Probability of Resolution of MUSIC and g-MUSIC: An Asymptotic Approach

David Schenck, Xavier Mestre, and Marius Pesavento

Abstract—In this article, the outlier production mechanism of the conventional Multiple Signal Classification (MUSIC) and the g-MUSIC Direction-of-Arrival (DoA) estimation technique is investigated using tools from Random Matrix Theory (RMT). A general Central Limit Theorem (CLT) is derived that allows to analyze the asymptotic stochastic behavior of eigenvector-based cost functions in the asymptotic regime where the number of snapshots and the number of antennas increase without bound at the same rate. Furthermore, this CLT is used to provide an accurate prediction of the resolution capabilities of the MUSIC and the g-MUSIC DoA estimation method. The finite dimensional distribution of the MUSIC and the g-MUSIC cost function is shown to be asymptotically jointly Gaussian distributed in this asymptotic regime.

Index Terms—MUSIC, g-MUSIC, DoA estimation, central limit theorem, random matrix theory, probability of resolution, performance analysis.

I. INTRODUCTION

Due to the vast variety of use cases, DoA estimation belongs to the most relevant research areas in signal processing. The applications range from radar and sonar to electric surveillance, seismology, astronomy and mobile communications [1]–[4]. Multiple DoA estimation techniques have been proposed in the literature. Among them, subspace-based DoA estimation techniques which are known to provide a good compromise between computational complexity and DoA estimation accuracy. This is mainly because these methods avoid multidimensional searches while providing relatively good performance. One of the most popular examples of subspace-based DoA estimation methods is MUSIC [5], which exploits the orthogonality between signal and noise subspaces by finding the DoAs that achieve the highest orthogonality between the array signature and the noise subspace of the sample covariance matrix. It is well known [6] that the noise space spanned by the sample covariance matrix is not a consistent estimate of the true one when both the sample size and the number of array elements become large but are still comparable in magnitude. Therefore, it is possible to come up with a refined MUSIC algorithm, usually referred to as g-MUSIC [6], which replaces the noise sample eigenvectors by

a consistent estimate of the true noise subspace. This provides a significant improvement in terms of both accuracy and resolution in the low sample size scenario, whereby the number of snapshots and the number of array elements have the same order of magnitude. In any case, both MUSIC and g-MUSIC are subspaced-based algorithms, and as such both suffer from the so-called breakdown or threshold effect. This effect is characterized by a rapid loss of resolution capabilities due to the systematic appearance of outliers in the DoA estimates when either the sample size (in snapshots per antennas) or the Signal-to-Noise Ratio (SNR) falls below a certain threshold. When this occurs, merging signal extrema in the null-spectrum of the DoA estimator leads to the appearance of outliers in the DoA estimates, as in this case at least one of the deepest local minima in the null-spectrum is not associated with a true source. The presence of outliers causes a severe performance breakdown in terms of DoA estimation accuracy which has great practical implications [7]. Moreover, the threshold effect and thus the production of outliers in the DoA estimates is not captured by standard statistical performance bounds such as the Cramer-Rao Bound (CRB) and therefore requires an estimator specific analysis.

The objective of this paper is to analytically characterize the breakdown effect (thus the outlier production mechanism) in the threshold region of MUSIC and g-MUSIC by studying the resolution capabilities of both DoA estimation algorithms. More specifically, we investigate the probability that these algorithms resolve two close sources in the asymptotic regime where both the sample size and the number of array elements tend to infinity at the same rate. Up to now, the literature has mainly focused on the performance characterization of the conventional MUSIC method in terms of both accuracy and resolution probability [8]–[14]. Most of the performance analyses rely on conventional large sample-size asymptotics, where the number of array elements is assumed to be fixed while the number of snapshots grows without bound. This asymptotic regime is not very suitable for characterizing the threshold performance, since loss of resolution occurs e.g. when the number of snapshots is not much larger (or even smaller) than the number of antennas. For this reason, we propose here to analyze this outlier production mechanism under the more appropriate setting where these two quantities are large but comparable in magnitude. This is indeed the setting that was considered in [15] to investigate the consistency of the DoA estimates obtained through MUSIC or g-MUSIC.

We extend the work in [15] and analyze the statistical fluctuations of the MUSIC and g-MUSIC cost functions, which are the key to understanding the outlier production mechanism that leads to loss of resolution. The approach is similar to the

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one followed in [16]–[18] to study the resolution capabilities of the recently introduced Partially Relaxed Deterministic Maximum Likelihood (PR-DML) algorithm. It also shares some ideas with [19], [20], where the resolution probability of the conventional Deterministic Maximum Likelihood (DML) and the Stochastic Maximum Likelihood (SML) method are derived through an asymptotic characterization of stochastic fluctuations of the corresponding multidimensional cost functions. In comparison to the PR-DML, DML and SML cost functions, the MUSIC and g-MUSIC cost functions involve eigenvectors of a random matrix and therefore require a fundamentally new asymptotic analysis. Furthermore, eigenvector-based cost functions are shown to be asymptotically jointly Gaussian distributed and a general CLT is provided that allows to characterize the asymptotic fluctuations of such cost functions. The asymptotic second order behavior is expressed as a double contour integral which is solved for both MUSIC and g-MUSIC. Finally, the derived asymptotic distribution of both cost functions is used to predict the probability of resolution of both subspace-based DoA estimation methods. In comparison to [21], we do not only analyze the asymptotic stochastic behavior of the g-MUSIC cost function but also of the conventional MUSIC cost function. Additionally, the detailed proofs for the second order asymptotic behavior of both cost functions are provided.

The original contributions of this article can be summarized as follows:

- We derive a CLT which states that eigenvector-based cost functions are asymptotically jointly Gaussian distributed for Gaussian distributed observations in the asymptotic regime where both sample size and array dimension go to infinity at the same rate. A general theorem is provided that completely specifies the asymptotic behavior of eigenvector-based cost functions in terms of (i) asymptotic deterministic behavior and (ii) fluctuations around this deterministic equivalent.
- We determine a set of conditions that guarantee that both MUSIC and g-MUSIC cost functions fluctuate around their asymptotic deterministic equivalents in this asymptotic regime.
- We particularize the above results to the MUSIC and g-MUSIC cost functions and derive a closed-form expression for the asymptotic probability of resolution of both DoA estimation methods. These expressions can be used to determine the probability of resolving closely spaced sources for a given array geometry, a sample volume per antenna and a scenario configuration.

The rest of the paper is organized as follows. Section II introduces the signal model that is assumed in the paper, while some important RMT fundamentals are then introduced in Section III. The conventional MUSIC and g-MUSIC DoA estimation techniques are presented in Section IV. The asymptotic stochastic analysis of these two cost functions and the corresponding probability of resolution of the associated methods are given in Section V. Section VI is devoted to the derivation of the asymptotic deterministic behavior of these two cost functions and the characterization of the correspond-ing fluctuations around it. Finally, these theoretical derivations are then validated by numerical experiments in Section VII and Section VIII concludes the paper.

Notation: Matrices are denoted by boldface uppercase letters $\boldsymbol{A}$, vectors are denoted by boldface lowercase letters $\boldsymbol{a}$, and scalars are denoted by regular letters a. Symbols $(\cdot)^{\mathrm{T}}$, $(\cdot)^{\mathrm{H}}$, $(\cdot)^{-1}$ and $(\cdot)^{1/2}$ denote the transpose, Hermitian transpose, inverse and the positive square root of the matrix argument, which is assumed positive semidefinite. The expectation operator is represented by $\mathbb{E}[\cdot]$. The trace operator is denoted by $\mathrm{tr}[\cdot]$, and the residue of a holomorphic complex function $f(\cdot)$ evaluated at b is denoted by $\mathrm{Res}[f(\cdot),\, b]$. $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm, and $\|\cdot\|$ is the spectral norm of the matrix argument.

II. SIGNAL MODEL

Consider a sensor array that is equipped with M sensors and K impinging narrowband signals with DoAs $\boldsymbol{\theta}=[\theta_1,...,\theta_K]^{\mathrm{T}}$ that lie within the field of view Θ of the array. The number of sources K is assumed to be known and smaller than the number of sensors $K<M$. The full-rank steering matrix is given by $\boldsymbol{A}(\boldsymbol{\theta})=[\boldsymbol{a}(\theta_1),...,\boldsymbol{a}(\theta_K)]^{\mathrm{T}}\in\mathbb{C}^{M\times K}$ where $\boldsymbol{a}(\theta_i)\in\mathbb{C}^M$ denotes the steering vector associated to the i-th source, which is assumed to be located at θ_i. Without loss of generality, we will assume that the array steering vector is normalized to have unit norm, that is $\|\boldsymbol{a}(\theta)\|=1$. The received baseband signal $\boldsymbol{y}(n)=[y_1(n),...,y_M(n)]^{\mathrm{T}}\in\mathbb{C}^M$ at time instant n is modeled as:

$$\boldsymbol{y}(n)=\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(n)+\boldsymbol{n}(n) \text{ for } n=1,...,N \tag{1}$$

where $\boldsymbol{s}(n)=[s_1(n),...,s_K(n)]^{\mathrm{T}}\in\mathbb{C}^K$ denotes the transmitted baseband source signal and $\boldsymbol{n}(n)$ represents the sensor noise. Assuming that both signal and noise vectors are statistically independent, zero-mean and circularly symmetric Gaussian vectors, the observation $\boldsymbol{y}(n)$ in (1) can be modeled as a zero-mean circularly symmetric Gaussian vector with covariance matrix $\boldsymbol{R}\in\mathbb{C}^{M\times M}$ given by

$$\boldsymbol{R}=\mathbb{E}\left[\boldsymbol{y}(n)\boldsymbol{y}(n)^{\mathrm{H}}\right]=\boldsymbol{A}\boldsymbol{R}_{\mathrm{s}}\boldsymbol{A}^{\mathrm{H}}+\sigma^2\boldsymbol{I}_M \tag{2}$$

where $\boldsymbol{R}_{\mathrm{s}}=\mathbb{E}[\boldsymbol{s}(n)\boldsymbol{s}(n)^{\mathrm{H}}]\in\mathbb{C}^{K\times K}$ is the covariance matrix of the transmitted source signal $\boldsymbol{s}(n)$ and $\sigma^2\boldsymbol{I}_M$ denotes the noise covariance. Let us consider the eigendecomposition of this covariance matrix, which can be expressed as

$$\boldsymbol{R}=\sum_{m=1}^{\bar{M}}\gamma_m\boldsymbol{E}_m\boldsymbol{E}_m^{\mathrm{H}}=\boldsymbol{E}\begin{bmatrix}\gamma_1\boldsymbol{I}_{K_1} & & \\ & \ddots & \\ & & \gamma_{\bar{M}}\boldsymbol{I}_{K_{\bar{M}}}\end{bmatrix}\boldsymbol{E}^{\mathrm{H}}. \tag{3}$$

Here, $\bar{M}\leq M$ denotes the total number of distinct eigenvalues, which are sorted in ascending order as $\gamma_1<\gamma_2<\cdots<\gamma_{\bar{M}}$, and K_m denotes the multiplicity of γ_m, $m=1,...,\bar{M}$. The eigenvectors associated to γ_m are grouped into an $M\times K_m$ matrix $\boldsymbol{E}_m$ of orthogonal columns that span the corresponding subspace and we let $\boldsymbol{E}=[\boldsymbol{E}_1,...,\boldsymbol{E}_{\bar{M}}]\in\mathbb{C}^{M\times M}$.

We also consider here the sample covariance matrix

$$\hat{\boldsymbol{R}}=\frac{1}{N}\sum_{n=1}^{N}\boldsymbol{y}(n)\boldsymbol{y}(n)^{\mathrm{H}}=\frac{1}{N}\boldsymbol{Y}\boldsymbol{Y}^{\mathrm{H}} \tag{4}$$

which has eigendecomposition given by

$$\hat{\boldsymbol{R}} = \sum_{m=1}^{M} \hat{\lambda}_m \hat{\boldsymbol{e}}_m \hat{\boldsymbol{e}}_m^{\mathrm{H}} = \hat{\boldsymbol{E}} \begin{bmatrix} \hat{\lambda}_1 & & \\ & \ddots & \\ & & \hat{\lambda}_M \end{bmatrix} \hat{\boldsymbol{E}}^{\mathrm{H}} \qquad (5)$$

where now $\hat{\lambda}_1 \leq \hat{\lambda}_2 \leq \cdots \leq \hat{\lambda}_M$ are the sample eigenvalues, $\hat{\boldsymbol{e}}_m$ denotes the eigenvector associated to $\hat{\lambda}_m$, $m=1,...,M$ and $\hat{\boldsymbol{E}} = [\hat{\boldsymbol{e}}_1,...,\hat{\boldsymbol{e}}_M]$. These sample eigenvalues are almost surely different, unless $N < M$, in which case we have a zero sample eigenvalue of multiplicity $M-N$. In this situation, $\hat{\boldsymbol{e}}_m$ for $m=1,...,M-N$, span the subspace associated to the zero sample eigenvalue. In the following section, we provide some interesting properties of the asymptotic behavior of sample eigenvalues and eigenvectors based on RMT results that will be extensively used throughout the paper.

III. RANDOM MATRIX THEORY PRELIMINARIES

Under the above statistical assumptions, the sample covariance matrix $\hat{\boldsymbol{R}}$ in (4) is a consistent estimator of the true one $\boldsymbol{R}$, provided that the number of antennas M is kept fixed while the number of samples grows without bound, $N \to \infty$. More formally, one can easily show that $\|\hat{\boldsymbol{R}} - \boldsymbol{R}\| \to 0$ with probability one under these asymptotic assumptions. As pointed out before, this asymptotic regime is often inappropriate in the sense that both M,N are typically comparable in magnitude. In this paper, we will therefore consider an asymptotic regime where these two quantities tend to infinity at the same rate.

Assumption 1. *The number of samples N is a function of the number of antennas M, that is $N = N(M)$ and $N(M) \to \infty$ as $M \to \infty$ in a way that $M/N(M) \to c$ for some constant $0 < c < \infty$.*

It turns out that under Assumption 1 the sample covariance matrix $\hat{\boldsymbol{R}}$ in (4) is not a consistent estimate of the true one in (2), in the sense that $\|\boldsymbol{R} - \hat{\boldsymbol{R}}\| \nrightarrow 0$. In particular, this shows that the eigenvalues and eigenvectors of $\hat{\boldsymbol{R}}$ do not really converge to the eigenvalues and eigenvectors of $\boldsymbol{R}$ when the dimensions of these matrices increase without bound. However, it is well known that, under some additional assumptions, the empirical eigenvalue distribution of the sample covariance matrix $\hat{\boldsymbol{R}}$ in (4) still shows a deterministic behavior in the large-dimensional regime. In order to formalize this observation, we need to introduce some additional technical assumptions.

Assumption 2. *The observations $\boldsymbol{y}(n)$ in (1), $n=1,...,N$ form a collection of independent circularly symmetric complex Gaussian vectors with zero-mean and covariance matrix $\boldsymbol{R}$, bounded in spectral norm. In particular, the quantities $\bar{M}$, $\gamma_1,...,\gamma_{\bar{M}}$ and $K_1,...,K_{\bar{M}}$ corresponding to the eigendecomposition of $\boldsymbol{R}$ in (3) may vary with M, but $\sup_M \gamma_{\bar{M}} < \infty$.*

We remark that the received signal $\boldsymbol{y}(n)$ for $n=1,...,N$ in (1) and the covariance matrix $\boldsymbol{R}$ in (2) satisfy Assumption 2 by default as long as the SNR is bounded. Under the above technical assumptions, the eigenvalues of the sample covariance matrix $\hat{\boldsymbol{R}}$ in (4) are asymptotically almost surely distributed as a non-random measure with density $q_M(x)$ [22]–[25]. Informally stated, the histogram of the eigenvalues of the sample covariance matrix tends to be shaped around $q_M(x)$

as M,N grow large, with probability one. This deterministic density is therefore the key to understanding the asymptotic behavior of the sample covariance matrix. In particular, one can show that, when $N > M$, the density $q_M(x)$ has compact support consisting of the union of S closed intervals, namely $\mathcal{S} = [x_1^-, x_1^+] \cup \cdots \cup [x_S^-, x_S^+]$ [24], [26], [27]. When $N \leq M$, the same description is valid but with the addition of the zero eigenvalue, i.e. $\{0\}$. The procedure to obtain $\mathcal{S}$ is as follows (see [25, Proposition 1] and also [26]). Consider the following function of the true covariance matrix

$$\Psi(\omega) = \frac{1}{N} \mathrm{tr}\big[\boldsymbol{R}^2 (\boldsymbol{R} - \omega \boldsymbol{I}_M)^{-2} \big]. \qquad (6)$$

The polynomial equation $\Psi(\omega) = 1$ has $2S$ solutions counting multiplicities, which can be denoted as $\{\omega_1^-, \omega_1^+, ..., \omega_S^-, \omega_S^+\}$. We then define $x_s^{\pm} = z(\omega_s^{\pm})$, $s=1,...,S$, where $z(\omega)$ is the transformation

$$z(\omega) = \omega \left(1 - \frac{1}{N} \mathrm{tr}\big[\boldsymbol{R}(\boldsymbol{R} - \omega \boldsymbol{I}_M)^{-1} \big] \right). \qquad (7)$$

Each eigenvalue of $\boldsymbol{R}$ can be univocally associated to one of the S intervals, in the sense that there exists a single interval $[\omega_s^-, \omega_s^+]$ that contains that particular eigenvalue. On the other hand, given a certain covariance matrix $\boldsymbol{R}$, the number of intervals of the support S increases with increasing N. Furthermore, there exists a minimum number of samples per antenna that guarantees that a certain interval $[x_s^-, x_s^+]$ is associated to a single eigenvalue of $\boldsymbol{R}$. In this paper, we will strongly rely on the assumption that the lowest eigenvalue of $\boldsymbol{R}$ is the only eigenvalue that belongs to the interval $[\omega_1^-, \omega_1^+]$ (see Assumption 3 below). This will allow us to analyze the behavior of subspace DoA detection techniques in large dimensional arrays.

Having reviewed some basic notions on the asymptotic spectral behavior of the sample covariance matrix, we are now in the position of introducing the MUSIC and g-MUSIC subspace DoA estimators in the large antenna regime.

IV. MUSIC AND G-MUSIC DIRECTION-OF-ARRIVAL ESTIMATION

The main idea behind the conventional MUSIC estimator is to exploit the fact that the eigenvectors associated to the noise subspace $\boldsymbol{E}_1$ of the true covariance matrix $\boldsymbol{R}$ in (3) are orthogonal to the steering vectors evaluated at the true DoAs of the received signals. Hence, we can consider a cost function

$$\bar{\eta}_{\mathrm{g}}(\theta) = \boldsymbol{a}(\theta)^{\mathrm{H}} \boldsymbol{E}_1 \boldsymbol{E}_1^{\mathrm{H}} \boldsymbol{a}(\theta). \qquad (8)$$

where we recall from (3) that $\boldsymbol{E}_1$ contains the $M-K$ eigenvectors associated to the smallest eigenvalue of $\boldsymbol{R}$. The DoAs of the received signals can be determined as the K distinct values of θ at which $\bar{\eta}_{\mathrm{g}}(\theta) = 0$ [6]. Since the noise subspace $\boldsymbol{E}_1$ is unknown in practice, the conventional MUSIC cost function is obtained by replacing the noise subspace $\boldsymbol{E}_1$ in (8) with the noise eigenvectors of the sample covariance matrix, namely

$$\hat{\eta}_{\mathrm{c}}(\theta) = \boldsymbol{a}(\theta)^{\mathrm{H}} \sum_{m=1}^{M-K} \hat{\boldsymbol{e}}_m \hat{\boldsymbol{e}}_m^{\mathrm{H}} \boldsymbol{a}(\theta) \qquad (9)$$

and the DoAs are determined by the K distinct values in θ where $\hat{\eta}_{\mathrm{c}}(\theta)$ in (9) attains its K deepest local minima [5].

Now, as a consequence of the fact that under Assumption 2 $\|\boldsymbol{R}-\hat{\boldsymbol{R}}\|\not\to 0$ one can generally expect that $|\hat{\eta}_{\mathrm{c}}(\theta)-\eta_{\mathrm{g}}(\theta)|\not\to 0$. With the help of RMT tools, one can however find a modification of the MUSIC cost function in (9) that is indeed consistent in this large dimensional regime. This is usually referred to as M,N-consistency, as opposed to the more conventional concept of N-consistency, which assumes a constant M. The modified cost function is usually referred to as g-MUSIC [6] and can be built from a proper combination of signal and noise subspaces, that is

$$\hat{\eta}_{\mathrm{g}}(\theta)=\sum_{m=1}^{M}\phi(m)\boldsymbol{a}(\theta)^{\mathrm{H}}\hat{\boldsymbol{e}}_m\hat{\boldsymbol{e}}_m^{\mathrm{H}}\boldsymbol{a}(\theta) \tag{10}$$

with real-valued weights

$$\phi(m)=\begin{cases}1+\sum_{k=M-K+1}^{M}\left(\frac{\hat{\lambda}_k}{\hat{\lambda}_m-\hat{\lambda}_k}-\frac{\hat{\mu}_k}{\hat{\lambda}_m-\hat{\mu}_k}\right), & m\leq M-K\\ -\sum_{k=1}^{M-K}\left(\frac{\hat{\lambda}_k}{\hat{\lambda}_m-\hat{\lambda}_k}-\frac{\hat{\mu}_k}{\hat{\lambda}_m-\hat{\mu}_k}\right), & m>M-K\end{cases}$$

where $\hat{\mu}_1\leq\hat{\mu}_2\leq\cdots\leq\hat{\mu}_M$ are the real-valued solutions (counting multiplicities) to the following equation in $\hat{\mu}$

$$\frac{1}{N}\sum_{k=1}^{M}\frac{\hat{\lambda}_k}{\hat{\lambda}_k-\hat{\mu}}=1.$$

The DoAs of the g-MUSIC estimator are determined by searching for the K deepest local minima of the cost function $\hat{\eta}_{\mathrm{g}}(\theta)$ in (10). In order to justify the superiority of this DoA estimation algorithm, we need to impose separation between the signal and noise subspaces in the asymptotic covariance $\boldsymbol{R}$. This is more formally stated in the following assumption.

Assumption 3. *We have $0<\inf_M K_1/M\leq\sup_M K_1/M<1$ and $\inf_M\gamma_1>0$. The eigenvalue γ_1 is the unique eigenvalue that is associated to the cluster with support $[x_1^-,x_1^+]$ for all M,N sufficiently large. Furthermore, there exists a deterministic ϱ and some small $\epsilon>0$, both independent of M, such that $\sup_M x_1^+ +\epsilon<\varrho<\inf_M x_2^- -\epsilon$.*

The first part of Assumption 3 guarantees that the noise subspace $\boldsymbol{E}_1$ does not vanish in the large dimensional regime when the number of antennas grows to infinity. Hence, the number of sources K is allowed to increase with M but in a way that $\limsup_M K/M<1$. Furthermore, the noise power $(\gamma_1=\sigma^2)$ can also vary with the number of antennas, as long as it does not vanish with $M\to\infty$. The second part of the assumption ensures that the eigenvalue cluster associated with the noise eigenvalue γ_1, denoted by $[x_1^-,x_1^+]$ is separated from the clusters of adjacent eigenvalues in the asymptotic eigenvalue distribution of the sample covariance matrix $\hat{\boldsymbol{R}}$. The existence of separation in the asymptotic eigenvalue distribution between the cluster associated to the smallest eigenvalue γ_1 and the clusters of adjacent eigenvalues depends on the DoAs, the SNR, as well as the number of snapshots N and the number of sensors M and can be verified using the procedure in [25, Section II] or [24, Section II-A]. With the aid of this separability assumption, we are now ready to describe the asymptotic behavior of both subspace-based cost functions.

