}^{\mathrm{H}} \left[ (1 - \mathrm{Tr}(\boldsymbol{Q}))^2 \boldsymbol{I} - (C_{\mathrm{e}} + 2\sqrt{N})^2 C_{\mathrm{e}}^2 \boldsymbol{Q}^{\mathrm{H}} \boldsymbol{Q} \right] \boldsymbol{t} \geq 0$ , and formulate an optimization problem

$$\begin{array}{l} \min_{\boldsymbol{t}} \quad f(\boldsymbol{t}) \quad (55) \\ \text{s.t.} \quad \|\boldsymbol{t}\|_2 = C_{\mathrm{t}}, \end{array}$$

where  $C_t$  is a positive constant. From (55), the minimum value of f(t) can be achieved as  $f(||t||_2 q_{\text{max}})$ , where  $q_{\text{max}}$  is an eigenvector corresponding to the maximum eigenvalue  $\lambda_{\text{max}}$ of  $Q^H Q$ . Hence, (54) is satisfied, if and only if, for arbitrary t, we have  $f(||t||_2 q_{\text{max}}) \ge 0$ .

Since  $\|\boldsymbol{Q}\|_2^2 = \lambda_{\max}$ , we can simplify  $f(\|\boldsymbol{t}\|_2 \boldsymbol{q}_{\max})$  as

$$f(\|\boldsymbol{t}\|_{2}\boldsymbol{q}_{\max}) = \|\boldsymbol{t}\|_{2}\boldsymbol{q}_{\max}^{\mathrm{H}}(1 - \operatorname{Tr}(\boldsymbol{Q}))^{2}\boldsymbol{I}\|\boldsymbol{t}\|_{2}\boldsymbol{q}_{\max} - \|\boldsymbol{t}\|_{2}\boldsymbol{q}_{\max}^{\mathrm{H}}(C_{\mathrm{e}} + 2\sqrt{N})^{2}C_{\mathrm{e}}^{2}\boldsymbol{Q}^{\mathrm{H}}\boldsymbol{Q}\|\boldsymbol{t}\|_{2}\boldsymbol{q}_{\max}$$
(56)

$$= \|\boldsymbol{t}\|_{2}^{2}(1 - \operatorname{Tr}(\boldsymbol{Q}))^{2} - \|\boldsymbol{t}\|_{2}^{2}(C_{\mathsf{e}} + 2\sqrt{N})^{2}C_{\mathsf{e}}^{2}\boldsymbol{q}_{\mathsf{max}}^{\mathsf{H}}\boldsymbol{Q}^{\mathsf{H}}\boldsymbol{Q}\boldsymbol{q}_{\mathsf{max}}$$
$$= \|\boldsymbol{t}\|_{2}^{2}(1 - \operatorname{Tr}(\boldsymbol{Q}))^{2} - \|\boldsymbol{t}\|_{2}^{2}(C_{\mathsf{e}} + 2\sqrt{N})^{2}C_{\mathsf{e}}^{2}\|\boldsymbol{Q}\|_{2}^{2}.$$

 $f(\|\boldsymbol{t}\|_2 \boldsymbol{q}_{\max}) \ge 0$  is equal to (56)  $\ge 0$  and implies that  $\operatorname{Tr}(\boldsymbol{Q}) + (C_e + 2\sqrt{N})C_e \|\boldsymbol{Q}\|_2 - 1 \le 0$ , which is the constraint in (19).

Finally, the constraint  $\text{Tr}(\mathbf{Q}) + (C_e + 2\sqrt{N})C_e \|\mathbf{Q}\|_2 - 1 \le 0$ in (19) is equal to the semidefinite matrix condition in (52), so the optimization problem (19) is a convex SDP problem.

