# Over-Parametrized Matrix Factorization in the Presence of Spurious Stationary Points 

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#### Abstract

Motivated by the emerging role of interpolating machines in signal processing and machine learning, this work considers the computational aspects of over-parametrized matrix factorization. In this context, the optimization landscape may contain spurious stationary points (SSPs), which are proved to be full-rank matrices. The presence of these SSPs means that it is impossible to hope for any global guarantees in over-parametrized matrix factorization. For example, when initialized at an SSP, the gradient flow will be trapped there forever. Nevertheless, despite these SSPs, we establish in this work that the gradient flow of the corresponding merit function converges to a global minimizer, provided that its initialization is rank-deficient and sufficiently close to the feasible set of the optimization problem. We numerically observe that a heuristic discretization of the proposed gradient flow, inspired by primaldual algorithms, is successful when initialized randomly. Our result is in sharp contrast with the local refinement methods which require an initialization close to the optimal set of the optimization problem. More specifically, we successfully avoid the traps set by the SSPs because the gradient flow remains rank-deficient at all times, and not because there are no SSPs nearby. The latter is the case for the local refinement methods. Moreover, the widely-used restricted isometry property plays no role in our main result.


Index Terms-rank-constrained matrix factorization, BurerMonteiro factorization, over-parametrization, interpolation, nonconvex optimization, stationary points

## I. Introduction

Rank-constrained matrix factorization from limited data is central to various applications in signal processing and machine learning [1], and more recently has also served as a platform to gain theoretical insight into unexplained phenomena in deep neural networks [2], [3], [4].
Despite recent strides, we face key theoretical questions in matrix factorization, particularly in the emerging areas that are motivated by the study of neural networks. The aim
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of this work is to take a step towards answering one such question.
To be concrete, we are interested in the computational (rather than statistical) aspects of solving the problem

$$
\begin{equation*}
\min _{U \in \mathbb{R}^{d \times p}}\|U\|_{\mathrm{F}}^{2} \text { subject to } \mathcal{A}\left(U U^{\top}\right)=b,\|U\| \leqslant \xi \tag{1}
\end{equation*}
$$

where $\mathcal{A}: \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{m}$ is a linear operator and $b \in \mathbb{R}^{m}$. Above, we have limited the spectral norm of $U$ to $\xi>0$ for technical convenience. By setting $\xi$ sufficiently large, we can practically ignore the constraint $\|U\| \leqslant \xi$ in (1).
Even though our interest in problem (1) is purely computational, the statistical significance of problem (1) can be motivated as follows.

- The (often nonconvex) problem (1) corresponds to the Burer-Monteiro factorization [5] of the (convex) nuclear norm minimization problem. This latter convex problem is at the heart of low-rank matrix sensing and completion, phase retrieval and quadratic sensing, and blind deconvolution, to name a few [6].
Compared to its convex analogue, which has $d(d+1) / 2$ optimization variables, problem (1) has $p d$ variables. Therefore, solving problem (1) can offer computational gains when $p$ is sufficiently small [6], [7].
- Solving problem (1) is equivalent to minimizing the empirical risk of the shallow linear network $x \rightarrow U U^{\top} x$ with weight decay regularization. The corresponding training data is collected in the vector $b$ [8]. Understanding linear neural networks, such as this one, is a necessary first step in studying (nonlinear) neural networks [9].
Kurzgesagt. In a nutshell, the contribution of this work is a gradient flow that provably solves problem (1) in the over-parametrized regime ( $p \gtrsim m / d$ ), whenever this flow is initialized at a rank-deficient matrix $U_{0}$ that is sufficiently close to the feasible set of problem (1).
In the over-parametrized regime, note that we cannot possibly hope for a global scheme for solving problem (1) because the feasible set of problem (1) may contain spurious stationary points. These spurious points can trap the gradient flow, when initialized arbitrarily.
It is then necessary to restrict the initialization $U_{0}$ in some way, and our particular choice of a "sufficiently feasible" $U_{0}$
has several precedents within the nonconvex optimization literature [4], [7], [10].
We will also describe, albeit less formally, how one can find an appropriate initialization for the gradient flow.
At the same time, our contribution is fundamentally different from the local refinement methods, within the signal processing literature: These methods rely on local strong convexity in a small neighborhood of an isolated global minimizer [6, Chapter 5].
To be specific, even though our gradient flow requires an initialization $U_{0}$ that is sufficiently close to the feasible set of problem (1), this initialization might be far from any global minimizer of problem (1). Moreover, a small neighborhood of the feasible set of problem (1) contains all spurious stationary points of problem (1), whereas a small neighborhood of an isolated global minimizer contains no other stationary points.
This work appears to be the first to study problem (1) in the over-parametrized regime ( $p \gtrsim m / d$ ). Therefore, to better motivate our contributions, let us first discuss below the concepts of interpolation and under/over-parametrization, both central to this work. Even though we are solely interested in the computational aspects of solving problem (1), whenever possible, we will also motivate the statistical significance of these concepts to enrich the presentation.


## A. Interpolation

A distinct feature of problem (1) is its interpolation property. That is, by design, any global minimizer $\bar{U}$ of problem (1) satisfies the equality constraints $\mathcal{A}\left(\bar{U} \bar{U}^{\top}\right)=b$. For example, if we interpret $x \rightarrow \bar{U} \bar{U}^{\top} x$ as a shallow linear network [9], this network perfectly interpolates its training data $b$ and thus achieves zero training error.
This interpolation property has attracted growing attention within machine learning, in part because modern deep neural networks appear to satisfy this property [11]. Consequently, better understanding the statistical and computational strengths of interpolating learning machines has become a major research target [12], [13]. For instance, stochastic gradient descent provably exhibits built-in variance reduction under the interpolation property [14].
To summarize, motivated by the success of modern neural networks, our interest in problem (1) partly stems from the need to better understand the computational aspects of interpolating learning machines, such as problem (1).
Moreover, through the lens of signal processing, the interpolation property of problem (1) indicates a subtle but important shift in statistical perspective, described next.
In the context of low-rank matrix recovery [1], let $b:=$ $\mathcal{A}\left(X^{\natural}\right)+e$ for a hidden model $X^{\natural} \in \mathbb{R}^{d \times d}$ and measurement noise vector $e \neq 0$. Alternatively, we can also let $b:=$ $\mathcal{A}\left(X^{\natural}+E\right)$, where the matrix $E \neq 0$ represents the model mismatch. In both cases, the equality constraints $\mathcal{A}\left(U U^{\top}\right)=$ $b$ in problem (1) replace the relaxed constraints $\| \mathcal{A}\left(U U^{\top}\right)-$ $b \| \leqslant \epsilon$, which is more common in signal processing [15].

This shift in statistical perspective imitates the success of deep neural networks, which often interpolate their training data and yet generalize well on test data [11]. Furthermore, whenever reliable prior knowledge about the probability distributions of the noise $e$ and mismatch $E$ is lacking, it might be wise to opt for problem (1), instead of incorporating the relaxed constraints with a (possibly) incorrect choice of norm $\|\cdot\|$ and $\epsilon$.

## B. Under- and Over-Parametrization

By counting the number of optimization variables in problem (1), we can distinguish two regimes: Under- and over-parametrized.
The focus of this work is on the over-parametrized regime of problem (1), which has never been studied before, to the best of our knowledge. To better appreciate the computational challenges of the over-parametrized regime, both regimes are juxtaposed together below.

1) (UNDER-PARAMETRIZED REGIME) When $p \lesssim m / d$, the landscape of problem (1) is often benign in the sense that problem (1), even though nonconvex, has no spurious stationary points [6]. Therefore, in this regime, problem (1) can be solved to global optimality with a range of algorithms, including the gradient descent algorithm and its perturbed variants [16], [17].
More specifically, often in the under-parametrized regime, every feasible point of problem (1) is a global minimizer. That is, the target function $\|U\|_{\mathrm{F}}^{2}$ in problem (1) is redundant and it suffices to minimize the feasibility gap $\left\|\mathcal{A}\left(U U^{\top}\right)-b\right\|_{2}^{2}$. The latter can be done successfully with, for example, the gradient descent algorithm [6], [18], [16].
As a toy example, in the left panel of Figure 1, the (discretized) gradient flow of the feasibility gap converges to the only feasible point of problem (1), which is unique up to a sign. This limit point is also evidently the global minimizer of problem (1) in this toy example.
From a signal processing perspective, the underparametrized regime of problem (1) is particularly well suited for low-rank matrix recovery in the absence of noise: Let $b:=\mathcal{A}\left(X^{\natural}\right)$ for a rank- $r$ matrix $X^{\natural} \in \mathbb{R}^{d \times d}$, and suppose that $\mathcal{A}$ is a generic linear operator. Then, with the choice of $p=r$, solving problem (1) uniquely recovers the true model $X^{\natural}$, provided that $m \gtrsim p d=r d$. Here, we have ignored logarithmic factors for simplicity [6].
2) (OVER-PARAMETRIZED REGIME) In contrast, when $p \gtrsim$ $m / d$, the feasible set of problem (1) may contain spurious stationary points which can trap first- or second-order optimization algorithms, including the gradient descent. The presence of spurious stationary points means that we cannot hope to globally solve problem (1). For example, if we initialize any first-order optimization algorithm at a spurious stationary point of problem (1), the algorithm will be trapped there forever!
In the over-parametrized regime, we can visualize the
computational challenges of solving problem (1) with a numerical example: In a precise sense, solving the (constrained) problem (1) is equivalent to minimizing its (smooth) merit function [19]. In the right panel of Figure 1, the (discretized) gradient flow of this merit function converges to a particular feasible point of problem (1) which is not a global minimizer.
This discouraging observation rules out the possibility of a global scheme for solving problem (1), but not all is lost. Indeed, the contribution of this work is a non-global (but also non-local) scheme for solving problem (1) in the overparametrized regime.
Let us next motivate the statistical significance of the overparametrized regime. First, recall that modern neural networks are highly over-parametrized [11]. In this sense, the overparametrized regime of problem (1) more faithfully represents the training of a neural network with weight decay, compared to the under-parametrized regime.
Second, from a signal processing perspective, the overparametrized regime can be motivated as follows. Consider the problem of low-rank matrix recovery in the presence of noise: Let $b:=\mathcal{A}\left(X^{\natural}\right)+e$ for a hidden rank- $r$ model $X^{\natural}$ and measurement noise $e \in \mathbb{R}^{m}$.
With this choice of $\mathcal{A}, b$ and $p=r$, problem (1) is not necessarily even feasible because of its interpolation property! That is, with the choice of $p=r$, we cannot necessarily find a matrix that satisfies the constraints of problem (1). A larger value of $p$ is often needed. In particular, in view of the Pataki's lemma [20], [21], it suffices to set $p \gtrsim \sqrt{2 m}$. (Under the mild assumption that $m \leqslant 2 d^{2}$, it is easy to verify that $p \gtrsim \sqrt{2 m} \Longrightarrow p \gtrsim m / d$. That is, $p \gtrsim \sqrt{2 m}$ falls well within the over-parametrized regime.) This observation motivates the study of problem (1) in the over-parametrized regime, from a signal processing viewpoint.
On the downside, the price that we pay for the interpolation property in problem (1) is a larger computational burden: Larger $p$ in the over-parametrized regime means more variables to optimize in problem (1). As a result, in the context of low-rank matrix recovery, where problem (1) is the BurerMonteiro factorization of a convex optimization problem [6], some but not necessarily all of the computational gains of the Burer-Monteiro factorization will be lost inevitably.

## C. Scope and Contributions

At last, we are now in position to fix the scope of this work: We are interested here in the computational (rather than statistical) aspects of solving the (interpolating) machine (1) in the over-parametrized regime $(p \gtrsim m / d)$. This has never been studied before, to the best of our knowledge.
Let us summarize below the contributions of this work in a simplified and, at times, inaccurate language. In short, this paper designs a gradient flow that provably solves the nonconvex problem (1) to global optimality in the overparametrized regime, even though the optimization landscape
may contain spurious stationary points. An informal statement of our main result is presented below.

Theorem 1 (Simplified main result, Theorem 21). Suppose that $p \gtrsim m / d$. Suppose also that problem (1) has a finite optimal value and a tight convex relaxation. We also consider the set

$$
\begin{equation*}
\mathcal{M}_{b}:=\left\{U: \mathcal{A}\left(U U^{\top}\right)=b,\|U\|<\xi\right\} \tag{2}
\end{equation*}
$$

which contains all (strictly) feasible matrices in problem (1). Suppose that $\mathcal{M}_{b}$ is a smooth submanifold of $\mathbb{R}^{d \times p}$.
Then there exists a gradient flow that almost surely converges in limit to a global minimizer of problem (1), provided that its initialization $U_{0} \in \mathbb{R}^{d \times p}$ is rank-deficient and sufficiently close to the manifold $\mathcal{M}_{b}$. (Such an initialization matrix exists under mild assumptions.)

Restricting the initialization is necessary above and we cannot hope to improve Theorem 1 and obtain global guarantees. Indeed, the feasible set of problem (1) may contain spurious stationary points in the over-parametrized regime which can trap the gradient flow, when initialized arbitrarily, see Figure 1.
At the same time, in Theorem 1, note that the neighborhood of the set $\mathcal{M}_{b}$ contains all spurious stationary points of problem (1). In that sense, Theorem 1 is not a local convergence result.
As a practical remark, we later empirically observe that a random initialization $U_{0}$ is often a good choice that avoids the worst-case scenario in the right panel of Figure 1.
A tight convex relaxation in Theorem 1 is a mild assumption which is met, for example, if we take $p \gtrsim \sqrt{2 m}$ [20], [21].
The manifold assumption in Theorem 1 is minimal in the sense that it corresponds to the weakest sufficient conditions under which there is in general any hope for the gradient flow to efficiently find a feasible matrix for problem (1), i.e., a matrix that satisfies the constraints of (1). This assumption has several precedents within the nonconvex optimization literature [4], [10], [22].
Theorem 1 radically departs from the established literature of rank-constrained matrix factorization. To begin, the restricted isometry property, defined later, which dominates the low-rank matrix recovery literature [1], [6], cannot hold in the over-parametrized regime of this paper and does not appear in Theorem 1. A review of the related work will follow shortly.
Technical novelty. The proof of Theorem 1 relies on a spectral argument that does not have any precedents within the matrix factorization literature, to the best of our knowledge. This argument builds on the observation that, perhaps remarkably, matrix rank does not increase along the gradient flow of the merit function. We have, however, used simpler versions of this argument in earlier works [4], [9].


Fig. 1: For a toy example, this figure visualizes the differences between the under- and over-parametrized regimes of problem (1). In particular, in this toy example, the feasible set of problem (1) contains a spurious stationary point in the over-parametrized regime (left panel) that traps the optimization algorithm, in a sharp contrast with the under-parametrized regime (right panel).
To be specific, we set $d=m=2, \xi=\infty$ and generate $\mathcal{A}, b$ randomly, as detailed in the code. $\mathcal{A}$ and $b$ are identical in both top and bottom panels. The left panel corresponds to $p=1$ whereas, in the right panel, we have $p=2$. The blue dot in the left panel is the only feasible point of problem (1), up to a sign, and also evidently its global minimizer. On the other hand, the blue surface in the right panel shows a section of the feasible set of problem (1). The three-dimensional mesh in the right panel was created with [23].
The left panel shows the trajectory of the (discretized) gradient flow of the feasibility gap $\left\|\mathcal{A}\left(U U^{\top}\right)-b\right\|_{2}^{2}$, whereas the right panel plots the (discretized) gradient flow of the merit function of problem (1), to be defined later. Recall that left and right panels correspond to $p=1$ and $p=2$, respectively. Accordingly, in the left panel, the gradient flow is initialized at $U_{0} \in \mathbb{R}^{d \times 1}$ whereas, in the right panel, the gradient flow is initialized at $\left[U_{0},\binom{0}{0}\right] \in \mathbb{R}^{d \times 2}$. The choice of initialization $U_{0}$ is detailed in the code.
In each panel, the limit point of the trajectory is a feasible point and also a stationary point of problem (1). In the left panel, the (discretized) gradient flow successfully finds the global minimizer whereas, in the right panel, the (discretized) gradient flow is trapped by a spurious stationary point. The stationary point in the right panel is spurious because the optimal value of problem (1) equals 0.24 in the right panel (obtained by CVX [24], [25]), whereas the (discretized) gradient flow reaches the suboptimal value of $4.22>0.24$. The MATLAB code will be available online.