Theorem 1. *Under Assumptions[1] 1-3 and for each $\theta\in\Theta$*

$$|\hat{\eta}_{\mathrm{c}}(\theta)-\bar{\eta}_{\mathrm{c}}(\theta)|\to 0 \tag{11}$$

$$|\hat{\eta}_{\mathrm{g}}(\theta)-\bar{\eta}_{\mathrm{g}}(\theta)|\to 0 \tag{12}$$

almost surely, where $\bar{\eta}_{\mathrm{c}}(\theta)$ and $\bar{\eta}_{\mathrm{g}}(\theta)$ are two deterministic equivalent objective functions defined as follows. The deterministic equivalent of the MUSIC cost function is defined as

$$\bar{\eta}_{\mathrm{c}}(\theta)=\sum_{m=1}^{\bar{M}}\psi(m)\boldsymbol{a}(\theta)^{\mathrm{H}}\boldsymbol{E}_m\boldsymbol{E}_m^{\mathrm{H}}\boldsymbol{a}(\theta) \tag{13}$$

with real-valued weights

$$\psi(m)=\begin{cases}1-\frac{1}{K_1}\sum_{r=2}^{\bar{M}}K_r\left(\frac{\gamma_1}{\gamma_r-\gamma_1}-\frac{\mu_1}{\gamma_r-\mu_1}\right), & m=1\\ \frac{\gamma_1}{\gamma_m-\gamma_1}-\frac{\mu_1}{\gamma_m-\mu_1}, & m\neq 1\end{cases}$$

where $\mu_1<\mu_2<\cdots<\mu_{\bar{M}}$ are the real-valued solutions to the following equation in μ

$$\frac{1}{N}\sum_{r=1}^{\bar{M}}\frac{K_r\gamma_r}{\gamma_r-\mu}=1. \tag{14}$$

The deterministic equivalent $\bar{\eta}_{\mathrm{g}}(\theta)$ of the g-MUSIC cost function $\hat{\eta}_{\mathrm{g}}(\theta)$ in (10) is defined in (8).

Proof. See [24, Theorem 2]. A sketch of the proof is provided in Section VI. ∎

Remark: The above theorem points out that only the g-MUSIC cost function is an M,N-consistent estimator of the originally intended cost function in (8). However, this does not need to have a direct translation into the consistency of the DoA estimates themselves. It was shown in [15] that for widely spaced sources and a Uniform Linear Array (ULA), both the conventional MUSIC and the g-MUSIC DoA estimators provide M,N-consistent DoA estimates in spite of the inherent inconsistency of the conventional MUSIC cost function. Hence, under certain circumstances, the position of the local minima of the conventional MUSIC cost function converge to the true DoAs, although the global cost function does not. Also, for closely spaced sources, both MUSIC methods provide M,N-consistent DoA estimates. However the g-MUSIC method provides M,N-consistent DoA estimates under lower asymptotic conditions on N, which explains the superiority in DoA estimation accuracy of g-MUSIC over conventional MUSIC in scenarios with closely spaced sources and limited sample size.

In order to analyze the probability of resolution of both MUSIC methods, we next focus on the characterization of the asymptotic fluctuations of both cost functions around the asymptotic deterministic equivalents in Theorem 1.

V. Main Result: Asymptotic Fluctuations of the MUSIC and g-MUSIC Cost Function

The objective of this section is to characterize the asymptotic fluctuations of both conventional MUSIC and g-MUSIC

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[1] Gaussianity is not necessary for this result, and can be replaced by a milder condition on the fourth order moments of the observations.

cost functions in (9) and (10) around their asymptotic deterministic equivalents. To that effect, we will describe the finite-dimensional asymptotic distribution of these cost functions evaluated at a constant number of L points. Let us therefore consider a fixed set of L directions given by $\bar{\boldsymbol{\theta}}=[\bar{\theta}_1,...,\bar{\theta}_L]^{\mathrm{T}}$ within the field of view Θ of the sensor array and denote

$$\hat{\boldsymbol{\eta}}_{\mathrm{c}}(\bar{\boldsymbol{\theta}})=[\hat{\eta}_{\mathrm{c}}(\bar{\theta}_1),...,\hat{\eta}_{\mathrm{c}}(\bar{\theta}_L)]^{\mathrm{T}} \quad (15)$$

$$\hat{\boldsymbol{\eta}}_{\mathrm{g}}(\bar{\boldsymbol{\theta}})=[\hat{\eta}_{\mathrm{g}}(\bar{\theta}_1),...,\hat{\eta}_{\mathrm{g}}(\bar{\theta}_L)]^{\mathrm{T}}, \quad (16)$$

where $\hat{\eta}_{\mathrm{c}}(\theta)$ and $\hat{\eta}_{\mathrm{g}}(\theta)$ are given in (9) and (10), respectively. In order to investigate the asymptotic behavior of the two vectors $\hat{\boldsymbol{\eta}}_{\mathrm{c}}(\bar{\boldsymbol{\theta}})$ and $\hat{\boldsymbol{\eta}}_{\mathrm{g}}(\bar{\boldsymbol{\theta}})$ we will also consider the two L-dimensional vectors $\bar{\boldsymbol{\eta}}_{\mathrm{c}}(\bar{\boldsymbol{\theta}})$ and $\bar{\boldsymbol{\eta}}_{\mathrm{g}}(\bar{\boldsymbol{\theta}})$ that contain the corresponding deterministic equivalents. These two vectors are respectively defined as (15)-(16) by replacing $\hat{\eta}_{\mathrm{c}}(\bar{\theta}_l)$ and $\hat{\eta}_{\mathrm{g}}(\bar{\theta}_l)$ with $\bar{\eta}_{\mathrm{c}}(\bar{\theta}_l)$ in (13) and $\bar{\eta}_{\mathrm{g}}(\bar{\theta}_l)$ in (8), where $l=1,...,L$.

In order to introduce the main result of this section, we need to introduce the asymptotic covariance matrices of the vectors in (15)-(16). Let $\omega(z)$ be defined by inverting the mapping in (7) as follows. When $z\in\mathbb{C}^+\doteq\{z\in\mathbb{C}:\mathrm{Im}[z]>0\}$, $\omega(z)$ is defined as the unique solution to

$$z=\omega(z)\left(1-\frac{1}{N}\sum_{r=1}^{\bar{M}}\frac{K_r\gamma_r}{\gamma_r-\omega(z)}\right) \quad (17)$$

in $\mathbb{C}^+$. When $z^*\in\mathbb{C}^+$, we take $\omega(z)=\omega^*(z^*)$. Finally, when $z\in\mathbb{R}$, we take $\omega(z)$ to be the unique solution to the above equation such that $\frac{1}{N}\sum_{r=1}^{\bar{M}}\frac{K_r\gamma_r^2}{|\gamma_r-\omega(z)|^2}\leq1$. It can be shown that $\omega(z)$ is well defined on all $\mathbb{C}$ and holomorphic on $\mathbb{C}\backslash\mathcal{S}$, with derivative

$$\omega'(z)=\frac{\partial\omega(z)}{\partial z}=\left(1-\frac{1}{N}\sum_{r=1}^{\bar{M}}\frac{K_r\gamma_r^2}{(\gamma_r-\omega(z))^2}\right)^{-1}. \quad (18)$$

Using tools from RMT it is shown in Section VI that both the conventional MUSIC and the g-MUSIC cost functions fluctuate around their asymptotic equivalents as Gaussian random vectors. In order to formulate this result, we now define the asymptotic covariance matrices of the corresponding random vectors in (15)-(16) after proper centering and normalization. Regarding the conventional MUSIC algorithm, we define

$$[\boldsymbol{\Gamma}_{\mathrm{c}}(\bar{\boldsymbol{\theta}})]_{p,q}=\sum_{r=1}^{\bar{M}}\sum_{k=1}^{\bar{M}}\Big(\xi_{\mathrm{c}}(r,k)\boldsymbol{a}(\bar{\theta}_p)^{\mathrm{H}}\boldsymbol{E}_r\boldsymbol{E}_r^{\mathrm{H}}\boldsymbol{a}(\bar{\theta}_q) \\ \times\boldsymbol{a}(\bar{\theta}_q)^{\mathrm{H}}\boldsymbol{E}_k\boldsymbol{E}_k^{\mathrm{H}}\boldsymbol{a}(\bar{\theta}_p)\Big) \quad (19)$$

for $p,q\in\{1,...,L\}$ where the real-valued weights

$$\xi_{\mathrm{c}}(r,k)=\begin{cases}\gamma_r\gamma_k\tilde{\xi}_{\mathrm{c}}(r,k) & M>N \\ \gamma_r\gamma_k\bar{\xi}_{\mathrm{c}}(r,k) & M\leq N\end{cases} \quad (20)$$

differ depending on the number of snapshots N and the number of sensors M. In the oversampled case ($M\leq N$) the weights are given by $\bar{\xi}_{\mathrm{c}}(r,k)$ in (21) whereas in the undersampled case ($M>N$) the weights are given by $\tilde{\xi}_{\mathrm{c}}(r,k)$ in (22), both at the top of the next page. The real-valued quantities μ_r for $r=1,...,\bar{M}$ are defined as in (14). We remark that $\mu_1=0$ if $M=N$.

Regarding the g-MUSIC cost function, we define

$$[\boldsymbol{\Gamma}_{\mathrm{g}}(\bar{\boldsymbol{\theta}})]_{p,q}=\sum_{r=1}^{\bar{M}}\sum_{k=1}^{\bar{M}}\Big(\xi_{\mathrm{g}}(r,k)\boldsymbol{a}(\bar{\theta}_p)^{\mathrm{H}}\boldsymbol{E}_r\boldsymbol{E}_r^{\mathrm{H}}\boldsymbol{a}(\bar{\theta}_q) \\ \times\boldsymbol{a}(\bar{\theta}_q)^{\mathrm{H}}\boldsymbol{E}_k\boldsymbol{E}_k^{\mathrm{H}}\boldsymbol{a}(\bar{\theta}_p)\Big) \quad (23)$$

for $p,q\in\{1,...,L\}$ where the real-valued weights $\xi_{\mathrm{g}}(r,k)$ are given by

$$\xi_{\mathrm{g}}(r,k)=-\frac{N}{K_1}\delta_{r=k=1}+\frac{2}{\pi}\int_{x_1^-}^{x_1^+}\frac{\gamma_r\gamma_k|\omega'(x)|^2\mathrm{Im}[\omega(x)]}{|\gamma_r-\omega(x)|^2|\gamma_k-\omega(x)|^2}\mathrm{d}x. \quad (24)$$

Having introduced these two asymptotic covariance matrices, we next introduce an additional assumption that essentially guarantees that the eigenvalues of the matrices $\boldsymbol{\Gamma}_{\mathrm{c}}(\bar{\boldsymbol{\theta}})$ in (19) and $\boldsymbol{\Gamma}_{\mathrm{g}}(\bar{\boldsymbol{\theta}})$ in (23) are contained in a compact interval of the positive real axis independent of M. This is necessary in order to guarantee that the two random cost functions asymptotically fluctuate around the corresponding deterministic equivalents.

Assumption 4. *Let $\boldsymbol{A}(\bar{\boldsymbol{\theta}})$ denote the $M\times L$ matrix that contains, stacked side by side, the steering vectors $\boldsymbol{a}(\theta)$ evaluated at the directions $\bar{\theta}_1,...,\bar{\theta}_L$. Then, if $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a matrix, we have*

$$\inf_M\sum_{m=2}^{\bar{M}}\lambda_{\min}^2\Big(\boldsymbol{A}^{\mathrm{H}}(\bar{\boldsymbol{\theta}})\boldsymbol{E}_m\boldsymbol{E}_m^{\mathrm{H}}\boldsymbol{A}(\bar{\boldsymbol{\theta}})\Big)>0.$$

This assumption is essentially pointing out that the matrix $\boldsymbol{A}(\bar{\boldsymbol{\theta}})$ must be full-rank, and its projection onto the different signal-subspaces $\boldsymbol{E}_2,...,\boldsymbol{E}_{\bar{M}}$ of the true covariance matrix $\boldsymbol{R}$ cannot vanish uniformly. Having introduced this last assumption, we are now in the position to introduce the main result of this paper.

Theorem 2. *Under Assumptions 1-4, we have*

$$\sqrt{N}\boldsymbol{\Gamma}_{\mathrm{c}}(\bar{\boldsymbol{\theta}})^{-1/2}\big(\hat{\boldsymbol{\eta}}_{\mathrm{c}}(\bar{\boldsymbol{\theta}})-\bar{\boldsymbol{\eta}}_{\mathrm{c}}(\bar{\boldsymbol{\theta}})\big)\overset{\mathcal{D}}{\to}\mathcal{N}(\boldsymbol{0},\boldsymbol{I}_L) \quad (25)$$

$$\sqrt{N}\boldsymbol{\Gamma}_{\mathrm{g}}(\bar{\boldsymbol{\theta}})^{-1/2}\big(\hat{\boldsymbol{\eta}}_{\mathrm{g}}(\bar{\boldsymbol{\theta}})-\bar{\boldsymbol{\eta}}_{\mathrm{g}}(\bar{\boldsymbol{\theta}})\big)\overset{\mathcal{D}}{\to}\mathcal{N}(\boldsymbol{0},\boldsymbol{I}_L) \quad (26)$$

where $\overset{\mathcal{D}}{\to}$ denotes convergence in distribution and where $\boldsymbol{\Gamma}_{\mathrm{c}}(\bar{\boldsymbol{\theta}})\in\mathbb{R}^{L\times L}$ is given in (19) and $\boldsymbol{\Gamma}_{\mathrm{g}}(\bar{\boldsymbol{\theta}})\in\mathbb{R}^{L\times L}$ is given in (23), respectively.

Proof. The proof is given in Section VI. ∎

Remark: The order of convergence in (25) and (26) is $\mathcal{O}(N^{-1/2})$. The real-valued integrals in (20) and (24) can be computed using numerical integration techniques such as the Riemann sum or the Simpson's rule.

Theorem 2 fully characterizes the asymptotic stochastic behavior of the MUSIC as well as the g-MUSIC cost function at any point within the Field of View (FoV) of the sensor array. With the aid of Theorem 2 we can approximate the probability density function of both MUSIC cost functions also under non-asymptotic conditions, which can be useful, e.g., for source detection or to quantify the resolution capabilities of the corresponding DoA estimator.

A. Probability of Resolution

The practical relevance of a DoA estimator highly depends on its computational complexity as well as its estimation accuracy. For the estimation accuracy of subspace-based DoA estimators like MUSIC and g-MUSIC the so called threshold effect is of major interest. The threshold effect describes an abrupt increase in the Mean Square Error (MSE) below a

$$\bar{\xi}_{\rm c}(r,k)=-\frac{N}{K_1}\frac{1}{\left(\frac{1}{N}K_1\gamma_1\right)^2}\left(1-\frac{1}{N}\sum_{m=2}^{\bar{M}}\frac{K_m\gamma_m}{\gamma_m-\gamma_1}\right)^2\delta_{r=k=1}-\frac{1}{\left(\frac{1}{N}K_1\gamma_1\right)^2}\left(1-\frac{1}{N}\sum_{m=2}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\gamma_1)^2}\right)\delta_{r=k=1}$$
$$-\frac{N}{K_1}\frac{1}{(\gamma_k-\gamma_1)^2}\delta_{r=1\neq k}-\frac{N}{K_1}\frac{1}{(\gamma_r-\gamma_1)^2}\delta_{k=1\neq r}+\frac{2}{\pi}\int_{x_1^-}^{x_1^+}\frac{1}{\left|1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m}{\gamma_m-\omega(x)}\right|^2}\frac{{\rm Im}[\omega(x)]}{|\gamma_r-\omega(x)|^2|\gamma_k-\omega(x)|^2}{\rm d}x$$
$$+2\frac{\mu_1}{\gamma_1}\frac{1}{\left(\frac{1}{N}K_1\gamma_1\right)^2}\left(1-\frac{1}{N}\sum_{m=2}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\gamma_1)(\gamma_m-\mu_1)}\right)\delta_{r=k=1}+2\frac{\mu_1}{\gamma_1-\mu_1}\frac{1}{\left(\frac{1}{N}K_1\gamma_1\right)^2}\left(1-\frac{1}{N}\sum_{m=2}^{\bar{M}}\frac{K_m\gamma_m}{\gamma_m-\gamma_1}\right)\delta_{r=k=1}$$
$$+2\frac{N}{K_1}\frac{1}{(\gamma_k-\gamma_1)\gamma_1}\frac{\mu_1}{\gamma_k-\mu_1}\delta_{r=1\neq k}+2\frac{N}{K_1}\frac{1}{(\gamma_r-\gamma_1)\gamma_1}\frac{\mu_1}{\gamma_r-\mu_1}\delta_{k=1\neq r}+\frac{\mu_1^2}{(\gamma_r-\mu_1)^2(\gamma_k-\mu_1)^2}\frac{1}{1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\mu_1)^2}}\tag{21}$$

$$\tilde{\xi}_{\rm c}(r,k)=\bar{\xi}_{\rm c}(r,k)-\frac{1}{\gamma_r\gamma_k}\frac{\mu_1}{(\gamma_r-\mu_1)(\gamma_k-\mu_1)}\frac{1}{\frac{1}{N}\sum_{n=1}^{\bar{M}}\frac{K_n}{\gamma_n-\mu_1}}-\frac{1}{\gamma_r\gamma_k}\frac{\mu_1^2}{(\gamma_r-\mu_1)(\gamma_k-\mu_1)}\frac{N}{N-M}\tag{22}$$

certain SNR threshold due to the production of outliers in the DoA estimates and can be characterized by analyzing the resolution capabilities of the underlying DoA estimator [2], [7], [28]. In the literature, several criteria to declare resolution between two neighboring sources have been introduced [29]–[32]. In case of a one-dimensional spectral search based DoA estimator, one may declare that two sources located at DoAs $\boldsymbol{\theta}=[\theta_1,\theta_2]^{\rm T}$ are resolved if both DoA estimation errors, namely $|\theta_1-\hat{\theta}_1|$ and $|\theta_2-\hat{\theta}_2|$ are smaller than $|\theta_1-\theta_2|/2$ [31]. A popular alternative is to evaluate the cost function at both true DoAs and to verify if the cost at the mid-angle $(\theta_1+\theta_2)/2$ is larger than at the true DoAs [32]. Hence, under the latter criterion resolution is declared if $\boldsymbol{u}^{\rm T}\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})<0$ where

$$\boldsymbol{u}=\begin{bmatrix}1/2\\1/2\\-1\end{bmatrix},\quad\bar{\boldsymbol{\theta}}=\begin{bmatrix}\theta_1\\\theta_2\\\frac{\theta_1+\theta_2}{2}\end{bmatrix},\quad\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})=\begin{bmatrix}\hat{\eta}(\theta_1)\\\hat{\eta}(\theta_2)\\\hat{\eta}\left(\frac{\theta_1+\theta_2}{2}\right)\end{bmatrix}$$

and $\hat{\eta}(\theta)$ denotes a generic cost function. Using the previously derived asymptotic stochastic behavior of the cost function vector $\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})$ in Theorem 2, the probability of resolution can be expressed as the cumulative distribution function [33]

$$P_{\rm res}={\rm Pr}\left(\boldsymbol{u}^{\rm T}\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})<0\right)=\int_{-\infty}^0 f_{\boldsymbol{u}^{\rm T}\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})}(x){\rm d}x\tag{27}$$

where $f_{\boldsymbol{u}^{\rm T}\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})}(x)$ denotes the pdf of the test quantity $\boldsymbol{u}^{\rm T}\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})$ that can be asymptotically approximated by a Gaussian distribution with law $\mathcal{N}(\boldsymbol{u}^{\rm T}\bar{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}}),N^{-1}\boldsymbol{u}^{\rm T}\boldsymbol{\Gamma}(\bar{\boldsymbol{\theta}})\boldsymbol{u})$, where $\bar{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})$ and $\boldsymbol{\Gamma}(\bar{\boldsymbol{\theta}})$ take the form in (13)-(19) or in (8)-(23) depending on the subspace method under evaluation [34].

VI. PROOF OF THEOREM 2

We begin by reviewing some standard arguments that allow to represent the MUSIC and g-MUSIC cost functions as contour integrals of the resolvent of the sample covariance matrix, which is a matrix-valued function of a complex variable $z\in\mathbb{C}\backslash\mathbb{R}$ defined as $\hat{\boldsymbol{Q}}(z)=(\hat{\boldsymbol{R}}-z\boldsymbol{I}_M)^{-1}$ [24], [25]. Observe that, using the eigenvalue decomposition of $\hat{\boldsymbol{R}}$ in (5) we are able to write

$$\hat{\boldsymbol{Q}}(z)=\sum_{m=1}^M\frac{1}{\hat{\lambda}_m-z}\hat{\boldsymbol{e}}_m\hat{\boldsymbol{e}}_m^{\rm H}$$

and therefore a direct application of the Cauchy integral theorem shows that

$$\sum_{m=1}^{M-K}\hat{\boldsymbol{e}}_m\hat{\boldsymbol{e}}_m^{\rm H}=\frac{1}{2\pi{\rm j}}\oint_{\mathcal{C}_z}\hat{\boldsymbol{Q}}(z){\rm d}z\tag{28}$$

where $\mathcal{C}_z$ is a clockwise oriented simple closed contour that encloses only the $K_1=M-K$ smallest eigenvalues of the sample covariance matrix $\hat{\boldsymbol{R}}$.

Now, it is well known that for all M,N sufficiently large, the $M-K$ smallest eigenvalues of $\hat{\boldsymbol{R}}$ are located inside $[x_1,x_1^+]$ plus $\{0\}$ if $M>N$ [26], [35]. Thanks to this fact, we can always deform the contour in (28) and make it independent of M. This deterministic contour may cross the real axis at the points ϱ (defined in Assumption 3) and any other point in the negative real axis. Consequently, one can investigate the asymptotic behavior of the eigenvectors of the sample covariance matrix by equivalently characterizing the asymptotic behavior of the resolvent $\hat{\boldsymbol{Q}}(z)$.

A. Asymptotic Equivalent of MUSIC and Derivation of g-MUSIC

Under Assumptions 1-2 and for fixed θ and $z\in\mathbb{C}\backslash\mathbb{R}$ we have

$$\left|\boldsymbol{a}^{\rm H}(\theta)\hat{\boldsymbol{Q}}(z)\boldsymbol{a}(\theta)-\frac{\omega(z)}{z}\boldsymbol{a}^{\rm H}(\theta)(\boldsymbol{R}-\omega(z)\boldsymbol{I}_M)^{-1}\boldsymbol{a}(\theta)\right|\to 0\tag{29}$$

almost surely [22], [23], where $\omega(z)$ is defined as in (17). A direct application of the dominated convergence theorem therefore shows that, since

$$\hat{\eta}_{\rm c}(\theta)=\frac{1}{2\pi{\rm j}}\oint_{\mathcal{C}_z}\boldsymbol{a}^{\rm H}(\theta)\hat{\boldsymbol{Q}}(z)\boldsymbol{a}(\theta){\rm d}z$$

we can write

$$\left|\hat{\eta}_{\rm c}(\theta)-\frac{1}{2\pi{\rm j}}\oint_{\mathcal{C}_z}\frac{\omega(z)}{z}\boldsymbol{a}^{\rm H}(\theta)(\boldsymbol{R}-\omega(z)\boldsymbol{I}_M)^{-1}\boldsymbol{a}(\theta){\rm d}z\right|\to 0.$$

The right hand side of the above difference can be shown to coincide with the expression in (13) after applying the change of variables $z=z(\omega)$ as given in (7) and solving the corresponding integral via Cauchy integration [24].