## APPENDIX B The Proof for Proposition 2

In the dual problem (13), with the atomic norm definition having a gain-phase error e in (10), the dual norm  $\|U\|_{\tilde{A}}^*$  can

be expressed as

$$\begin{split} \|\boldsymbol{U}\|_{\tilde{\mathcal{A}}}^{*} &= \sup_{\|\boldsymbol{X}\|_{\tilde{\mathcal{A}}} \leq 1} \langle \boldsymbol{X}, \boldsymbol{U} \rangle \tag{57} \\ &\stackrel{(a)}{=} \sup_{\substack{\|\boldsymbol{b}\|_{1} \leq 1\\ \theta_{k} \in [0, 2\pi) \\ \|\boldsymbol{d}_{k}\|_{2} \leq C_{c}}} \left\langle \sum_{k=0}^{K-1} b_{k} (\operatorname{diag}\{\boldsymbol{e}\} + \boldsymbol{I}) \boldsymbol{a}(\theta_{k}) \boldsymbol{d}_{k}^{\mathrm{T}}, \boldsymbol{U} \right\rangle \\ &= \sup_{\substack{\|\boldsymbol{b}\|_{1} \leq 1\\ \theta_{k} \in [0, 2\pi) \\ \|\boldsymbol{d}_{k}\|_{2} \leq C_{c}}} \sum_{k=0}^{K-1} \langle b_{k} (\operatorname{diag}\{\boldsymbol{e}\} + \boldsymbol{I}) \boldsymbol{a}(\theta_{k}) \boldsymbol{d}_{k}^{\mathrm{T}}, \boldsymbol{U} \rangle \\ &\stackrel{(b)}{=} \sup_{\substack{\|\boldsymbol{b}\|_{1} \leq 1\\ \theta_{k} \in [0, 2\pi) \\ \|\boldsymbol{d}_{k}\|_{2} \leq C_{c}}} \sum_{k=0}^{K-1} \mathcal{R} \left\{ \boldsymbol{d}_{k}^{\mathrm{T}} \boldsymbol{U}^{\mathrm{H}} b_{k} (\operatorname{diag}\{\boldsymbol{e}\} + \boldsymbol{I}) \boldsymbol{a}(\theta_{k}) \right\} \\ &\stackrel{(c)}{=} \sup_{\substack{\|\boldsymbol{b}\|_{1} \leq 1\\ \theta_{k} \in [0, 2\pi) \\ \|\boldsymbol{e}\|_{2} \leq C_{c}}} \sum_{k=0}^{K-1} b_{k} \left\| \boldsymbol{U}^{\mathrm{H}} (\operatorname{diag}\{\boldsymbol{e}\} + \boldsymbol{I}) \boldsymbol{a}(\theta) \right\|_{2} \\ &\stackrel{(c)}{=} \sup_{\substack{\|\boldsymbol{b}\|_{1} \leq 1\\ \theta_{k} \in [0, 2\pi) \\ \|\boldsymbol{e}\|_{2} \leq C_{c}}} \left\| \boldsymbol{U}^{\mathrm{H}} [\boldsymbol{e} + \boldsymbol{a}(\theta)] \right\|_{2} \|\boldsymbol{b}\|_{1} \\ &\stackrel{(d)}{=} \sup_{\substack{\|\boldsymbol{e}\|_{1} \leq C_{c}\\ \theta \in [0, 2\pi)}} \left\| \boldsymbol{U}^{\mathrm{H}} [\boldsymbol{e} + \boldsymbol{a}(\theta)] \right\|_{2}, \\ &\stackrel{\|\boldsymbol{e}\|_{2} \leq C_{c}\\ \theta \in [0, 2\pi)} \right\| \boldsymbol{U}^{\mathrm{H}} [\boldsymbol{e} + \boldsymbol{a}(\theta)] \right\|_{2}, \end{aligned}$$

where (a) is from the definition of atomic norm with gainphase errors, (b) is obtained with the definition of inner product between matrices, and (c) is given by

$$\boldsymbol{d}_{k} = \frac{\boldsymbol{U}^{\mathrm{T}}(\mathrm{diag}\{\boldsymbol{e}\}^{\mathrm{H}} + \boldsymbol{I})\boldsymbol{a}^{*}(\boldsymbol{\theta}_{k})}{\left\|\boldsymbol{U}^{\mathrm{T}}(\mathrm{diag}\{\boldsymbol{e}\}^{\mathrm{H}} + \boldsymbol{I})\boldsymbol{a}^{*}(\boldsymbol{\theta}_{k})\right\|_{2}}.$$
 (58)