Fig. 2: With a numerical example, this figure illustrates the importance of explicit regularization in problem (1). That is, this figure shows that implicit regularization might fail in general. More specifically, we generated a random rank-1 matrix $X^{\natural} \in \mathbb{R}^{d \times d}$ with $d=20$ and $\left\|X^{\natural}\right\|_{\mathrm{F}}=1$. We also set $m=4 r d=40$ and chose the linear operator $\mathcal{A}: \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{m}$ randomly and then set $b:=\mathcal{A}\left(X^{\sharp}\right)$.
For $p=10$ and with a random initialization, we then tried to recover $X^{\sharp}$ with two approaches: (1) First, by following the trajectory of the (discretized) gradient flow of the feasibility gap $\left\|\mathcal{A}\left(U U^{\top}\right)-b\right\|_{2}^{2}$ with no explicit regularization of the variable $U \in \mathbb{R}^{d \times p}$. (2) Second, by following the (discretized) gradient flow of the merit function of problem (1), which is explicitly regularized by $\|U\|_{\mathrm{F}}^{2}$.
The left and right panels show the feasibility gap and recovery error, respectively. Notably, the first approach, which only relies on the implicit regularization of the gradient flow, fails to recover $X^{\sharp}$. The MATLAB code will be made available with the paper.

## D. Approach and Organization of the Paper

Let us now outline our approach to establish Theorem 1. After a formal setup in Sections III-V, we first study the nonconvex geometry of problem (1) in Section VI. We then review the local and global optimality conditions for problem (1) in Sections VII and VIII, respectively. In particular, we show that any rank-deficient second-order stationary point (SOSP) of problem (1) is also its global minimizer [7]. Finally, in Section IX, we introduce a merit (or exact penalty) function [19], denoted by $h_{\gamma}$, that allows us to reformulate problem (1) as a smooth and unconstrained optimization problem.
The main technical contribution of this work appears in Section X. Theorem 21 therein is the detailed version of the simplified Theorem 1 and can be summarized as follows: Once initialized rank-deficient and sufficiently close to the feasible set of problem (1), the gradient flow of $h_{\gamma}$ converges to a rank-deficient SOSP of problem (1) and thus solves problem (1) to global optimality. Theorem 21 is then followed by several remarks that justify our assumptions.
Lastly, in Section XI, we introduce a heuristic discretization of the gradient flow of $h_{\gamma}$. We then show the potential of this algorithm for solving problem (1) with a numerical example. However, we leave a comprehensive study of this algorithm as a future research question.

## II. Related Work

We are not aware of any prior literature on solving problem (1) in the over-parametrized regime ( $p \gtrsim m / d$ ). Let us instead review a number of closely related works.
If we move the constraints in problem (1) to its target function (in the form of a penalty term), then [26, Theorem 17] provides a high-level procedure for solving the penalized problem that might not converge in polynomial time or might not converge at all; the procedure might hop from one local minimizer to another, without ever visiting a global minimizer.
An additional concern is that, in nonconvex optimization, often the solutions of the penalized problem approach feasibility only in the limit as the penalty weight grows increasingly large [19, Section 17.1].
Indeed, polynomial time convergence to a global (rather than local) minimizer and the constrained nature of problem (1) both will pose significant technical challenges for us.
We should also mention the literature of implicit regularization, which attempts to recover a planted low-rank matrix $X^{\sharp} \in \mathbb{R}^{d \times d}$ by following the gradient flow of the feasibility gap $\left\|\mathcal{A}\left(U U^{\top}\right)-b\right\|_{2}^{2}$, where $U \in \mathbb{R}^{d \times d}$ and $b:=\mathcal{A}\left(X^{\sharp}\right)$. Often, this gradient flow is initialized near the origin [2], [3], [4], [27], and the literature of implicit regularization relies heavily on the restricted isometry property (RIP) of the operator $\mathcal{A}$ [1].
In contrast, we do not assume the RIP in this work and the gradient flow can be initialized anywhere sufficiently close
to the feasible set of (factor), rather than near the origin. We also consider the more general case where $U \in \mathbb{R}^{d \times p}$ with $p \gtrsim m / d$, rather than the square factorization $U \in \mathbb{R}^{d \times d}$. Perhaps most importantly, problem (1) is explicitly regularized by $\|U\|_{\mathrm{F}}^{2}$. This regularization directly promotes a low-rank solution, in contrast with the literature of implicit regularization which lacks any explicit regularization.
In Figure 2, the importance of explicit regularization is visualized with a numerical example in the context of low-rank matrix recovery. Implicit regularization fails spectacularly in this figure. Lastly, the technical machinery involved in this work is fundamentally different from that in the literature of implicit regularization.
It is also worth noting that we can adapt the literature of over-parametrized linear networks to our situation: This literature is concerned with solving the optimization problem $\min _{U}\left\|\mathcal{A}\left(U U^{\top}\right)-b\right\|_{2}^{2}$ for a particular choice of the linear operator $\mathcal{A}$. Here, the common approach in [8] is often criticized as "lazy training" [28], because it linearizes the map $U \rightarrow U U^{\top}$ near a global minimizer of $\min _{U}\left\|\mathcal{A}\left(U U^{\top}\right)-b\right\|_{2}^{2}$ and then replaces the nonconvex target function with its local convex approximation.
The approach in [8] thus relies on an initialization near a global minimizer of $\min _{U}\left\|\mathcal{A}\left(U U^{\top}\right)-b\right\|_{2}^{2}$, which can be generated randomly when the map $U \rightarrow U U^{\top}$ is sufficiently over-parametrized. We refer to Definition 2 and Theorem 1 in [8] for more information. Outside of this lazy training regime, we are only aware of the recent work [9] for linear neural networks. In contrast, our Theorem 1 relies on an initialization near the feasible set (rather than the optimal set) of problem (1). Lazy training is also highly popular in the context of nonlinear neural networks, e.g., [29].
We also mention the very over-parametrized regime of $p \gtrsim$ $d$. In this regime and in the context of nonlinear shallow neural networks, the landscape of the feasibility gap is known not have any spurious SOSPs [30], [31]. We can translate this fact to our setup: If $p \geqslant d$, then it is not difficult to verify from (4) and Definition 10 that any full-rank SOSP of problem (1) is also a global minimizer of problem (1). Moreover, any rank-deficient SOSP of problem (1) is a global minimizer of problem (1) by Proposition 14.
In analogy with [30], [31], we thus obtain the following result: When $p \geqslant d$, problem (1) does not have any spurious SOSPs. This very over-parametrized regime ( $p \geqslant d$ ) is evidently not interesting in the context of rank-constrained matrix factorization, and is not studied in this work.
Problem (1) can also be interpreted as the Burer-Monteiro factorization [5] of a particular semi-definite program (SDP). More specifically, if we replace the target function of problem (1) with $\left\langle C, U U^{\top}\right\rangle$ for a generic symmetric matrix $C$, then the new optimization problem is known not have any spurious SOSPs when $p \gtrsim \sqrt{2 m}$ [7, Theorem 2].
However, the randomness of the cost matrix $C$ means that [7, Theorem 2] is not applicable to our problem. Moreover, $p \gtrsim$
$\sqrt{2 m} \Longrightarrow p \gtrsim m / d$, under the mild assumption that $m \leqslant$ $2 d^{2}$. In words, the over-parametrized regime ( $p \gtrsim m / d$ ) studied in this paper absorbs $p \gtrsim \sqrt{2 m}$ as a special case.
Lastly, to read about matrix factorization in the underparametrized regime ( $p \lesssim m / d$ ), we wish to refer the reader to the survey $[6$, Section 9] and the references therein.

## III. Nonconvex Optimization Problem

For symmetric matrices $\left\{A_{i}\right\}_{i=1}^{m} \subset \mathbb{R}^{d \times d}$, let us consider the linear operator $\mathcal{A}: \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{m}$, defined as

$$
\begin{equation*}
\mathcal{A}(X):=\left[\left\langle A_{1}, X\right\rangle, \cdots,\left\langle A_{m}, X\right\rangle\right]^{\top} . \tag{3}
\end{equation*}
$$

We can now formally introduce the nonconvex optimization problem at the center of this work. For $\xi>0$, integer $p$ and a vector $b \in \mathbb{R}^{m}$, we will study the optimization problem

$$
\min _{U \in \mathbb{R}^{d \times p}}\|U\|_{\mathrm{F}}^{2} \text { subject to } \mathcal{A}\left(U U^{\top}\right)=b,\|U\| \leqslant \xi, \quad \text { (factor) }
$$

where $\|U\|$ denotes the spectral norm of the matrix $U$. Above, we have limited the spectral norm to $\xi$ for technical convenience. For a sufficiently large $\xi$, we can practically ignore the constraint $\|U\| \leqslant \xi$ above.
Even though we are purely interested in the computational aspects of solving (factor), the statistical significance of this problem was motivated in the introduction: (factor) arises as the Burer-Monteiro factorization of the (convex) nuclear norm minimization problem. Alternatively, (factor) can be interpreted as regularized empirical risk minimization for training a shallow linear network.
It is worth noting that, in the context of low-rank matrix recovery, the literature also offers some alternatives which can be statistically superior to the learning machine (factor) and its convex relaxation [32], [33].

## IV. A Convex Relaxation

It is also be helpful to introduce a convex relaxation of (factor), specified by the semi-definite program

$$
\min _{X \in \mathbb{R}^{d \times d}} \operatorname{tr}(X) \text { subject to } \mathcal{A}(X)=b, 0 \preccurlyeq X \preccurlyeq \xi^{2} I_{d} \text {, (SDP) }
$$

where $I_{d} \in \mathbb{R}^{d \times d}$ is the identity matrix, and $\leqslant$ denotes the positive semi-definite (PSD) pseudo-order. Indeed, it is easy to obtain (SDP) by relaxing the rank restriction in (factor).
In the other direction, we can also interpret (factor) as the Burer-Monteiro factorization of (SDP). In particular, (SDP) explicitly enforces the PSD constraint $(X \geqslant 0)$ which was implicit in (factor). Moreover, if $p$ is sufficiently small, (factor) would have fewer optimization variables compared to (SDP) and can offer computational savings [6], [7].
Note that the restriction to PSD matrices does not reduce the generality of (SDP). That is, if we choose $\mathcal{A}^{\prime}: \mathbb{R}^{d_{1} \times d_{2}} \rightarrow \mathbb{R}^{m}$ and $b^{\prime} \in \mathbb{R}^{m}$ appropriately, the optimization problem

$$
\min _{Z \in \mathbb{R}^{d_{1} \times d_{2}}}\|Z\|_{*} \text { subject to } \mathcal{A}^{\prime}(Z)=b^{\prime},\|Z\| \leqslant \xi
$$

reduces to (SDP). Above, $\|\cdot\|_{*}$ stands for the nuclear norm. For completeness, we record this observation below.

Remark 2 (Restriction to PSD matrices). The restriction in (SDP) to PSD matrices is, for our purposes, without any loss of generality, because a matrix $Z \in \mathbb{R}^{d_{1} \times d_{2}}$ with $\|Z\| \leqslant \xi$ can be mapped or lifted to a PSD matrix $X$ via the map

$$
Z \rightarrow X:=\left[\begin{array}{cc}
\xi^{2} I_{d_{1}} & Z \\
Z^{\top} & I_{d_{2}}
\end{array}\right]
$$

For future reference, we also record that a matrix $\bar{X} \in \mathbb{R}^{d \times d}$ is a (global) minimizer of problem (SDP) if there exists a dual certificate $\bar{\lambda} \in \mathbb{R}^{m}$ such that

$$
\begin{align*}
& I_{d}-\mathcal{A}^{*}(\bar{\lambda}) \geqslant 0, \quad\left(I_{d}-\mathcal{A}^{*}(\bar{\lambda})\right) \bar{X}=0 \\
& 0 \leqslant \bar{X}<\xi^{2} I_{d}, \quad \mathcal{A}(\bar{X})=b \tag{4}
\end{align*}
$$

Above, $\mathcal{A}^{*}: \mathbb{R}^{m} \rightarrow \mathbb{R}^{d \times d}$ is the adjoint of the linear operator $\mathcal{A}$ in (3), defined as

$$
\begin{equation*}
\mathcal{A}^{*}(\lambda):=\sum_{i=1}^{m} \lambda_{i} A_{i} \tag{5}
\end{equation*}
$$

and $\lambda_{i}$ is the $i^{\text {th }}$ coordinate of the vector $\lambda \in \mathbb{R}^{m}$.

## V. TARGET

After introducing the nonconvex problem (factor) and its convex relaxation (SDP) in Sections III and IV, we are now in position to formally state the target of this work.
First, let us record two key assumptions. The first assumption below ensures that the convex relaxation of (factor) is tight, i.e., (SDP) has $a$ minimizer with rank at most $p$. The second assumption below reflects the limited sampling budget that is common in signal processing. If we regard $x \rightarrow U U^{\top} x$ as a shallow linear network with weight sharing, the second assumption below also corresponds to the over-parametrized regime prevalent in neural networks [9], [34].

## Assumption 3 (Matrix factorization).

(i) (Equivalence) (factor) and (SDP) have the same optimal value, which is assumed to be finite throughout.
(ii) (Over-parametrized) $m<p d$ if $p<d / 2$ and $m<$ $d(d+1) / 2$ otherwise.
Assumption 3.(i) is not too restrictive: For example, consider a planted matrix $X^{\sharp}$ such that $\operatorname{rank}\left(X^{\sharp}\right) \leqslant p$ and set $b:=\mathcal{A}\left(X^{\sharp}\right)$. Then it is not difficult to verify that Assumption 3.(i) is fulfilled when the linear operator $\mathcal{A}$ satisfies the RIP of order $2 r$ [1]. As another example, for an arbitrary linear operator $\mathcal{A}: \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{m}$, we can use the Pataki's lemma to verify that Assumption 3.(i) holds whenever $p \gtrsim \sqrt{2 m}$ [20][21, Theorem 6.1]. As mentioned in Section II, $p \gtrsim \sqrt{2 m}$ does not violate Assumption 3.(ii), under the mild assumption that $m=O\left(d^{2}\right)$. Let us now state our target.

Target 4. Suppose that Assumption 3 is fulfilled. Our aim is to solve (factor) to global optimality.

We will soon see that Target 4 is out of the reach for the current literature of matrix factorization. The main contribution of this paper is to achieve Target 4: Sections VIIX will help us develop the necessary tools to achieve Target 4 and our main result is then presented in Section X.
For technical convenience, let us also define the functions $f: \mathbb{R}^{d \times p} \rightarrow \mathbb{R}, g: \mathbb{R}^{d \times p} \rightarrow \mathbb{R}^{m}$ and $G: \mathbb{R}^{d \times p} \rightarrow \mathbb{R}$ as

$$
\begin{align*}
& f(U):=\frac{1}{2}\|U\|_{\mathrm{F}}^{2}, \quad g(U):=\frac{1}{2}\left(A\left(U U^{\top}\right)-b\right) \\
& G(U):=\frac{1}{2}\|g(U)\|_{2}^{2} \tag{6}
\end{align*}
$$

Note that (factor) has the same minimizers as the optimization problem $\min _{U} f(U)$ subject to $g(U)=0$ and $\|U\| \leqslant \xi$. We can interpret $G$ in (6) as the (scaled) feasibility gap associated with (factor). The new functions $f, g, G$ will frequently appear in our analysis throughout.