A very similar idea can be used to derive the g-MUSIC cost function. We begin by expressing the cost function that we want to estimate using again the Cauchy integral, that is

$$\bar{\eta}_{\rm g}(\theta)=\frac{1}{2\pi{\rm j}}\oint_{\mathcal{C}_\omega}\boldsymbol{a}^{\rm H}(\theta)(\boldsymbol{R}-\omega\boldsymbol{I}_M)^{-1}\boldsymbol{a}(\theta){\rm d}\omega$$

where now $\mathcal{C}_\omega$ is a clockwise oriented simple closed contour that encloses γ_1 and no other eigenvalue. In particular, it can be shown that [6] $z \mapsto \omega(z)$ in (17) can be used as parametrization of such a contour, in the sense that we can choose $\mathcal{C}_\omega = \omega(\mathcal{C}_z)$ with $\mathcal{C}_z$ as defined above. It can be seen that the generated contour has the properties that we are looking for [6], and consequently we can express

$$\bar{\eta}_{\mathrm{g}}(\theta) = \frac{1}{2\pi\mathrm{j}} \oint_{\mathcal{C}_z} \boldsymbol{a}^{\mathrm{H}}(\theta)(\boldsymbol{R} - \omega(z)\boldsymbol{I}_M)^{-1}\boldsymbol{a}(\theta)\omega'(z)\mathrm{d}z \qquad (30)$$

with $\omega'(z)$ as in (18). Now, observe that the above contour does not depend on M, so that invoking again the dominated convergence theorem we can find an M,N-consistent estimator of $\bar{\eta}_{\mathrm{g}}(\theta)$ by simply replacing the integrand above with a corresponding M,N-consistent estimate. Using (29) we readily see that we only need to find an M,N-consistent estimator of $\omega(z)$ and $\omega'(z)$. An M,N-consistent estimator of $\omega(z)$ can be obtained by using the well-known fact that [22], [23], under Assumptions 1-2 we have

$$\left| \frac{1}{M}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z) \right] - \frac{\omega(z)}{z}\frac{1}{M}\mathrm{tr}\left[(\boldsymbol{R} - \omega(z)\boldsymbol{I}_M)^{-1} \right] \right| \to 0$$

almost surely for fixed $z \in \mathbb{C} \backslash \mathbb{R}$. Using the fact that $\omega(z)$ is a solution to the equation in (17), this can be reformulated as

$$\left| \omega(z) - z\left(1 - \frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{R}}(\hat{\boldsymbol{R}} - z\boldsymbol{I}_M)^{-1} \right] \right)^{-1} \right| \to 0$$

(see [19, Lemma 8]). Therefore, the right hand side of the above difference, which will be denoted as $\hat{\omega}(z)$, is an M,N-consistent estimator of $\omega(z)$. Furthermore, the M,N-consistent estimator of $\omega'(z)$ can be obtained by using the fact that convergence of holomorphic functions imply the convergence of their derivatives, so that

$$\hat{\omega}'(z) = \frac{\partial \hat{\omega}(z)}{\partial z} = \frac{\hat{\omega}(z)}{z} + \frac{\hat{\omega}(z)^2}{z}\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{R}}\left(\hat{\boldsymbol{R}} - z\boldsymbol{I}_M \right)^{-2} \right]$$

is an M,N-consistent estimator of $\omega'(z)$. We can therefore obtain the M,N-consistent estimator of the g-MUSIC cost function by inserting these estimators into the Cauchy integral in (30), that is

$$\hat{\eta}_{\mathrm{g}}(\theta) = \frac{1}{2\pi\mathrm{j}} \oint_{\mathcal{C}_z} \boldsymbol{a}^{\mathrm{H}}(\theta)\hat{\boldsymbol{Q}}(z)\boldsymbol{a}(\theta)\frac{z}{\hat{\omega}(z)}\hat{\omega}'(z)\mathrm{d}z. \qquad (31)$$

The contour integral in (31) is solved in closed-form in [6, Theorem 2] (also see [25, Theorem 3]) using conventional residue calculus, which yields the expression in (10).

B. Asymptotic Fluctuations

Let us now consider again the fluctuations of these two cost functions at a set of L fixed distinct angles, $\bar{\theta}_1,...,\bar{\theta}_L$. Using the previously derived asymptotic equivalents of the MUSIC cost function we can express

$$\sqrt{N}\left(\hat{\eta}_{\mathrm{c}}(\bar{\theta}_l) - \bar{\eta}_{\mathrm{c}}(\bar{\theta}_l) \right) = \frac{1}{2\pi\mathrm{j}} \oint_{\mathcal{C}_z} \sqrt{N}\boldsymbol{a}^{\mathrm{H}}(\bar{\theta}_l)\left(\hat{\boldsymbol{Q}}(z) - \bar{\boldsymbol{Q}}(z) \right)\boldsymbol{a}(\bar{\theta}_l)\mathrm{d}z$$

where we have introduced $\bar{\boldsymbol{Q}}(z) = \frac{\omega(z)}{z}(\boldsymbol{R} - \omega(z)\boldsymbol{I}_M)^{-1}$. Similarly, for the g-MUSIC cost function we have

$$\sqrt{N}\left(\hat{\eta}_{\mathrm{g}}(\bar{\theta}_l) - \bar{\eta}_{\mathrm{g}}(\bar{\theta}_l) \right)$$
$$= \frac{1}{2\pi\mathrm{j}} \oint_{\mathcal{C}_z} \sqrt{N}\boldsymbol{a}^{\mathrm{H}}(\bar{\theta}_l)\left(\frac{z\hat{\omega}'(z)}{\hat{\omega}(z)}\hat{\boldsymbol{Q}}(z) - \frac{z\omega'(z)}{\omega(z)}\bar{\boldsymbol{Q}}(z) \right)\boldsymbol{a}(\bar{\theta}_l)\mathrm{d}z.$$

It can be seen that both expressions take the form

$$\sqrt{N}\left(\hat{\eta}(\bar{\theta}_l) - \bar{\eta}(\bar{\theta}_l) \right)$$
$$= \frac{1}{2\pi\mathrm{j}} \oint_{\mathcal{C}_z} \sqrt{N}\boldsymbol{a}^{\mathrm{H}}(\bar{\theta}_l)\left(\hat{h}(z)\hat{\boldsymbol{Q}}(z) - \bar{h}(z)\bar{\boldsymbol{Q}}(z) \right)\boldsymbol{a}(\bar{\theta}_l)\mathrm{d}z \qquad (32)$$

where $\hat{\eta}(\theta)$ is a generic cost function with deterministic equivalent $\bar{\eta}(\theta)$. In case of the conventional MUSIC cost function we have $\hat{h}(z) = 1$ and $\bar{h}(z) = 1$ whereas in case of the g-MUSIC cost function $\hat{h}(z) = \frac{z\hat{\omega}'(z)}{\hat{\omega}(z)}$ and $\bar{h}(z) = \frac{z\omega'(z)}{\omega(z)}$. It is well known [36], [37] that the statistic in (32) asymptotically fluctuates as a Gaussian random variable with zero-mean and positive variance. However, we are interested in the more general case of the asymptotic joint distribution of a collection of random variables, namely

$$\sqrt{N}\left(\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}}) - \bar{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}}) \right) = \begin{bmatrix} \sqrt{N}\left(\hat{\eta}(\bar{\theta}_1) - \bar{\eta}(\bar{\theta}_1) \right) \\ \vdots \\ \sqrt{N}\left(\hat{\eta}(\bar{\theta}_L) - \bar{\eta}(\bar{\theta}_L) \right) \end{bmatrix} \qquad (33)$$

where $\sqrt{N}(\hat{\eta}(\bar{\theta}_l) - \bar{\eta}(\bar{\theta}_l)$ for $l = 1,...,L$ is defined in (32). The second order asymptotic behavior in Theorem 2 is derived by establishing pointwise convergence of the characteristic function of the statistic $\sqrt{N}\left(\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}}) - \bar{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}}) \right)$ in (33) to the characteristic function of a Gaussian distributed random variable. Moreover, by the Cramér-Wold device [38] it is sufficient to establish that the one-dimensional projection of the statistic in (33), namely

$$\sum_{l=1}^{L} \sqrt{N}w_l\left(\hat{\eta}(\bar{\theta}_l) - \bar{\eta}(\bar{\theta}_l) \right) = \sqrt{N}\boldsymbol{w}^{\mathrm{T}}\left(\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}}) - \bar{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}}) \right) \qquad (34)$$

is asymptotically Gaussian distributed for any collection of real-valued bounded quantities $\boldsymbol{w} = [w_1,...,w_L]^{\mathrm{T}} \in \mathbb{R}^L$ to show that (33) is asymptotically jointly Gaussian distributed. Additionally, by Lévy's continuity Theorem convergence in distribution of a set of random variables can be proven by establishing pointwise convergence of the characteristic functions. Let $\chi(r)$ be defined as $\chi(r) = \exp\left(\mathrm{j}r\sqrt{N}\boldsymbol{w}^{\mathrm{T}}\left(\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}}) - \bar{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}}) \right) \right)$ and let $\mathbb{E}[\chi(r)]$ be the corresponding characteristic function of the one-dimensional projection in (34). The target is to study the asymptotic behavior of the characteristic function $\mathbb{E}[\chi(r)]$ in the asymptotic regime where $M,N \to \infty$ at the same rate by establishing convergence towards the characteristic function of a Gaussian random variable with zero-mean and covariance $\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Gamma}(\bar{\boldsymbol{\theta}})\boldsymbol{w}$, that is $\mathbb{E}[\chi(r)] - \bar{\chi}(r) \to 0$ where

$$\bar{\chi}(r) = \exp\left(-r^2\frac{\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Gamma}(\bar{\boldsymbol{\theta}})\boldsymbol{w}}{2} \right). \qquad (35)$$

In (35), $\boldsymbol{\Gamma}(\bar{\boldsymbol{\theta}}) \in \mathbb{R}^{L \times L}$ characterizes the second order asymptotic behavior of the statistic $\sqrt{N}(\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}}) - \bar{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}}))$ in (33), which is the quantity of interest. In the asymptotic analysis of the characteristic function $\mathbb{E}[\chi(r)]$ of the one-dimensional projection $\sqrt{N}\boldsymbol{w}^{\mathrm{T}}(\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}}) - \bar{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}}))$ we rely on the integration by parts formula [39]–[41] and the Nash-Poincaré inequality [40]–[42]. It can be shown that the second order asymptotic behavior of the statistic in (34) is computed as follows.

Theorem 3. *Consider an $L \times L$ matrix $\boldsymbol{\Gamma}(\bar{\boldsymbol{\theta}})$ the elements of which are given by*

$$\left[\boldsymbol{\Gamma}(\bar{\boldsymbol{\theta}})\right]_{p,q} = \frac{1}{2\pi\mathrm{j}}\frac{1}{2\pi\mathrm{j}} \oint_{\mathcal{C}_{\omega_1}} \oint_{\mathcal{C}_{\omega_2}} \bar{h}(z(\omega_1))\bar{h}(z(\omega_2))\frac{\partial z(\omega_1)}{\partial \omega_1}\frac{\partial z(\omega_2)}{\partial \omega_2}$$
$$\times \frac{\omega_1}{z(\omega_1)}\frac{\omega_2}{z(\omega_2)}\frac{\Upsilon_{p,q}(\omega_1,\omega_2)}{1-\Omega(\omega_1,\omega_2)}\mathrm{d}\omega_2 \mathrm{d}\omega_1 \tag{36}$$

for $p,q \in \{1,...,L\}$, where $\mathcal{C}_{\omega_1}$, $\mathcal{C}_{\omega_2}$ are two clockwise oriented simple closed contours enclosing only the smallest eigenvalue γ_1 of $\boldsymbol{R}$,

$$\Upsilon_{p,q}(\omega_1,\omega_2) = \boldsymbol{a}(\bar{\theta}_p)^{\mathrm{H}}(\boldsymbol{R}-\omega_1\boldsymbol{I}_M)^{-1}\boldsymbol{R}(\boldsymbol{R}-\omega_2\boldsymbol{I}_M)^{-1}\boldsymbol{a}(\bar{\theta}_q)$$
$$\times \boldsymbol{a}(\bar{\theta}_q)^{\mathrm{H}}(\boldsymbol{R}-\omega_2\boldsymbol{I}_M)^{-1}\boldsymbol{R}(\boldsymbol{R}-\omega_1\boldsymbol{I}_M)^{-1}\boldsymbol{a}(\bar{\theta}_p) \tag{37}$$

and where

$$\Omega(\omega_1,\omega_2) = \frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}(\boldsymbol{R}-\omega_1\boldsymbol{I}_M)^{-1}\boldsymbol{R}(\boldsymbol{R}-\omega_2\boldsymbol{I}_M)^{-1}\right]. \tag{38}$$

Function $z(\omega)$ is defined in (7) and its first order derivative can be computed as

$$\frac{\partial z(\omega)}{\partial \omega} = 1 - \frac{1}{N}\sum_{r=1}^{\bar{M}}\frac{K_r\gamma_r^2}{(\gamma_r-\omega)^2}. \tag{39}$$

Let Assumptions 1-3 hold true and assume additionally that the eigenvalues of $\boldsymbol{\Gamma}(\bar{\boldsymbol{\theta}})$ are all located in a compact interval of the positive real axis independent of M. Then, $\sqrt{N}\boldsymbol{\Gamma}(\bar{\boldsymbol{\theta}})^{-1/2}(\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})-\bar{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}}))$ in (33) converges in law to a standardized multivariate Gaussian distribution.

Proof. A detailed proof is provided in [43]. $\blacksquare$

With the help of Theorem 3 above, the proof of Theorem 2 follows directly once we have been able to (i) compute the integral that defines the asymptotic covariance matrix for these two cost functions and (ii) prove that the maximum (resp. minimum) eigenvalue of these two covariance matrices is bounded (resp. bounded away from zero).

Let us first consider the computation of the two asymptotic covariance matrices, which follows from solving the integral in (36) for $\bar{h}(z(\omega))=1$ (conventional MUSIC) and for $\bar{h}(z(\omega))=\frac{z(\omega)}{\omega}\frac{\partial \omega(z)}{\partial z}$ (g-MUSIC). To simplify the computation of the asymptotic covariance, we express $\boldsymbol{\Gamma}(\bar{\boldsymbol{\theta}})$ in (36) as

$$\boldsymbol{\Gamma}(\bar{\boldsymbol{\theta}}) = \sum_{r=1}^{\bar{M}}\sum_{k=1}^{\bar{M}}\xi(r,k)\boldsymbol{A}^{\mathrm{H}}(\bar{\boldsymbol{\theta}})\boldsymbol{E}_r\boldsymbol{E}_r^{\mathrm{H}}\boldsymbol{A}(\bar{\boldsymbol{\theta}})\odot\left(\boldsymbol{A}^{\mathrm{H}}(\bar{\boldsymbol{\theta}})\boldsymbol{E}_k\boldsymbol{E}_k^{\mathrm{H}}\boldsymbol{A}(\bar{\boldsymbol{\theta}})\right)^{\mathrm{T}} \tag{40}$$

where $\odot$ denotes element-wise product and where the coefficients $\xi(r,k)$ are defined as

$$\xi(r,k) = \frac{1}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{\omega_1}}\bar{h}(z(\omega_1))\frac{\omega_1}{z(\omega_1)}\frac{\partial z(\omega_1)}{\partial \omega_1}\frac{\gamma_r\gamma_k\mathcal{I}_{r,k}(\omega_1)}{(\gamma_r-\omega_1)(\gamma_k-\omega_1)}\mathrm{d}\omega_1, \tag{41}$$

with

$$\mathcal{I}_{r,k}(\omega_1) = \frac{1}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{\omega_2}}\bar{h}(z(\omega_2))\frac{\omega_2}{z(\omega_2)}\frac{\partial z(\omega_2)}{\partial \omega_2}\frac{\frac{1}{(\gamma_r-\omega_2)(\gamma_k-\omega_2)}}{1-\Omega(\omega_1,\omega_2)}\mathrm{d}\omega_2. \tag{42}$$

These two integrals are solved in Appendices A and B for the conventional MUSIC and the g-MUSIC cost functions, leading to the expressions in (19) and (23) respectively.

To obtain an upper bound on the spectral norm of $\boldsymbol{\Gamma}(\bar{\boldsymbol{\theta}})$, we consider the expression in (40) and observe that the

coefficients $\xi(r,k)$ are all real-valued and positive. The fact that they are real-valued follows from the property $z(\omega^*)=z^*(\omega)$ in (7) as well as the fact that $\bar{h}(z^*)=\bar{h}^*(z)$ for both definitions of this function. The fact that the coefficients $\xi(r,k)$ in (41) are non-negative can be shown by using the following results.

Lemma 1. *Let $\Omega(\omega_1,\omega_2)$ be defined as in (38). Then,*

$$\sup_M \sup_{(z_1,z_2)\in\mathcal{C}_z\times\mathcal{C}_z}|\Omega(\omega(z_1),\omega(z_2))| < 1. \tag{43}$$

Proof. By the Cauchy-Schwarz inequality, we have $|\Omega(\omega(z_1),\omega(z_2))|^2 \leq \tilde{\Psi}(\omega(z_1))\tilde{\Psi}(\omega(z_1))$ with

$$\tilde{\Psi}(\omega) = \frac{1}{N}\sum_{m=1}^{\bar{M}}K_m\frac{\gamma_m^2}{|\gamma_m-\omega|^2}. \tag{44}$$

Hence, it is sufficient to prove that $\sup_M\sup_{z\in\mathcal{C}_z}|\tilde{\Psi}(\omega(z))| < 1$. Let us first consider $z \in \mathbb{C}^+$. Taking imaginary parts on both sides of (17) we see that

$$\tilde{\Psi}(\omega(z)) = 1 - \frac{\mathrm{Im}[z]}{\mathrm{Im}[\omega(z)]}$$

where we recall that $\mathrm{Im}[\omega(z)] > 0$ when $\mathrm{Im}[z] > 0$. This implies that $\sup_M\tilde{\Psi}(\omega(z)) \leq 1$ and $\inf_M\mathrm{Im}[\omega(z)] > 0$. In order to see that the inequality $\sup_M\tilde{\Psi}(\omega(z)) \leq 1$ must be strict, we reason by contradiction. Assume that the equality holds, and consider a subsequence M' such that $\tilde{\Psi}(\omega(z)) \to 1$, implying $\mathrm{Im}[\omega(z)] \to \infty$, as $M' \to \infty$. Now, using the fact that $|\gamma_m-\omega(z)|^2 \geq \mathrm{Im}[w(z)]^2$ we see from the definition of $\tilde{\Psi}(\omega(z))$ in (44) that $\tilde{\Psi}(\omega(z)) \leq \mathrm{Im}[\omega(z)]^{-2}\frac{1}{N}\sum_{m=1}^{\bar{M}}K_m\gamma_m^2 \to 0$ along that subsequence, leading to contradiction.

Consider now $z \in \mathcal{C}_z \cap \mathbb{R}$ and observe that this means that either $z < 0$ or $z = \varrho$ in Assumption 3, which are the two crossing points of the contour on the real axis. Assume first that $z = \varrho$ and consider ω_1^+ as right hand side of the first cluster of the support $\mathcal{S}$ and ω_2^- as the left hand side of the second cluster such that $\gamma_1 < \omega_1^+ < \omega(\varrho) < \omega_2^- < \gamma_2$ (the case $z < 0$ follows by similar arguments). It can be observed that $\tilde{\Psi}(\omega)$ in (44) is a strongly convex function since the second order derivative w.r.t. ω

$$\tilde{\Psi}''(\omega) = \frac{6}{N}\sum_{r=1}^{\bar{M}}\frac{K_r\gamma_r^2}{(\gamma_r-\omega)^4} \tag{45}$$

is lower bounded by a positive quantity q, namely

$$q = \inf_M\inf_{\omega\in(\omega_1^+,\omega_2^-)}\tilde{\Psi}''(\omega) > \frac{3}{4}\frac{1}{N}\sum_{r=1}^{\bar{M}}\frac{K_r\gamma_r^2}{\gamma_r^4+\gamma_2^4} > 0.$$

By strong convexity it follows that

$$\tilde{\Psi}(\omega_2^-)-\tilde{\Psi}(\omega(\varrho)) \geq \tilde{\Psi}'(\omega(\varrho))(\omega_2^--\omega(\varrho))+\frac{q}{2}(\omega_2^--\omega(\varrho))^2$$
$$\tilde{\Psi}(\omega_1^+)-\tilde{\Psi}(\omega(\varrho)) \geq -\tilde{\Psi}'(\omega(\varrho))(\omega(\varrho)-\omega_1^+)+\frac{q}{2}(\omega_1^+-\omega(\varrho))^2.$$

Using the fact that $\tilde{\Psi}(\omega_2^-)=1$ and $\tilde{\Psi}(\omega_1^+)=1$ we obtain

$$1-\tilde{\Psi}(\omega(\varrho)) \geq \begin{cases} \tilde{\Psi}'(\omega(\varrho))(\omega_2^--\omega(\varrho))+\frac{q}{2}(\omega_2^--\omega(\varrho))^2 \\ -\tilde{\Psi}'(\omega(\varrho))(\omega(\varrho)-\omega_1^+)+\frac{q}{2}(\omega_1^+-\omega(\varrho))^2. \end{cases}$$

Hence, $\inf_M 1-\tilde{\Psi}(\omega(\varrho)) > \frac{q}{2}\min\left[(\omega_2^--\omega(\varrho))^2,(\omega_1^+-\omega(\varrho))^2\right] > 0$ and therefore $\sup_M\tilde{\Psi}(\omega(\varrho)) < 1$. The boundedness of $\tilde{\Psi}(\omega(z))$ at the second crossing point with the real axis $z < 0$ follows by similar reasoning. $\blacksquare$

The fact that $\Omega(\omega_1,\omega_2)$ is uniformly bounded on $\mathcal{C}_\omega \times \mathcal{C}_\omega$ implies that we can take a power series expansion of the term $(1-\Omega(\omega_1,\omega_2))^{-1}$ and write

$$\xi(r,k)=\gamma_r\gamma_k\sum_{n\geq0}\frac{1}{N^n}\sum_{r_1+\ldots+r_{\bar{M}}=n}\binom{n}{r_1,\ldots,r_{\bar{M}}}\prod_{m=1}^{\bar{M}}K_m^{r_m}\gamma_m^{2r_m}$$
$$\times\left(\frac{1}{2\pi j}\oint_{\mathcal{C}_z}\frac{\omega(z)}{z}\frac{\bar{h}(z)}{(\gamma_r-\omega(z))(\gamma_k-\omega(z))}\prod_{m=1}^{\bar{M}}\frac{1}{(\gamma_m-\omega(z))^{r_m}}dz\right)^2$$
(46)

where we employed the multinomial theorem to factor out the different powers $\Omega^n(\omega_1,\omega_2)$, $n\geq0$. An immediate consequence of the above decomposition is the fact that the coefficients $\xi(r,k)$ are non-negative. This will be useful in order to determine the appropriate lower bound on the corresponding covariance matrix.

Lemma 2. *Under Assumptions 1-3, we have* $\sup_M\|\mathbf{\Gamma}_c(\bar{\boldsymbol{\theta}})\|< +\infty$ *and* $\sup_M\|\mathbf{\Gamma}_g(\bar{\boldsymbol{\theta}})\|<+\infty$.

Proof. Consider the general expression in (40) and assume that we are able to find a positive constant κ such that $\sup_M\sup_{r,k}\xi(r,k)<\kappa$. It will immediately follow that

$$\|\mathbf{\Gamma}(\bar{\boldsymbol{\theta}})\|\leq\kappa\left\|\sum_{r=1}^{\bar{M}}\sum_{k=1}^{\bar{M}}\boldsymbol{A}^{\mathrm{H}}(\bar{\boldsymbol{\theta}})\boldsymbol{E}_r\boldsymbol{E}_r^{\mathrm{H}}\boldsymbol{A}(\bar{\boldsymbol{\theta}})\odot\left(\boldsymbol{A}^{\mathrm{H}}(\bar{\boldsymbol{\theta}})\boldsymbol{E}_k\boldsymbol{E}_k^{\mathrm{H}}\boldsymbol{A}(\bar{\boldsymbol{\theta}})\right)^{\mathrm{T}}\right\|$$
$$=\kappa\left\|\boldsymbol{A}^{\mathrm{H}}(\bar{\boldsymbol{\theta}})\boldsymbol{A}(\bar{\boldsymbol{\theta}})\odot\left(\boldsymbol{A}^{\mathrm{H}}(\bar{\boldsymbol{\theta}})\boldsymbol{A}(\bar{\boldsymbol{\theta}})\right)^{\mathrm{T}}\right\|\leq L\kappa$$

where in the last inequality we have used the fact that the steering vectors have unit norm and the spectral norm of a matrix is upper bounded by its trace. Hence, we only need to find an upper bound on the coefficients. Following the definition of these coefficients in (46) we can readily see that

$$\xi(r,k)\leq\frac{\gamma_r\gamma_k}{1-\bar{\Omega}}\left(\frac{1}{2\pi j}\oint_{\mathcal{C}_z}|\bar{h}(z)|\frac{|\omega(z)|}{|z|}\frac{1}{|\gamma_r-\omega(z)||\gamma_k-\omega(z)|}|dz|\right)^2$$
(47)

where $\bar{\Omega}=\sup_{M,(\omega_1,\omega_2)\in\mathcal{C}_\omega\times\mathcal{C}_\omega}\Omega(\omega_1,\omega_2)<1$. Now, from the proof of Lemma 1 it directly follows that $\inf_{M,\omega\in\mathcal{C}_\omega}|\omega(z)-\gamma_k|> 0$ for $k=1,\ldots,\bar{M}$. On the other hand, taking real and imaginary part on both sides of (17) it can be shown by contradiction that for bounded $|z|<\infty$ we have $\sup_M|\omega(z)/z|<\infty$.