For the equation (d), we formulate  $e' \triangleq \operatorname{diag}(e)a(\theta) \in \mathbb{C}^{N \times 1}$ , where we use the steering vector  $a(\theta)$  ( $\theta \in [0, 2\pi)$ ) and a vector  $e \in \mathbb{C}^{N \times 1}$  ( $||e||_2 \leq C_e$ ). Then, the  $\ell_2$  norm of e' can be simplified as

$$\begin{aligned} \|\boldsymbol{e}'\|_{2}^{2} &= \|\operatorname{diag}(\boldsymbol{e})\boldsymbol{a}(\theta)\|_{2}^{2} \\ &= \boldsymbol{e}^{\mathrm{H}}\operatorname{diag}(\boldsymbol{a}^{*}(\theta))\operatorname{diag}(\boldsymbol{a}(\theta))\boldsymbol{e} = \|\boldsymbol{e}\|_{2}^{2} \leq C_{\mathrm{e}}^{2}. \end{aligned}$$
(59)

Therefore, we have

$$\sup_{\substack{\theta_{k} \in [0,2\pi) \\ \|\boldsymbol{e}\|_{2} \leq C_{e}}} \left\| \boldsymbol{U}^{\mathrm{H}} \underbrace{\operatorname{diag}\{\boldsymbol{e}\}\boldsymbol{a}(\theta)}_{\boldsymbol{e}'} + \boldsymbol{U}^{\mathrm{H}}\boldsymbol{a}(\theta) \right\|_{2}$$
$$= \sup_{\substack{\theta_{k} \in [0,2\pi) \\ \|\boldsymbol{e}'\|_{2} \leq C_{e}}} \left\| \boldsymbol{U}^{\mathrm{H}} \left[\boldsymbol{e}' + \boldsymbol{a}(\theta)\right] \right\|_{2}.$$
(60)

Then, the following equality in (d) can be obtained, where we just reuse the notation e instead of e' in (60) to avoid introducing additional symbol e'.

Then, for the constraint  $\|U\|_{\tilde{\mathcal{A}}}^* \leq \tau$ , we build the following positive semidefinite matrix

$$\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{U} \\ \boldsymbol{U}^{\mathrm{H}} & \boldsymbol{W} \end{bmatrix} \succeq \boldsymbol{0}, \tag{61}$$

where Q and W are the Hermitian matrices. With the Schur complement, when W is invertible, the matrix is positive semidefinite if and only if we have

$$\boldsymbol{Q} \succeq \boldsymbol{0}, \tag{62}$$

$$\boldsymbol{Q} - \boldsymbol{U}\boldsymbol{W}^{-1}\boldsymbol{U}^{\mathrm{H}} \succeq \boldsymbol{0}. \tag{63}$$

Therefore, for any vector  $t \in \mathbb{C}^{N \times 1}$ , we have  $t^{\mathrm{H}}Qt \geq t^{\mathrm{H}}UW^{-1}U^{\mathrm{H}}t$ . By selecting  $W = \tau^{2}I_{P}$ , we obtain

$$\left\|\boldsymbol{U}^{\mathrm{H}}\boldsymbol{t}\right\|_{2}^{2} \leq \tau^{2}\boldsymbol{t}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{t}.$$
(64)