## VI. Nonconvex Geometry

In Section III, we introduced (factor) as the nonconvex optimization program at the center of this work. Now, as our first step towards Target 4, we study in this section the geometry of the feasible set of (factor).
In Figure 1, the blue dot and the blue surface visualize parts of the feasible set of (factor) for two toy examples. For brevity, we will denote the interior of the feasible set of (factor) by $\mathcal{M}_{b}$. That is,

$$
\begin{aligned}
\mathcal{M}_{b} & :=\left\{U: \mathcal{A}\left(U U^{\top}\right)=b,\|U\|<\xi\right\} \subset \mathbb{R}^{d \times p} \\
& =\{U: g(U)=0,\|U\|<\xi\} . \quad(\text { see (6)) } \quad \text { (manifold) }
\end{aligned}
$$

In particular, note that (factor) has the same minimizers as the optimization problem $\min _{U} f(U)$ subject to $U \in \operatorname{cl}\left(\mathcal{M}_{b}\right)$, where $\operatorname{cl}\left(\mathcal{M}_{b}\right)$ denotes the closure of $\mathcal{M}_{b}$.
After recalling (6), we also observe that (factor) finds the distance from the origin to the set $\mathcal{M}_{b}$. As a minor transgression, in our qualitative discussions, we will occasionally refer to $\mathcal{M}_{b}$ as the feasible set of (factor), even thought $\mathcal{M}_{b}$ is the interior of that feasible set, strictly speaking.
In the rest of this section, we will study the geometry of the set $\mathcal{M}_{b}$. For future reference, let us record that the (total) derivative of $g$ at $U$ is the linear operator $\mathrm{D} g(U): \mathbb{R}^{d \times p} \rightarrow$ $\mathbb{R}^{m}$, specified as

$$
\begin{equation*}
\mathrm{D} g(U)[\Delta]:=\mathcal{A}\left(\Delta U^{\top}\right) \tag{7}
\end{equation*}
$$

and its adjoint $(\mathrm{D} g(U))^{*}: \mathbb{R}^{m} \rightarrow \mathbb{R}^{d \times p}$ is defined as

$$
\begin{equation*}
(\mathrm{D} g(U))^{*}[\delta]:=\mathcal{A}^{*}(\delta) \cdot U=\sum_{i=1}^{m} \delta_{i} A_{i} U \tag{8}
\end{equation*}
$$

where $\mathcal{A}^{*}$ was defined in (5) and $\delta_{i}$ is the $i^{\text {th }}$ entry of the vector $\delta \in \mathbb{R}^{m}$. To obtain (7), we leveraged the fact
that $\left\{A_{i}\right\}_{i=1}^{m}$ are symmetric matrices. Also for later reference, the neighborhood of the set $\mathcal{M}_{b}$ is defined as follows.
Definition 5 (NEIGHBORHOOD OF $\mathcal{M}_{b}$ ). The neighborhood of radius $\rho>0$ of the set $\mathcal{M}_{b}$ in (manifold) is the (closed) set

$$
\left\{U: \operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho\right\} \subset \mathbb{R}^{d \times p}, \quad \text { (neighborhood) }
$$

where

$$
\begin{align*}
\operatorname{dist}\left(U, \mathcal{M}_{b}\right) & :=\inf \left\{\left\|U-U^{\prime}\right\|_{\mathrm{F}}: U^{\prime} \in \mathcal{M}_{b}\right\} \\
& =\min \left\{\left\|U-U^{\prime}\right\|_{\mathrm{F}}: U^{\prime} \in \operatorname{cl}\left(\mathcal{M}_{b}\right)\right\} \\
& =\operatorname{dist}\left(U, \operatorname{cl}\left(\mathcal{M}_{b}\right)\right) \tag{metric}
\end{align*}
$$

is the distance from the matrix $U$ to the set $\mathcal{M}_{b}$ and its closure. For brevity, we often use the term $\rho$-neighborhood of $\mathcal{M}_{b}$ throughout. We will also often say $U$ is sufficiently close to $\mathcal{M}_{b}$ to mean that $\operatorname{dist}\left(U, \mathcal{M}_{b}\right)$ is sufficiently small.

As claimed above, the $\rho$-neighborhood of $\mathcal{M}_{b}$ is indeed a closed set because $\operatorname{dist}\left(U, \mathcal{M}_{b}\right)$ is a continuous function of $U$ and $[0, \rho]$ is a closed interval. A short remark follows next to justify our choice of metric in Definition 5.
Remark 6 (InVARIANCE OF THE METRIC). The metric in Definition 5 is invariant under rotation from right. That is, for any $U \in \mathbb{R}^{d \times p}$ and $R \in \mathcal{O}_{p}$, it holds that

$$
\begin{align*}
\operatorname{dist}\left(U R, \mathcal{M}_{b}\right) & =\inf _{U^{\prime} \in \mathcal{M}_{b}}\left\|U R-U^{\prime}\right\|_{\mathrm{F}} \\
& =\inf _{U^{\prime} \in \mathcal{M}_{b}}\left\|U R-U^{\prime} R\right\|_{\mathrm{F}} \\
& =\inf _{U^{\prime} \in \mathcal{M}_{b}}\left\|U-U^{\prime}\right\|_{\mathrm{F}} \\
& =\operatorname{dist}\left(U, \mathcal{M}_{b}\right) \tag{9}
\end{align*}
$$

Above, $\mathcal{O}_{p}=\left\{R^{\prime}: R^{\prime \top} R^{\prime}=I_{p}\right\} \subset \mathbb{R}^{p \times p}$ is the orthogonal group and $I_{p} \in \mathbb{R}^{p \times p}$ is the identity matrix. Moreover, the second line above holds because $U^{\prime} \in \mathcal{M}_{b}$ if and only if $U^{\prime} R \in \mathcal{M}_{b}$, see (manifold). The third line above holds by the rotational invariance of the Frobenius norm.

We are now in position to state the central assumption of this work, which also subsumes the earlier Assumption 3. Assumption 7 is similar to [22, Assumption 1.1] or [4]. As clarified later, Assumption 7 enables us to efficiently find feasible points of (factor) which is evidently necessary for solving (factor).
Assumption 7 (Manifold). Assumption 3 is fulfilled and it also holds that

$$
\begin{equation*}
\operatorname{rank}(\mathrm{D} g(U))=m \tag{10}
\end{equation*}
$$

for every matrix $U$ in a sufficiently small neighborhood of the set $\mathcal{M}_{b}$, i.e., the matrices $\left\{A_{i} U\right\}_{i=1}^{m}$ are linearly independent for every matrix $U$ that is sufficiently close to the set $\mathcal{M}_{b}$.

Under Assumption 7, note that $\mathcal{M}_{b}$ is a closed embedded submanifold of $\mathbb{R}^{d \times p}$ of co-dimension $m$, see [35, Corollary 5.24]. As suggested by Assumption 7, we will frequently
use the qualifiers "sufficiently small" and "sufficiently close" throughout this work. This decision is justified below.

Remark 8 (SUfficiently small / Close). A conservative lower bound for the radius of the neighborhood in Assumption 7 is given by [4, Proposition 13].
However, the lower bound in [4] is of little practical value because it involves certain geometric attributes of the set $\mathcal{M}_{b}$ in (manifold) which are difficult to estimate.
In view of this practical limitation and also to avoid any unnecessary clutter, we opted not to precisely specify the neighborhood size in Assumption 7.
Instead, Assumption 7 and most of our results are stated for a "sufficiently small" neighborhood of the set $\mathcal{M}_{b}$ with respect to (metric). We will revisit this issue later on. Note also that the set $\mathcal{M}_{b}$ and its neighborhood are often large sets, i.e., our results are not local.

Before closing this section, we collect below the geometric properties of $\mathcal{M}_{b}$ : It is not difficult to verify that the normal space of the smooth manifold $\mathcal{M}_{b} \subset \mathbb{R}^{d \times p}$ at the matrix $U \in \mathcal{M}_{b}$ can be identified with the linear subspace

$$
\begin{align*}
\mathcal{N}_{U} \mathcal{M}_{b} & :=\operatorname{range}\left((\mathrm{D} g(U))^{*}\right)  \tag{11}\\
& =\operatorname{range}\left(\left\{A_{i} U\right\}_{i=1}^{m}\right) \subset \mathbb{R}^{d \times p} \tag{8}
\end{align*}
$$

The tangent space of $\mathcal{M}_{b}$ at $U \in \mathcal{M}_{b}$ immediately follows from the fundamental theorem of linear algebra. That is,

$$
\begin{equation*}
\mathcal{T}_{U} \mathcal{M}_{b}:=\operatorname{null}(\mathrm{D} g(U)), \tag{12}
\end{equation*}
$$

where null $(\mathrm{D} g(U))$ denotes the null space (or kernel) of the linear operator $\mathrm{D} g(U)$ in (7).
In view of (11), the orthogonal projection onto the tangent space at $U \in \mathcal{M}_{b}$ is the linear operator $\mathcal{P}_{\mathcal{T}_{U} \mathcal{M}_{b}}: \mathbb{R}^{d \times p} \rightarrow$ $\mathcal{T}_{U} \mathcal{M}_{b}$ that maps the matrix $\Delta \in \mathbb{R}^{d \times p}$ to

$$
\begin{align*}
\mathcal{P}_{\mathcal{T}_{U} \mathcal{M}_{b}}(\Delta) & :=\Delta-\left((\mathrm{D} g(U))^{*} \circ\left((\mathrm{D} g(U))^{*}\right)^{\dagger}\right)[\Delta] \\
& =\Delta-(\mathrm{D} g(U))^{*}\left[\left((\mathrm{D} g(U))^{*}\right)^{\dagger}[\Delta]\right] \\
& =: \Delta-\mathcal{A}^{*}(\lambda(U, \Delta)) \cdot U, \quad(\operatorname{see}(8)) \tag{13}
\end{align*}
$$

where, for brevity, above we defined

$$
\lambda(U, \Delta):=\left((\mathrm{D} g(U))^{*}\right)^{\dagger}[\Delta] . \quad \text { (multipliers) }
$$

In (13), ○ represents the composition of two operators and $\dagger$ stands for the Moore-Penrose pseudo-inverse. The vector $\lambda(U, U)$ in (multipliers) plays a key role in the ensuing arguments and we close this section with a technical observation about this quantity.
Lemma 9 (Derivative of Lagrange multipliers). Suppose that Assumption 3. (ii) holds. For a matrix $U \in \mathbb{R}^{d \times p}$, $\lambda(U, U)$ in (multipliers) has the closed-form expression

$$
\begin{align*}
& \lambda(U, U)=K(U)^{\dagger} \mathcal{A}\left(U U^{\top}\right) \\
& K(U):=\left[\left\langle A_{i} U, A_{j} U\right\rangle\right]_{i, j=1}^{m} \in \mathbb{R}^{m \times m} \tag{14}
\end{align*}
$$

Suppose now that Assumption 7 holds. Then, $\lambda(U, U)$ is an analytic function of $U$ on a sufficiently small neighborhood of $\mathcal{M}_{b}$ in (manifold). For completeness, the directional derivative of $\lambda(U, U)$ with respect to $U$ is given explicitly by (51) in the supplementary material.

## VII. Local Optimality

In Section VI, we studied the geometry of the feasible set of (factor). See $\mathcal{M}_{b}$ in (manifold). As our next step towards Target 4, we will review in this section the sufficient conditions for local optimality in (factor).
To begin, recall that the manifold gradient of $f$ in (6) at the matrix $U \in \mathcal{M}_{b}$ is defined as the projection of the (Euclidean) gradient of $f$ onto the tangent space of the manifold at $U$ [36]. That is,

$$
\begin{align*}
\nabla_{\mathcal{M}_{b}} f(U) & :=\mathcal{P}_{\mathcal{T}_{U} \mathcal{M}_{b}}(\nabla f(U)) \\
& =\mathcal{P}_{\mathcal{T}_{U} \mathcal{M}_{b}}(U) \quad(\text { see }(6)) \\
& =\left(I_{d}-\mathcal{A}^{*}(\lambda(U, U)) \cdot U \in \mathbb{R}^{d \times p}\right. \tag{15}
\end{align*}
$$

where the last line above follows from (13). Above, $\nabla$ stands for (Euclidean) gradient and $\nabla_{\mathcal{M}_{b}}$ denotes the manifold gradient for $\mathcal{M}_{b}$.
For (factor), the matrix $\bar{U} \in \mathcal{M}_{b}$ is a first-order stationary point if the manifold gradient of $f$ vanishes at $\bar{U}$. More specifically, after recalling the notation in (6) and (multipliers), we record the following.
Definition 10 (FOSP). Consider a matrix $\bar{U} \in \mathbb{R}^{d \times p}$ such that $\|\bar{U}\|<\xi$. This matrix $\bar{U}$ is a first-order stationary point (FOSP) of (factor) if

$$
\begin{align*}
& g(\bar{U})=\frac{1}{2}\left(A\left(\bar{U} \cdot \bar{U}^{\top}\right)-b\right)=0 \\
& \nabla_{\mathcal{M}_{b}} f(\bar{U})=\left(I_{d}-\mathcal{A}^{*}(\lambda(\bar{U}, \bar{U})) \cdot \bar{U}=0\right. \tag{16}
\end{align*}
$$

Moving on to second-order optimality, we denote the manifold Hessian of $f$ in (6) at the matrix $U \in \mathcal{M}_{b}$ with the bilinear map $\nabla_{\mathcal{M}_{b}}^{2} f(U): \mathcal{T}_{U} \mathcal{M}_{b} \times \mathcal{T}_{U} \mathcal{M}_{b} \rightarrow \mathbb{R}$ [36]. This bilinear operator maps $[\Delta, \Delta] \in \mathbb{R}^{d \times p} \times \mathbb{R}^{d \times p}$ to the scalar

$$
\begin{align*}
& \nabla_{\mathcal{M}_{b}}^{2} f(U)[\Delta, \Delta] \\
& :=\left(\nabla^{2} f(U)-\sum_{i=1}^{m} \lambda_{i}(U, U) \cdot \nabla^{2} g_{i}(U)\right)[\Delta, \Delta]  \tag{17}\\
& =\|\Delta\|_{\mathrm{F}}^{2}-\left\langle\mathcal{A}^{*}(\lambda(U, U)), \Delta \Delta^{\top}\right\rangle, \quad \text { (see (6),(8)) }
\end{align*}
$$

for $\Delta \in \mathcal{T}_{U} \mathcal{M}_{b}$. Above, $\nabla^{2}$ stands for (Euclidean) Hessian. Also, $\lambda_{i}(U, U)$ and $g_{i}(U)$ are the $i^{\text {th }}$ coordinates of the vectors $\lambda(U, U)$ and $g(U)$, respectively.
For (factor), a second-order stationary point has a PSD manifold Hessian, as detailed below.
Definition 11 (SOSP). Consider a matrix $\bar{U} \in \mathbb{R}^{d \times p}$ such that $\|\bar{U}\|<\xi$. This matrix $\bar{U}$ is a second-order stationary
point (SOSP) of (factor) if, in addition to (16), the manifold Hessian $\nabla_{\mathcal{M}_{b}}^{2} f(\bar{U})$ in (17) is a PSD linear operator:

$$
\begin{equation*}
\left\langle\Delta,\left(I_{d}-\mathcal{A}^{*}(\lambda(\bar{U}, \bar{U}))\right) \Delta\right\rangle \geqslant 0, \quad \text { if } \Delta \in \mathcal{T}_{\bar{U}} \mathcal{M}_{b} \tag{18}
\end{equation*}
$$

Above, $\mathcal{T}_{\bar{U}} \mathcal{M}_{b}$ is the tangent space of the manifold $\mathcal{M}_{b}$ at the matrix $\bar{U}$, see (12).

## VIII. Global Optimality

In Section VII, we reviewed the local optimality conditions for (factor), e.g., see Definition 11 of an SOSP. In general, however, not every SOSP of (factor) is a global minimizer. That is, some SOSPs might be local minimizers or non-strict saddle points of (factor). We label such points as the spurious SOSPs of the optimization problem (factor).

Definition 12 (Spurious SOSP). An SOSP of (factor) is spurious if it is not a global minimizer of (factor).

In the under-parametrized regime ( $p \lesssim m / d$ ), a generic linear operator $\mathcal{A}$ in (3) often satisfies the RIP of order $2 p$ [1]. Based on this observation, it can be shown that every feasible point of (factor) is also a global minimizer of (factor). In other words, (factor) does not have any spurious SOSPs in the under-parametrized regime for a generic operator $\mathcal{A}$.
In this under-parametrized regime, even though nonconvex, (factor) can be solved to global optimality by a variety of first- or second-order optimization algorithms, including the gradient descent [16].
While the focus of this work is on the over-parametrized regime ( $p \gtrsim m / d$ ), the under-parametrized regime of (factor) is also reviewed in the arXiv version of this paper for completeness, see also [6].
Unlike the under-parametrized regime and its benign optimization landscape, it is far more difficult in general to solve (factor) in the over-parametrized regime:

Remark 13 (Spurious SOSPs). Under Assumption 3.(ii), we see by counting the degrees of freedom that the linear operator $\mathcal{A}$ in (3) cannot satisfy the RIP of order $2 p$, unlike the under-parametrized regime.
(factor) might therefore have spurious SOSPs which could trap a first- or second-order optimization algorithm, including the gradient descent. A toy example of this pathological situation appeared earlier in Figure 1 (right panel).
This discouraging observation rules out the possibility of a global scheme for solving problem (1). For instance, initialized at a spurious stationary point, gradient descent remains there forever.
Fortunately, not all is lost. In the remainder of this work, we will devise a gradient flow that solves (factor) to global optimality and achieves Target 4, when initialized within a "capture neighborhood" of the feasible set of (factor).
Lastly, note that the feasible set and its neighborhood are often large sets, i.e., our results are not local.

Before closing this section, let us recall a sufficient condition for global optimality of an SOSP, see [7], [26].
Proposition 14 (Global optimality). Any rank-deficient SOSP $\bar{U}$ of (factor) with $\|\bar{U}\|<\xi$ is also a global minimizer of (factor). Moreover, $\bar{U} \bar{U}^{\top}$ is a (global) minimizer of (SDP). By rank-deficient, we mean that $\bar{U}$ is a singular matrix.