This directly shows that the coefficients $\xi(r,k)$ are bounded for the MUSIC cost function, which has $\bar{h}(z)=1$. In order to prove boundedness of the g-MUSIC covariance $\mathbf{\Gamma}_g(\bar{\boldsymbol{\theta}})$ we only need to show that $\sup_M|\omega'(z)|<\infty$ since $\bar{h}(z)=\frac{z}{\omega(z)}\omega'(z)$ and $\frac{z}{\omega(z)}$ cancels out with $\frac{\omega(z)}{z}$ in (47). The upper bound $\sup_M|\omega'(z)|<\infty$ follows from the fact that, by the triangular inequality, $\sup_M|\omega'(z)|\leq\sup_M(1-|\tilde{\Psi}(\omega(z))|)^{-1}<1$. This completes the proof of Lemma 2. ∎

We finally conclude the proof of Theorem 2 by showing that the smallest eigenvalue of the two asymptotic covariance matrices is bounded away from zero.

Lemma 3. *Under Assumptions 1-4, we have* $\inf_M\lambda_{\min}(\mathbf{\Gamma}_c(\bar{\boldsymbol{\theta}}))>0$ *and* $\inf_M\lambda_{\min}(\mathbf{\Gamma}_g(\bar{\boldsymbol{\theta}}))>0$, *where* $\lambda_{\min}(\cdot)$ *denotes the minimum eigenvalue of a matrix.*

Proof. In order to proof this lemma, we consider again the general expression of any of these two matrices that is given in

(40). By inserting the series expansion of $\xi(r,k)\geq0$ in (46) into (40) we obtain an expression of $\mathbf{\Gamma}(\bar{\boldsymbol{\theta}})$ as a linear combination of positive semidefinite matrices with non-negative coefficients. Hence, the original covariance can be lower bounded (in the ordering of positive semidefinite matrices) by selecting any of the terms of this expansion. In the case of MUSIC we can select the term $n=0$ in (46) along the sum $k=r\geq2$ in (40), leading to the lower bound

$$\mathbf{\Gamma}_c(\bar{\boldsymbol{\theta}})\geq\sum_{k=2}^{\bar{M}}v_c(k)\boldsymbol{A}^{\mathrm{H}}(\bar{\boldsymbol{\theta}})\boldsymbol{E}_k\boldsymbol{E}_k^{\mathrm{H}}\boldsymbol{A}(\bar{\boldsymbol{\theta}})\odot\left(\boldsymbol{A}^{\mathrm{H}}(\bar{\boldsymbol{\theta}})\boldsymbol{E}_k\boldsymbol{E}_k^{\mathrm{H}}\boldsymbol{A}(\bar{\boldsymbol{\theta}})\right)^{\mathrm{T}}$$

where

$$v_c(k)=\left(\frac{1}{2\pi j}\oint_{\mathcal{C}_z^-}\frac{\omega(z)}{z}\frac{\gamma_k}{(\gamma_k-\omega(z))^2}dz\right)^2. \quad(48)$$

Likewise, for the g-MUSIC cost function we can select the term $n=1$ in (46) along the sum $k=r\geq2$ in (40), so that

$$\mathbf{\Gamma}_g(\bar{\boldsymbol{\theta}})\geq\sum_{k=2}^{\bar{M}}v_g(k)\boldsymbol{A}^{\mathrm{H}}(\bar{\boldsymbol{\theta}})\boldsymbol{E}_k\boldsymbol{E}_k^{\mathrm{H}}\boldsymbol{A}(\bar{\boldsymbol{\theta}})\odot\left(\boldsymbol{A}^{\mathrm{H}}(\bar{\boldsymbol{\theta}})\boldsymbol{E}_k\boldsymbol{E}_k^{\mathrm{H}}\boldsymbol{A}(\bar{\boldsymbol{\theta}})\right)^{\mathrm{T}}$$

where now

$$v_g(k)=\frac{1}{N}\sum_{j=1}^{\bar{M}}K_j\gamma_j^2\left(\frac{1}{2\pi j}\oint_{\mathcal{C}_\omega^-}\frac{\omega}{z(\omega)}\frac{\gamma_k}{(\gamma_k-\omega)^2(\gamma_j-\omega)}d\omega\right)^2. \quad(49)$$

Using the fact that, for any two positive semidefinite matrices $\boldsymbol{A},\boldsymbol{B}$, we have $\lambda_{\min}(\boldsymbol{A}\odot\boldsymbol{B})\geq\lambda_{\min}(\boldsymbol{A})\lambda_{\min}(\boldsymbol{B})$ [44], we see that the lemma will follow from Assumption 4 if we are able to show that $\inf_{M,k\geq2}v_c(k)>0$ and $\inf_{M,k\geq2}v_g(k)>0$.

In case of $v_g(k)$ in (49), we have

$$v_g(k)=\frac{K_1}{N}\frac{\gamma_1^2\gamma_k^2}{(\gamma_1-\gamma_k)^4}\delta_{k\geq2}+\frac{1}{N}\sum_{j=2}^{\bar{M}}K_j\frac{\gamma_j^2\gamma_k^2}{(\gamma_1-\gamma_j)^4}\delta_{k=1}$$
$$\geq\frac{K_1}{N}\frac{\gamma_1^4}{(2\|\boldsymbol{R}\|)^4}\delta_{k\geq2}+\frac{M-K_1}{N}\frac{\gamma_1^4}{(2\|\boldsymbol{R}\|)^4}\delta_{k=1}$$

and the lower bound follows from the fact that $\inf_M\gamma_1> 0$, $\sup_M\|\boldsymbol{R}\|<\infty$ and $0<\inf_M K_1/M\leq\sup_M K_1/M<1$. Furthermore, the closed-form solution for $v_c(k)$ in (48) is given by

$$v_c(k)=\gamma_k^2(\Phi_k(\mu_1)-\Phi_k(\gamma_1))^2\delta_{k\geq2}$$
$$+\left(\gamma_1\sum_{m=2}^{\bar{M}}(\Phi_1(\mu_m)-\Phi_1(\gamma_m))\right)^2\delta_{k=1} \quad(50)$$

where we have introduced $\Phi_k(x)=\frac{x}{(\gamma_k-x)^2}$. Consider first the lower bound of $v_c(k)$ in (50) for $k\geq2$ in the oversampled case ($M\leq N$). Since $\Phi_k(x)$ is a convex function on the region $0< x<\gamma_k$, we can upper bound $\Phi_k(\gamma_1)>\Phi_k(\mu_1)+\Phi_k'(\mu_1)(\gamma_1-\mu_1)$ where $\Phi_k'(x)$ is the derivative of $\Phi_k(x)$ w.r.t. x. This shows that

$$(\Phi_k(\gamma_1)-\Phi_k(\mu_1))^2\geq(\Phi_k'(\mu_1))^2(\gamma_1-\mu_1)^2\geq\frac{(\gamma_1-\mu_1)^2}{\gamma_k^4}\geq\frac{(\gamma_1-\mu_1)^2}{\|\boldsymbol{R}\|^4}$$

where we have used the fact that in the oversampled case $\mu_1\geq0$ and therefore $\Phi_k'(\mu_1)\geq\Phi_k'(0)=1/\gamma_k^2$. On the other hand, to see that $\inf_M|\gamma_1-\mu_1|>0$ we simply recall from the definition of μ_1 in (14) that we can write

$$\gamma_1-\mu_1=\left(1-\frac{1}{N}\sum_{j=2}^{\bar{M}}K_j\frac{\gamma_j}{\gamma_j-\mu_1}\right)^{-1}\frac{K_1\gamma_1}{N}>\frac{K_1\gamma_1}{N}$$

where in the last equation we used the fact that $\mu_1 < \gamma_1 < \gamma_j$ for $j = 2,...,\bar{M}$ and therefore the second term in the denominator is always positive. The fact that $\inf_M |\mu_1 - \gamma_1| > 0$ follows the boundedness of the spectral norm of $\boldsymbol{R}$ and Assumption 1. This proves that in the oversampled case $v_c(k)$ in (50) for $k \geq 2$ is lower bounded by a constant independent of M.

Let us now consider the lower bound of $v_c(k)$ in (50) in the undersampled case ($M > N$) for $k \geq 2$. In the undersampled case we know that $\mu_1 < 0 < \gamma_1$ and therefore

$$\phi_k(\gamma_1) - \phi_k(\mu_1) > \frac{\gamma_1}{(\gamma_k - \gamma_1)^2} > \frac{\gamma_1}{(2\|\boldsymbol{R}\|)^2} = \frac{\gamma_1}{4\|\boldsymbol{R}\|^2}$$

where we have used the fact that $-\phi_k(\mu_1) > 0$. Consequently, the lower bound on $v_c(k)$ in (50) for $k \geq 2$ in the undersampled case directly follows from the fact that $\inf_M \gamma_1 > 0$ and $\sup_M \|\boldsymbol{R}\| < \infty$. This concludes the proof of Lemma 3. ∎

VII. SIMULATION RESULTS

In this Section the predicted probability of resolution in (27) is compared to the simulated one. Consider a scenario with $K = 2$ sources that are located at $\boldsymbol{\theta} = [45°, 50°]^T$ and a ULA that is equipped with $M = 15$ sensors. Since we consider only non-asymptotic scenarios with a finite number of sensors, we refrain from normalizing the steering vectors in all simulations. The transmitted signals are zero-mean with unit power and the SNR is given by $SNR = 1/\sigma^2$. The simulations are carried out for correlated signals with correlation coefficient $\rho = 0.95$ as well as uncorrelated signals. The separation boundary is defined as the lowest SNR or the smallest angular separation between both sources that provides separation between the eigenvalue cluster that is associated to the noise eigenvalue γ_1 and remaining eigenvalue clusters. Hence, the separation boundary is given by the smallest SNR or the smallest angular separation between both sources that allows to differentiate between noise and signal subspace and therefore satisfies Assumption 3. All simulations are conducted for 10000 Monte-Carlo trials.

In Figure 1, 2 and 3 the probability of resolution is depicted for different SNRs and for $N = 10$ (undersampled case), $N = 15$ (special case where $M = N$) and $N = 100$ (oversampled case) snapshots, respectively. In all three scenarios our prediction of the probability of resolution in (27) is very accurate since it is very close to the simulated one. Even in case of correlated sources and limited number of snapshots the proposed forecast of the probability of resolution in (27) provides a remarkably accurate description of the threshold effect. Furthermore, in all three scenarios the g-MUSIC DoA estimation method shows superior resolution capabilities than the conventional MUSIC technique.

In Figure 4 the probability of resolution is depicted for different numbers of snapshots N. The SNR is fixed to $SNR = 6dB$ and the source signals are uncorrelated. The proposed analytic expression for the probability of resolution very accurately describes the empirical simulated probability of resolution of MUSIC as well as g-MUSIC. Furthermore, g-MUSIC outperforms conventional MUSIC especially under harsh conditions with limited number of snapshots.
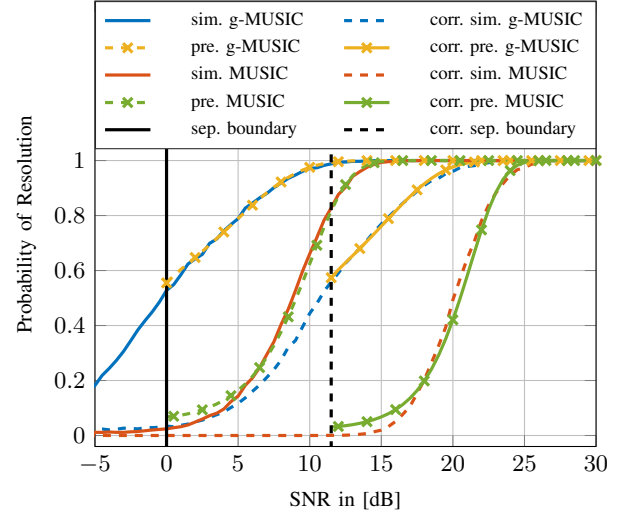

Fig. 1: Uncorrelated and Correlated Sources, Correlation Coefficient $\rho = 0.95$, $M = 15$ Sensors, $N = 10$ Snapshots
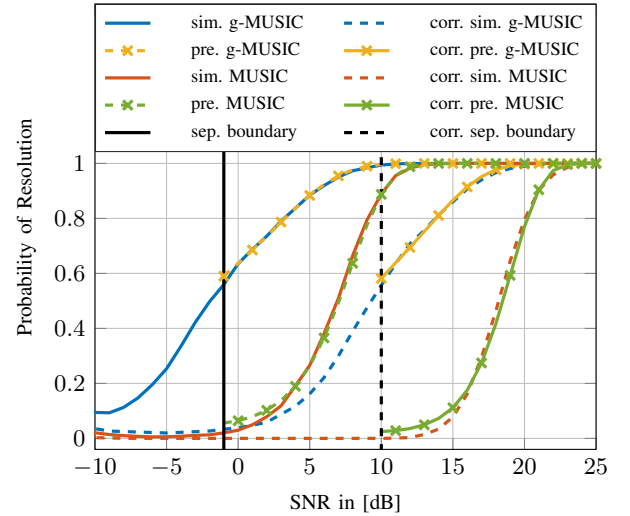

Fig. 2: Uncorrelated and Correlated Sources, Correlation Coefficient $\rho = 0.95$, $M = 15$ Sensors, $N = 15$ Snapshots

In Figure 5 the probability of resolution is depicted for different angular separation $\Delta\vartheta$ between two sources. The SNR is fixed to $SNR = 2dB$ and a total of $N = 15$ snapshots are considered. The transmitted signals are uncorrelated and the two sources are located at $\boldsymbol{\theta} = [45°, 45° + \Delta\vartheta]^T$ with $\Delta\vartheta \in [0.2°, 10°]$. Furthermore, a ULA with $M = 15$ antennas is used. It can be observed that the proposed prediction of the probability of resolution is almost identical to the actual one. Especially in difficult scenarios with low SNR and closely spaced sources g-MUSIC is superior to conventional MUSIC.

VIII. CONCLUSION

In this article the asymptotic stochastic behavior of the conventional MUSIC and the g-MUSIC cost function is investigated in the asymptotic regime where the number of snapshots and the number of sensors go to infinity at the same rate. Using tools from RMT the finite dimensional
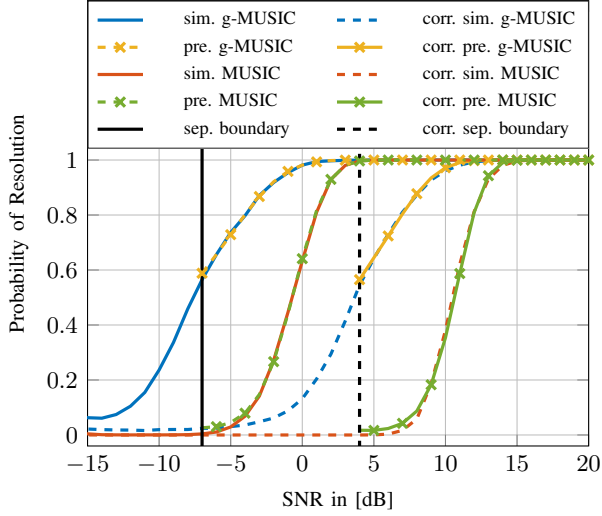
Fig. 3: Uncorrelated and Correlated Sources, Correlation Coefficient $\rho=0.95$, $M=15$ Sensors, $N=100$ Snapshots
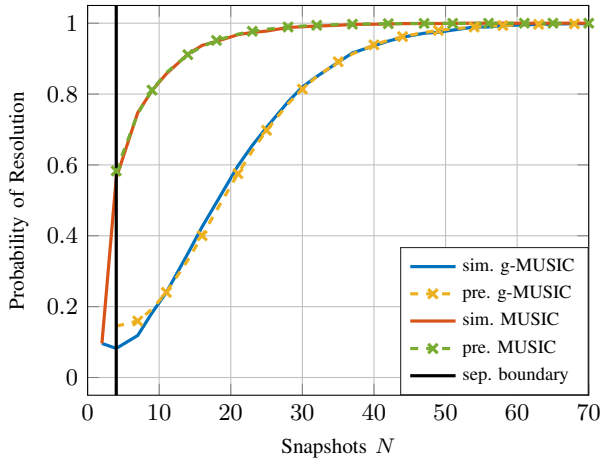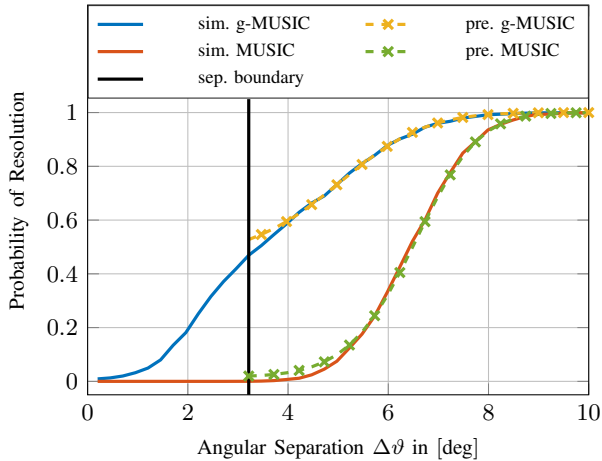

Fig. 4: Uncorrelated Sources, $M=15$ Sensors, SNR=6dB

Fig. 5: Uncorrelated Sources Located at $\boldsymbol{\theta}=[45°,45°+\Delta\vartheta]^{\mathrm{T}}$, $M=15$ Sensors, $N=15$ Snapshots, SNR=2dB

distribution of the random MUSIC and g-MUSIC cost function is derived and shown to be asymptotically jointly Gaussian distributed. Furthermore, the resolution capabilities of both MUSIC DoA estimation methods is analyzed based on the asymptotic stochastic behavior of their cost functions. An analytic expression for the probability of resolution is provided that allows to predict the probability of resolution in the threshold region and thus provides an accurate description of the outlier production mechanism.

APPENDIX A
DETERMINATION OF THE ASYMPTOTIC COVARIANCE OF THE MUSIC COST FUNCTION

In case of the conventional MUSIC cost function $\bar{h}(z(\omega_2))=1$. Hence, the complex contour integral in (42) yields

$$\mathcal{I}_{r,k}(\omega_1)=\frac{1}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{\omega_2}}\frac{\omega_2}{z(\omega_2)}\frac{\partial z(\omega_2)}{\partial\omega_2}\frac{\frac{1}{(\gamma_r-\omega_2)(\gamma_k-\omega_2)}}{1-\Omega(\omega_1,\omega_2)}\mathrm{d}\omega_2 \quad (51)$$

and can be solved in closed-form by applying conventional residue calculus [45]. A closed-form expression for $\mathcal{I}_{r,k}(\omega_1)$ is obtained by summing the residues of the integrand in (51) evaluated at all singularities that lie within the complex contour $\mathcal{C}_{\omega_2}$. To begin with, we consider the oversampled case where $M\leq N$. It can be seen, that the integrand of $\mathcal{I}_{r,k}(\omega_1)$ in (51) exhibits three different types of singularities that lie inside the complex contour $\mathcal{C}_{\omega_2}$. The first type of singularities corresponds to the poles of $\frac{\omega_2}{z(\omega_2)}$, which are denoted by $\mu_1<\mu_2<\cdots<\mu_{\bar{M}}$ in (14). The corresponding residue w.r.t. ω_2 evaluated at μ_t yields

$$\mathrm{Res}\left[\frac{\omega_2}{z(\omega_2)}\frac{\partial z(\omega_2)}{\partial\omega_2}\frac{\frac{1}{(\gamma_r-\omega_2)(\gamma_k-\omega_2)}}{1-\Omega(\omega_1,\omega_2)},\ \mu_t\right]=\frac{\frac{\mu_t}{(\gamma_r-\mu_t)(\gamma_k-\mu_t)}}{1-\Omega(\omega_1,\mu_t)} \quad (52)$$

where we have used $\Omega(\omega_1,\omega_2)$ in (38), and $\frac{\partial z(\omega)}{\partial\omega}$ in (39). However, only μ_1 lies within the contour $\mathcal{C}_{\omega_2}$ which is why only the residue evaluated at μ_1 contributes to the solution of $\mathcal{I}_{r,k}(\omega_1)$ in (51).

The second type of singularities corresponds to the roots of $1-\Omega(\omega_1,\omega_2)$ where $\Omega(\omega_1,\omega_2)$ is given in (38). The complex-valued roots are denoted by $\varphi_r(\omega_1)$ for $r=1,...,\bar{M}$ and sorted according to their real-part in ascending order $\mathrm{Re}[\varphi_1(\omega_1)]<\cdots<\mathrm{Re}[\varphi_{\bar{M}}(\omega_1)]$ and given by the solutions of the polynomial equation in $\varphi(\omega_1)$

$$\frac{1}{N}\sum_{l=1}^{\bar{M}}\frac{K_l\gamma_l^2}{(\gamma_l-\omega_1)(\gamma_l-\varphi(\omega_1))}=1. \quad (53)$$

The corresponding residue evaluated at $\varphi_t(\omega_1)$ is given by

$$\mathrm{Res}\left[\frac{\omega_2}{z(\omega_2)}\frac{\partial z(\omega_2)}{\partial\omega_2}\frac{\frac{1}{(\gamma_r-\omega_2)(\gamma_k-\omega_2)}}{1-\Omega(\omega_1,\omega_2)},\ \varphi_t(\omega_1)\right]$$
$$=\frac{\varphi_t(\omega_1)-\omega_1}{(\gamma_r-\varphi_t(\omega_1))(\gamma_k-\varphi_t(\omega_1))}\frac{1}{1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m}{\gamma_m-\varphi_t(\omega_1)}}. \quad (54)$$

Lemma 4. *Assuming that ω_1 is located outside the contour generated by the parameterization $x\mapsto\omega_1(x)$ in [25, Remark 3], there exists exactly one solution of the equation in (53) that is enclosed by the contour $\mathcal{C}_{\omega_2}$, namely $\varphi_1(\omega_1)$.*

Proof. Since ω_1 is located outside the contour $\tilde{\Psi}(\omega_1)<1$ where $\tilde{\Psi}(\omega)$ is given in (44). It can be observed that for a fixed ω_1 the

$\varphi_r(\omega_1)$ values are the zeros of the function $f(\varphi)=1-\Omega(\omega_1,\varphi)$ where $\Omega(\omega_1,\omega_2)$ is given in (38). By Cauchy-Schwarz inequality it follows that, if $\varphi\in\mathcal{C}_{\omega_2}$, we have $|1-f(\varphi)|^2\leq\tilde{\Psi}(\omega_1)\tilde{\Psi}(\varphi)<1$ (see Lemma 1). It can also be seen that $f(\varphi)$ has no singularities or zeros lying directly on the contour $\mathcal{C}_{\omega_2}$, so that by Rouché's theorem $f(\varphi)$ has the same number of zeros and poles inside $\mathcal{C}_{\omega_2}$. The poles of $f(\varphi)$ are located at the true eigenvalues γ_r for $r=1,...,\bar{M}$. However, only the smallest eigenvalue γ_1 is enclosed by $\mathcal{C}_{\omega_2}$. Hence, it follows that only one zero of $1-f(\varphi)$ is enclosed by the contour $\mathcal{C}_{\omega_2}$, namely $\varphi_1(\omega_1)$. ∎

Consequently, only the residue evaluated at $\varphi_1(\omega_1)$ contributes to the solution of $\mathcal{I}_{r,k}(\omega_1)$ in (51).