When the gain-phase errors are considered, we formulate  $t = a(\theta) + e$ , and we have

$$\begin{aligned} \left\| \boldsymbol{U}^{\mathrm{H}} \left[ \boldsymbol{e} + \boldsymbol{a}(\theta) \right] \right\|_{2}^{2} &\leq \tau^{2} \left[ \boldsymbol{e} + \boldsymbol{a}(\theta) \right]^{\mathrm{H}} \boldsymbol{Q} \left[ \boldsymbol{e} + \boldsymbol{a}(\theta) \right] \tag{65} \\ &= \tau^{2} \left( \boldsymbol{e}^{\mathrm{H}} \boldsymbol{Q} \boldsymbol{e} + 2\mathcal{R} \{ \boldsymbol{e}^{\mathrm{H}} \boldsymbol{Q} \boldsymbol{a}(\theta) \} + \boldsymbol{a}^{\mathrm{H}}(\theta) \boldsymbol{Q} \boldsymbol{a}(\theta) \right) \\ &\leq \tau^{2} \left( C_{\mathrm{e}} \frac{(\boldsymbol{Q} \boldsymbol{e})^{\mathrm{H}}}{\|\boldsymbol{Q} \boldsymbol{e}\|_{2}} \boldsymbol{Q} \boldsymbol{e} + 2\mathcal{R} \{ \boldsymbol{e}^{\mathrm{H}} \boldsymbol{Q} \boldsymbol{a}(\theta) \} + \boldsymbol{a}^{\mathrm{H}}(\theta) \boldsymbol{Q} \boldsymbol{a}(\theta) \right) \\ &\leq \tau^{2} \left( C_{\mathrm{e}} \| \boldsymbol{Q} \boldsymbol{e} \|_{2} + 2C_{\mathrm{e}} \| \boldsymbol{Q} \boldsymbol{a}(\theta) \|_{2} + \boldsymbol{a}^{\mathrm{H}}(\theta) \boldsymbol{Q} \boldsymbol{a}(\theta) \right) \\ &= \tau^{2} \left( C_{\mathrm{e}} \sqrt{\|\boldsymbol{Q} \boldsymbol{e}\|_{2}^{2}} + 2C_{\mathrm{e}} \sqrt{\|\boldsymbol{Q} \boldsymbol{a}(\theta)\|_{2}^{2}} + \boldsymbol{a}^{\mathrm{H}}(\theta) \boldsymbol{Q} \boldsymbol{a}(\theta) \right) \end{aligned}$$

Since  $\|\boldsymbol{e}\|_2 \leq C_{\text{e}}$  and  $\boldsymbol{Q} \succeq 0$ , we have  $\|\boldsymbol{Q}\boldsymbol{e}\|_2^2 \leq C_{\text{e}}^2 \|\boldsymbol{Q}\|_2^2$ , where  $\|\boldsymbol{Q}\|_2 \triangleq \lambda_{\max}(\boldsymbol{Q})$ , and  $\lambda_{\max}(\boldsymbol{Q})$  is the largest singular value of  $\boldsymbol{Q}$ . Additionally, we have  $\|\boldsymbol{Q}\boldsymbol{a}(\theta)\|_2^2 \leq N \|\boldsymbol{Q}\|_2^2$ . Therefore, we can simplified (65) as

$$\left\|\boldsymbol{U}^{\mathrm{H}}\left[\boldsymbol{e}+\boldsymbol{a}(\theta)\right]\right\|_{2}^{2} \leq \tau^{2} \left[ (C_{\mathrm{e}}+2\sqrt{N})C_{\mathrm{e}}\|\boldsymbol{Q}\|_{2}+\boldsymbol{a}^{\mathrm{H}}(\theta)\boldsymbol{Q}\boldsymbol{a}(\theta) \right]$$
(66)

When Q satisfies the following condition

$$\sum_{n} Q_{n,n+k} = 0 \ (k \neq 0)$$

$$Tr(\boldsymbol{Q}) + (C_{e} + 2\sqrt{N})C_{e} \|\boldsymbol{Q}\|_{2} - 1 \le 0,$$
(67)

we have  $\|\boldsymbol{U}^{\mathrm{H}}[\boldsymbol{e} + \boldsymbol{a}(\theta)]\|_{2}^{2} \leq \tau^{2}$ . Therefore, substitute into (57) and the constraint  $\|\boldsymbol{U}\|_{\tilde{\mathcal{A}}}^{*} \leq \tau$  is satisfied, then, the dual problem (13) can be simplified as the SDP problem in (19).