## IX. Merit Function

Our next step towards Target 4 is as follows: While (factor) is a constrained optimization program, we introduce in this section a merit (or exact penalty) function that allows us to reformulate (factor) as a smooth (and unconstrained) optimization program.
The main contribution of this paper, as we will see shortly, is using the gradient flow of this merit function to solve (factor).
To begin, for $\gamma>0$, let $L_{\gamma}: \mathbb{R}^{d \times p} \times \mathbb{R}^{m} \rightarrow \mathbb{R}$ denote the (scaled) augmented Lagrangian [19] associated with (factor), defined as

$$
\begin{equation*}
L_{\gamma}\left(U, \lambda^{\prime}\right):=f(U)-\left\langle g(U), \lambda^{\prime}\right\rangle+\frac{\gamma}{2}\|g(U)\|_{2}^{2} \tag{AL}
\end{equation*}
$$

where $f$ and $g$ were specified in (6). The augmented Lagrangian has two remarkable properties that are listed in the next proposition, inspired by [19, Theorem 17.5].
Loosely speaking, Proposition 15 posits that the augmented Lagrangian encodes the optimality criteria of (factor). The result below also allows us to interpret $\lambda(\bar{U}, \bar{U})$ in (multipliers) as the (dual) optimal Lagrange multipliers for a feasible matrix $\bar{U} \in \mathcal{M}_{b}$ in (manifold).

Proposition 15 (Augmented Lagrangian). Suppose that Assumption 7 holds. Consider a matrix $\bar{U} \in \mathbb{R}^{d \times p}$ that is sufficiently close to $\mathcal{M}_{b}$ and satisfies $\|\bar{U}\|<\xi$. For $\gamma>0$, the following statements are true:
(i) $\nabla_{1} L_{\gamma}(\bar{U}, \lambda(\bar{U}, \bar{U}))=0$ implies that $\bar{U}$ is an FOSP of (factor) and, in particular, $\bar{U} \in \mathcal{M}_{b}$.
(ii) If, in addition, $\nabla_{1}^{2} L_{\gamma}(\bar{U}, \lambda(\bar{U}, \bar{U}))[\Delta, \Delta] \geqslant 0$ for every tangent direction $\Delta \in \mathcal{T}_{\bar{U}} \mathcal{M}_{b}$ in (12), then $\bar{U}$ is also an SOSP of (factor).
Above, $\nabla_{1} L_{\gamma}$ and $\nabla_{1}^{2} L_{\gamma}$ are the (Euclidean) gradient and Hessian of $L_{\gamma}$ with respect to its first argument, respectively. For example, $\nabla_{1} L_{\gamma}(\bar{U}, \lambda(\bar{U}, \bar{U}))$ is the gradient of $L_{\gamma}(\cdot, \cdot)$ with respect to its first argument and evaluated at the $\operatorname{pair}(\bar{U}, \lambda(\bar{U}, \bar{U}))$.

Inspired by the properties of the augmented Lagrangian, let us now consider the function $h_{\gamma}: \mathbb{R}^{d \times p} \rightarrow \mathbb{R}$, defined as

$$
\begin{equation*}
h_{\gamma}(U):=L_{\gamma}(U, \lambda(U, U)), \tag{merit}
\end{equation*}
$$

also known as the Fletcher's augmented Lagrangian [19]. Before we uncover the key property of $h_{\gamma}$ and its namesake, let us first record that $h_{\gamma}$ is an analytic function.

Lemma 16 (Derivative of $h_{\gamma}$ ). Suppose that Assumption 7 holds. Then, $h_{\gamma}(U)$ in (merit) is an analytic function of $U$
on a sufficiently small neighborhood of $\mathcal{M}_{b}$ in (manifold). Its derivative is specified as

$$
\begin{align*}
\nabla h_{\gamma}(U) & =\left(I_{d}-\mathcal{A}^{*}(\lambda(U, U))\right) U+\frac{\gamma}{2} \mathcal{A}^{*}\left(\mathcal{A}\left(U U^{\top}\right)-b\right) U \\
& -\frac{1}{2}(\mathrm{D} \lambda(U, U))^{*}\left[\mathcal{A}\left(U U^{\top}\right)-b\right] \tag{19}
\end{align*}
$$

where $I_{d} \in \mathbb{R}^{d \times d}$ is the identity matrix. Above, $\lambda(U, U)$ and its (total) derivative $\mathrm{D} \lambda(U, U)$ were both defined in Lemma 9.

The next lemma asserts that, when $\gamma$ is sufficiently large, $h_{\gamma}$ is a merit (or exact penalty) function for (factor), i.e., we can focus on minimizing the smooth function $h_{\gamma}$, instead of the constrained problem (factor).
While we are not aware of a precise precedent, the proposition below is similar to [37, Proposition 4.22].
Proposition 17 (Merit function). Suppose that Assumption 7 holds. Consider a matrix $\bar{U} \in \mathbb{R}^{d \times p}$ that is sufficiently close to $\mathcal{M}_{b}$ and satisfies $\|\bar{U}\|<\xi$, see (factor) and (manifold). For a sufficiently large $\gamma$, the following statements are true:
(i) If $\bar{U}$ is an FOSP of $h_{\gamma}$, then $\bar{U}$ is also an FOSP of (factor) and, in particular, $\bar{U} \in \mathcal{M}_{b}$.
(ii) If, in addition, $\bar{U}$ is an SOSP of $h_{\gamma}$, then $\bar{U}$ is also an SOSP of (factor).

We are now ready and fully equipped to reach Target 4.

## X. Main Result

In Sections VI-VIII, we studied the nonconvex geometry and optimality conditions of (factor). In Section IX, we then introduced $h_{\gamma}$, a (smooth) merit function for (factor).
We will establish in this section that the gradient flow for the merit function $h_{\gamma}$, when initialized properly, converges almost surely to a global minimizer of (factor) and thus achieves Target 4, without getting trapped by any spurious SOSPs present in the feasible set of (factor).
The main result of this work is summarized in Theorem 21 and the rest of this section is devoted to the proof of this theorem. We begin with an outline of the proof below.
Proof sketch of Theorem 21. At a high-level, we will take the following steps in the remainder of this section to ultimately prove Theorem 21. We will establish in this section that
(i) Rank does not increase along the gradient flow of $h_{\gamma}$.
(ii) The gradient flow does not escape from a "capture neighborhood" around the feasible set $\mathcal{M}_{b}$ of (factor).
(iii) When initialized rank-deficient and within this capture neighborhood, the gradient flow of $h_{\gamma}$ converges to a rank-deficient stationary point of $h_{\gamma}$.
(iv) Finally, because $h_{\gamma}$ is a merit function for (factor), any rank-deficient limit point of the gradient flow is, in fact, a global minimizer of (factor).
Let us now turn to the details. For an initialization $U_{0} \in$ $\mathbb{R}^{d \times p}$, the gradient flow of $h_{\gamma}$ in (merit) is specified as

$$
\dot{U}(t)=-\nabla h_{\gamma}(U(t)), \quad U(0)=U_{0}, \quad \text { (gradient flow) }
$$

where we used the shorthand $\dot{U}(t)=\mathrm{d} U(t) / \mathrm{d} t$.
The first lemma in this section posits that rank does not increase along (gradient flow), as long as the flow remains sufficiently close to the feasible set of (factor).

Lemma 18 (RANK of GRADIENT FLOW). Suppose that Assumption 7 holds. For a sufficiently small $\rho_{0}>0$, suppose also that the initialization $U_{0} \in \mathbb{R}^{d \times p}$ of (gradient flow) satisfies $\operatorname{dist}\left(U_{0}, \mathcal{M}_{b}\right)<\rho_{0}$.
Let $\tau \in(0, \infty)$ (if it exists) denote the smallest number such that $\operatorname{dist}\left(U(\tau), \mathcal{M}_{b}\right)=\rho_{0}$. Then it holds that

$$
\begin{equation*}
\operatorname{rank}(U(t)) \leqslant \operatorname{rank}\left(U_{0}\right), \quad \text { if } t \in[0, \tau] \tag{20}
\end{equation*}
$$

Proof sketch of Lemma 18. This claim follows from writing the analytic singular value decomposition (SVD) of $U(t) U(t)^{\top}$, then taking the derivative of this SVD with respect to time $t$, and finally establishing that any zero singular value of $U(t) U(t)^{\top}$ remains zero afterwards.
To successfully apply Lemma 18 for all times $t$, we must ensure that (gradient flow) never escapes the $\rho_{0}$ neighborhood of the feasible set $\mathcal{M}_{b}$ of (factor). To that end, we need the following lemma, which uses the remaining freedom in Lemma 18 in order to tighten the inequality $\operatorname{dist}\left(U(t), \mathcal{M}_{b}\right) \leqslant \rho_{0}$ to obtain the new inequality $\operatorname{dist}\left(U(t), \mathcal{M}_{b}\right) \leqslant \rho_{0} / 2$, for every $t \in[0, \tau]$.

Lemma 19 (Flow remains nearby). Suppose that Assumption 7 holds. Suppose also that $\xi$ in (factor) and $\gamma$ in (gradient flow) are sufficiently large, and that $\rho_{0}>0$ in Lemma 18 is sufficiently small. Suppose lastly that the initialization $U_{0} \in \mathbb{R}^{d \times p}$ of (gradient flow) is sufficiently close to $\mathcal{M}_{b}$, see (manifold). Then it holds that

$$
\begin{equation*}
\operatorname{dist}\left(U(t), \mathcal{M}_{b}\right) \leqslant \rho_{0} / 2, \quad \text { if } t \in[0, \tau] \tag{21}
\end{equation*}
$$

where $\tau$ was defined in Lemma 18.
Proof sketch of Lemma 19. Moving along the trajectory of (gradient flow) evidently reduces $h_{\gamma}$. Intuitively, when $\gamma$ is large enough, moving along this trajectory also reduces the feasibility gap $\frac{1}{2}\|g(U)\|_{2}^{2}$, see (6), (AL) and (merit).
In the proof, we first quantify the above observation, i.e., formally establish that the feasibility gap does not increase along (gradient flow) when $\gamma$ is sufficiently large.
The remaining technical challenge is then translating the above observation into a statement about the distance between $U(t)$ to the manifold $\mathcal{M}_{b}$.
In the remainder of this section, for the sake of brevity, we freely invoke earlier lemmas and propositions without restating their assumptions.
Recalling the definition of $\tau$ in Lemma 18, it immediately follows from Lemma 19 that

$$
\begin{equation*}
\operatorname{dist}\left(U(t), \mathcal{M}_{b}\right) \leqslant \rho_{0} / 2, \quad \text { if } t \geqslant 0 \tag{22}
\end{equation*}
$$

That is, (gradient flow) always remains near the feasible set $\mathcal{M}_{b}$ of (factor), as desired. The key observation in (22)
will enable us to prove the convergence of (gradient flow), after recalling the Łojasiewicz's Theorem [38], [39].

Theorem 20 (ŁOJASIEWICZ's THEOREM). If $h^{\prime}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is an analytic function and the curve $[0, \infty) \rightarrow \mathbb{R}^{n}, t \rightarrow z(t)$ is bounded and solves the gradient flow $\dot{z}(t)=-\nabla h^{\prime}(z)$, then this curve converges to an FOSP of $h^{\prime}$.

To apply Theorem 20, we proceed as follows. When $\rho_{0}>0$ is sufficiently small, recall from Lemma 16 that $h_{\gamma}$ is an analytic function on the set $\left\{U: \operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho_{0} / 2\right\}$. Let $h^{\prime}: \mathbb{R}^{d \times p} \rightarrow \mathbb{R}$ be the analytic continuation of $h_{\gamma}$ from the neighborhood $\left\{U: \operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho_{0} / 2\right\}$ to $\mathbb{R}^{d \times p}$.
Recall also from (manifold) and (22) that (gradient flow) is bounded and solves $\dot{U}(t)=-\nabla h^{\prime}(U(t))$. Therefore, by Theorem 20, (gradient flow) converges to an FOSP of $h^{\prime}$, denoted by $\bar{U}$. To reiterate, $\bar{U}$ is both the limit point of (gradient flow) and an FOSP of $h^{\prime}$.
By (22) and continuity of dist, the limit point $\bar{U}$ also satisfies $\operatorname{dist}\left(\bar{U}, \mathcal{M}_{b}\right) \leqslant \rho_{0} / 2$. By construction of $h^{\prime}$ as the analytic continuation of $h_{\gamma}$, we then observe that $\bar{U}$ is also an FOSP of $h_{\gamma}$. To summarize, we have proved so far that (gradient flow) converges to an FOSP of $h_{\gamma}$, which we have denoted by $\bar{U}$.
Moreover, by Theorem 3 of [16], the FOSP $\bar{U}$ of $h_{\gamma}$ is almost surely also an SOSP of $h_{\gamma}$ (rather than just an FOSP). After recalling Proposition 17 about the relationship between $h_{\gamma}$ and (factor), it follows that $\bar{U}$ is also an SOSP of (factor), provided that $\gamma$ is sufficiently large.
Suppose lastly that the initialization $U_{0}$ of (gradient flow) is rank-deficient, i.e., we have $\operatorname{rank}\left(U_{0}\right)<p$. Then, using Lemma 18 about the rank along the trajectory and also using the fact that $\left\{U: \operatorname{rank}(U) \leqslant \operatorname{rank}\left(U_{0}\right)\right\}$ is a closed set, we find that the limit point $\bar{U}$ of (gradient flow) also satisfies

$$
\begin{equation*}
\operatorname{rank}(\bar{U}) \leqslant \operatorname{rank}\left(U_{0}\right)<p \tag{23}
\end{equation*}
$$

We have established that the limit point $\bar{U}$ of (gradient flow) is almost surely a rank-deficient SOSP of (factor).
Finally, by combining Proposition 14 and (23), we conclude that the SOSP $\bar{U}$ of (factor) is almost surely a global minimizer of (factor). The main result below summarizes our findings and achieves Target 4.

Theorem 21 (Main ReSUlT). Suppose that Assumption 7 holds. Suppose also that $\xi$ in (factor) and $\gamma$ in (gradient flow) are sufficiently large. Suppose lastly that the initialization $U_{0} \in \mathbb{R}^{d \times p}$ of (gradient flow) is rank-deficient and sufficiently close to $\mathcal{M}_{b}$, see (manifold).
Then (gradient flow) almost surely converges to a global minimizer $\bar{U}$ of (factor). Moreover, $\bar{U} \bar{U}^{\top}$ is a global minimizer of (SDP). Above, the notion of distance to $\mathcal{M}_{b}$ was made precise in Definition 5 and Remark 8.

Theorem 21 is a theoretical (rather than practical) recipe for successful over-parametrized matrix factorization. In particular, note that the operator $\mathcal{A}$ in (3) is not required
above to satisfy the RIP, which dominates the literature of matrix sensing [1].
We also note that the existence of the initialization prescribed in Theorem 21 can be ensured under the same mild conditions that were listed after Assumption 3. In the remainder of this section, we justify the assumptions made in Theorem 21.

Remark 22 (Assumption 7 IS MINIMAL). Assumption 7 corresponds to the standard constraint qualifications for (factor). More specifically, Assumption 7 corresponds to the weakest sufficient conditions under which the KKT conditions [19], [40] are necessary for global optimality in (factor), similar to [22, Section 5] or [4].
Without Assumption 7, in general there cannot be be any hope of efficiently finding a matrix $U$ that satisfies the constraints $\mathcal{A}\left(U U^{\top}\right)=b$ of (factor).
That is, without Assumption 7, the feasibility gap $G$ in (6) is not necessarily dominated by its gradient $(\nabla G(U)=$ $0 \nRightarrow G(U)=0)$. In this scenario, a first-order optimization algorithm cannot in general find a feasible matrix $U$ (a matrix $U$ that satisfies $G(U)=0$ ). Such peculiarities are not uncommon in nonconvex optimization [41].
From this perspective, Assumption 7 is minimal in order to achieve Target 4.
Even though Assumption 7 has several precedents within the nonconvex optimization literature [4], [7], [10], [37], we note that verifying this assumption is often difficult in practice. In this sense, Theorem 21 should be regarded as a theoretical result that sheds light, for the first time, on the nonconvex geometry of over-parametrized matrix factorization.

Remark 23 (Initialization). In Theorem 21, we cannot obtain global guarantees because the feasible set of (factor) may contain spurious stationary points that can trap the gradient flow with an arbitrary initialization, see Figure 1.
It is then necessary to restrict the initialization in some way, e.g., the initilization near the feasible set in Theorem 21.