The third type of singularities corresponds to the poles at the true eigenvalues γ_t for $t=1,...,\bar{M}$. However, we have to distinguish between poles of order one and order two. The residue with respect to ω_2 evaluated at γ_t yields

$$\mathrm{Res}\left[\frac{\omega_2}{z(\omega_2)}\frac{\partial z(\omega_2)}{\partial\omega_2}\frac{\frac{1}{(\gamma_r-\omega_2)(\gamma_k-\omega_2)}}{1-\Omega(\omega_1,\omega_2)},\ \gamma_t\right]$$
$$=\begin{cases}\frac{N}{K_t\gamma_t}\frac{\gamma_t-\omega_1}{\gamma_k-\gamma_t}, & \text{for } t=r\neq k\\ \frac{N}{K_t\gamma_t}\frac{\gamma_t-\omega_1}{\gamma_r-\gamma_t}, & \text{for } t=k\neq r\\ \alpha_t(\omega_1), & \text{for } r=k=t\end{cases} \tag{55}$$

where

$$\alpha_t(\omega_1)=\frac{(\gamma_t-\omega_1)}{\left(\frac{1}{N}K_t\gamma_t\right)^2}\left(1-\frac{1}{N}\sum_{\substack{m=1\\m\neq t}}^{\bar{M}}\frac{K_m\gamma_m}{\gamma_m-\gamma_t}\right)$$
$$+\frac{(\gamma_t-\omega_1)^2}{\gamma_t\left(\frac{1}{N}K_t\gamma_t\right)^2}\left(1-\frac{1}{N}\sum_{\substack{m=1\\m\neq t}}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\omega_1)(\gamma_m-\gamma_t)}\right). \tag{56}$$

It can be observed that only the noise eigenvalue γ_1 is enclosed by the contour $\mathcal{C}_{\omega_2}$. Correspondingly, the closed-form solution for the complex contour integral $\mathcal{I}_{r,k}(\omega_1)$ in (51) is obtained by taking the negative sum (negative sum because of the negatively oriented contour $\mathcal{C}_{\omega_2}$) of the residues in (52), (54) and (55) evaluated at all singularities that lie inside the complex contour $\mathcal{C}_{\omega_2}$ and yields

$$\mathcal{I}_{r,k}(\omega_1)=-\frac{\mu_1}{(\gamma_r-\mu_1)(\gamma_k-\mu_1)}\frac{1}{1-\Omega(\omega_1,\mu_1)}-\alpha_1(\omega_1)\delta_{r=k=1}$$
$$-\frac{\varphi_1(\omega_1)-\omega_1}{(\gamma_r-\varphi_1(\omega_1))(\gamma_k-\varphi_1(\omega_1))}\frac{1}{1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m}{\gamma_m-\varphi_1(\omega_1)}}$$
$$-\frac{N}{K_1\gamma_1}\frac{\gamma_1-\omega_1}{\gamma_k-\gamma_1}\delta_{r=1\neq k}-\frac{N}{K_1\gamma_1}\frac{\gamma_1-\omega_1}{\gamma_r-\gamma_1}\delta_{k=1\neq r}. \tag{57}$$

In the following we substitute the closed-form expression for $\mathcal{I}_{r,k}(\omega_1)$ in (57) into the general expression for the real-valued weights $\xi(r,k)$ in (41) such that

$$\frac{1}{\gamma_r\gamma_k}\xi_{\mathrm{c}}(r,k)=\frac{-1}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{\omega_1}}\frac{\omega_1}{z(\omega_1)}\frac{\partial z(\omega_1)}{\partial\omega_1}\frac{1}{(\gamma_r-\omega_1)(\gamma_k-\omega_1)}$$
$$\times\left(\alpha_1(\omega_1)\delta_{r=k=1}+\frac{\mu_1}{(\gamma_r-\mu_1)(\gamma_k-\mu_1)}\frac{1}{1-\Omega(\omega_1,\mu_1)}\right.$$
$$+\frac{N}{K_1\gamma_1}\frac{\gamma_1-\omega_1}{\gamma_k-\gamma_1}\delta_{r=1\neq k}+\frac{N}{K_1\gamma_1}\frac{\gamma_1-\omega_1}{\gamma_r-\gamma_1}\delta_{k=1\neq r}$$
$$+\left.\frac{\varphi_1(\omega_1)-\omega_1}{(\gamma_r-\varphi_1(\omega_1))(\gamma_k-\varphi_1(\omega_1))}\frac{1}{1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m}{\gamma_m-\varphi_1(\omega_1)}}\right)\mathrm{d}\omega_1 \tag{58}$$

where we have used $\bar{h}(z(\omega_1))=1$, $\Omega(\omega_1,\omega_2)$ in (38) and $\alpha_t(\omega_1)$ in (56). The integrand in (58) exhibits four different types of singularities. The first group of singularities corresponds to the poles that are located at the true eigenvalues γ_t. Hence, the residue of the first part of the integrand in (58) evaluated at γ_t is given by

$$\mathrm{Res}\left[\frac{\omega_1}{z(\omega_1)}\frac{\partial z(\omega_1)}{\partial\omega_1}\frac{\alpha_t(\omega_1)}{(\gamma_r-\omega_1)(\gamma_k-\omega_1)}\delta_{r=k=t},\ \gamma_t\right]$$
$$=-\frac{N}{K_t}\frac{1}{\left(\frac{1}{N}K_t\gamma_t\right)^2}\left(1-\frac{1}{N}\sum_{\substack{m=1\\m\neq t}}^{\bar{M}}\frac{K_m\gamma_m}{\gamma_m-\gamma_t}\right)^2\delta_{r=k=t}$$
$$-\frac{1}{\left(\frac{1}{N}K_t\gamma_t\right)^2}\left(1-\frac{1}{N}\sum_{\substack{m=1\\m\neq t}}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\gamma_t)^2}\right)\delta_{r=k=t} \tag{59}$$

whereas the residue of the second part of the integrand in (58) yields

$$\mathrm{Res}\left[\frac{\frac{\mu_t}{(\gamma_r-\mu_t)(\gamma_k-\mu_t)}}{1-\Omega(\omega_1,\mu_t)}\frac{\frac{\omega_1}{z(\omega_1)}\frac{\partial z(\omega_1)}{\partial\omega_1}}{(\gamma_r-\omega_1)(\gamma_k-\omega_1)},\ \gamma_t\right]$$
$$=\begin{cases}\frac{1}{\frac{1}{N}K_t\gamma_t}\frac{\mu_t}{\gamma_k-\mu_t}\frac{1}{\gamma_k-\gamma_t}, & \text{for } t=r\neq k\\ \frac{1}{\frac{1}{N}K_t\gamma_t}\frac{\mu_t}{\gamma_r-\mu_t}\frac{1}{\gamma_r-\gamma_t}, & \text{for } t=k\neq r\\ \beta_t, & \text{for } t=k=r\end{cases} \tag{60}$$

where

$$\beta_t=\frac{1}{\left(\frac{1}{N}K_t\gamma_t\right)^2}\frac{\mu_t}{\gamma_t-\mu_t}\left(1-\frac{1}{N}\sum_{\substack{m=1\\m\neq t}}^{\bar{M}}\frac{K_m\gamma_m}{\gamma_m-\gamma_t}\right)$$
$$+\frac{\mu_t}{\gamma_t\left(\frac{1}{N}K_t\gamma_t\right)^2}\left(1-\frac{1}{N}\sum_{\substack{m=1\\m\neq t}}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\gamma_t)(\gamma_m-\mu_t)}\right).$$

The residue of the third part of the integrand in (58) evaluated at γ_t is computed as follows

$$\mathrm{Res}\left[\frac{N}{K_t\gamma_t}\frac{\omega_1}{z(\omega_1)}\frac{\partial z(\omega_1)}{\partial\omega_1}\frac{\frac{\gamma_t-\omega_1}{\gamma_k-\gamma_t}\delta_{r=t\neq k}+\frac{\gamma_t-\omega_1}{\gamma_r-\gamma_t}\delta_{k=t\neq r}}{(\gamma_r-\omega_1)(\gamma_k-\omega_1)},\ \gamma_t\right]$$
$$=-\frac{N}{K_t}\frac{1}{(\gamma_k-\gamma_t)^2}\delta_{r=t\neq k}-\frac{N}{K_t}\frac{1}{(\gamma_r-\gamma_t)^2}\delta_{k=t\neq r}. \tag{61}$$

Since only the noise eigenvalue γ_1 is enclosed by the contour $\mathcal{C}_{\omega_1}$, only the residues in (59), (60) and (61) evaluated at γ_1 contribute to the result of $\xi_{\mathrm{c}}(r,k)$ in (58).

The second type of singularities belongs to the poles of $\frac{\omega_1}{z(\omega_1)}$, which are located at the μ-values that are defined in (14). The corresponding residue of the first part of the integrand in (58) evaluated at μ_t is given by

$$\mathrm{Res}\left[\frac{\omega_1}{z(\omega_1)}\frac{\partial z(\omega_1)}{\partial\omega_1}\frac{\alpha_t(\omega_1)}{(\gamma_r-\omega_1)(\gamma_k-\omega_1)}\delta_{r=k=t},\ \mu_t\right]$$
$$=\frac{\mu_t}{\gamma_t-\mu_t}\frac{1}{\left(\frac{1}{N}K_t\gamma_t\right)^2}\left(1-\frac{1}{N}\sum_{\substack{m=1\\m\neq t}}^{\bar{M}}\frac{K_m\gamma_m}{\gamma_m-\gamma_t}\right)\delta_{r=k=t}$$
$$+\frac{\mu_t}{\gamma_t\left(\frac{1}{N}K_t\gamma_t\right)^2}\left(1-\frac{1}{N}\sum_{\substack{m=1\\m\neq t}}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\mu_t)(\gamma_m-\gamma_t)}\right)\delta_{r=k=t}. \tag{62}$$

Furthermore, the residues of the second and third part of the integrand in (58) evaluated at μ_t are given by

$$\text{Res}\left[\frac{\frac{\mu_t}{(\gamma_r-\mu_t)(\gamma_k-\mu_t)}}{1-\Omega(\omega_1,\mu_t)}\frac{\frac{\omega_1}{z(\omega_1)}\frac{\partial z(\omega_1)}{\partial\omega_1}}{(\gamma_r-\omega_1)(\gamma_k-\omega_1)},\mu_t\right]$$
$$=\frac{\mu_t^2}{(\gamma_r-\mu_t)^2(\gamma_k-\mu_t)^2}\frac{1}{\left(1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\mu_t)^2}\right)}$$

(63)

and

$$\text{Res}\left[\frac{N}{K_t\gamma_t}\frac{\omega_1}{z(\omega_1)}\frac{\partial z(\omega_1)}{\partial\omega_1}\frac{\frac{\gamma_t-\omega_1}{\gamma_k-\gamma_t}\delta_{r=t\neq k}+\frac{\gamma_t-\omega_1}{\gamma_r-\gamma_t}\delta_{k=t\neq r}}{(\gamma_r-\omega_1)(\gamma_k-\omega_1)},\mu_t\right]$$
$$=\frac{N}{K_t}\frac{1}{(\gamma_k-\gamma_t)\gamma_t}\frac{\mu_t}{\gamma_k-\mu_t}\delta_{r=t\neq k}+\frac{N}{K_t}\frac{1}{(\gamma_r-\gamma_t)\gamma_t}\frac{\mu_t}{\gamma_r-\mu_t}\delta_{k=t\neq r}.$$

(64)

However, only μ_1 is enclosed by the contour $\mathcal{C}_{\omega_1}$ which is why only the residues in (62), (63) and (64) evaluated at μ_1 contribute to the final result of $\xi_c(r,k)$ in (58).

The third type of singularities of the integrand in (58) is given by the solutions to the polynomial equation $1=\Omega(\omega_1,\mu_t)$ in ω_1 where $\Omega(\omega_1,\omega_2)$ is defined in (38). However, using Cauchy's argument principle it can be shown that the solutions to $1=\Omega(\omega_1,\mu_t)$ are located outside the contour $\mathcal{C}_{\omega_1}$ and therefore do not contribute to the solution of $\xi_c(r,k)$ in (58). The following lemma is in line.

Lemma 5. *Function $f(\omega_1)=1-\Omega(\omega_1,\mu_t)$ with $\Omega(\omega_1,\omega_2)$ in (38) does not exhibit any zeros inside the contour $\mathcal{C}_{\omega_1}$.*

Proof. Let P denote the number of poles and Z the number of zeros of $f(\omega_1)$ that are located inside the contour $\mathcal{C}_{\omega_1}$, then according to Cauchy's argument principle

$$-\frac{1}{2\pi j}\oint_{\mathcal{C}_{\omega_1}}\frac{f'(\omega_1)}{f(\omega_1)}d\omega_1=Z-P,$$

(65)

where the first order derivative of $f(\omega_1)$ w.r.t. ω_1 is given by

$$f'(\omega_1)=\frac{\partial f(\omega_1)}{\partial\omega_1}=-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\omega_1)^2(\gamma_m-\mu_t)}.$$

(66)

Substituting $f(\omega_1)=1-\Omega(\omega_1,\mu_t)$ and its first order derivative in (66) into the argument principle in (65) yields

$$\frac{1}{2\pi j}\oint_{\mathcal{C}_{\omega_1}}\frac{\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m^2+\gamma_m-\gamma_m}{(\gamma_m-\omega_1)^2(\gamma_m-\mu_t)}}{1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\omega_1)(\gamma_m-\mu_t)}}\frac{\mu_t-\omega_1}{\mu_t-\omega_1}d\omega_1$$
$$=\frac{1}{2\pi j}\oint_{\mathcal{C}_{\omega_1}}\frac{1}{\mu_t-\omega_1}\frac{\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m^2(\mu_t-\gamma_m)}{(\gamma_m-\omega_1)^2(\gamma_m-\mu_t)}}{1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\omega_1)(\gamma_m-\mu_t)}}d\omega_1$$
$$+\frac{1}{2\pi j}\oint_{\mathcal{C}_{\omega_1}}\frac{1}{\mu_t-\omega_1}\frac{\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m^2(\gamma_m-\omega_1)}{(\gamma_m-\omega_1)^2(\gamma_m-\mu_t)}}{1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\omega_1)(\gamma_m-\mu_t)}}d\omega_1$$

where we have added and subtracted γ_m and multiplied and divided by $\mu_t-\omega_1$. Next, we add and subtract 1 to obtain

$$\frac{1}{2\pi j}\oint_{\mathcal{C}_{\omega_1}}\frac{1}{\mu_t-\omega_1}\frac{1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\omega_1)^2}}{1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\omega_1)(\gamma_m-\mu_t)}}d\omega_1$$
$$+\frac{1}{2\pi j}\oint_{\mathcal{C}_{\omega_1}}\frac{1}{\mu_t-\omega_1}\frac{\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\omega_1)(\gamma_m-\mu_t)}-1}{1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\omega_1)(\gamma_m-\mu_t)}}d\omega_1$$

(67)

$$=\frac{1}{2\pi j}\oint_{\mathcal{C}_{\omega_1}}\left[-\frac{1}{z_1}\left(1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\omega_1)^2}\right)-\frac{1}{\mu_t-\omega_1}\right]d\omega_1$$

where we have used

$$z_1=(\omega_1-\mu_t)\left[1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\omega_1)(\gamma_m-\mu_t)}\right]$$

which follows by subtracting $0=\mu_t\left(1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m}{\gamma_m-\mu_t}\right)$ from the definition of z_1 in (17). Applying the change of variables $d\omega_1=\frac{\partial\omega(z_1)}{\partial z_1}dz_1$ in (67) one obtains

$$-\frac{1}{2\pi j}\oint_{\mathcal{C}_{z_1}}\frac{1}{z_1}dz_1-\frac{1}{2\pi j}\oint_{\mathcal{C}_{\omega_1}}\frac{1}{\mu_t-\omega_1}d\omega_1$$
$$=\text{Res}\left[\frac{1}{z_1},0\right]+\text{Res}\left[\frac{1}{\mu_t-\omega_1},\mu_t\right]=1-1=0$$

for (65). Furthermore, it can be observed that in the oversampled case if $\mathcal{C}_{z_1}$ encloses zero, also $\mathcal{C}_{\omega_1}$ encloses zero. Correspondingly, $f(\omega_1)$ exhibits a single zero at $\omega_1=0$ that is enclosed by the contour $\mathcal{C}_{\omega_1}$ such that $Z=1$ and a single pole at $\omega=\gamma_1$ such that $P=1$. Consequently, except of the zero at $\omega=0$ no other zero of $f(\omega_1)$ is located inside the contour $\mathcal{C}_{\omega_1}$. ∎

The fourth and last type of singularities of the integrand in (58) is located at zero. The corresponding residue with respect to ω_1 evaluated at zero is given by

$$\text{Res}\left[\frac{1}{(\gamma_r-\omega_1)(\gamma_k-\omega_1)}\frac{\frac{\mu_1}{(\gamma_r-\mu_1)(\gamma_k-\mu_1)}}{1-\Omega(\omega_1,\mu_1)}\frac{\omega_1}{z(\omega_1)}\frac{\partial z(\omega_1)}{\partial\omega_1},0\right]$$
$$=-\frac{1}{\gamma_r\gamma_k}\frac{\mu_1}{(\gamma_r-\mu_1)(\gamma_k-\mu_1)}\frac{1}{\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m}{\gamma_m-\mu_1}}.$$

(68)

It remains to compute the complex contour integral that belongs to the last part in (58)

$$\frac{-1}{2\pi j}\oint_{\mathcal{C}_{\omega_1}}\frac{(\varphi_1(\omega_1)-\omega_1)\frac{\omega_1}{z(\omega_1)}\frac{\partial z(\omega_1)}{\partial\omega_1}\frac{1}{(\gamma_r-\omega_1)(\gamma_k-\omega_1)}}{(\gamma_r-\varphi_1(\omega_1))(\gamma_k-\varphi_1(\omega_1))\left(1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m}{\gamma_m-\varphi_1(\omega_1)}\right)}d\omega_1$$

(69)

which can be solved in part numerically and in part in closed-form. The numerical part is obtained by applying the change of variables $d\omega_1=\frac{\partial\omega(z_1)}{\partial z_1}dz_1$ to (69) and parameterizing the complex contour through $z\mapsto\omega(z)$ for $z\in[x_1^-,x_1^+]$ as proposed in [25, Section IV] and given by

$$-\frac{1}{\pi}\int_{x_1^-}^{x_1^+}\text{Im}\left[\frac{1}{\left(1-\frac{1}{N}\sum_{n=1}^{\bar{M}}\frac{K_n\gamma_n}{\gamma_n-\omega(x)}\right)\left(1-\frac{1}{N}\sum_{n=1}^{\bar{M}}\frac{K_n\gamma_n}{\gamma_n-\varphi_1(\omega(x))}\right)}\right.$$
$$\left.\times\frac{\varphi_1(\omega(x))-\omega(x)}{(\gamma_r-\omega(x))(\gamma_r-\varphi_1(\omega(x)))(\gamma_k-\omega(x))(\gamma_k-\varphi_1(\omega(x)))}\right]dx$$
$$=\frac{2}{\pi}\int_{x_1^-}^{x_1^+}\frac{1}{\left|1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m}{\gamma_m-\omega(x)}\right|^2}\frac{\text{Im}[\omega(x)]}{|\gamma_r-\omega(x)|^2|\gamma_k-\omega(x)|^2}dx.$$

(70)

The following lemma was used to simplify the real-valued integral.

Lemma 6. *By parameterizing the contour $\mathcal{C}_{\omega_1}$ as proposed in [25, Section IV]*

$$\frac{1}{N}\sum_{r=1}^{\bar{M}}\frac{K_r\gamma_r^2}{|\gamma_r-\omega(x)|^2}=1$$

(71)

for $x\in\mathcal{S}$ and it can be shown that

$$\lim_{\omega_1\to\omega(x)}\varphi_t(\omega_1)=\omega(x)^*,\quad\text{for }x\in[x_t^-,x_t^+].$$

(72)

Proof. Let $u(x)$ and $v(x)$ denote the real and imaginary parts of $\omega(x)$ in (17), i.e., $\omega(x)=u(x)+jv(x)$. It is shown in [24, Proposition 2] that $v(x)>0$ for $x\in\mathring{\mathcal{S}}\equiv(x_1^-,x_1^+)\cup\cdots\cup(x_S^-,x_S^+)$. Taking imaginary parts on both sides of (17) one obtains

$$0=v(x)\left(1-\frac{1}{N}\sum_{r=1}^{\bar{M}}\frac{K_r\gamma_r^2}{(\gamma_r-u(x))^2+v(x)^2}\right)$$

which implies that

$$1=\frac{1}{N}\sum_{r=1}^{\bar{M}}\frac{K_r\gamma_r^2}{(\gamma_r-u(x))^2+v(x)^2}=\frac{1}{N}\sum_{r=1}^{\bar{M}}\frac{K_r\gamma_r^2}{|\gamma_r-\omega(x)|^2} \quad (73)$$

for $x\in\mathring{\mathcal{S}}$. From [25, Proposition 1] it follows that (73) also holds for the boundaries of the clusters $x\in\{x_1^-,x_1^+,...,x_S^-,x_S^+\}$. Furthermore, the convergence result in (72) follows as a direct consequence of (71) and the definition of $\varphi_t(\omega_1)$ in (53). $\blacksquare$

In the undersampled case ($M>N$) it can be observed that the contour obtained through the parameterization $z\mapsto\omega(z)$ for $z\in[x_1^-,x_1^+]$ encloses zero but not μ_1 since $\omega(0)=\mu_1<\omega(x_1^-)<0<\omega(x_1^+)$. However, since μ_1 is enclosed by $\mathcal{C}_{\omega_1}$ we compute the residue of the integrand in (69) with respect to ω_1 and evaluate it at μ_1

$$\text{Res}\left[\frac{(\varphi_1(\omega_1)-\omega_1)\frac{\omega_1}{z(\omega_1)}\frac{\partial z(\omega_1)}{\partial\omega_1}\frac{1}{(\gamma_r-\omega_1)(\gamma_k-\omega_1)}\delta_{M>N}}{(\gamma_r-\varphi_1(\omega_1))(\gamma_k-\varphi_1(\omega_1))\left(1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m}{\gamma_m-\varphi_1(\omega_1)}\right)}, \mu_1\right]$$
$$=-\frac{1}{\gamma_r\gamma_k}\frac{\mu_1^2}{(\gamma_r-\mu_1)(\gamma_k-\mu_1)}\frac{N}{N-M}\delta_{M>N}. \quad (74)$$

Furthermore, in the oversampled case ($M<N$) it can be observed that the contour obtained through the parameterization $z\mapsto\omega(z)$ for $z\in[x_1^-,x_1^+]$ encloses μ_1, however it does not enclose zero since $0<\omega(x_1^-)<\mu_1<\omega(x_1^+)$ although zero is enclosed by the contour. Therefore, we compute the residue of the integrand in (69) with respect to ω_1 and evaluate it at zero

$$\text{Res}\left[\frac{(\varphi_1(\omega_1)-\omega_1)\frac{\omega_1}{z(\omega_1)}\frac{\partial z(\omega_1)}{\partial\omega_1}\frac{1}{(\gamma_r-\omega_1)(\gamma_k-\omega_1)}\delta_{M<N}}{(\gamma_r-\varphi_1(\omega_1))(\gamma_k-\varphi_1(\omega_1))\left(1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m}{\gamma_m-\varphi_1(\omega_1)}\right)}, 0\right]$$
$$=\frac{1}{\gamma_r\gamma_k}\frac{\mu_1}{(\gamma_r-\mu_1)(\gamma_k-\mu_1)}\frac{1}{\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m}{\gamma_m-\mu_1}}\delta_{M<N}. \quad (75)$$

Consequently, the solution of the contour integral in (69) is obtained by summing the intermediate results in (70), (74) and (75) and yields

$$\frac{-1}{2\pi j}\oint_{\mathcal{C}_{\omega_1}}\frac{(\varphi_1(\omega_1)-\omega_1)\frac{\omega_1}{z(\omega_1)}\frac{\partial z(\omega_1)}{\partial\omega_1}\frac{1}{(\gamma_r-\omega_1)(\gamma_k-\omega_1)}}{(\gamma_r-\varphi_1(\omega_1))(\gamma_k-\varphi_1(\omega_1))\left(1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m}{\gamma_m-\varphi_1(\omega_1)}\right)}d\omega_1$$
$$=\frac{2}{\pi}\int_{x_1^-}^{x_1^+}\frac{1}{\left|1-\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m}{\gamma_m-\omega(x)}\right|^2}\frac{\text{Im}[\omega(x)]}{|\gamma_r-\omega(x)|^2|\gamma_k-\omega(x)|^2}dx$$
$$-\frac{1}{\gamma_r\gamma_k}\frac{\mu_1^2}{(\gamma_r-\mu_1)(\gamma_k-\mu_1)}\frac{N}{N-M}\delta_{M>N}$$
$$+\frac{1}{\gamma_r\gamma_k}\frac{\mu_1}{(\gamma_r-\mu_1)(\gamma_k-\mu_1)}\frac{1}{\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m}{\gamma_m-\mu_1}}\delta_{M<N}. \quad (76)$$

Finally, the expression of the real-valued weights $\xi_c(r,k)$ in (20) is obtained by taking the sum of the residues in (59), (60), (61), (62), (63), (64), (68) evaluated at all singularities

that lie inside the contour $\mathcal{C}_{\omega_1}$ namely $\{0,\gamma_1,\mu_1\}$ and adding (76).