# APPENDIX C The Proof for Proposition 4

The entries of N follow the zero-mean Gaussian distribution with the variance being  $\sigma_N^2$ . The dual norm of N with the definition in (57) can be simplified as

$$\mathcal{E}\left\{\|\boldsymbol{N}\|_{\tilde{\mathcal{A}}}^{*}\right\} = \mathcal{E}\left\{\sup_{\substack{\|\boldsymbol{e}\|_{2} \leq C_{e} \\ \boldsymbol{\theta} \in [0, 2\pi)}} \left\|\boldsymbol{N}^{\mathrm{H}}[\boldsymbol{e} + \boldsymbol{a}(\boldsymbol{\theta})]\right\|_{2}\right\}$$
$$\leq \mathcal{E}\left\{\sup_{\|\boldsymbol{e}\|_{2} \leq C_{e}} \left\|\boldsymbol{N}^{\mathrm{H}}\boldsymbol{e}\right\|_{2}\right\} + \mathcal{E}\left\{\sup_{\boldsymbol{\theta} \in [0, 2\pi)} \left\|\boldsymbol{N}^{\mathrm{H}}\boldsymbol{a}(\boldsymbol{\theta})\right\|_{2}\right\}$$
$$\leq C_{e}\mathcal{E}\left\{\|\boldsymbol{N}\|_{2}\right\} + \sigma_{N}\sqrt{4NP\ln N}, \tag{68}$$

where  $\mathcal{E}\left\{\sup_{\theta \in [0,2\pi)} \|\mathbf{N}^{\mathsf{H}} \boldsymbol{a}(\theta)\|_{2}\right\} \leq \sigma_{\mathsf{N}} \sqrt{4NP \ln N}$  is obtained from Lemma 5.1 of [55]. Additionally, we can obtain the upper bound of  $\mathcal{E}\left\{\|\mathbf{N}\|_{2}\right\}$  as two types. The first type is

formulated based on  $\mathcal{E} \{ \|N\|_2 \} \leq \mathcal{E} \{ \|N\|_F \}$ . Since  $\|N\|_F$  follows chi distribution  $\|N\|_F \sim \chi_{NP}$ , we can obtain

$$\mathcal{E}\left\{\left\|\boldsymbol{N}\right\|_{F}\right\} = \sqrt{2}\sigma_{N}\frac{\Gamma((NP+1)/2)}{\Gamma(NP/2)},\tag{69}$$

where the gamma function is defined as  $\Gamma(x) \triangleq \int_0^\infty z^{x-1} e^{-z} dz$ . Then, we have

$$\mathcal{E}\left\{\left\|\boldsymbol{N}\right\|_{2}\right\} \leq \mathcal{E}\left\{\left\|\boldsymbol{N}\right\|_{F}\right\} = \sqrt{2}\sigma_{N}\frac{\Gamma((NP+1)/2)}{\Gamma(NP/2)} \triangleq \mathrm{bd}_{1}.$$
(70)

With the probability more than  $1-2e^{-t^2/2}$ , the second type of upper bound can be formulated as

$$\mathcal{E}\left\{\left\|\boldsymbol{N}\right\|_{2}\right\} \leq \left(\sqrt{N} + \sqrt{P} + t\right)\sigma_{N} \triangleq \mathrm{bd}_{2},$$
 (71)

where the upper bound is obtained from Theorem 5.35 of [62]. Finally, the upper bound of  $\mathcal{E}\left\{\|N\|_{\tilde{\mathcal{A}}}^*\right\}$  can be obtained as

$$\mathcal{E}\left\{\|\boldsymbol{N}\|_{\mathcal{A}}^{*}\right\} \leq \min\left\{\mathrm{bd}_{1},\mathrm{bd}_{2}\right\}C_{\mathrm{e}} + \sigma_{\mathrm{N}}\sqrt{4NP\ln N}.$$
 (72)