Formally, Theorem 21 is a "capture theorem", common in nonconvex optimization literature [4], [7], [10], which predicates on an initialization within a specific "capture neighborhood" of the feasibility problem $\min _{U} G(U)$, see (6).
In Theorem 21, this capture neighborhood coincides with a sufficiently small neighborhood of the feasible set $\mathcal{M}_{b}$ of (factor), i.e., Theorem 21 applies only when (gradient flow) is initialized near $\mathcal{M}_{b}$.
We do not provide a provable scheme for finding a sufficiently feasible initialization for (gradient flow). In that sense, Theorem 21 should not be viewed as a practical initialization scheme for (gradient flow) but rather a theoretical result about the nonconvex geometry of (factor).
Nevertheless, as an important practical remark, we later empirically observe that a random initialization $U_{0}$ is often a good choice that avoids the worst-case scenario in the right panel of Figure 1.

Remark 24 (LOCAL REFINEMENT RESULTS). Capture theorems, including Theorem 21, are fundamentally different from local refinement results that appear within the signal processing literature [6, Chapter 5].
Indeed, note that the local refinement results rely on an initialization within a small neighborhood of an isolated global minimizer, in which the target function is locally strongly convex.
In contrast, even though Theorem (21) requires a sufficiently feasible initialization, this initialization might be far from any global minimizer of (factor).
Moreover, a small neighborhood of the feasible set contains all spurious stationary points of (factor), whereas a small neighborhood of a global minimizer will contain no other stationary points, by design.
In other words, even though Theorem 21 requires a sufficiently feasible initialization, (gradient flow) might have to travel far within the capture neighborhood and avoid the spurious stationary points, before eventually reaching a global minimizer.

Remark 25 (SUFFICIENTLY SMALL/CLOSE/LARGE). Adding to the earlier Remark 8, we note that the requirements in Theorem 21 on $\xi, \gamma, U_{0}$ involve certain geometric attributes of $\mathcal{M}_{b}$ in (manifold) which are difficult to estimate.
Even though the requirements on the initialization $U_{0}$ are specified precisely in the proofs, we chose not to present them in the body of the paper because of their little added value and to avoid any unnecessary clutter.

## XI. Discretization

It is not difficult to verify that (gradient flow) converges at the rate of $1 / t$. Its limit point is almost surely a global minimizer of (factor), by virtue of Theorem 21.
The literature often focuses on flows rather than their discretization, for the sake of simplicity and insight [2], [4], [9], [42]. Nevertheless, discretization of (gradient flow) is an important computational consideration, which we now discuss in this section.
We will not pursue an explicit Euler (or forward) discretization of (gradient flow). We do so to avoid stability concerns about the derivative of $\lambda(U, U)$, see Lemma 9.
Instead of a forward discretization, we consider here a heuristic discretization of (gradient flow). Our heuristic discretization below is inspired by [43] in the context of optimization with orthogonality constraints.
In short, at iteration $k$, we move along the direction $-\nabla_{1} L_{\gamma}\left(U_{k}, \lambda_{k}\right)$, where $\lambda_{k}:=\lambda\left(U_{k}, U_{k}\right)$. Recall that $\lambda(\cdot, \cdot)$ was defined in (14) and $\nabla_{1} L_{\gamma}$ is the partial derivative of the augmented Lagrangian in (AL) with respect to its first argument.
The details are presented in Algorithm 1. The convergence analysis of Algorithm 1 is an important and nontrivial research question that lies beyond of the scope of this
theoretical paper. Nevertheless, we next present a numerical example to showcase the potential of Algorithm 1 for solving (factor).

Input: Symmetric $d \times d$ matrices $\left\{A_{i}\right\}_{i=1}^{m}$ and the corresponding operator $\mathcal{A}$ in (3), vector $b \in \mathbb{R}^{m}$, integer $p$ such that $p d \geqslant m$, initialization $U_{0} \in \mathbb{R}^{d \times p}$, positive penalty weight $\gamma$ and positive step sizes $\left\{\eta_{k}\right\}_{k}$.

Set $k=0$. Until convergence, repeat

1) Update the dual variables as $\lambda_{k}:=\lambda\left(U_{k}, U_{k}\right)$, see (14).
2) Update the primal variables as

$$
\begin{aligned}
U_{k+1} & =\left(1-\eta_{k}\right) U_{k} \\
& +\sum_{i=1}^{m} \eta_{k}\left(\lambda_{k, i}-\frac{\gamma}{2}\left(\left\langle A_{i}, U_{k} U_{k}^{\top}\right\rangle-b_{i}\right)\right) A_{i} U_{k}
\end{aligned}
$$

where $\lambda_{k, i}$ and $b_{i}$ are the $i^{\text {th }}$ entries of the vectors $\lambda_{k}$ and $b$, respectively.
3) $k \leftarrow k+1$

Algorithm 1: Discretization of (gradient flow)

## XII. Numerical Example

This section presents a small numerical example that shows the potential of Algorithm 1 for over-parametrized matrix factorization. A comprehensive numerical study remains as a future research target, alongside developing a convergence theory for Algorithm 1.
Recall the setup of (factor). In our numerical example, we set $d=15, m=30, p=\lceil\sqrt{2 m}\rceil, \xi \gg 1$. invoke the Pataki's lemma [20][21, Theorem 6.1] to verify that Assumption 3 is fulfilled. (In particular, we are indeed in the over-parametrized regime, see Assumption 3.(ii).)
We also choose the linear operator $\mathcal{A}: \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{m}$ and the vector $b \in \mathbb{R}^{m}$ both randomly. More specifically, the upper triangle entries of every matrix $A_{i} \in \mathbb{R}^{d \times d}$ are independently drawn from the zero-mean and unit-variance Gaussian distribution. Similarly, $b$ is a standard Gaussian random vector. Moreover, $\left\{A_{i}\right\}_{i=1}^{m}$ and $b$ are independent from one another.
Because Assumption 3.(i) is fulfilled, (SDP) is a tight relaxation of (factor). In particular, the optimal value of (factor) coincides with the optimal value of the convex problem (SDP). As a benchmark, we can use CVX [24], [25] to solve (SDP) and obtain the common optimal value of (SDP) and (factor).
We then attempt to solve (factor) with Algorithm 1, where we set the penalty weight and step sizes to $\gamma=100$ and $\eta_{k}=$ $2 \cdot 10^{-5}$ for every $k$. We use three different initializations for Algorithm 1, detailed below:
(i) A deterministic initialization, where $U_{0} \in \mathbb{R}^{d \times p}$ is filled by zeros and ones.
(ii) A "partial oracle" initialization, where $U_{0}$ contains only one correct column of a global minimizer of (factor). The remaining entries of $U_{0}$ are all set to one. Here, we can obtain a global minimizer of (factor) by taking the square root of CVX's output of (SDP).
(iii) $U_{0}$ is a standard Gaussian random matrix.

In all cases, we then normalize $U_{0}$ to ensure that $\left\|U_{0}\right\|_{\mathrm{F}}=3$. This last step is for convenience and allows us to use the same step size for all three initializations.
Figure 3 shows the feasibility gap and the target value of (factor) across the iterations of Algorithm 1, using the above three initializations. For comparison, the optimal value of problem (factor) is shown with a dashed line. The MATLAB code will be made available with the paper.
Algorithm 1 with both the deterministic and partial oracle initializations converges to stationary points of $h_{\gamma}$ but neither of these two limit points is feasible for (factor).
That is, Algorithm 1 fails for both initializations to produce an output that satisfies the constraints in (factor). In both cases, note that the output of Algorithm 1 is not a spurious stationary point of (factor).
The failure of Algorithm 1 with these two initializations hints at the complex landscape of the merit function $h_{\gamma}$ and, in turn, the difficulty of solving (factor) in the overparametrized regime.
Nevertheless, Algorithm 1 with a generic initialization successfully solves (factor) to global optimality in Figure 3. Remarkably, our experience was that a generic initialization always avoids the worst-case scenarios, such as the right panel of Figure 1 or the first two initializations in Figure 3. This observation is briefly discussed in the next section.
It is also worth noting that we found it helpful in our simulations to stabilize Algorithm 1 by replacing $K\left(U_{k}\right)$ in (14) by $K\left(U_{k}\right)+10^{-9} I_{m}$, where $I_{m}$ is the identity matrix. Lastly, note that it is difficult to verify the manifold requirement for $\mathcal{M}_{b}$ or to numerically identify its capture neighborhood. In this sense, Theorem 21 should be regarded as a theoretical contribution that sheds light, for the first time, on the nonconvex geometry of (factor) in the overparametrized regime, rather than a practical initialization scheme for Algorithm 1. In practice, a generic initialization seems to be an excellent choice for Algorithm 1.

## XIII. Final Thoughts

This work raises a few intriguing questions. First, as mentioned earlier, a stable and provable discretization of (gradient flow) is an interesting and nontrivial future research question. In particular, the convergence analysis of Algorithm 1 remains an open problem.
Second, in our numerical examples, recall that we came across both spurious stationary points of (factor) and infeasible stationary points of its merit function $h_{\gamma}$, see the right panel of Figures 1 and 3, respectively. These observations


Fig. 3: For the numerical example of Section XII, this figure shows the feasibility gap (top panel) and the target value (bottom panel) of (factor) across the iterations of Algorithm 1, using three different initializations.
suggest that (factor) is a difficult problem to solve in the over-parametrized regime.
At the same time, what explains the surprising success of Algorithm 1 when the linear operator $\mathcal{A}$ and the initialization are both generic? Answering this interesting question might require new technical tools beyond our toolbox in this paper.

## Appendix A <br> Proof of Lemma 19

An important ingredient of the proof is the following technical lemma, which states that the feasibility gap is non-increasing along (gradient flow).

Lemma 26 (Flow remains nearly feasible). Suppose that the same assumptions made in Lemma 19 are fulfilled. Then it holds that

$$
\begin{equation*}
\left\|g\left(U_{t}\right)\right\|_{2} \leqslant\left\|g\left(U_{0}\right)\right\|_{2}, \quad \text { if } t \in[0, \tau] \tag{24}
\end{equation*}
$$

where $g$ was defined in (6).
Before proving Lemma 26 in a later appendix, let us here complete the proof of Lemma 19. The technical challenge ahead is translating the upper bound in (24) into an upper bound on the distance from $U_{t}$ to the set $\mathcal{M}_{b}$ in (manifold). We first write that

$$
\begin{align*}
\left\|g\left(U_{0}\right)\right\|_{2} & \geqslant\left\|g\left(U_{t}\right)\right\|_{2} \quad(\text { see Lemma 26) } \\
& =\frac{1}{2}\left\|\mathcal{A}\left(U_{t} U_{t}^{\top}\right)-b\right\|_{2} \quad(\text { see (6)) } \tag{25}
\end{align*}
$$

for every $t \in[0, \tau]$. To lower bound the last norm above, the idea is to replace $b$ above with the image of a particular point in $\mathcal{M}_{b}$ under the map $U \rightarrow \mathcal{A}\left(U U^{\top}\right)$, see (manifold). That particular point is the projection of $U_{t}$ onto $\mathcal{M}_{b}$, as detailed next: When $\xi$ is sufficiently large, let $V_{t} \in \mathcal{M}_{b}$ denote a projection of $U_{t}$ onto $\mathcal{M}_{b}$ in (manifold). That is, when $\xi$ is sufficiently large, there exists $V_{t} \in \mathcal{M}_{b}$ such that

$$
\begin{align*}
\left\|U_{t}-V_{t}\right\|_{\mathrm{F}} & =\operatorname{dist}\left(U_{t}, \mathcal{M}_{b}\right) \\
& \leqslant\left\|U_{t}-V\right\|_{\mathrm{F}}, \quad \text { if } V \in \mathcal{M}_{b} \tag{26}
\end{align*}
$$

where the second inequality above follows from (metric). Recall also that the normal space of the smooth manifold $\mathcal{M}_{b}$ was specified in (11). In particular, note that $U_{t}-V_{t} \in$ $\mathcal{N}_{V_{t}} \mathcal{M}_{b}$ by (26). Equivalently, there exists a vector $\alpha_{t} \in \mathbb{R}^{m}$ such that

$$
\begin{equation*}
U_{t}=V_{t}+\left(\mathrm{D} g\left(V_{t}\right)\right)^{*}\left[\alpha_{t}\right] \tag{27}
\end{equation*}
$$

It follows that

$$
\begin{align*}
\operatorname{dist}\left(U_{t}, \mathcal{M}_{b}\right) & =\left\|U_{t}-V_{t}\right\|_{\mathrm{F}} \quad(\text { see (26)) } \\
& =\left\|\left(\mathrm{D} g\left(V_{t}\right)\right)^{*}\left[\alpha_{t}\right]\right\|_{\mathrm{F}} \quad(\text { see (27)) } \\
& \leqslant\left\|\left(\mathrm{D} g\left(V_{t}\right)\right)^{*}\right\| \cdot\left\|\alpha_{t}\right\|_{2} \\
& =\left\|\mathrm{D} g\left(V_{t}\right)\right\| \cdot\left\|\alpha_{t}\right\|_{2} \\
& \leqslant \xi\|\mathcal{A}\| \cdot\left\|\alpha_{t}\right\|_{2} \tag{28}
\end{align*}
$$

The last line above follows from the observation that

$$
\begin{align*}
\left\|\mathrm{D} g\left(V_{t}\right)\right\| & \leqslant\left\|V_{t}\right\|\|\mathcal{A}\| \quad(\text { see }(7)) \\
& \leqslant \xi\|\mathcal{A}\| . \quad\left((\text { manifold }) \text { and } V_{t} \in \mathcal{M}_{b}\right) \tag{29}
\end{align*}
$$

Under Assumption 7, we can also write a converse for (28). That is,

$$
\begin{array}{rlr}
\operatorname{dist}\left(U_{t}, \mathcal{M}_{b}\right) & =\left\|U_{t}-V_{t}\right\|_{\mathrm{F}} \quad(\text { see }(26)) \\
& =\left\|\left(\mathrm{D} g\left(V_{t}\right)\right)^{*}\left[\alpha_{t}\right]\right\|_{2} \quad(\text { see }(27)) \\
& \geqslant \sigma_{m}\left(\mathrm{D} g\left(V_{t}\right)\right) \cdot\left\|\alpha_{t}\right\|_{2} \quad(m \leqslant p d) \\
& \geqslant \sigma_{m}\left(\mathcal{M}_{b}\right) \cdot\left\|\alpha_{t}\right\|_{2} . \quad\left(V_{t} \in \mathcal{M}_{b}\right) \tag{30}
\end{array}
$$

For brevity, above we set

$$
\begin{equation*}
\sigma_{m}\left(\mathcal{M}_{b}\right):=\min \left\{\sigma_{m}(\mathrm{D} g(U)): U \in \operatorname{cl}\left(\mathcal{M}_{b}\right)\right\}>0 \tag{31}
\end{equation*}
$$