In case of the g-MUSIC cost function $\bar{h}(z(\omega_2))=\frac{z(\omega_2)}{\omega_2}\frac{\partial\omega(z_2)}{\partial z_2}$. Hence the complex contour integral in (42) simplifies as follows

$$\mathcal{I}_{r,k}(\omega_1)=\frac{1}{2\pi j}\oint_{\mathcal{C}_{\omega_2}}\frac{1}{(\gamma_r-\omega_2)(\gamma_k-\omega_2)}\frac{1}{1-\Omega(\omega_1,\omega_2)}d\omega_2 \quad (77)$$

which can be solved in closed-form using conventional residue calculus [45]. It can be observed that the integrand in (77) exhibits two different types of singularities. The first type of singularities belongs to the roots of $1-\Omega(\omega_1,\omega_2)$ where $\Omega(\omega_1,\omega_2)$ is given in (38). The complex-valued roots are denoted by $\varphi_t(\omega_1)$ for $t=1,...,\bar{M}$ and defined in (53). The residue w.r.t. ω_2 evaluated at $\varphi_t(\omega_1)$ is given by

$$\text{Res}\left[\frac{\frac{1}{(\gamma_r-\omega_2)(\gamma_k-\omega_2)}}{1-\Omega(\omega_1,\omega_2)}, \varphi_t(\omega_1)\right]=\frac{\frac{-1}{(\gamma_r-\varphi_t(\omega_1))(\gamma_k-\varphi_t(\omega_1))}}{\frac{1}{N}\sum_{r=1}^{\bar{M}}\frac{K_r\gamma_r^2}{(\gamma_r-\omega_1)(\gamma_r-\varphi_t(\omega_1))^2}}. \quad (78)$$

However, it can be seen that only the residue evaluated at $\varphi_1(\omega_1)$ contributes to the solution of $\mathcal{I}_{r,k}(\omega_1)$ in (77) as it is the only singularity of the roots of $1-\Omega(\omega_1,\omega_2)$ that is enclosed by the contour $\mathcal{C}_{\omega_2}$.

The second type of singularities belongs to the poles of the integrand in (77) at γ_t for $t=1,...,\bar{M}$. The corresponding residue w.r.t. ω_2 evaluated at γ_t is given by

$$\text{Res}\left[\frac{\frac{1}{(\gamma_r-\omega_2)(\gamma_k-\omega_2)}}{1-\Omega(\omega_1,\omega_2)}, \gamma_t\right]=\begin{cases}\frac{\gamma_t-\omega_1}{\frac{1}{N}K_t\gamma_t^2} & \text{for } r=k=t\\0 & \text{for } r\neq k.\end{cases} \quad (79)$$

It can be observed that only the first eigenvalue γ_1 is enclosed by the contour $\mathcal{C}_{\omega_2}$. Hence, only the residue evaluated at γ_1 contributes to the closed-form expression of $\mathcal{I}_{r,k}(\omega_1)$ in (77) which is obtained by taking the negative sum (negative due to the negatively orientated contour) of the residue in (78) evaluated at $\varphi_1(\omega_1)$ and the residue in (79) evaluated at γ_1

$$\mathcal{I}_{r,k}(\omega_1)=\frac{\frac{1}{(\gamma_r-\varphi_1(\omega_1))(\gamma_k-\varphi_1(\omega_1))}}{\frac{1}{N}\sum_{r=1}^{\bar{M}}\frac{K_r\gamma_r^2}{(\gamma_r-\omega_1)(\gamma_r-\varphi_1(\omega_1))^2}}-\frac{\gamma_1-\omega_1}{\frac{1}{N}K_1\gamma_1^2}. \quad (80)$$

Substituting the closed-form expression for $\mathcal{I}_{r,k}(\omega_1)$ in (80) into the general expression of $\xi_g(r,k)$ in (41) and using $\bar{h}(z(\omega_1))=\frac{z(\omega_1)}{\omega_1}\frac{\partial\omega(z_1)}{\partial z_1}$ yields

$$\xi_g(r,k)=\frac{1}{2\pi j}\oint_{\mathcal{C}_{\omega_1}}\frac{\frac{\gamma_r\gamma_k}{(\gamma_r-\omega_1)(\gamma_k-\omega_1)(\gamma_r-\varphi_1(\omega_1))(\gamma_k-\varphi_1(\omega_1))}}{\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\omega_1)(\gamma_m-\varphi_1(\omega_1))^2}}d\omega_1$$
$$-\frac{1}{2\pi j}\oint_{\mathcal{C}_{\omega_1}}\frac{1}{(\gamma_r-\omega_1)(\gamma_k-\omega_1)}\frac{\gamma_1-\omega_1}{\frac{1}{N}K_1\gamma_1^2}d\omega_1. \quad (81)$$

It can be observed that the integrand of the second contour integral in (81) exhibits a single singularity at γ_1 that lies within the contour $\mathcal{C}_{\omega_1}$. The corresponding residue w.r.t. ω_1 evaluated at γ_1 is given by

$$\text{Res}\left[\frac{(\gamma_t-\omega_1)\frac{1}{\frac{1}{N}K_t\gamma_t^2}}{(\gamma_r-\omega_1)(\gamma_k-\omega_1)}, \gamma_t\right]=\begin{cases}\frac{-1}{\frac{1}{N}K_t\gamma_t^2}, & \text{for } r=k=t\\0, & r\neq k.\end{cases} \quad (82)$$

Hence the closed-form solution for the second integral in (81) is given by the residue in (82) evaluated at γ_1 for $r=k=1$ or zero otherwise. The first contour integral in (81) can be solved numerically by applying the change of variables $d\omega=\frac{\partial\omega(x)}{\partial x}dx$ and parameterizing the contour $\mathcal{C}_{\omega_1}$ by concatenation of $\omega(x)$ and $\omega(x)^*$ as proposed in [25, Section IV] such that

$$\frac{1}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{\omega_1}}\frac{\frac{1}{(\gamma_r-\omega_1)(\gamma_k-\omega_1)(\gamma_r-\varphi_1(\omega_1))(\gamma_k-\varphi_1(\omega_1))}}{\frac{1}{N}\sum_{m=1}^{\bar{M}}\frac{K_m\gamma_m^2}{(\gamma_m-\omega_1)(\gamma_m-\varphi_1(\omega_1))^2}}d\omega_1$$
$$=\frac{1}{\pi}\int_{x_1^-}^{x_1^+}\mathrm{Im}\left[\frac{\frac{1}{|\gamma_r-\omega(x)|^2|\gamma_k-\omega(x)|^2}}{\frac{1}{N}\sum_{r=1}^{\bar{M}}\frac{K_r\gamma_r^2}{|\gamma_r-\omega(x)|^2(\gamma_r-\omega(x)^*)}}\frac{\partial\omega(x)}{\partial x}\right]dx \quad (83)$$

where we have used Lemma 6 to simplify the expression. The following Lemma allows to further simplify the real-valued integral in (83).

Lemma 7. *By parameterizing the contour $\mathcal{C}_\omega$ through concatenation of $\omega(x)$ in (17) and $\omega(x)^*$ as proposed in [25, Section IV] the following equality holds*

$$\frac{1}{N}\sum_{r=1}^{\bar{M}}\frac{K_r\gamma_r^2}{|\gamma_r-\omega(x)|^2(\gamma_r-\omega(x))^*}=\frac{1}{2\mathrm{j}\mathrm{Im}[\omega(x)]\omega'(x)^*} \quad (84)$$

where $\omega'(x)$ is defined in (18).

Proof. Let $u(x)$ and $v(x)$ denote the real and imaginary parts of $\omega(x)$ in (17), i.e., $\omega(x)=u(x)+\mathrm{j}v(x)$. Using the expression in (71) we can express the inverse of the first order derivative of $\omega(x)$ with respect to x in (18) as

$$\frac{1}{\omega'(x)}=\frac{1}{N}\sum_{r=1}^{\bar{M}}\frac{K_r\gamma_r^2[(\gamma_r-\omega(x))^2-|\gamma_r-\omega(x)|^2]}{(\gamma_r-\omega(x))^2|\gamma_r-\omega(x)|^2} \quad (85)$$
$$=\frac{1}{N}\sum_{r=1}^{\bar{M}}\frac{-K_r\gamma_r^2[2\mathrm{j}v(x)(\gamma_r-u(x))+2v(x)^2]}{|\gamma_r-\omega(x)|^2[(\gamma_r-u(x))^2-v(x)^2-2\mathrm{j}v(x)(\gamma_r-u(x))]}.$$

Taking the complex conjugate of $\frac{1}{\omega'(x)}$ in (85) and multiplying by $\frac{1}{2\mathrm{j}v(x)}$ one obtains the following expression for $\frac{1}{2\mathrm{j}v(x)\omega'(x)^*}$

$$\frac{1}{N}\sum_{r=1}^{\bar{M}}\frac{K_r\gamma_r^2[(\gamma_r-u(x))+\mathrm{j}v(x)]}{|\gamma_r-\omega(x)|^2[(\gamma_r-u(x))^2+2\mathrm{j}v(x)(\gamma_r-u(x))-v(x)^2]}$$

which is identical to the one in (84). ∎

Using the equalities established in (84) the real-valued integral in (83) can equivalently be expressed as

$$\frac{2}{\pi}\int_{x_1^-}^{x_1^+}\frac{|\omega'(x)|^2\mathrm{Im}[\omega(x)]}{|\gamma_r-\omega(x)|^2|\gamma_k-\omega(x)|^2}dx. \quad (86)$$

Finally, the solution for $\xi_\mathrm{g}(r,k)$ in (24) is obtained by summing the residue in (82) for $t=1$ and (86).

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SUPPLEMENTARY MATERIAL

This document provides some additional proofs of results used in the article "Probability of Resolution of MUSIC and G-MUSIC: An Asymptotic Approach". More specifically the proof of the Central Limit Theorem in Theorem 3 is provided.

Notation: Let x, x_1 and x_2 be three random quantities then $\mathbb{E}[x]$ denotes the expectation of x, $\mathrm{var}[x]$ denotes the variance of x and $\mathrm{cov}[x_1, x_2]$ denotes the covariance between x_1 and x_2. Furthermore, let $x^{(\circ)} = x - \mathbb{E}[x]$.

APPENDIX C
PROOF OF THEOREM 3

In the following a more general form of Theorem 3 is derived. Instead of focusing on the noise subspace estimator only the asymptotic stochastic behavior of the estimator for the subspace that is associated to the eigenvalue γ_m for $m = 1,...,\bar{M}$ is derived. Let $\mathcal{K}_m$ denote a set of indexes

$$\mathcal{K}_m = \left\{ \sum_{r=1}^{m-1} K_r + 1, \sum_{r=1}^{m-1} K_r + 2, ..., \sum_{r=1}^{m} K_r \right\}.$$

The cardinality of the set $\mathcal{K}_m$ is equal to the multiplicity of the corresponding eigenvalue γ_m, namely K_m. It is assumed that the cluster with support $[x_s^-, x_s^+]$ that is associated to γ_m is well separated from the rest of the eigenvalue distribution and that γ_m is the unique eigenvalue that is associated to this cluster (subspace separability condition holds). Let $\mathcal{C}_z^{(m)}$ denote a negatively (clockwise) orientated contour that encloses the sample eigenvalues $\{\hat{\lambda}_k : k \in \mathcal{K}_m\}$ that correspond to the true and distinct eigenvalue γ_m only. Next, consider the random cost function $\hat{\eta}_l^{(m)}$ that belongs to the estimator of the subspace that is associated to the eigenvalue γ_m and its deterministic equivalent $\bar{\eta}_l^{(m)}$

$$\hat{\eta}_l^{(m)} = \frac{1}{2\pi\mathrm{j}} \oint_{\mathcal{C}_z^{(m)}} \hat{h}(z)\hat{m}_l(z)\mathrm{d}z, \qquad \bar{\eta}_l^{(m)} = \frac{1}{2\pi\mathrm{j}} \oint_{\mathcal{C}_z^{(m)}} \bar{h}(z)\bar{m}_l(z)\mathrm{d}z, \tag{83}$$

where l denotes the point of evaluation of the cost function. Furthermore, for the M,N-consistent estimator of the subspace that is associated to the m-th distinct eigenvalue we define $\hat{h}(z) = \frac{z}{\hat{\omega}(z)} \frac{\partial \hat{\omega}(z)}{\partial z}$ and $\bar{h}(z) = \frac{z}{\omega(z)} \frac{\partial \omega(z)}{\partial z}$ where

$$\frac{z}{\hat{\omega}(z)} = 1 - \frac{1}{N} \sum_{r=1}^{M} \frac{\hat{\lambda}_r}{\hat{\lambda}_r - z}, \qquad \frac{\partial \hat{\omega}(z)}{\partial z} = \frac{\hat{\omega}(z)}{z} + \frac{\hat{\omega}(z)^2}{z} \frac{1}{N} \sum_{r=1}^{M} \frac{\hat{\lambda}_r}{(\hat{\lambda}_r - z)^2},$$

and

$$\frac{z}{\omega(z)} = 1 - \frac{1}{N} \sum_{r=1}^{\bar{M}} \frac{K_r \gamma_r}{\gamma_r - \omega(z)}, \qquad \frac{\partial \omega(z)}{\partial z} = \left(1 - \frac{1}{N} \sum_{r=1}^{\bar{M}} \frac{K_r \gamma_r^2}{(\gamma_r - \omega(z))^2} \right)^{-1}.$$

On the other hand, in case of the N-consistent estimator we instead have $\hat{h}(z) = 1$ and $\bar{h}(z) = 1$. Furthermore, the Stieltjes transform and its deterministic equivalent are defined as

$$\hat{m}_l(z) = \boldsymbol{v}_l^{\mathrm{H}} \hat{\boldsymbol{Q}}(z) \boldsymbol{v}_l, \qquad \bar{m}_l(z) = \boldsymbol{v}_l^{\mathrm{H}} \bar{\boldsymbol{Q}}(z) \boldsymbol{v}_l \tag{84}$$

$$\hat{\boldsymbol{Q}}(z) = \left(\hat{\boldsymbol{R}} - z\boldsymbol{I}_M \right)^{-1}, \qquad \bar{\boldsymbol{Q}}(z) = \frac{\omega(z)}{z} (\boldsymbol{R} - \omega(z)\boldsymbol{I}_M)^{-1}, \tag{85}$$

where $\omega(z)$ is given by the unique solution to the following equation

$$z = \omega(z) \left(1 - \frac{1}{N} \mathrm{tr} \left[\boldsymbol{R} (\boldsymbol{R} - \omega(z)\boldsymbol{I}_M)^{-1} \right] \right), \tag{86}$$

on $\{\omega(z) \in \mathbb{C}^+ : \mathrm{Im}(\omega(z)) \geq 0\}$ for $z \in \mathbb{C}^+$ and $\boldsymbol{v}_l \in \mathbb{C}^M$ is a deterministic $M \times 1$ vector which depends on the point of evaluation l of the cost function. The cost function is evaluated at L distinct points where $\hat{\boldsymbol{\eta}}^{(m)} = [\hat{\eta}_1^{(m)},...,\hat{\eta}_L^{(m)}]^{\mathrm{T}} \in \mathbb{R}^L$ denotes the corresponding real-valued $L \times 1$ cost function vector and $\bar{\boldsymbol{\eta}}^{(m)} = [\bar{\eta}_1^{(m)},...,\bar{\eta}_L^{(m)}]^{\mathrm{T}} \in \mathbb{R}^L$ its deterministic equivalent, respectively. Subtracting the deterministic equivalent $\bar{\eta}_l^{(m)}$ from the random cost function $\hat{\eta}_l^{(m)}$ in (83) yields

$$\eta_l^{(m)} = \sqrt{N} \left(\hat{\eta}_l^{(m)} - \bar{\eta}_l^{(m)} \right) = \frac{\sqrt{N}}{2\pi\mathrm{j}} \oint_{\mathcal{C}_z^{(m)}} \left(\hat{h}(z)\hat{m}_l(z) - \bar{h}(z)\bar{m}_l(z) \right) \mathrm{d}z. \tag{87}$$

It is shown in [37] that the statistic in (87) asymptotically fluctuates as a Gaussian random variable with zero-mean and positive variance. However, we are interested in the more general case of the asymptotic joint distribution of a collection of random variables, namely

$$\sqrt{N} \left(\hat{\boldsymbol{\eta}}^{(m)} - \bar{\boldsymbol{\eta}}^{(m)} \right) = \begin{bmatrix} \sqrt{N}(\hat{\eta}_1^{(m)} - \bar{\eta}_1^{(m)}) \\ \vdots \\ \sqrt{N}(\hat{\eta}_L^{(m)} - \bar{\eta}_L^{(m)}) \end{bmatrix} \tag{88}$$

where $\sqrt{N}(\hat{\eta}(\bar{\theta}_l) - \bar{\eta}(\bar{\theta}_l))$ for $l = 1,...,L$ is defined in (87). In order to derive the Central Limit Theorem (CLT) we have to establish pointwise convergence of the characteristic function of the statistic $\sqrt{N}(\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}}) - \bar{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}}))$ in (88) towards the characteristic

function of a Gaussian random variable. According to the Cramér-Wold device it is sufficient to show that the one-dimensional projection

$$\eta^{(m)} = \sum_{p=1}^{L} w_p \eta_p^{(m)} = \sum_{p=1}^{L} \sqrt{N} w_p \left(\hat{\eta}_p^{(m)} - \bar{\eta}_p^{(m)}\right) = \sum_{p=1}^{L} w_p \frac{\sqrt{N}}{2\pi \mathrm{j}} \oint_{\mathcal{C}_z^{(m)}} \left(\hat{h}(z)\hat{m}_p(z) - \bar{h}(z)\bar{m}_p(z)\right) \mathrm{d}z, \tag{89}$$

is asymptotically Gaussian distributed for any collection of real-valued quantities w_l, $l = 1,...,L$ to proof that $\sqrt{N}(\hat{\boldsymbol{\eta}}^{(m)} - \bar{\boldsymbol{\eta}}^{(m)})$ in (88) is asymptotically jointly Gaussian distributed. Let $\chi(r)$ be defined as

$$\chi(r) = \exp\left(\mathrm{j}r\eta^{(m)}\right) = \exp\left(\mathrm{j}r\sqrt{N}\sum_{p=1}^{L} w_p \left(\hat{\eta}_p^{(m)} - \bar{\eta}_p^{(m)}\right)\right) = \exp\left(\mathrm{j}r\sum_{p=1}^{L} w_p \frac{\sqrt{N}}{2\pi \mathrm{j}} \oint_{\mathcal{C}_z^{(m)}} \left(\hat{h}(z)\hat{m}_p(z) - \bar{h}(z)\bar{m}_p(z)\right) \mathrm{d}z\right)$$
$$= \exp\left(\mathrm{j}r\sum_{p=1}^{L} w_p \frac{\sqrt{N}}{2\pi \mathrm{j}} \oint_{\mathcal{C}_z^{(m)}} \boldsymbol{v}_p^{\mathrm{H}} \left(\hat{h}(z)\hat{\boldsymbol{Q}}(z) - \bar{h}(z)\bar{\boldsymbol{Q}}(z)\right) \boldsymbol{v}_p \mathrm{d}z\right) \tag{90}$$

and let $\mathbb{E}[\chi(r)]$ be the corresponding characteristic function of the random one-dimensional projection $\eta^{(m)}$ in (89). In the following we analyze the asymptotic behavior of the characteristic function $\mathbb{E}[\chi(r)]$ in (90) in the asymptotic regime where $M,N \to \infty$ at the same rate. In fact, we approximate the characteristic function $\mathbb{E}[\chi(r)]$ in (90) with the characteristic function $\bar{\chi}(r)$ of a normal distributed random variable

$$\bar{\chi}(r) = \exp\left(\mathrm{j}r\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\mu} - r^2 \frac{\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{w}}{2}\right), \tag{91}$$

with mean $\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\mu}$ and covariance $\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{w}$ where $\boldsymbol{\mu} \in \mathbb{R}^L$, $\boldsymbol{\Gamma} \in \mathbb{R}^{L \times L}$, and $\boldsymbol{w} = [w_1,...,w_L]^{\mathrm{T}} \in \mathbb{R}^L$. Hence, the target is to determine $\boldsymbol{\mu}$ and $\boldsymbol{\Gamma}$ such that $\mathbb{E}[\chi(r)] - \bar{\chi}(r) \to 0$ pointwise in r for $M,N \to \infty$ at the same rate. Assuming that $\boldsymbol{\Gamma}$ has bounded spectral norm and that the infimum of the smallest eigenvalue of $\boldsymbol{\Gamma}$ is bounded away from zero uniformly in M we have

$$\boldsymbol{\Gamma}^{-1/2}\left(\sqrt{N}(\hat{\boldsymbol{\eta}}^{(m)} - \bar{\boldsymbol{\eta}}^{(m)}) - \boldsymbol{\mu}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_L), \tag{92}$$

for $M,N \to \infty$ at the same rate ($M/N \to c$, $0 < c < \infty$) where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution. In order to derive $\boldsymbol{\mu}$ and $\boldsymbol{\Gamma}$ in (92) we choose a deterministic parameterization of the contour $\mathcal{C}_z^{(m)}$ that is independent on the sample eigenvalues an M and N. Therefore, it is assumed that there exists a deterministic ϱ^- and ϱ^+ for the cluster $[x_s^-, x_s^+]$ that is associated to γ_m such that

$$\varrho^- < 0 < \sup_{M,N} x_s^+ < \varrho^+ < \inf_{M,N} x_{s+1}^-, \quad \text{for } s = 1$$
$$\sup_{M,N} x_{s-1}^+ < \varrho^- < \inf_{M,N} x_s^- \leq \sup_{M,N} x_s^+ < \varrho^+ < \inf_{M,N} x_{s+1}^-, \quad \text{for } 1 < s < S$$
$$\sup_{M,N} x_{s-1}^+ < \varrho^- < \inf_{M,N} x_s^- \leq \sup_{M,N} x_s^+ < \varrho^+, \quad \text{for } s = S.$$