## Appendix D

THE ENTRIES OF FISHER INFORMATION MATRIX For the Fisher information  $\boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_{1,1} & \boldsymbol{F}_{1,2} \\ \boldsymbol{F}_{2,1} & \boldsymbol{F}_{2,2} \end{bmatrix}$ , the entries can be obtained as follows:

• The  $k_1, k_2$ -th entry of Fisher information matrix  $F_{1,1}$  can be obtained as

$$F_{k_1,k_2}^{1,1} = \frac{\partial \ln \det\{\boldsymbol{C}\}}{\partial \theta_{k_1} \partial \theta_{k_2}} + \mathcal{E}\left\{\frac{\partial \boldsymbol{y}^{\mathsf{H}} \boldsymbol{C}^{-1} \boldsymbol{y}}{\partial \theta_{k_1} \partial \theta_{k_2}}\right\}, \quad (73)$$

where the first term can be obtained as

$$\frac{\partial \ln \det\{\boldsymbol{C}\}}{\partial \theta_{k_1} \partial \theta_{k_2}} = \operatorname{Tr}\left\{\frac{\partial \boldsymbol{C}^{-1} \frac{\partial \boldsymbol{C}}{\partial \theta_{k_1}}}{\partial \theta_{k_2}}\right\}$$
(74)
$$= \operatorname{Tr}\left\{\frac{\partial \boldsymbol{C}^{-1}}{\partial \theta_{k_2}} \frac{\partial \boldsymbol{C}}{\partial \theta_{k_1}}\right\} + \operatorname{Tr}\left\{\boldsymbol{C}^{-1} \frac{\partial \boldsymbol{C}}{\partial \theta_{k_1} \partial \theta_{k_2}}\right\},$$

and the second term is

$$\mathcal{E}\left\{\frac{\partial \boldsymbol{y}^{\mathrm{H}}\boldsymbol{C}^{-1}\boldsymbol{y}}{\partial \theta_{k_{1}}\partial \theta_{k_{2}}}\right\} = \mathrm{Tr}\left\{\frac{\partial \boldsymbol{C}^{-1}}{\partial \theta_{k_{1}}\partial \theta_{k_{2}}}\boldsymbol{C}\right\}.$$
 (75)

Therefore,  $F_{k_1,k_2}^{1,1}$  can be simplified as

$$F_{k_1,k_2}^{1,1} = \operatorname{Tr}\left\{ \boldsymbol{C}^{-1} \frac{\partial \boldsymbol{C}}{\partial \theta_{k_1}} \boldsymbol{C}^{-1} \frac{\partial \boldsymbol{C}}{\partial \theta_{k_2}} \right\}.$$
 (76)

• The  $k_1, k_2$ -th entry of Fisher information matrix  $F_{1,2}$  can be obtained as

$$F_{k_1,k_2}^{1,2} = \frac{\partial \ln \det\{\boldsymbol{C}\}}{\partial \theta_{k_1} \partial g_{k_2}} + \mathcal{E}\left\{\frac{\partial \boldsymbol{y}^{\mathsf{H}} \boldsymbol{C}^{-1} \boldsymbol{y}}{\partial \theta_{k_1} \partial g_{k_2}}\right\}, \quad (77)$$

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where the first term is simplified as

$$\frac{\partial \ln \det\{\boldsymbol{C}\}}{\partial \theta_{k_1} \partial g_{k_2}} = \operatorname{Tr} \left\{ \frac{\partial \boldsymbol{C}^{-1} \frac{\partial \boldsymbol{C}}{\partial \theta_{k_1}}}{\partial g_{k_2}} \right\}$$

$$= \operatorname{Tr} \left\{ \frac{\partial \boldsymbol{C}^{-1}}{\partial g_{k_2}} \frac{\partial \boldsymbol{C}}{\partial \theta_{k_1}} \right\} + \operatorname{Tr} \left\{ \boldsymbol{C}^{-1} \frac{\partial \boldsymbol{C}}{\partial \theta_{k_1} \partial g_{k_2}} \right\},$$
(78)

and the second term is

$$\mathcal{E}\left\{\frac{\partial \boldsymbol{y}^{\mathrm{H}}\boldsymbol{C}^{-1}\boldsymbol{y}}{\partial \theta_{k_{1}}\partial g_{k_{2}}}\right\} = \mathrm{Tr}\left\{\frac{\partial \boldsymbol{C}^{-1}}{\partial \theta_{k_{1}}\partial g_{k_{2}}}\boldsymbol{C}\right\}.$$
 (79)