Note that $\sigma_{m}\left(\mathcal{M}_{b}\right)$ above is positive under Assumption 7 because $\mathcal{M}_{b}$ in (manifold) is bounded. The two inequalities
in (28) and (30) relate $\operatorname{dist}\left(U_{t}, \mathcal{M}_{b}\right)$ to $\alpha_{t}$. These two inequalities will be useful for us later in the proof.
To continue, note also that we have $\mathcal{A}\left(V_{t} V_{t}^{\top}\right)=b$ because $V_{t} \in \mathcal{M}_{b}$ by construction, see also (manifold). Using this last observation, we now revisit (25) and write that

$$
\begin{align*}
& 2\left\|g\left(U_{0}\right)\right\|_{2} \\
& \geqslant\left\|\mathcal{A}\left(U_{t} U_{t}^{\top}\right)-b\right\|_{2} \quad(\text { see }(25)) \\
& =\left\|\mathcal{A}\left(U_{t} U_{t}^{\top}-V_{t} V_{t}^{\top}\right)\right\| \quad(\text { see }(\text { manifold })) \\
& = \\
& \quad \| 2 \mathcal{A}\left(\left(\mathrm{D} g\left(V_{t}\right)\right)^{*}\left[\alpha_{t}\right] V_{t}^{\top}\right)  \tag{32}\\
& \\
& \quad+\mathcal{A}\left(\left(\mathrm{D} g\left(V_{t}\right)\right)^{*}\left[\alpha_{t}\right] \cdot\left(\left(\mathrm{D} g\left(V_{t}\right)\right)^{*}\left[\alpha_{t}\right]\right)^{\top}\right) \|_{2}
\end{align*}
$$

where we used (27) in the last identity above. There, we also benefited from the symmetry of $\left\{A_{i}\right\}_{i=1}^{m}$ in (3). By applying the reverse triangle inequality to the last line above, it follows from (32) that

$$
\begin{align*}
2\left\|g\left(U_{0}\right)\right\|_{2} & \geqslant 2\left\|\mathcal{A}\left(\left(\mathrm{D} g\left(V_{t}\right)\right)^{*}\left[\alpha_{t}\right] V_{t}^{\top}\right)\right\|_{2} \\
& -\left\|\mathcal{A}\left(\left(\mathrm{D} g\left(V_{t}\right)\right)^{*}\left[\alpha_{t}\right] \cdot\left(\left(\mathrm{D} g\left(V_{t}\right)\right)^{*}\left[\alpha_{t}\right]\right)^{\top}\right)\right\|_{2} \\
& \geqslant 2\left\|\mathcal{A}\left(\left(\mathrm{D} g\left(V_{t}\right)\right)^{*}\left[\alpha_{t}\right] V_{t}^{\top}\right)\right\|_{2} \\
& -\|\mathcal{A}\|\left\|\cdot \mathrm{D} g\left(V_{t}\right)\right\|^{2} \cdot\left\|\alpha_{t}\right\|_{2}^{2} \\
& =2\left\|\left(\left(\mathrm{D} g\left(V_{t}\right)\right) \circ\left(\mathrm{D} g\left(V_{t}\right)\right)^{*}\right)\left[\alpha_{t}\right]\right\|_{2} \\
& -\|\mathcal{A}\| \cdot\left\|\mathrm{D} g\left(V_{t}\right)\right\|^{2} \cdot\left\|\alpha_{t}\right\|_{2}^{2}, \quad \text { (see (7)) } \tag{33}
\end{align*}
$$

where $\circ$ denotes the composition of two operators. Recalling the fact that $V_{t} \in \mathcal{M}_{b}$ by construction, the last line above can be lower bounded as

$$
\begin{align*}
2\left\|g\left(U_{0}\right)\right\|_{2} \geqslant & 2 \sigma_{m}\left(\mathcal{M}_{b}\right)^{2}\left\|\alpha_{t}\right\|_{2} \\
& -\|\mathcal{A}\| \cdot \xi^{2}\|\mathcal{A}\|^{2} \cdot\left\|\alpha_{t}\right\|_{2}^{2} \quad \text { (see (29), (31)) } \\
\geqslant & \sigma_{m}\left(\mathcal{M}_{b}\right)^{2}\left\|\alpha_{t}\right\|_{2} \tag{34}
\end{align*}
$$

where the last line above holds if $\|\alpha\|_{2}$ is sufficiently small, i.e., the last line above holds if

$$
\begin{equation*}
\left\|\alpha_{t}\right\|_{2} \leqslant \frac{\sigma_{m}\left(\mathcal{M}_{b}\right)^{2}}{\xi^{2}\|\mathcal{A}\|^{3}} \tag{35}
\end{equation*}
$$

In view of (28) and (30), it follows from (34) and (35) that

$$
\begin{align*}
& \operatorname{dist}\left(U_{t}, \mathcal{M}_{b}\right) \leqslant \frac{\sigma_{m}\left(\mathcal{M}_{b}\right)^{3}}{\xi^{2}\|\mathcal{A}\|^{3}} \\
& \Longrightarrow \operatorname{dist}\left(U_{t}, \mathcal{M}_{b}\right) \leqslant \frac{2 \xi\|\mathcal{A}\|\left\|g\left(U_{0}\right)\right\|_{2}}{\sigma_{m}\left(\mathcal{M}_{b}\right)^{2}} \tag{36}
\end{align*}
$$

Recalling the assumptions of Lemma 19, note that $\operatorname{dist}\left(U_{t}, \mathcal{M}_{b}\right) \leqslant \rho_{0}$ for every $t \in[0, \tau]$. Suppose that $\rho_{0}$ is sufficiently small, i.e., take

$$
\begin{equation*}
\rho_{0} \leqslant \frac{\sigma_{m}\left(\mathcal{M}_{b}\right)^{3}}{\xi^{2}\|\mathcal{A}\|^{3}} \tag{37}
\end{equation*}
$$

Then, (36) immediately implies that

$$
\begin{equation*}
\operatorname{dist}\left(U_{t}, \mathcal{M}_{b}\right) \leqslant \frac{2 \xi\|\mathcal{A}\| \cdot\left\|g\left(U_{0}\right)\right\|_{2}}{\sigma_{m}\left(\mathcal{M}_{b}\right)^{2}}, \quad \text { if } t \in[0, \tau] \tag{38}
\end{equation*}
$$

Let us now rephrase the right-hand side of (38). Specifically, we next upper bound $\left\|g\left(U_{0}\right)\right\|_{2}$ above by the initial distance to the manifold, i.e., $\operatorname{dist}\left(U_{0}, \mathcal{M}_{b}\right)$. Recall from (26) that $V_{0} \in \mathcal{M}_{b}$ denotes a projection of $U_{0}$ on $\mathcal{M}_{b}$ in (manifold). In particular, $V_{0} \in \mathcal{M}_{b}$ implies that $\mathcal{A}\left(V_{0} V_{0}^{\top}\right)=b$ by (manifold). Using this last observation and (26), we bound $\left\|g\left(U_{0}\right)\right\|_{2}$ as

$$
\begin{align*}
\left\|g\left(U_{0}\right)\right\|_{2} & =\frac{1}{2}\left\|\mathcal{A}\left(U_{0} U_{0}^{\top}\right)-b\right\|_{2} \quad(\text { see (6)) } \\
& =\frac{1}{2}\left\|\mathcal{A}\left(U_{0} U_{0}^{\top}-V_{0} V_{0}^{\top}\right)\right\|_{2} \quad \text { (see (manifold)) } \\
& \leqslant \frac{1}{2}\|\mathcal{A}\| \cdot\left\|U_{0} U_{0}^{\top}-V_{0} V_{0}^{\top}\right\|_{\mathrm{F}} \\
& \leqslant \frac{1}{2}\|\mathcal{A}\| \cdot\left(\left\|U_{0}\right\|+\left\|V_{0}\right\|\right) \cdot\left\|U_{0}-V_{0}\right\|_{\mathrm{F}} \\
& \leqslant\|\mathcal{A}\| \xi \cdot\left\|U_{0}-V_{0}\right\|_{\mathrm{F}} \\
& =\|\mathcal{A}\| \xi \cdot \operatorname{dist}\left(U_{0}, \mathcal{M}_{b}\right) \tag{39}
\end{align*}
$$

where the second-to-last line above assumes that $\xi$ is sufficiently large, i.e., $\xi \geqslant\left\|U_{0}\right\|$. The second-to-last line in (39) also uses the fact that $V_{0} \in \mathcal{M}_{b}$ in (manifold) and, in particular, $\left\|V_{0}\right\| \leqslant \xi$. By combining (38) and (39), we arrive at

$$
\operatorname{dist}\left(U_{t}, \mathcal{M}_{b}\right) \leqslant \frac{2 \xi^{2}\|\mathcal{A}\|^{2} \operatorname{dist}\left(U_{0}, \mathcal{M}_{b}\right)}{\sigma_{m}\left(\mathcal{M}_{b}\right)^{2}}, \quad \text { if } t \in[0, \tau]
$$

provided that $\xi$ is sufficiently large. By setting $\operatorname{dist}\left(U_{0}, \mathcal{M}_{b}\right)$ sufficiently small, i.e., by taking

$$
\begin{equation*}
\operatorname{dist}\left(U_{0}, \mathcal{M}_{b}\right) \leqslant \frac{\sigma_{m}\left(\mathcal{M}_{b}\right)^{2} \rho_{0}}{4 \xi^{2}\|\mathcal{A}\|^{2}} \tag{40}
\end{equation*}
$$

we can ensure that

$$
\begin{equation*}
\operatorname{dist}\left(U_{t}, \mathcal{M}_{b}\right) \leqslant \rho_{0} / 2, \quad \text { if } t \in[0, \tau] \tag{41}
\end{equation*}
$$

Above, $\rho_{0}$ was defined in Lemma 18. This completes the proof of Lemma 19.

## Appendix B

Proof of Lemma 26

Recall that $\rho_{0}$ denotes the radius of the neighborhood in Lemma 18. Recall also from (6), (8) and (19) that

$$
\begin{align*}
\nabla h_{\gamma}\left(U_{t}\right) & =\left(I-\mathcal{A}^{*}\left(\lambda\left(U_{t}, U_{t}\right)\right)\right) U_{t}+\gamma\left(\mathrm{D} g\left(U_{t}\right)\right)^{*}\left[g\left(U_{t}\right)\right] \\
& -\left(\mathrm{D} \lambda\left(U_{t}, U_{t}\right)\right)^{*}\left[g\left(U_{t}\right)\right] \tag{42}
\end{align*}
$$

for every $t \in[0, \tau]$. Recall from (6) the definition of the (scaled) feasibility gap $G: \mathbb{R}^{d \times p} \rightarrow \mathbb{R}$. To study the evolution of the feasibility gap along (gradient flow), we write that

$$
\begin{align*}
& \frac{\mathrm{d} G\left(U_{t}\right)}{\mathrm{d} t} \\
& =\left\langle\nabla G\left(U_{t}\right), \dot{U}_{t}\right\rangle \quad \text { (chain rule) } \\
& =-\left\langle\nabla G\left(U_{t}\right), \nabla h_{\gamma}\left(U_{t}\right)\right\rangle \quad \text { (see (gradient flow)) } \\
& =-\left\langle\left(\mathrm{D} g\left(U_{t}\right)\right)^{*}\left[g\left(U_{t}\right)\right], \nabla h_{\gamma}\left(U_{t}\right)\right\rangle \quad((6) \text { and chain rule) } \\
& =-\left\langle\left(\mathrm{D} g\left(U_{t}\right)\right)^{*}\left[g\left(U_{t}\right)\right],\left(I_{d}-\mathcal{A}^{*}\left(\lambda\left(U_{t}, U_{t}\right)\right)\right) U_{t}\right\rangle \\
& -\gamma\left\|\left(\mathrm{D} g\left(U_{t}\right)\right)^{*}\left[g\left(U_{t}\right)\right]\right\|_{\mathrm{F}}^{2} \\
& +\left\langle\left(\mathrm{D} g\left(U_{t}\right)\right)^{*}\left[g\left(U_{t}\right)\right],\left(\mathrm{D} \lambda\left(U_{t}, U_{t}\right)\right)^{*}\left[g\left(U_{t}\right)\right]\right\rangle \quad(\text { see (42)) } \\
& \leqslant\left\|\mathrm{D} g\left(U_{t}\right)\right\| \cdot\left\|g\left(U_{t}\right)\right\|_{2} \cdot\left\|I_{d}-\mathcal{A}^{*}\left(\lambda\left(U_{t}, U_{t}\right)\right)\right\| \cdot\left\|U_{t}\right\|_{\mathrm{F}} \\
& -\gamma\left(\sigma_{m}\left(\mathrm{D} g\left(U_{t}\right)\right)\right)^{2} \cdot\left\|g\left(U_{t}\right)\right\|_{2}^{2} \quad \\
& +\left\|\mathrm{D} g\left(U_{t}\right)\right\| \cdot\left\|\mathrm{D} \lambda\left(U_{t}, U_{t}\right)\right\| \cdot\left\|g\left(U_{t}\right)\right\|_{2}^{2} \quad(\text { Cauchy-Schwarz) } \\
& =\sqrt{2 G\left(U_{t}\right)} \cdot\left(\left\|\mathrm{D} g\left(U_{t}\right)\right\| \cdot\left\|I_{d}-\mathcal{A}^{*}\left(\lambda\left(U_{t}, U_{t}\right)\right)\right\| \cdot\left\|U_{t}\right\|_{\mathrm{F}}\right. \\
& -\gamma\left(\sigma_{m}\left(\mathrm{D} g\left(U_{t}\right)\right)\right)^{2} \sqrt{2 G\left(U_{t}\right)} \\
& \left.+\left\|\mathrm{D} g\left(U_{t}\right)\right\| \cdot\left\|\mathrm{D} \lambda\left(U_{t}, U_{t}\right)\right\| \sqrt{2 G\left(U_{t}\right)}\right) . \quad(\text { see (6)) } \tag{43}
\end{align*}
$$

To control the terms in the last identity above, we make two observations:

1) Recall that both $\operatorname{cl}\left(\mathcal{M}_{b}\right)$ in (manifold) and its neighborhood $\left\{U: \operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho_{0}\right\}$ are compact sets. Because $\rho_{0}<\rho$ by design, note also that $g$ in (6) and $\lambda$ are both continuously-differentiable functions on $\left\{U: \operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho_{0}\right\}$, see Lemma 9. Here, $\rho$ is the radius of the neighborhoods both in Assumption 7 and Lemma 9.
It follows that the functions $U \rightarrow\|\mathrm{D} g(U)\|, U \rightarrow$ $\left\|I_{d}-\mathcal{A}^{*}(\lambda(U, U))\right\|, U \rightarrow\|U\|_{\mathrm{F}}$, and $U \rightarrow\|\mathrm{D} \lambda(U, U)\|$ are all bounded on the set $\left\{U: \operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho_{0}\right\}$. Consequently, the corresponding terms in the last identity of (43) are bounded on the interval $[0, \tau]$.
2) Moreover, Assumption 7 and $\rho_{0}<\rho$ together imply that the function $U \rightarrow \sigma_{m}(\mathrm{D} g(U))$ is bounded away from zero on the set $\left\{U: \operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho_{0}\right\}$. Consequently, the corresponding term in the last identity of (43) is bounded away from zero on the interval $[0, \tau]$.
If $G\left(U_{t_{0}}\right)=0$ for $t_{0} \in[0, \tau]$, we find from (43) that $G\left(U_{t}\right)=$ 0 for every $t \in\left[t_{0}, \tau\right]$. That is, once feasible, (gradient flow) remains feasible afterwards. On the other hand, in view of the above two observations, we see that $\mathrm{d} G\left(U_{t}\right) / \mathrm{d} t<0$ for every $t \in\left[0, t_{0}\right)$, provided that $\gamma$ is sufficiently large. We conclude that $\mathrm{d} G\left(U_{t}\right) / \mathrm{d} t<0$ for every $t \in[0, \tau]$, provided that $\gamma$ is sufficiently large. This completes the proof of Lemma 26.

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# Supplementary Material 

## Appendix A <br> Proof of Lemma 9

In order to find an explicit expression for $\lambda(U, U)$ in (multipliers), we first calculate $\left((\mathrm{D} g(U))^{*}\right)^{\dagger}$ as follows. Under Assumption 3.(ii) and after recalling (8), note that $\left((\mathrm{D} g(U))^{*}\right)^{\dagger}: \mathbb{R}^{d \times p} \rightarrow \mathbb{R}^{m}$ is specified as

$$
\begin{align*}
& \left((\mathrm{D} g(U))^{*}\right)^{\dagger}[\Delta]:=(K(U))^{\dagger} \mathcal{A}\left(\Delta U^{\top}\right) \\
& K(U):=\left[\left\langle A_{i} U, A_{j} U\right\rangle\right]_{i, j=1}^{m} \in \mathbb{R}^{m \times m} \tag{44}
\end{align*}
$$

for $\Delta \in \mathbb{R}^{d \times p}$. We then recall the definition of $\lambda(U, U)$ in (multipliers) to write that

$$
\begin{align*}
\lambda(U, U) & =\left((\mathrm{D} g(U))^{*}\right)^{\dagger}[U] \quad(\text { see (multipliers) }) \\
& =(K(U))^{\dagger} \mathcal{A}\left(U U^{\top}\right), \quad(\text { see }(44)) \tag{45}
\end{align*}
$$

which proves (14).
We next prove the second claim in Lemma 9. Recall that Assumption 7 holds and let $\rho$ denote the radius of the neighborhood specified in Assumption 7. That is,

$$
\begin{equation*}
\operatorname{rank}(\mathrm{D} g(U))=m, \quad \text { if } \operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho \tag{46}
\end{equation*}
$$

Equivalently, by definition of $(\mathrm{D} g(U))^{*}$ in (8), the matrices $\left\{A_{i} U\right\}_{i=1}^{m}$ are linearly independent for every $U$ such that $\operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho$. From (44) and (46), it immediately follows that

$$
\begin{equation*}
\operatorname{rank}(K(U))=m, \quad \text { if } \operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho \tag{47}
\end{equation*}
$$

Note that $K(U)$ in (45) is an analytic function of $U$ in $\mathbb{R}^{d \times p}$. By (47) and the boundedness of $\mathcal{M}_{b}$ in (manifold), $(K(U))^{-1}$ is also an analytic function of $U$ on the set

$$
\begin{equation*}
\mathcal{M}_{b, \rho}:=\left\{U: \operatorname{dist}\left(U, \mathcal{M}_{b}\right)<\rho\right\} \tag{48}
\end{equation*}
$$