This allows to deform the contour such that it becomes independent on M and N. The negatively oriented deterministic contour $\mathcal{C}_z^{(m)}$ may cross the real axis at the points ϱ^- and ϱ^+ and can e.g. be described by the boundaries of the rectangle

$$\mathbb{R}_y^{(m)} = \left\{z \in \mathbb{C} : \varrho^- \leq \mathrm{Re}(z) \leq \varrho^+, \; |\mathrm{Im}(z)| \leq y\right\} \tag{93}$$

where $y > 0$ and $\mathcal{C}_z^{(m)} = \partial \mathbb{R}_y^{(m)}$. Unfortunately, the random variable $\eta^{(m)}$ in (89) does not need to have a characteristic function for all M and N as there might exist realizations for which the singularities of $\hat{h}(z)\hat{\boldsymbol{Q}}(z)$ become dangerously close to the contour $\mathcal{C}_z^{(m)}$ or even lie on $\mathcal{C}_z^{(m)}$. Therefore, we need to consider the points where $\hat{h}(z)\hat{\boldsymbol{Q}}(z)$ is not holomorphic, namely $\hat{\lambda}_k$ for $k = 1,...,M$ and in case of the M,N-consistent subspace estimator (the g-MUSIC case) where

$$\hat{h}(z) = \frac{z}{\hat{\omega}(z)} \frac{\partial \hat{\omega}(z)}{\partial z} = 1 + \hat{\omega}(z) \frac{1}{N} \mathrm{tr}\left[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}^2(z)\right] = 1 + \frac{z}{1 - \frac{1}{N}\sum_{m=1}^{M} \frac{\hat{\lambda}_m}{\hat{\lambda}_m - z}} \frac{1}{N} \mathrm{tr}\left[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}^2(z)\right], \tag{94}$$

also the points $\hat{\mu}_k$ for $k = 1,...,M$ which are the solutions (counting multiplicities) to

$$1 = \frac{1}{N} \sum_{m=1}^{M} \frac{\hat{\lambda}_m}{\hat{\lambda}_m - z} = \frac{1}{N} \sum_{m=[M-N]^+ + 1}^{M} \frac{\hat{\lambda}_m}{\hat{\lambda}_m - z}, \tag{95}$$

where

$$[M-N]^+ = \begin{cases} M-N & \text{for } M > N \\ 0 & \text{for } M \leq N. \end{cases}$$

Furthermore, it can be observed that the $\hat{\mu}$-values satisfy the interlacing property such that $0 < \hat{\mu}_1 < \hat{\lambda}_1 < \hat{\mu}_2 < \cdots < \hat{\lambda}_{M-1} < \hat{\mu}_M < \hat{\lambda}_M$ for $M < N$, $0 = \hat{\mu}_1 < \hat{\lambda}_1 < \hat{\mu}_2 < \cdots < \hat{\lambda}_{M-1} < \hat{\mu}_M < \hat{\lambda}_M$ for $M = N$, and $0 = \hat{\mu}_1 = \hat{\lambda}_1 = \cdots = \hat{\mu}_{M-N} = \hat{\lambda}_{M-N} = \hat{\mu}_{M-N+1} < \hat{\lambda}_{M-N+1} < \hat{\mu}_{M-N+2} < \cdots < \hat{\lambda}_{M-1} < \hat{\mu}_M < \hat{\lambda}_M$ for $M > N$. It can be observed that, for all M and N sufficiently large, the roots $\{\hat{\mu}_k : k \in \mathcal{K}_m\}$ of the polynomial in (95) and the sample eigenvalues $\{\hat{\lambda}_k : k \in \mathcal{K}_m\}$ always lie inside the

contour $\mathcal{C}_z^{(m)}$ [25, Appendix IV]. For those realizations of M and N where either one of the sample eigenvalues $\hat{\lambda}_k$ or one of the $\hat{\mu}$-values become dangerously close to the contour $\mathcal{C}_z^{(m)}$ or even lie on $\mathcal{C}_z^{(m)}$, we will follow the approach in [19], [40], [41] and consider an equivalent representation of $\eta^{(m)}$ in (89) that is guaranteed to have a characteristic function for all M and N. For this purpose we introduce $\mathcal{S}_\epsilon = \{x \in \mathbb{R} : \text{dist}(x,\mathcal{S}) \leq \epsilon\}$ for positive $\epsilon > 0$. It is assumed that ϵ is small enough such that $\mathcal{S}_\epsilon$ does not intersect with the deterministic contour $\partial\mathbb{R}_y^{(m)}$ in (93). Furthermore, let ϕ denote a smooth function $\phi : \mathbb{R} \to [0,1]$ such that $\phi(x) = 1$ for $x \in \mathcal{S}_\epsilon$ and $\phi(x) = 0$ for $x \in \mathbb{R}\backslash\mathcal{S}_{2\epsilon}$. We will write $\phi_M = \tilde{\phi}_M\check{\phi}_M$ where

$$\tilde{\phi}_M = \prod_{k\in\mathcal{K}_m} \phi\left(\hat{\lambda}_k\right), \qquad \text{and} \qquad \check{\phi}_M = \prod_{k\in\mathcal{K}_m} \phi(\hat{\mu}_k). \tag{96}$$

By [26], [41] and [25, Appendix IV] we know that $\tilde{\phi}_M = 1$ and $\check{\phi}_M = 1$ with probability one for all M and N sufficiently large. Furthermore, $\tilde{\phi}_M$ and $\check{\phi}_M$ can almost surely be extended to a continuous function, such that the integration by parts formula is applicable [41, Appendix IV] and $\phi_M = \tilde{\phi}_M\check{\phi}_M = 1$ for all M and N sufficiently large. Correspondingly, we may represent $\eta^{(m)}$ in (89) as

$$\eta^{(m)} = \sum_{p=1}^{L} w_p\eta_p^{(m)}\phi_M = \sum_{p=1}^{L}\sqrt{N}w_p\left(\hat{\eta}_p^{(m)}-\bar{\eta}_p^{(m)}\right)\phi_M = \sum_{p=1}^{L}w_p\frac{\sqrt{N}}{2\pi\text{j}}\oint_{\mathcal{C}_z^{(m)}}\left(\hat{h}(z)\hat{m}_p(z)-\bar{h}(z)\bar{m}_p(z)\right)\phi_M\text{d}z, \tag{97}$$

almost surely for M,N sufficiently large. Please note that the characteristic in (97) exists for every realization and every possible M and N. From now on we will therefore consider the definition of $\eta^{(m)}$ in (97).

During this proof, we strongly rely on the integration by parts formula for Gaussian functionals as well as the Nash-Poincaré in [19, Proposition 2, supplementary material].

Remark: In the following $\mathcal{O}(N^{-k})$ denotes a general bivariate complex function that is bounded by $\epsilon(z_1,z_2)N^{-k}$, where $\epsilon(z_1,z_2)$ does not depend on N and is such that

$$\sup_{(z_1,z_2)\in\mathcal{C}_{z_1}^{(m)}\times\mathcal{C}_{z_2}^{(m)}} \|\epsilon(z_1,z_2)\| < \infty. \tag{98}$$

The function $\epsilon(z_1,z_2)$ may differ from one line to another and may be matrix valued in which case (98) is understood as the spectral norm. Furthermore, $\mathcal{O}(N^{-\mathbb{N}})$ should be understood as bivariate complex function that can be written as $\mathcal{O}(N^{-l})$ for every $l \in \mathbb{N}$ (see [19, Remark 1, supplementary material]).

Proposition 3. *Assume that, for each fixed $z \in \mathbb{C}$, function $\Omega(\boldsymbol{X},\boldsymbol{X}^*,z) : \mathbb{R}^{2MN} \to \mathbb{C}$ is continuously differentiable with polynomially bounded derivatives and partial derivatives. The regularizer ϕ_M is a continuous function with polynomially bounded partial derivatives. Then, for $M,N \to \infty$ at the same rate and if*

$$\sup_{z\in\mathcal{C}_z^{(m)}} \mathbb{E}\left[|\Omega(\boldsymbol{X},\boldsymbol{X}^*,z)\phi_M|^2\right] < C$$

for some positive deterministic C independent on M we have

$$\mathbb{E}[\Omega(\boldsymbol{X},\boldsymbol{X}^*,z)\phi_M^r] = \mathbb{E}[\Omega(\boldsymbol{X},\boldsymbol{X}^*,z)\phi_M]+\mathcal{O}\left(N^{-\mathbb{N}}\right),$$

for any $r \in \mathbb{N}$ and also

$$\mathbb{E}\left[\Omega(\boldsymbol{X},\boldsymbol{X}^*,z)\frac{\partial\phi_M}{\partial X_{ij}}\right] = \mathcal{O}\left(N^{-\mathbb{N}}\right).$$

This allows to ignore the presence of the regularizer ϕ_M in the asymptotic analysis up to an error term of order $\mathcal{O}(N^{-\mathbb{N}})$ which will be irrelevant for our purpose.

Proof. See [19, Proposition 2, supplementary material]. ∎

Moreover, it can be seen that the characteristic function of $\eta^{(m)}$ in (97), namely $\mathbb{E}[\chi(r)] = \mathbb{E}\left[\exp\left(\text{j}r\eta^{(m)}\right)\right]$ is a differentiable function of r with derivative

$$\begin{aligned}
\frac{\partial\mathbb{E}[\chi(r)]}{\partial r} &= \text{j}\mathbb{E}\left[\eta^{(m)}\chi(r)\phi_M\right] = \text{j}\sum_{p=1}^{L}w_p\sqrt{N}\mathbb{E}\left[\left(\hat{\eta}_p^{(m)}-\bar{\eta}_p^{(m)}\right)\chi(r)\phi_M\right] \\
&= \text{j}\sum_{p=1}^{L}w_p\frac{\sqrt{N}}{2\pi\text{j}}\oint_{\mathcal{C}_{z_1}^{(m)}}\mathbb{E}\left[\left(\hat{h}(z_1)\hat{m}_p(z_1)-\bar{h}(z_1)\bar{m}_p(z_1)\right)\chi(r)\phi_M\right]\text{d}z_1 \\
&= \text{j}\sum_{p=1}^{L}w_p\frac{\sqrt{N}}{2\pi\text{j}}\oint_{\mathcal{C}_{z_1}^{(m)}}\mathbb{E}\left[\boldsymbol{v}_p^{\text{H}}\left(\hat{h}(z_1)\hat{\boldsymbol{Q}}(z_1)-\bar{h}(z_1)\bar{\boldsymbol{Q}}(z_1)\right)\boldsymbol{v}_p\chi(r)\phi_M\right]\text{d}z_1.
\end{aligned} \tag{99}$$

In parallel we compute the first order derivative of the characteristic function $\bar{\chi}(r)$ in (91) with respect to r

$$\frac{\partial\bar{\chi}(r)}{\partial r} = \left(\text{j}\boldsymbol{w}^{\text{T}}\boldsymbol{\mu}-r\boldsymbol{w}^{\text{T}}\boldsymbol{\Gamma}\boldsymbol{w}\right)\bar{\chi}(r). \tag{100}$$

Instead of approximating the characteristic function $\mathbb{E}[\chi(r)] = \mathbb{E}\left[\exp\left(\mathrm{j}r\eta^{(m)}\right)\right]$ with $\bar{\chi}(r)$ in (91) such that $\mathbb{E}[\chi(r)] - \bar{\chi}(r) \to 0$, one can utilize the first order derivatives in (99) and (100) to establish that

$$\frac{\partial \mathbb{E}[\chi(r)]}{\partial r} - \frac{\partial \bar{\chi}(r)}{\partial r} \to 0, \tag{101}$$

for $M, N \to \infty$ at the same rate. Noting that $\chi(0) = 1$, it can be seen that the integration constant is zero which allows to compute $\boldsymbol{\mu}$ and $\boldsymbol{\Gamma}$ from (101). The quantity inside the expectation in (99) can be studied using the resolvent identity

$$z\hat{\boldsymbol{Q}}(z) = \hat{\boldsymbol{Q}}(z)\hat{\boldsymbol{R}} - \boldsymbol{I}_M, \tag{102}$$

which immediately follows from the definition of the resolvent $\hat{\boldsymbol{Q}}(z) = (\hat{\boldsymbol{R}} - z\boldsymbol{I}_M)^{-1}$. According to Assumption 2 it is possible to express the sample covariance matrix in terms of the true covariance matrix as $\hat{\boldsymbol{R}} = \boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^H}{N}\boldsymbol{R}^{H/2}$ where $\boldsymbol{X} = [\boldsymbol{x}_1, ..., \boldsymbol{x}_N]$ denotes an $M \times N$ matrix of independent identically distributed Gaussian random variables with law $\mathcal{CN}(0,1)$. The (i,j)-th entry of $\boldsymbol{X}$ is denoted by X_{ij} and the sample covariance matrix $\hat{\boldsymbol{R}}$ can equivalently be expressed as

$$\hat{\boldsymbol{R}} = \sum_{i=1}^{M}\sum_{j=1}^{N} X_{ij}\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^H}{N}\boldsymbol{R}^{H/2}, \tag{103}$$

where $\boldsymbol{e}_i$ denotes a column vector with all zeros except the i-th entry which is one. Throughout the proof we will be working with with the first order derivative of the sample covariance matrix $\hat{\boldsymbol{R}}$ and the resolvent $\hat{\boldsymbol{Q}}(z)$ with respect to X_{ij} and its complex conjugate X_{ij}^*:

$$\frac{\partial}{\partial X_{ij}}\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^H}{N}\boldsymbol{R}^{H/2} = \boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^H}{N}\boldsymbol{R}^{H/2}, \qquad \frac{\partial}{\partial X_{ij}^*}\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^H}{N}\boldsymbol{R}^{H/2} = \boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^T}{N}\boldsymbol{R}^{H/2}, \tag{104}$$

$$\frac{\partial \hat{\boldsymbol{Q}}(z)}{\partial X_{ij}} = -\hat{\boldsymbol{Q}}(z)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^H}{N}\boldsymbol{R}^{H/2}\hat{\boldsymbol{Q}}(z), \qquad \frac{\partial \hat{\boldsymbol{Q}}(z)}{\partial X_{ij}^*} = -\hat{\boldsymbol{Q}}(z)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^T}{N}\boldsymbol{R}^{H/2}\hat{\boldsymbol{Q}}(z). \tag{105}$$

We begin by expressing the quantity of interest $\sqrt{N}\mathbb{E}\left[\boldsymbol{v}_p^H\left(\hat{h}(z_1)\hat{\boldsymbol{Q}}(z_1) - \bar{h}(z_1)\bar{\boldsymbol{Q}}(z_1)\right)\boldsymbol{v}_p\chi(r)\phi_M\right]$ in (99) as

$$\sqrt{N}\mathbb{E}\left[\boldsymbol{v}_p^H\left(\hat{h}(z_1)\hat{\boldsymbol{Q}}(z_1) - \bar{h}(z_1)\bar{\boldsymbol{Q}}(z_1)\right)\boldsymbol{v}_p\chi(r)\phi_M\right] = \sqrt{N}\mathbb{E}\left[\boldsymbol{v}_p^H\left(\hat{h}(z_1)\hat{\boldsymbol{Q}}(z_1) - \bar{h}(z_1)\bar{\boldsymbol{Q}}(z_1) - \bar{h}(z_1)\hat{\boldsymbol{Q}}(z_1) + \bar{h}(z_1)\hat{\boldsymbol{Q}}(z_1)\right)\boldsymbol{v}_p\chi(r)\phi_M\right]$$

$$= \sqrt{N}\mathbb{E}\left[\left(\hat{h}(z_1) - \bar{h}(z_1)\right)\boldsymbol{v}_p^H\hat{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p\chi(r)\phi_M\right] \tag{106}$$

$$+ \sqrt{N}\bar{h}(z_1)\mathbb{E}\left[\boldsymbol{v}_p^H\left(\hat{\boldsymbol{Q}}(z_1) - \bar{\boldsymbol{Q}}(z_1)\right)\boldsymbol{v}_p\chi(r)\phi_M\right], \tag{107}$$

where we have added and subtracted $\bar{h}(z_1)\hat{\boldsymbol{Q}}(z_1)$ and used the fact that $\bar{h}(z_1)$ is deterministic. The expectation in (106) can be upper bounded according to the following lemma.

Lemma 8. *As* $M, N \to \infty$ *at the same rate, the quantity in* (106) *can be approximated by*

$$\sqrt{N}\mathbb{E}\left[\left(\hat{h}(z_1) - \bar{h}(z_1)\right)\boldsymbol{v}_p^H\hat{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p\chi(r)\phi_M\right] = \begin{cases} 0 & \text{for MUSIC} \\ \mathcal{O}\left(N^{-1/2}\right) & \text{for G-MUSIC.} \end{cases} \tag{108}$$

Proof. See Appendix D. ∎

Next, we analyze the quantity in (107).

Lemma 9. *As* $M, N \to \infty$ *at the same rate, the quantity in* (107) *can be approximated by*

$$\sqrt{N}\bar{h}(z_1)\mathbb{E}\left[\boldsymbol{v}_p^H\left(\hat{\boldsymbol{Q}}(z_1) - \bar{\boldsymbol{Q}}(z_1)\right)\boldsymbol{v}_p\chi(r)\phi_M\right]$$

$$= \mathrm{j}r\bar{h}(z_1)\sum_{q=1}^{L}\frac{w_q}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{z_2}^{(m)}}\frac{z_1}{\omega(z_1)}\frac{z_2}{\omega(z_2)}\bar{h}(z_2)\frac{\boldsymbol{v}_p^H\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\boldsymbol{v}_q^H\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p}{1 - \gamma(z_1, z_2)}\mathbb{E}[\chi(r)]\mathrm{d}z_2 + \mathcal{O}\left(N^{-1/2}\right). \tag{109}$$

Proof. See Appendix F. ∎

Consequently, the quantity of interest can be expressed as

$$\sqrt{N}\mathbb{E}\left[\boldsymbol{v}_p^H\left(\hat{h}(z_1)\hat{\boldsymbol{Q}}(z_1) - \bar{h}(z_1)\bar{\boldsymbol{Q}}(z_1)\right)\boldsymbol{v}_p\chi(r)\phi_M\right]$$

$$= \mathrm{j}r\bar{h}(z_1)\sum_{q=1}^{L}\frac{w_q}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{z_2}^{(m)}}\frac{z_1}{\omega(z_1)}\frac{z_2}{\omega(z_2)}\bar{h}(z_2)\frac{\boldsymbol{v}_p^H\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\boldsymbol{v}_q^H\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p}{1 - \gamma(z_1, z_2)}\mathbb{E}[\chi(r)]\mathrm{d}z_2 + \mathcal{O}\left(N^{-1/2}\right). \tag{110}$$

Substituting (110) into $\frac{\partial \mathbb{E}[\chi(r)]}{\partial r}$ in (99) yields

$$
\begin{aligned}
\frac{\partial \mathbb{E}[\chi(r)]}{\partial r} &= \mathrm{j} \sum_{p=1}^{L} \frac{w_p}{2\pi\mathrm{j}} \oint_{\mathcal{C}_{z_1}^{(m)}} \sqrt{N} \mathbb{E}\Big[\boldsymbol{v}_p^{\mathrm{H}}\big(\hat{h}(z_1)\hat{\boldsymbol{Q}}(z_1) - \bar{h}(z_1)\bar{\boldsymbol{Q}}(z_1)\big)\boldsymbol{v}_p \chi(r)\phi_M\Big]dz_1 \\
&= -r \sum_{p=1}^{L}\sum_{q=1}^{L} \frac{w_p}{2\pi\mathrm{j}}\frac{w_q}{2\pi\mathrm{j}} \oint_{\mathcal{C}_{z_1}^{(m)}} \oint_{\mathcal{C}_{z_2}^{(m)}} \frac{z_1}{\omega(z_1)}\frac{z_2}{\omega(z_2)}\bar{h}(z_1)\bar{h}(z_2)\frac{\boldsymbol{v}_p^{\mathrm{H}}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\boldsymbol{v}_q^{\mathrm{H}}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p}{1-\gamma(z_1,z_2)}dz_2 dz_1 \mathbb{E}[\chi(r)] + \mathcal{O}\big(N^{-1/2}\big).
\end{aligned}
\tag{111}
$$

Comparing (111) with (100) as proposed in (101), it can be seen that $\boldsymbol{\mu} = \boldsymbol{0}$ and

$$
[\boldsymbol{\Gamma}]_{p,q} = \frac{1}{2\pi\mathrm{j}}\frac{1}{2\pi\mathrm{j}} \oint_{\mathcal{C}_{z_1}^{(m)}} \oint_{\mathcal{C}_{z_2}^{(m)}} \frac{z_1}{\omega(z_1)}\frac{z_2}{\omega(z_2)}\bar{h}(z_1)\bar{h}(z_2)\frac{\boldsymbol{v}_p^{\mathrm{H}}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\boldsymbol{v}_q^{\mathrm{H}}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p}{1-\gamma(z_1,z_2)}dz_2 dz_1,
\tag{112}
$$

where $p,q \in \{1,...,L\}$. Substituting $\bar{\boldsymbol{Q}}(z)$ from (85) into (112) and applying the change of variables $\mathrm{d}z = \frac{\partial z(\omega)}{\partial \omega}\mathrm{d}\omega$ we obtain

$$
[\boldsymbol{\Gamma}]_{p,q} = \frac{1}{2\pi\mathrm{j}}\frac{1}{2\pi\mathrm{j}} \oint_{\mathcal{C}_{\omega_1}^{(m)}} \oint_{\mathcal{C}_{\omega_2}^{(m)}} \bar{h}(z(\omega_1))\bar{h}(z(\omega_2))\frac{\omega_1}{z(\omega_1)}\frac{\omega_2}{z(\omega_2)}\frac{\partial z(\omega_1)}{\partial \omega_1}\frac{\partial z(\omega_2)}{\partial \omega_2}\frac{\Upsilon_{p,q}(\omega_1,\omega_2)}{1-\Omega(\omega_1,\omega_2)}\mathrm{d}\omega_2 \mathrm{d}\omega_1
\tag{113}
$$

where $\mathcal{C}_{\omega}^{(m)} = \omega(\mathcal{C}_z^{(m)})$ denotes a negatively orientated contour that encloses γ_m only and

$$
\Omega(\omega_1,\omega_2) = \frac{1}{N}\mathrm{tr}\big[\boldsymbol{R}(\boldsymbol{R}-\omega_1\boldsymbol{I}_M)^{-1}\boldsymbol{R}(\boldsymbol{R}-\omega_2\boldsymbol{I}_M)^{-1}\big],
$$
$$
\Upsilon_{p,q}(\omega_1,\omega_2) = \boldsymbol{v}_p^{\mathrm{H}}(\boldsymbol{R}-\omega_1\boldsymbol{I}_M)^{-1}\boldsymbol{R}(\boldsymbol{R}-\omega_2\boldsymbol{I}_M)^{-1}\boldsymbol{v}_q\boldsymbol{v}_q^{\mathrm{H}}(\boldsymbol{R}-\omega_2\boldsymbol{I}_M)^{-1}\boldsymbol{R}(\boldsymbol{R}-\omega_1\boldsymbol{I}_M)^{-1}\boldsymbol{v}_p.
$$

In order to obtain the expression given in Theorem 3 of the article we choose $m=1$ and $\boldsymbol{v}_l = \boldsymbol{a}(\bar{\theta}_l)$ for $l = 1,...,L$.

APPENDIX D
PROOF OF LEMMA 8

It can be observed that in case of the conventional N-consistent estimator (the MUSIC case)

$$
\sqrt{N}\mathbb{E}\Big[\big(\hat{h}(z_1)-\bar{h}(z_1)\big)\boldsymbol{v}_p^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p\chi(r)\phi_M\Big] = 0,
$$

since $\hat{h}(z_1)-\bar{h}(z_1) = 0$. However, in case of the M,N-consistent estimator (the g-MUSIC case) we have to study the asymptotic behavior of $(\hat{h}(z)-\bar{h}(z))$ since $\hat{h}(z_1)-\bar{h}(z_1) \neq 0$. Using the Cauchy-Schwarz inequality we can upper-bound

$$
\begin{aligned}
\Big|\mathbb{E}\Big[\big(\hat{h}(z_1)-\bar{h}(z_1)\big)\boldsymbol{v}_p^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p\chi(r)\phi_M\Big]\Big|^2 &\leq \mathbb{E}\Big[\big|\hat{h}(z_1)-\bar{h}(z_1)\big|^2\phi_M\Big]\mathbb{E}\Big[\big|\boldsymbol{v}_p^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p\chi(r)\big|^2\phi_M\Big] \\
&= \mathbb{E}\Big[\big|\hat{h}(z_1)-\bar{h}(z_1)\big|^2\phi_M\Big]\mathbb{E}\Big[\big|\boldsymbol{v}_p^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p\big|^2|\chi(r)|^2\phi_M\Big] \\
&= \mathbb{E}\Big[\big|\hat{h}(z_1)-\bar{h}(z_1)\big|^2\phi_M\Big]\mathbb{E}\Big[\big|\boldsymbol{v}_p^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p\big|^2\phi_M\Big],
\end{aligned}
\tag{114}
$$

where we have used $|\chi(r)|^2 = 1$. Using the definition of the variance of a random variable we can express $\mathbb{E}\Big[\big|\hat{h}(z_1)-\bar{h}(z_1)\big|^2\phi_M\Big]$ in (114) as follows

$$
\mathbb{E}\Big[\big|\hat{h}(z_1)-\bar{h}(z_1)\big|^2\phi_M\Big] = \mathrm{var}\Big[(\hat{h}(z_1)-\bar{h}(z_1))\phi_M\Big] + \mathbb{E}\Big[(\hat{h}(z_1)-\bar{h}(z_1))\phi_M\Big]^2.
\tag{115}
$$

The asymptotic stochastic behavior of $\hat{h}(z)$ is analyzed next.