Therefore,  $F_{k_1,k_2}^{1,2}$  can be simplified as

$$F_{k_1,k_2}^{1,2} = \operatorname{Tr}\left\{\frac{\partial \boldsymbol{C}}{\partial \theta_{k_1}} \boldsymbol{C}^{-1} \frac{\partial \boldsymbol{C}}{\partial g_{k_2}} \boldsymbol{C}^{-1}\right\}.$$
 (80)

• Similarly, we can get the  $k_1, k_2$ -th entry of Fisher information matrix  $F_{2,1}$  as

$$F_{k_1,k_2}^{2,1} = -\mathcal{E}\left\{ \frac{\partial \ln f(\boldsymbol{y};\boldsymbol{\theta},\boldsymbol{g})}{\partial g_{k_1} \partial \theta_{k_2}} \middle| \boldsymbol{\theta}, \boldsymbol{g} \right\}$$
$$= \operatorname{Tr}\left\{ \frac{\partial \boldsymbol{C}}{\partial g_{k_1}} \boldsymbol{C}^{-1} \frac{\partial \boldsymbol{C}}{\partial \theta_{k_2}} \boldsymbol{C}^{-1} \right\}.$$
(81)

• The  $k_1, k_2$ -th entry of Fisher information matrix  $\boldsymbol{F}_{2,2}$  can be obtained as

$$F_{k_1,k_2}^{2,2} = \operatorname{Tr}\left\{ \boldsymbol{C}^{-1} \frac{\partial \boldsymbol{C}}{\partial g_{k_1}} \boldsymbol{C}^{-1} \frac{\partial \boldsymbol{C}}{\partial g_{k_2}} \right\}.$$
 (82)

From the entries of block matrices  $F_{1,1}, F_{1,2}, F_{2,1}$  and  $F_{2,2}$ , we can find that the expressions of  $\frac{\partial C}{\partial \theta_k}$  and  $\frac{\partial C}{\partial g_k}$  must be calculated, so we can obtain the expressions as follows:

• For  $\frac{\partial C}{\partial \theta_{h}}$ , we have

$$\frac{\partial \boldsymbol{C}}{\partial \theta_k} = \left( \boldsymbol{I} \otimes \boldsymbol{G} \frac{\partial \boldsymbol{A}}{\partial \theta_k} \right) \boldsymbol{B} (\boldsymbol{I} \otimes (\boldsymbol{G} \boldsymbol{A})^{\mathrm{H}}) + (\boldsymbol{I} \otimes \boldsymbol{G} \boldsymbol{A}) \boldsymbol{B} \left( \boldsymbol{I} \otimes \frac{\partial \boldsymbol{A}^{\mathrm{H}}}{\partial \theta_k} \boldsymbol{G}^{\mathrm{H}} \right)$$
(83)

where  $\frac{\partial A}{\partial \theta_k}$  is expressed as  $\frac{\partial A}{\partial \theta_k} = \left[\mathbf{0}, \frac{\partial a(\theta_k)}{\partial \theta_k}, \mathbf{0}\right]$ , and  $\frac{\partial a(\theta_k)}{\partial \theta_k} \text{ can be obtained easily.}$ • For  $\frac{\partial C}{\partial g_k}$ , we can obtain

$$rac{\partial oldsymbol{C}}{\partial g_k} = \left(oldsymbol{I} \otimes rac{\partial oldsymbol{G}}{\partial g_k}oldsymbol{A}
ight)oldsymbol{B}(oldsymbol{I} \otimes (oldsymbol{G}oldsymbol{A})^{ ext{H}}),$$

where  $\frac{\partial G}{\partial q_k}$  can be obtained easily.

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