In view of (45), $\lambda(U, U)$ is also an analytic function of $U$ on the set $\mathcal{M}_{b, \rho}$, which proves the second claim in Lemma 9.
We next prove the third and final claim in Lemma 9. To compute the derivative of $\lambda(U, U)$ with respect to $U$, we begin by computing the (total) derivatives of $K(U)$ and $(K(U))^{-1}$, see (44). For $\Delta \in \mathbb{R}^{d \times p}$, note that the directional derivative of $K(U)$ at $U$ and along the direction $\Delta$ is given by

$$
\begin{equation*}
\mathrm{D} K(U)[\Delta]=2\left[\left\langle A_{i} \Delta, A_{j} U\right\rangle\right]_{i, j=1}^{m}=: 2 \widetilde{K}(U, \Delta) \tag{49}
\end{equation*}
$$

To compute the directional derivative of $(K(U))^{-1}$ at $U \in$ $\mathcal{M}_{b, \rho}$ and along $\Delta \in \mathbb{R}^{n \times p}$, we compute the directional
derivative of both sides of the identity $K(U) \cdot(K(U))^{-1}=$ $I_{m}$, along a direction $\Delta \in \mathbb{R}^{d \times p}$. That is,

$$
\mathrm{D} K(U)[\Delta] \cdot(K(U))^{-1}+K(U) \cdot \mathrm{D}(K(U))^{-1}[\Delta]=0
$$

which, after rearranging, yields that

$$
\begin{align*}
& \mathrm{D}(K(U))^{-1}[\Delta] \\
& =-(K(U))^{-1} \cdot \mathrm{D} K(U)[\Delta] \cdot(K(U))^{-1} \\
& =-2(K(U))^{-1} \cdot \widetilde{K}(U, \Delta) \cdot(K(U))^{-1} \cdot \quad(\text { see }(49)) \tag{50}
\end{align*}
$$

Having computed in (50) the directional derivative of $(K(U))^{-1}$ at $U \in \mathcal{M}_{b, \rho}$, we are now ready to compute the derivative of $\lambda(U, U)$ with respect to $U$ as follows. Using the definition of $\lambda(U, U)$ in (45) and for a direction $\Delta \in \mathbb{R}^{d \times p}$, the directional derivative of $\lambda(U, U)$ along $\Delta$ is given by

$$
\begin{align*}
& \mathrm{D} \lambda(U, U)[\Delta]  \tag{51}\\
& =\mathrm{D}(K(U))^{-1}[\Delta] \cdot \mathcal{A}\left(U U^{\top}\right)+2(K(U))^{-1} \mathcal{A}\left(\Delta U^{\top}\right) \\
& =-2(K(U))^{-1} \widetilde{K}(U, \Delta)(K(U))^{-1} \mathcal{A}\left(U U^{\top}\right) \\
& \quad+2(K(U))^{-1} \mathcal{A}\left(\Delta U^{\top}\right) \quad(\text { see }(50)) \\
& =2(K(U))^{-1}\left(-\widetilde{K}(U, \Delta)(K(U))^{-1} \mathcal{A}\left(U U^{\top}\right)+\mathcal{A}\left(\Delta U^{\top}\right)\right) .
\end{align*}
$$

This completes the proof of Lemma 9.

## Appendix B

Proof of Proposition 15

Recalling (manifold), let us fix $U \in \mathbb{R}^{d \times p}$ such that $\operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho$ and $\|U\|<\xi$, where $\rho$ is the radius of the neighborhood in Assumption 7. Recall also (AL). For $\lambda^{\prime} \in \mathbb{R}^{m}$, note that the gradient of the augmented Lagrangian with respect to its first argument is specified as

$$
\begin{align*}
& \nabla_{1} L_{\gamma}\left(U, \lambda^{\prime}\right) \\
& =\nabla f(U)-(\mathrm{D} g(U))^{*}\left[\lambda^{\prime}\right]+\gamma(\mathrm{D} g(U))^{*}[g(U)] \tag{52}
\end{align*}
$$

With the choice of $\lambda^{\prime}=\lambda(U, U)$ from (multipliers), we rewrite (52) as

$$
\begin{align*}
& \nabla_{1} L_{\gamma}(U, \lambda(U, U)) \\
& =\nabla f(U)-(\mathrm{D} g(U))^{*}[\lambda(U, U)]+\gamma(\mathrm{D} g(U))^{*}[g(U)] \\
& =\nabla f(U)-\left((\mathrm{D} g(U))^{*} \circ\left((\mathrm{D} g(U))^{*}\right)^{\dagger}\right)[U] \\
& \quad+\gamma(\mathrm{D} g(U))^{*}[g(U)] \quad(\text { see }(\text { multipliers })) \\
& =\nabla f(U)-\left((\mathrm{D} g(U))^{*} \circ\left((\mathrm{D} g(U))^{*}\right)^{\dagger}\right)[\nabla f(U)] \\
& \quad+\gamma(\mathrm{D} g(U))^{*}[g(U)] \quad(\text { see }(6)) \\
& =\left(\mathrm{Id}-(\mathrm{D} g(U))^{*} \circ\left((\mathrm{D} g(U))^{*}\right)^{\dagger}\right)[\nabla f(U)] \\
& \quad+\gamma(\mathrm{D} g(U))^{*}[g(U)] \tag{53}
\end{align*}
$$

where Id is the shorthand for the identity map. Note that the two terms in the last line above are in fact orthogonal to one another; one is in the range of the operator $(\mathrm{D} g(U))^{*}$ and the other is orthogonal to range $\left((\mathrm{D} g(U))^{*}\right)$. In particular, $\nabla_{1} L_{\gamma}(U, \lambda(U, U))=0$ implies that both

$$
\begin{align*}
& \left(\operatorname{Id}-(\mathrm{D} g(U))^{*} \circ\left((\mathrm{D} g(U))^{*}\right)^{\dagger}\right)[\nabla f(U)]=0, \\
& \text { and }(\mathrm{D} g(U))^{*}[g(U)]=0 . \tag{54}
\end{align*}
$$

Moreover, recall the earlier assumption that $\operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho$, where $\rho$ is the radius of the neighborhood of $\mathcal{M}_{b}$ in Assumption 7. From this assumption, it follows that $\mathrm{D} g(U)$ : $\mathbb{R}^{d \times p} \rightarrow \mathbb{R}^{m}$ is a rank- $m$ linear operator. In particular, the operator $\mathrm{D} g(U)^{*}$ has a trivial null-space. This observation allows us to simplify the second identity in (54). More specifically, we find that both

$$
\begin{align*}
& \left(\operatorname{Id}-(\mathrm{D} g(U))^{*} \circ\left((\mathrm{D} g(U))^{*}\right)^{\dagger}\right)[\nabla f(U)]=0 \\
& \text { and } g(U)=0 . \tag{55}
\end{align*}
$$

Note that $g(U)=0$ above and the earlier assumption that $\|U\|<\xi$ together imply that $U \in \mathcal{M}_{b}$, see (manifold). In view of (13) and (15), we also identify the first expression above as $\nabla_{\mathcal{M}_{b}} f(U)$. We can therefore rewrite (55) as

$$
\begin{equation*}
\nabla_{\mathcal{M}_{b}} f(U)=0 \text { and } g(U)=0 \text { and }\|U\|<\xi \tag{56}
\end{equation*}
$$

That is, in view of Definition 10, $U$ is an FOSP of (factor). This proves the first item of Proposition 15.
To prove the second item in Proposition 15, let $U$ be an FOSP of (factor). For $\lambda^{\prime} \in \mathbb{R}^{m}$, note that the Hessian of the augmented Lagrangian with respect to its first argument is the bilinear operator specified as

$$
\begin{align*}
\nabla_{1}^{2} L_{\gamma}\left(U, \lambda^{\prime}\right)= & \nabla^{2} f(U)-\sum_{i=1}^{m}\left(\lambda_{i}^{\prime}-\gamma g_{i}(U)\right) \nabla^{2} g_{i}(U) \\
& +\gamma(\mathrm{D} g(U))^{*} \circ \mathrm{D} g(U) \quad(\text { see }(\mathrm{AL})) \\
= & \nabla^{2} f(U)-\sum_{i=1}^{m} \lambda_{i}^{\prime} \nabla^{2} g_{i}(U) \\
& +\gamma(\mathrm{D} g(U))^{*} \circ \mathrm{D} g(U), \quad(\text { see }(16)) \tag{57}
\end{align*}
$$

where $\lambda_{i}^{\prime}$ and $g_{i}(U)$ are the $i^{\text {th }}$ coordinates of the vectors $\lambda^{\prime} \in \mathbb{R}^{m}$ and $g(U)$, respectively. In the second line above,
we used the fact that $g(U)=0$ for the FOSP $U$. For the choice of $\lambda^{\prime}=\lambda(U, U)$ from (multipliers), we reach

$$
\begin{array}{r}
\nabla_{1}^{2} L_{\gamma}(U, \lambda(U, U))=\nabla^{2} f(U)-\sum_{i=1}^{m} \lambda_{i}(U, U) \nabla^{2} g_{i}(U) \\
+\gamma(\mathrm{D} g(U))^{*} \circ \mathrm{D} g(U) \tag{58}
\end{array}
$$

where $\lambda_{i}(U, U)$ is the $i^{\text {th }}$ coordinate of the vector $\lambda(U, U)$. Let $\mathcal{P}_{\mathcal{T}_{U} \mathcal{M}_{b}}$ denote the projection onto the tangent space $\mathcal{T}_{U} \mathcal{M}_{b}$. Suppose also that $\nabla_{1}^{2} L_{\gamma}(U, \lambda(U, U))[\Delta, \Delta] \geqslant 0$ for every tangent direction $\Delta \in \mathcal{T}_{U} \mathcal{M}_{b}$. It then follows that

$$
\begin{align*}
0 & \leqslant \mathcal{P}_{\mathcal{T}_{U} \mathcal{M}_{b}} \circ \nabla_{1}^{2} L_{\gamma}(U, \lambda(U, U)) \circ \mathcal{P}_{\mathcal{T}_{U} \mathcal{M}_{b}} \\
= & \mathcal{P}_{\mathcal{T}_{U} \mathcal{M}_{b}} \circ\left(\nabla^{2} f(U)-\sum_{i=1}^{m} \lambda_{i}(U, U) \nabla^{2} g_{i}(U)\right. \\
& \left.\quad+\gamma(\mathrm{D} g(U))^{*} \circ \mathrm{D} g(U)\right) \circ \mathcal{P}_{\mathcal{T}_{U} \mathcal{M}_{b}} \quad \text { (see (58)) } \\
& =\mathcal{P}_{\mathcal{T}_{U} \mathcal{M}_{b}} \circ\left(\nabla^{2} f(U)-\sum_{i=1}^{m} \lambda_{i}(U, U) \nabla^{2} g_{i}(U)\right) \circ \mathcal{P}_{\mathcal{T}_{U} \mathcal{M}_{b}} \\
= & \mathcal{P}_{\mathcal{T}_{U} \mathcal{M}_{b}} \circ \nabla_{\mathcal{M}_{b}}^{2} f(U) \circ \mathcal{P}_{\mathcal{T}_{U} \mathcal{M}_{b}} . \tag{59}
\end{align*}
$$

To obtain the second identity above, we used (11), and the orthogonality of tangent and normal spaces. In view of Definition 11, we conclude from (59) that $U$ is an SOSP of (factor). This proves the second item in Proposition 15 and completes the proof of Lemma 15.

## Appendix C <br> Proof of Lemma 16

Recall from Lemma 9 that $\lambda(U, U)$ is an analytic function of $U$ on the set $\left\{U: \operatorname{dist}\left(U, \mathcal{M}_{b}\right)<\rho\right\}$, where $\rho$ is the radius of the neighborhood in Assumption 7. It follows that $h_{\gamma}$ in (merit) is also an analytic function of $U$ in the same set. See also (6) and (AL) to review the notation used in this paragraph.
To derive an expression for the derivative of $h_{\gamma}$ in (19), we calculate the directional derivative of $h_{\gamma}$ at $U$ such that $\operatorname{dist}\left(U, \mathcal{M}_{b}\right)<\rho$ and along the direction $\Delta \in \mathbb{R}^{d \times p}$, i.e.,

$$
\begin{aligned}
& \mathrm{D} h_{\gamma}(U)[\Delta] \\
& =\mathrm{D} L_{\gamma}(U, \lambda(U, U))[\Delta] \quad \text { (see (merit)) } \\
& =\mathrm{D} f(U)[\Delta]-\left\langle(\mathrm{D} g(U))^{*}[\lambda(U, U)-\gamma g(U)], \Delta\right\rangle \\
& \quad-\left\langle(\mathrm{D} \lambda(U, U))^{*}[g(U)], \Delta\right\rangle \quad \text { (see (AL)) } \\
& =\left\langle\left(I_{d}-\mathcal{A}^{*}(\lambda(U, U))\right) U, \Delta\right\rangle+\frac{\gamma}{2}\left\langle\mathcal{A}^{*}\left(\mathcal{A}\left(U U^{\top}\right)-b\right) U, \Delta\right\rangle \\
& \quad-\frac{1}{2}\left\langle(\mathrm{D} \lambda(U, U))^{*}\left[\mathcal{A}\left(U U^{\top}\right)-b\right], \Delta\right\rangle . \text { (see (6), (8)) }
\end{aligned}
$$

Above, the derivative of $\lambda(U, U)$ with respect to $U$ was computed in Lemma 9. This completes the proof of Lemma 16.

## Appendix D <br> Proof of Proposition 17

The proof is similar to that of Proposition 15. Recalling (manifold), we fix $U \in \mathbb{R}^{d \times p}$ such that $\operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho^{\prime}$ and $\|U\|<\xi$. Here, $\rho^{\prime}>0$ is sufficiently small. To be specific, we assume that $\rho^{\prime}<\rho$, where $\rho$ is the radius of the neighborhood in Assumption 7. Our starting point is the expression for $\nabla h_{\gamma}(U)$ in (19). By comparing this expression with (6), (7) and (13), we can rewrite (19) as

$$
\begin{align*}
& \left.\nabla h_{\gamma}(U)=\left(\mathrm{Id}-(\mathrm{D} g(U))^{*}\right) \circ\left((\mathrm{D} g(U))^{*}\right)^{\dagger}\right)[\nabla f(U)] \\
& \quad+\gamma(\mathrm{D} g(U))^{*}[g(U)]-(\mathrm{D} \lambda(U, U))^{*}[g(U)] \tag{60}
\end{align*}
$$

Note that the second term on the right-hand side above is in the range of the operator $(\mathrm{D} g(U))^{*}$, whereas the first term on the right-hand side above is orthogonal to this range. For brevity, let $\mathcal{P}_{U}:=\left((\mathrm{D} g(U))^{*}\right) \circ\left((\mathrm{D} g(U))^{*}\right)^{\dagger}$ denote the orthogonal projection onto the subspace range $\left((\mathrm{D} g(U))^{*}\right)$. After projecting both sides of (60) onto range $\left((\mathrm{D} g(U))^{*}\right)$ :

$$
\begin{align*}
& \mathcal{P}_{U}\left[\nabla h_{\gamma}(U)\right] \\
& \left.=\gamma(\mathrm{D} g(U))^{*}[g(U)]-\left(\mathcal{P}_{U} \circ(\mathrm{D} \lambda(U, U))^{*}\right)\right)[g(U)] \\
& =: O_{\gamma}(U)[g(U)] \tag{61}
\end{align*}
$$

Below we make two observations about (61).

1) It follows from the boundedness of $\mathcal{M}_{b}$ in (manifold) that the neighborhood $\left\{U: \operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho^{\prime}\right\}$ is a compact set. In view of Assumption 7, we therefore have that

$$
\begin{equation*}
\min \left\{\sigma_{m}(\mathrm{D} g(U)): \operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho^{\prime}\right\}>0 \tag{62}
\end{equation*}
$$

where $\rho^{\prime}$ was specified earlier.
2) Recall from the first item above that $\left\{U: \operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant\right.$ $\left.\rho^{\prime}\right\}$ is a compact set. Recall also from Lemma 9 that $\mathrm{D} \lambda$ is a continuous function on the this set because, earlier in this appendix, we specified that $\rho^{\prime}<\rho$. Consequently,

$$
\begin{align*}
& \left.\max \left\{\| \mathcal{P}_{U} \circ(\mathrm{D} \lambda(U, U))^{*}\right) \|: \operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho^{\prime}\right\} \leqslant \\
& \left.\max \left\{\|(\mathrm{D} \lambda(U, U))^{*}\right) \|: \operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho^{\prime}\right\}<\infty, \tag{63}
\end{align*}
$$

where the first inequality above uses the fact that $\mathcal{P}_{U}$ is an orthogonal projection.
In view of (62) and (63), for sufficiently large $\gamma$, we conclude:

$$
\begin{equation*}
\min \left\{\sigma_{m}\left(O_{\gamma}(U)\right): \operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho^{\prime}\right\}>0 \tag{64}
\end{equation*}
$$

In particular, the operator $O_{\gamma}(U)$ has a trivial null space for every $U$ such that $\operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho^{\prime}$.
Next, let us consider $\bar{U}$ such that $\operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho^{\prime}$ and $\|\bar{U}\|<\xi$. Suppose also that $\bar{U}$ is an FOSP of $h_{\gamma}$, i.e., $\nabla h_{\gamma}(\bar{U})=0$. It follows from (61) that $O_{\gamma}(\bar{U})[g(\bar{U})]=0$. Since we just established in (64) that $O_{\gamma}(\bar{U})$ has a trivial null space, $O_{\gamma}(\bar{U})[g(\bar{U})]=0$ in turn implies that $g(\bar{U})=0$. Combined with the assumption that $\|\bar{U}\|<\xi$, we reach that $\bar{U} \in \mathcal{M}_{b}$, see (6) and (manifold). From this point, by following the same steps as in the proof of Proposition 15,
we find that $\bar{U}$ is an FOSP of (factor). This proves the first item in Proposition 17.
To prove the second item in Proposition 17, for the same $\bar{U}$ as in the above paragraph, we can use (AL) and (merit) to calculate the (Euclidean) Hessian of $h_{\gamma}$ at $\bar{U}$ as

$$
\begin{gather*}
\nabla^{2} h_{\gamma}(\bar{U})=\nabla^{2} f(\bar{U})-\sum_{i=1}^{m}\left(\lambda_{i}(\bar{U})-\gamma g_{i}(\bar{U})\right) \nabla^{2} g_{i}(\bar{U}) \\
\quad-(\mathrm{D} g(\bar{U}))^{*} \circ \mathrm{D} \lambda(\bar{U})-(\mathrm{D} \lambda(\bar{U}))^{*} \circ \mathrm{D} g(\bar{U}) \\
+\gamma(\mathrm{D} g(\bar{U}))^{*} \circ \mathrm{D} g(\bar{U})-\sum_{i=1}^{m} g_{i}(\bar{U}) \nabla^{2} \lambda_{i}(\bar{U}) \\
=\nabla^{2} f(\bar{U})-\sum_{i=1}^{m} \lambda_{i}(\bar{U}) \nabla^{2} g_{i}(\bar{U}) \\
\quad-(\mathrm{D} g(\bar{U}))^{*} \circ \mathrm{D} \lambda(\bar{U})-(\mathrm{D} \lambda(\bar{U}))^{*} \circ \mathrm{D} g(\bar{U}) \\
\quad+\gamma(\mathrm{D} g(\bar{U}))^{*} \circ \mathrm{D} g(\bar{U}), \quad(\text { see }(16)) \tag{65}
\end{gather*}
$$

where the the second identity above uses the fact that $\bar{U}$ is an FOSP of (factor); in particular, $g(\bar{U})=0$ by (16).
Next, let us assume that $\nabla^{2} h_{\gamma}(\bar{U}) \geqslant 0$. It follows that $\mathcal{P}_{\mathcal{T}_{\bar{U}} \mathcal{M}_{b}} \circ \nabla^{2} h_{\gamma}(\bar{U}) \circ \mathcal{P}_{\bar{\tau}_{\bar{U}} \mathcal{M}_{b}} \geqslant 0$, where $\mathcal{P}_{\mathcal{T}_{\bar{U}} \mathcal{M}_{b}}$ is the orthogonal projection onto the tangent space of $\mathcal{M}_{b}$ at $\bar{U}$ :

$$
\begin{align*}
& 0 \leqslant \mathcal{P}_{\mathcal{T}_{U} \mathcal{M}_{b}} \circ \nabla^{2} h_{\gamma}(\bar{U}) \circ \mathcal{P}_{\mathcal{T}_{\bar{U}} \mathcal{M}_{b}} \\
&= \mathcal{P}_{\mathcal{T}_{\bar{U} \mathcal{M}_{b}} \circ\left(\nabla^{2} f(\bar{U})-\sum_{i=1}^{m} \lambda_{i}(\bar{U}) \nabla^{2} g_{i}(\bar{U})\right.} \\
& \quad-(\mathrm{D} g(\bar{U}))^{*} \circ \mathrm{D} \lambda(\bar{U})-(\mathrm{D} \lambda(\bar{U}))^{*} \circ \mathrm{D} g(\bar{U}) \\
&\left.+\gamma(\mathrm{D} g(\bar{U}))^{*} \circ \mathrm{D} g(\bar{U})\right) \circ \mathcal{P}_{\mathcal{T}_{\bar{U}} \mathcal{M}_{b}} \quad \text { (see (65)) } \\
&= \mathcal{P}_{\mathcal{T}_{\bar{U} \mathcal{M}_{b}} \circ\left(\nabla^{2} f(\bar{U})-\sum_{i=1}^{m} \lambda_{i}(\bar{U}) \nabla^{2} g_{i}(\bar{U})\right) \circ \mathcal{P}_{\mathcal{T}_{\bar{U}} \mathcal{M}_{b}}}^{=} \\
&=\mathcal{P}_{\mathcal{T}_{\bar{U} \mathcal{M}_{b}} \circ \nabla_{\mathcal{M}_{b}}^{2} f(\bar{U}) \circ \mathcal{P}_{\mathcal{T}_{\bar{U}} \mathcal{M}_{b}}, \quad \text { (see (17)) }} \quad \tag{66}
\end{align*}
$$

where, in the second identity above, we used the fact that the tangent space at $\bar{U} \in \mathcal{M}_{b}$ is orthogonal to range $\left((\mathrm{D} g(\bar{U}))^{*}\right)$, see (11). From the last line above, we conclude that $\bar{U}$ is also an SOSP of (factor), see Definition 11. This proves the second item and completes the proof of Proposition 17.

## Appendix E <br> Proof of Lemma 18

Throughout the remaining proofs, we will often show the dependence on time $t$ as a subscript. For example, we will write $U_{t}$ instead of $U(t)$. In this particular proof, we assume that the radius $\rho_{0}$ of the neighborhood specified in Lemma 18 is strictly smaller than $\rho$, where $\rho$ is the radius of the neighborhood in Assumption 7 and Lemma 9. That is, $\rho_{0}<\rho$. Recall from (6) and (19) that

$$
\begin{align*}
\nabla h_{\gamma}\left(U_{t}\right)= & \left(I_{d}-\mathcal{A}^{*}\left(\lambda\left(U_{t}, U_{t}\right)\right)\right) U_{t}+\gamma \mathcal{A}^{*}\left(g\left(U_{t}\right)\right) U_{t} \\
& -\left(\mathrm{D} \lambda\left(U_{t}, U_{t}\right)\right)^{*}\left[g\left(U_{t}\right)\right], \tag{67}
\end{align*}
$$

for every $t \in[0, \tau]$. Let us also define

$$
\begin{equation*}
X_{t}:=U_{t} U_{t}^{\top} \in \mathbb{R}^{d \times d}, \quad \text { if } t \in[0, \tau] . \tag{68}
\end{equation*}
$$

Note that the above flow in $\mathbb{R}^{d \times d}$ satisfies

$$
\begin{align*}
X_{0}= & U_{0} U_{0}^{\top},  \tag{69}\\
\dot{X}_{t}= & \dot{U}_{t} U_{t}^{\top}+U_{t} \dot{U}_{t}^{\top} \quad(\text { see }(68)) \\
= & -\nabla h_{\gamma}\left(U_{t}\right) U_{t}^{\top}-U_{t}\left(\nabla h_{\gamma}\left(U_{t}\right)\right)^{\top}(\text { see }(\text { gradient flow) }) \\
= & -\left(I_{d}-\mathcal{A}^{*}\left(\lambda\left(U_{t}, U_{t}\right)\right)\right) X_{t}-\gamma \mathcal{A}^{*}\left(g\left(U_{t}\right)\right) X_{t} \\
+ & \left(\mathrm{D} \lambda\left(U_{t}, U_{t}\right)\right)^{*}\left[g\left(U_{t}\right)\right] U_{t}^{\top}-X_{t}\left(I_{d}-\mathcal{A}^{*}\left(\lambda\left(U_{t}, U_{t}\right)\right)\right) \\
& \quad-\gamma X_{t} \mathcal{A}^{*}\left(g\left(U_{t}\right)\right)+U_{t}\left(\left(\mathrm{D} \lambda\left(U_{t}, U_{t}\right)\right)^{*}\left[g\left(U_{t}\right)\right]\right)^{\top},
\end{align*}
$$

where the last identity above uses (67) and (68). In the last identity above, we also used the fact that $\left\{A_{i}\right\}_{i=1}^{m}$ are symmetric matrices, see (3). The next technical result establishes that the flow (69) has an analytic singular value decomposition (SVD).

Lemma 27 (AnAlytic SVD). Suppose that the assumptions made in Lemma 18 are fulfilled. Then the flow (69) has the analytic SVD

$$
\begin{equation*}
X_{t} \stackrel{\mathrm{SVD}}{=} V_{t} S_{t} V_{t}^{\top}, \quad \text { if } t \in[0, \tau] \tag{70}
\end{equation*}
$$

where $V_{t} \in \mathbb{R}^{d \times d}$ is an orthonormal basis and the diagonal matrix $S_{t} \in \mathbb{R}^{d \times d}$ contains the singular values of $X_{t}$ in no particular order. Moreover, $V_{t}$ and $S_{t}$ are analytic functions of $t$ on the interval $[0, \tau]$.

Proof. Recall from Lemma 16 that $h_{\gamma}(U)$ is an analytic function of $U$ in the set $\left\{U: \operatorname{dist}\left(U, \mathcal{M}_{b}\right) \leqslant \rho_{0}\right\}$, where $\rho_{0}$ was specified in the beginning of this appendix. By construction, (gradient flow) remains in the $\rho_{0}$-neighborhood of $\mathcal{M}_{b}$ on the interval $[0, \tau]$. That is, $\operatorname{dist}\left(U_{t}, \mathcal{M}_{b}\right) \leqslant \rho_{0}$ for every $t \in[0, \tau]$. It therefore follows from Theorem 1.1 in [44] that $U_{t}$ is an analytic function of $t$ on the interval $[0, \tau]$. Consequently, $X_{t}=U_{t} U_{t}^{\top}$ is also an analytic function of $t$ on the interval $[0, \tau]$, see (68). In view of Theorem 1 in [45], it follows that $X_{t}$ has an analytic SVD, as claimed.

By comparing (68) and (70), we record another simple technical lemma for later use.

Lemma 28 (DECOMPOSITION). Suppose that the assumptions made in Lemma 18 are fulfilled. Then there exist $\left\{R_{t}\right\}_{t} \subset \mathbb{R}^{d \times p}$ such that

$$
\begin{equation*}
U_{t}=V_{t} \sqrt{S_{t}} R_{t}, \quad \text { if } t \in[0, \tau] \tag{71}
\end{equation*}
$$

where the nonzero rows of $R_{t} \in \mathbb{R}^{d \times p}$ are orthonormal.
Proof. Since $V_{t}$ is an orthonormal basis by Lemma 27, we let $U_{t}=V_{t} Q_{t}$ for a matrix $Q_{t} \in \mathbb{R}^{d \times p}$. It follows from (68) and (70) that $Q_{t} Q_{t}^{\top}=S_{t}$. For notational convenience, suppose that only the first $l$ diagonal entries of the diagonal matrix $S_{t}$ are nonzero, for an integer $l \leqslant p$. We let $S_{t, l} \in \mathbb{R}^{l \times l}$
denote the corresponding submatrix of $S_{t}$. We can then write that

$$
Q_{t} Q_{t}^{\top}=\left[\begin{array}{cc}
S_{t, l} & 0_{l \times(d-l)}  \tag{72}\\
0_{(d-l) \times l} & 0_{(d-l) \times(d-l)}
\end{array}\right] .
$$

It follows from (72) that

$$
\begin{equation*}
Q_{t, l} Q_{t, l}^{\top}=S_{t, l}, \quad Q_{t, l^{+}}=0 \tag{73}
\end{equation*}
$$

where $Q_{t, l} \in \mathbb{R}^{l \times p}$ is the row submatrix of $Q_{t}$ that corresponds to its first $l$ rows. Similarly, $Q_{t, l^{+}} \in \mathbb{R}^{(d-l) \times p}$ contains the remaining rows of $Q_{t}$. It follows from (73) that the rows of $Q_{t, l}$ are orthogonal to one another. That is, $Q_{t, l}=\sqrt{S_{t, l}} R_{t, l}$ for a matrix $R_{t, l} \in \mathbb{R}^{l \times p}$ with orthonormal rows. Here, $\sqrt{S_{t, l}}$ is well-defined because the diagonal matrix $S_{t, l}$ contains the positive singular values of $X_{t}$, see Lemma 27. In turn it follows that $Q_{t}=\sqrt{S_{t}} R_{t}$, where we set the remaining row subset of $R_{t} \in \mathbb{R}^{d \times p}$ to zero.

In view of Lemma 27, we take the derivative with respect to $t$ of both sides of (70) to find that

$$
\begin{equation*}
\dot{X}_{t}=\dot{V}_{t} S_{t} V_{t}^{\top}+V_{t} \dot{S}_{t} V_{t}^{\top}+V_{t} S_{t} \dot{V}_{t}^{\top}, \quad \text { if } t \in[0, \tau] \tag{74}
\end{equation*}
$$

By multiplying both sides above by $V_{t}^{\top}$ and $V_{t}$ from left and right, we reach

$$
\begin{equation*}
V_{t}^{\top} \dot{X}_{t} V_{t}=V_{t}^{\top} \dot{V}_{t} S_{t}+\dot{S}_{t}+S_{t} \dot{V}_{t}^{\top} V_{t}, \quad \text { if } t \in[0, \tau] \tag{75}
\end{equation*}
$$

On the right-hand side above, we used the fact that $V_{t}$ is an orthonormal basis by Lemma 27, i.e., $V_{t}^{\top} V_{t}=I_{d}$. Taking the derivative of both sides of the last identity also yields

$$
\begin{equation*}
\dot{V}_{t}^{\top} V_{t}+V_{t}^{\top} \dot{V}_{t}=0, \quad \text { if } t \in[0, \tau] \tag{76}
\end{equation*}
$$

That is, $V_{t}^{\top} \dot{V}_{t}$ is a skew-symmetric matrix. In particular, both $\dot{V}_{t}^{\top} V_{t}$ and $V_{t}^{\top} \dot{V}_{t}$ are hollow matrices, i.e., both matrices have zero diagonal entries. By taking the diagonal part of both sides of (75), we thus arrive at

$$
\begin{equation*}
\dot{s}_{t, i}=v_{t, i}^{\top} \dot{X}_{t} v_{t, i}, \quad \text { if } t \in[0, \tau] \tag{77}
\end{equation*}
$$

where $s_{t, i}$ is the $i^{\text {th }}$ singular value of $X_{t}$ and $v_{t, i} \in \mathbb{R}^{d}$ is the corresponding singular vector. By substituting above the expression for $\dot{X}_{t}$ from (69), we find that

$$
\begin{align*}
\dot{s}_{t, i}=- & 2 s_{t, i} \cdot v_{t, i}^{\top}\left(I_{d}-\mathcal{A}^{*}\left(\lambda\left(U_{t}, U_{t}\right)\right)\right) v_{t, i} \\
& -2 \gamma s_{t, i} \cdot v_{t, i}^{\top} \mathcal{A}^{*}\left(g\left(U_{t}\right)\right) v_{t, i} \\
& +2 \sqrt{s_{t, i}} \cdot v_{t, i}^{\top}\left(\mathrm{D} \lambda\left(U_{t}, U_{t}\right)\right)^{*}\left[g\left(U_{t}\right)\right] \cdot R_{t}^{\top} e_{i} \tag{78}
\end{align*}
$$

for every $t \in[0, \tau]$. Above, we used (69), (71) and (77). Above, we also used multiple times the fact that $\left(s_{t, i}, v_{t, i}\right)$ is by definition a pair of singular value and its corresponding singular vector for $X_{t}$. Also, $e_{i} \in \mathbb{R}^{d}$ in (78) stands for the $i^{\text {th }}$ canonical vector. That is, only the $i^{\text {th }}$ entry of $e_{i}$ is nonzero and that entry equals one.
In view of the evolution of singular values, specified by (78), it is evident that

$$
\begin{equation*}
\operatorname{rank}\left(U_{t}\right)=\operatorname{rank}\left(X_{t}\right) \leqslant \operatorname{rank}\left(X_{0}\right)=\operatorname{rank}\left(U_{0}\right) \tag{79}
\end{equation*}
$$

for every $t \in[0, \tau]$. This completes the proof of Lemma 18. The two identities above follow from (68).