Lemma 10. *As $M,N \to \infty$ at the same rate,*

$$
\mathbb{E}\Big[\hat{h}(z)\phi_M\Big] = \bar{h}(z) + \mathcal{O}\big(N^{-1}\big)
$$

and the variance decays with

$$
\mathrm{var}\Big[\hat{h}(z)\phi_M\Big] = \mathcal{O}\big(N^{-2}\big).
$$

Proof. See Appendix E ∎

According to Lemma 10 and (115), $\mathbb{E}\Big[\big|\hat{h}(z_1)-\bar{h}(z_1)\big|^2\phi_M\Big] = \mathcal{O}\big(N^{-2}\big)$. Similarly, we can express the second term on the right hand side of (114) as

$$
\mathbb{E}\Big[\big|\boldsymbol{v}_p^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p\phi_M\big|^2\Big] = \mathrm{var}\Big[\boldsymbol{v}_p^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p\phi_M\Big] + \mathbb{E}\Big[\boldsymbol{v}_p^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p\phi_M\Big]^2 = \mathcal{O}\big(N^{-1}\big) + \mathcal{O}(1) = \mathcal{O}(1)
$$

such that (114) decays with

$$
\Big|\mathbb{E}\Big[\big(\hat{h}(z_1)-\bar{h}(z_1)\big)\boldsymbol{v}_p^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p\chi(r)\phi_M\Big]\Big|^2 \leq \mathbb{E}\Big[\big|\hat{h}(z_1)-\bar{h}(z_1)\big|^2\phi_M\Big]\mathbb{E}\Big[\big|\boldsymbol{v}_p^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p\chi(r)\big|^2\phi_M\Big] = \mathcal{O}\big(N^{-2}\big),
$$

and consequently, the quantity of interest decays with

$$
\sqrt{N}\mathbb{E}\Big[\big(\hat{h}(z_1)-\bar{h}(z_1)\big)\boldsymbol{v}_p^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p\chi(r)\phi_M\Big] = \mathcal{O}\big(N^{-1/2}\big).
$$

APPENDIX E
PROOF OF LEMMA 10

Computing the first order derivative of

$$\frac{z}{\hat{\omega}(z)} = 1 - \frac{1}{N}\mathrm{tr}\Big[\hat{R}\hat{Q}(z)\Big] = 1 - \frac{1}{N}\mathrm{tr}\Big[\hat{R}\big(\hat{R} - zI_M\big)^{-1}\Big], \tag{116}$$

w.r.t. z, we can express

$$\hat{\omega}'(z) = \frac{\partial\hat{\omega}(z)}{\partial z} = \frac{\hat{\omega}(z)}{z} + \frac{\hat{\omega}(z)^2}{z}\frac{1}{N}\mathrm{tr}\Big[\hat{R}\hat{Q}(z)^2\Big] = \frac{\hat{\omega}(z)}{z} + \frac{\hat{\omega}(z)^2}{z}\frac{1}{N}\mathrm{tr}\Big[\hat{R}\big(\hat{R} - zI_M\big)^{-2}\Big], \tag{117}$$

such that

$$\hat{h}(z) = \frac{z}{\hat{\omega}(z)}\frac{\partial\hat{\omega}(z)}{\partial z} = 1 + \hat{\omega}(z)\frac{1}{N}\mathrm{tr}\Big[\hat{R}\hat{Q}(z)^2\Big].$$

Similarly, we compute the first order derivative of

$$\frac{z}{\omega(z)} = 1 - \frac{z}{\omega(z)}\frac{1}{N}\mathrm{tr}\big[R\bar{Q}(z)\big] = 1 - \frac{1}{N}\mathrm{tr}\big[R(R - \omega(z)I_M)^{-1}\big], \tag{118}$$

w.r.t. z and obtain

$$\begin{aligned}
\omega'(z) = \frac{\partial\omega(z)}{\partial z} &= \left(\frac{z}{\omega(z)} - \omega(z)\frac{1}{N}\mathrm{tr}\big[R(R - \omega(z)I_M)^{-2}\big]\right)^{-1}\\
&= \left(\frac{z}{\omega(z)} - \omega(z)\frac{1}{N}\frac{z^2}{\omega(z)^2}\mathrm{tr}\big[R\bar{Q}(z)^2\big]\right)^{-1}\\
&= \frac{\omega(z)}{z}\left(1 - z\frac{1}{N}\mathrm{tr}\big[R\bar{Q}(z)^2\big]\right)^{-1},
\end{aligned} \tag{119}$$

such that

$$\bar{h}(z) = \frac{z}{\omega(z)}\frac{\partial\omega(z)}{\partial z} = \left(1 - z\frac{1}{N}\mathrm{tr}\big[R\bar{Q}(z)^2\big]\right)^{-1}.$$

A. Variance

The variance of $\hat{h}(z) = 1 + \hat{\omega}(z)\frac{1}{N}\mathrm{tr}\Big[\hat{R}\hat{Q}(z)^2\Big]\phi_M$ is identical to $\mathrm{var}\Big[\hat{\omega}(z)\frac{1}{N}\mathrm{tr}\Big[\hat{R}\hat{Q}(z)^2\Big]\phi_M\Big]$ which can be upper-bounded using the common bound

$$\mathrm{var}\Big[\hat{\omega}(z)\frac{1}{N}\mathrm{tr}\Big[\hat{Q}(z)^2\hat{R}\Big]\phi_M\Big] \leq 2\|\hat{\omega}(z)\phi_M\|^2\mathrm{var}\Big[\frac{1}{N}\mathrm{tr}\Big[\hat{Q}(z)^2\hat{R}\Big]\phi_M\Big] + 2\left\|\frac{1}{N}\mathrm{tr}\Big[\hat{Q}(z)^2\hat{R}\Big]\phi_M\right\|^2\mathrm{var}[\hat{\omega}(z)\phi_M]. \tag{120}$$

According to Lemma 15,

$$\mathrm{var}\Big[\frac{1}{N}\mathrm{tr}\Big[\hat{Q}(z)^2\hat{R}\phi_M\Big]\Big] = \mathcal{O}\big(N^{-2}\big).$$

Next, the variance of $\hat{\omega}(z)$ in (116) is upper-bounded by

$$\begin{aligned}
\mathrm{var}[\hat{\omega}(z)\phi_M] &= \mathbb{E}\big[|\hat{\omega}(z) - \mathbb{E}[\hat{\omega}(z)]|^2\phi_M\big] = \mathbb{E}\big[|\hat{\omega}(z) - \omega(z) + \omega(z) - \mathbb{E}[\hat{\omega}(z)]|^2\phi_M\big]\\
&\leq 2\mathbb{E}\big[|\hat{\omega}(z) - \omega(z)|^2\phi_M\big] + 2\mathbb{E}\big[|\omega(z) - \mathbb{E}[\hat{\omega}(z)]|^2\phi_M\big],
\end{aligned} \tag{121}$$

where we have added and subtracted $\omega(z)$ and applied Jensen's inequality. Inserting the expressions of $\omega(z)$ in (118) and $\hat{\omega}(z)$ in (116) into (121) yields

$$\begin{aligned}
\mathrm{var}[\hat{\omega}(z)\phi_M] \leq {} &2\mathbb{E}\left[\left|\frac{z}{1 - \frac{1}{N}\mathrm{tr}\Big[\hat{R}\hat{Q}(z)\Big]} - \frac{z}{1 - \frac{1}{N}\mathrm{tr}[RQ(z)]}\right|^2\phi_M\right]\\
&+ 2\mathbb{E}\left[\left|\frac{z}{1 - \frac{1}{N}\mathrm{tr}[RQ(z)]} - \mathbb{E}\left[\frac{z}{1 - \frac{1}{N}\mathrm{tr}\Big[\hat{R}\hat{Q}(z)\Big]}\right]\right|^2\phi_M\right],
\end{aligned} \tag{122}$$

where $Q(z) = (R - \omega(z)I_M)^{-1}$ and $\hat{Q}(z) = (\hat{R} - zI_M)^{-1}$. Before the upper-bounds for the first and second term on the right hand side of (122) are computed it is shown that

$$\frac{\phi_M}{\left|1 - \frac{1}{N}\mathrm{tr}\Big[\hat{R}\hat{Q}(z)\Big]\right|} \tag{123}$$

is bounded on $z \in \mathcal{C}_z^{(m)}$ for all M,N sufficiently large which is essential in order to proof that (122) is bounded. Using the fact that the roots $\hat{\mu}_k$ are the solutions to the polynomial equation $1 = \frac{1}{N}\sum_{r=1}^{M}\frac{\hat{\lambda}_r}{\hat{\lambda}_r - z} = \frac{1}{N}\sum_{r=[M-N]^++1}^{M}\frac{\hat{\lambda}_r}{\hat{\lambda}_r - z}$ we can establish the following identity

$$\prod_{k=[M-N]^++1}^{M}(\hat{\mu}_k - z) = \prod_{k=[M-N]^++1}^{M}(\hat{\lambda}_k - z) - \frac{1}{N}\sum_{r=[M-N]^++1}^{M}\hat{\lambda}_r \prod_{\substack{k=[M-N]^++1 \\ k \neq r}}^{M}(\hat{\lambda}_k - z).$$

which allows us to express $1 - \frac{1}{N}\text{tr}\left[\hat{R}\hat{Q}(z)\right]$ as

$$1 - \frac{1}{N}\sum_{m=1}^{M}\frac{\hat{\lambda}_m}{\hat{\lambda}_m - z} = \frac{\prod_{k=[M-N]^++1}^{M}(\hat{\mu}_k - z)}{\prod_{k=[M-N]^++1}^{M}(\hat{\lambda}_k - z)}. \tag{124}$$

Furthermore, using the representation of $1 - \frac{1}{N}\text{tr}\left[\hat{R}\hat{Q}(z)\right]$ in (124) we can establish that

$$\left|1 - \frac{1}{N}\sum_{m=1}^{M}\frac{\hat{\lambda}_m}{\hat{\lambda}_m - z}\right|^{-1}\phi_M = \frac{\prod_{k=[M-N]^++1}^{M}\left|\hat{\lambda}_k - z\right|}{\prod_{k=[M-N]^++1}^{M}|\hat{\mu}_k - z|}\phi_M \leq \left(\frac{\|\boldsymbol{R}\|_{\text{F}} + |z| + \epsilon}{\text{dist}(z,\mathcal{S})}\right)^{\min(M,N)}. \tag{125}$$

Due to the regularizer ϕ_M in (96) we have a continuous function $\phi_M\left(1 - \frac{1}{N}\text{tr}\left[\hat{R}\hat{Q}(z)\right]\right)^{-1}$ in (125) over a compact support that always has a minimum and a maximum and is therefore bounded. In the following the upper-bound of (123) on $z \in \mathcal{C}_z^{(m)}$ is denoted by $\kappa(z) = \left(\frac{\|\boldsymbol{R}\|_{\text{F}} + |z| + \epsilon}{\text{dist}(z,\mathcal{S})}\right)^{\min(M,N)}$. With this the first term on the right hand side of (122) can be upper-bounded by

$$
\begin{aligned}
\mathbb{E}\left[|\hat{\omega}(z) - \omega(z)|^2\phi_M\right] &= \mathbb{E}\left[\left|\frac{z}{1 - \frac{1}{N}\text{tr}\left[\hat{R}\hat{Q}(z)\right]} - \frac{z}{1 - \frac{1}{N}\text{tr}[\boldsymbol{R}\boldsymbol{Q}(z)]}\right|^2\phi_M\right] \\
&= |z|^2\mathbb{E}\left[\left|\frac{1 - \frac{1}{N}\text{tr}[\boldsymbol{R}\boldsymbol{Q}(z)] - 1 + \frac{1}{N}\text{tr}\left[\hat{R}\hat{Q}(z)\right]}{\left(1 - \frac{1}{N}\text{tr}[\boldsymbol{R}\boldsymbol{Q}(z)]\right)\left(1 - \frac{1}{N}\text{tr}\left[\hat{R}\hat{Q}(z)\right]\right)}\right|^2\phi_M\right] \\
&= |z|^2\mathbb{E}\left[\left|\frac{\frac{1}{N}\text{tr}\left[\hat{R}\hat{Q}(z)\right] - \frac{1}{N}\text{tr}[\boldsymbol{R}\boldsymbol{Q}(z)]}{\left(1 - \frac{1}{N}\text{tr}[\boldsymbol{R}\boldsymbol{Q}(z)]\right)\left(1 - \frac{1}{N}\text{tr}\left[\hat{R}\hat{Q}(z)\right]\right)}\right|^2\phi_M\right] \\
&= |\omega(z)|^2\mathbb{E}\left[\left|\frac{\frac{1}{N}\text{tr}\left[\hat{R}\hat{Q}(z)\right] - \frac{1}{N}\text{tr}[\boldsymbol{R}\boldsymbol{Q}(z)]}{1 - \frac{1}{N}\text{tr}\left[\hat{R}\hat{Q}(z)\right]}\right|^2\phi_M\right] \\
&\leq \sup_{z \in \mathcal{C}_z^{(m)}}|\omega(z)\kappa(z)|^2\mathbb{E}\left[\left|\frac{1}{N}\text{tr}\left[\hat{R}\hat{Q}(z)\right] - \frac{1}{N}\text{tr}[\boldsymbol{R}\boldsymbol{Q}(z)]\right|^2\phi_M\right] \\
&= \sup_{z \in \mathcal{C}_z^{(m)}}|\omega(z)\kappa(z)|^2\left(\text{var}\left[\frac{1}{N}\text{tr}\left[\hat{R}\hat{Q}(z)\phi_M\right]\right] + \left|\mathbb{E}\left[\left(\frac{1}{N}\text{tr}\left[\hat{R}\hat{Q}(z)\right] - \frac{1}{N}\text{tr}[\boldsymbol{R}\boldsymbol{Q}(z)]\right)\phi_M\right]\right|^2\right) \\
&= \sup_{z \in \mathcal{C}_z^{(m)}}|\omega(z)\kappa(z)|^2\left(\text{var}\left[\frac{1}{N}\text{tr}\left[\hat{R}\hat{Q}(z)\phi_M\right]\right] + \mathcal{O}\left(N^{-2}\right)\right) \\
&= \sup_{z \in \mathcal{C}_z^{(m)}}|\omega(z)\kappa(z)|^2\mathcal{O}\left(N^{-2}\right), \tag{126}
\end{aligned}
$$

where we have used (159) and (161) in Lemma 13. Similarly, the second term on the right hand side of (122) can be upper-bounded by

$$
\begin{aligned}
\mathbb{E}\big[|\omega(z)-\mathbb{E}[\hat{\omega}(z)]|^2\phi_M\big] &= \mathbb{E}\left[\left|\left|\frac{z}{1-\frac{1}{N}\mathrm{tr}[\boldsymbol{R}\boldsymbol{Q}(z)]}-\mathbb{E}\left[\frac{z}{1-\frac{1}{N}\mathrm{tr}\big[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\big]}\right]\right|\right|^2\phi_M\right] \\
&= \left|\mathbb{E}\left[\left(\frac{z}{1-\frac{1}{N}\mathrm{tr}[\boldsymbol{R}\boldsymbol{Q}(z)]}-\frac{z}{1-\frac{1}{N}\mathrm{tr}\big[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\big]}\right)\phi_M\right]\right|^2 \\
&= |z|^2\left|\mathbb{E}\left[\frac{1-\frac{1}{N}\mathrm{tr}\big[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\big]-1+\frac{1}{N}\mathrm{tr}[\boldsymbol{R}\boldsymbol{Q}(z)]}{\big(1-\frac{1}{N}\mathrm{tr}[\boldsymbol{R}\boldsymbol{Q}(z)]\big)\big(1-\frac{1}{N}\mathrm{tr}\big[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\big]\big)}\phi_M\right]\right|^2 \\
&= |\omega(z)|^2\left|\mathbb{E}\left[\frac{\frac{1}{N}\mathrm{tr}[\boldsymbol{R}\boldsymbol{Q}(z)]-\frac{1}{N}\mathrm{tr}\big[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\big]}{1-\frac{1}{N}\mathrm{tr}\big[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\big]}\phi_M\right]\right|^2 \\
&\le \sup_{z\in\mathcal{C}_z^{(m)}}|\omega(z)\kappa(z)|^2\left|\mathbb{E}\left[\left(\frac{1}{N}\mathrm{tr}[\boldsymbol{R}\boldsymbol{Q}(z)]-\frac{1}{N}\mathrm{tr}\big[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\big]\right)\phi_M\right]\right|^2 \\
&= \sup_{z\in\mathcal{C}_z^{(m)}}|\omega(z)\kappa(z)|^2\mathcal{O}\big(N^{-2}\big) \tag{127}
\end{aligned}
$$

where we have used (159) in Lemma 13. Correspondingly, (126) and (127) both decay with $\mathcal{O}\big(N^{-2}\big)$ and the variance of $\hat{\omega}(z)$ in (122) decays with $\mathrm{var}[\hat{\omega}(z)\phi_M] = \mathcal{O}\big(N^{-2}\big)$. Furthermore, since $\hat{\omega}(z)\phi_M$ and $\frac{1}{N}\mathrm{tr}\big[\hat{\boldsymbol{Q}}(z)^2\hat{\boldsymbol{R}}\big]\phi_M$ in (120) are bounded on $z\in\mathcal{C}_z^{(m)}$ by definition, the variance in (120) decays with $\mathcal{O}\big(N^{-2}\big)$. Hence, the variance of $\hat{h}(z)$ also decays with $\mathrm{var}\big[\hat{h}(z)\phi_M\big] = \mathcal{O}\big(N^{-2}\big)$.

B. Expectation

Using Lemma 15 and the definition of $\omega'(z)$ in (119) it can be shown that the expectation of $\hat{h}(z)$ is given by

$$
\begin{aligned}
\mathbb{E}\big[\hat{h}(z)\phi_M\big] &= \mathbb{E}\left[\left(1+\hat{\omega}(z)\frac{1}{N}\mathrm{tr}\big[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)^2\big]\right)\phi_M\right] \\
&= 1+\omega(z)\frac{z^2}{\omega(z)^2}\frac{1}{N}\mathrm{tr}\big[\bar{\boldsymbol{Q}}(z)^2\boldsymbol{R}\big]\omega'(z)+\mathcal{O}(N^{-1}) \\
&= \omega'(z)\left(\frac{1}{\omega'(z)}+\frac{z^2}{\omega(z)}\frac{1}{N}\mathrm{tr}\big[\bar{\boldsymbol{Q}}(z)^2\boldsymbol{R}\big]\right)+\mathcal{O}(N^{-1}) \\
&= \omega'(z)\left(\frac{z}{\omega(z)}-\frac{z^2}{\omega(z)}\frac{1}{N}\mathrm{tr}\big[\bar{\boldsymbol{Q}}(z)^2\boldsymbol{R}\big]+\frac{z^2}{\omega(z)}\frac{1}{N}\mathrm{tr}\big[\bar{\boldsymbol{Q}}(z)^2\boldsymbol{R}\big]\right)+\mathcal{O}(N^{-1}) \\
&= \frac{z}{\omega(z)}\omega'(z)+\mathcal{O}(N^{-1}) \\
&= \bar{h}(z)+\mathcal{O}(N^{-1}),
\end{aligned}
$$

where $\bar{h}(z) = \frac{z}{\omega(z)}\frac{\partial\omega(z)}{\partial z}$.

APPENDIX F
PROOF OF LEMMA 9

To begin with, we develop the expectation $\mathbb{E}\big[\hat{\boldsymbol{Q}}(z)\hat{\boldsymbol{R}}\chi(r)\phi_M\big]$ by using the expression of the sample covariance matrix in (103) and by applying the integration by parts formula

$$
\mathbb{E}\big[\hat{\boldsymbol{Q}}(z)\hat{\boldsymbol{R}}\chi(r)\phi_M\big] = \sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[X_{ij}\hat{\boldsymbol{Q}}(z)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\chi(r)\phi_M\right] = \sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\frac{\partial}{\partial X_{ij}^*}\hat{\boldsymbol{Q}}(z)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\chi(r)\phi_M\right], \tag{128}
$$

where the first order derivative with respect to X_{ij}^* is given by

$$
\frac{\partial}{\partial X_{ij}^*}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\chi(r)\phi_M
$$
$$
=-\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\chi(r)\phi_M+\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\chi(r)\phi_M+\mathcal{O}\left(N^{-\mathbb{N}}\right)
$$
$$
+\mathrm{j}r\sum_{q=1}^{L}w_q\frac{\sqrt{N}}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{z_2}^{(m)}}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{v}_q^{\mathrm{H}}\left[\left(\frac{\partial}{\partial X_{ij}^*}\hat{h}(z_2)\right)\hat{\boldsymbol{Q}}(z_2)+\hat{h}(z_2)\left(\frac{\partial}{\partial X_{ij}^*}\hat{\boldsymbol{Q}}(z_2)\right)\right]\boldsymbol{v}_q\chi(r)\phi_M\mathrm{d}z_2
$$
$$
=-\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\chi(r)\phi_M+\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\chi(r)\phi_M+\mathcal{O}\left(N^{-\mathbb{N}}\right)
$$
$$
-\mathrm{j}r\sum_{q=1}^{L}w_q\frac{\sqrt{N}}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{z_2}^{(m)}}\hat{h}(z_2)\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\chi(r)\phi_M\mathrm{d}z_2
$$
$$
+\mathrm{j}r\sum_{q=1}^{L}w_q\frac{\sqrt{N}}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{z_2}^{(m)}}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{v}_q^{\mathrm{H}}\left(\frac{\partial}{\partial X_{ij}^*}\hat{h}(z_2)\right)\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\chi(r)\phi_M\mathrm{d}z_2. \tag{129}
$$

The first order derivative of $\hat{h}(z_2)$ in (94) with respect to X_{ij}^* yields

$$
\frac{\partial}{\partial X_{ij}^*}\hat{h}(z_2)=\frac{\partial}{\partial X_{ij}^*}\left(1+\hat{\omega}(z_2)\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z_2)^2\right]\right)=\frac{\partial}{\partial X_{ij}^*}\left(1+\frac{z_2\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z_2)^2\right]}{1-\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z_2)\right]}\right)
$$
$$
=\hat{\omega}(z_2)\frac{1}{N}\sum_{m=1}^{M}\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_2)^2\boldsymbol{e}_m
$$
$$
-\hat{\omega}(z_2)\frac{2}{N}\sum_{m=1}^{M}\boldsymbol{e}_m^{\mathrm{T}}\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z_2)^3\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{e}_m \tag{130}
$$
$$
+\frac{\hat{\omega}(z_2)^2}{z_2}\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z_2)^2\right]\frac{1}{N}\sum_{m=1}^{M}\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{e}_m
$$
$$
-\frac{\hat{\omega}(z_2)^2}{z_2}\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z_2)^2\right]\frac{1}{N}\sum_{m=1}^{M}\boldsymbol{e}_m^{\mathrm{T}}\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{e}_m.
$$

Substituting the first oder derivative of $\hat{h}(z_2)$ in (130) into (129) one obtains

$$
\frac{\partial}{\partial X_{ij}^*}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\chi(r)\phi_M
$$
$$
=-\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\chi(r)\phi_M+\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\chi(r)\phi_M+\mathcal{O}\left(N^{-\mathbb{N}}\right)
$$
$$
-\mathrm{j}r\sum_{q=1}^{L}w_q\frac{\sqrt{N}}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{z_2}^{(m)}}\hat{h}(z_2)\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\chi(r)\phi_M\mathrm{d}z_2
$$
$$
+\mathrm{j}r\sum_{q=1}^{L}w_q\frac{\sqrt{N}}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{z_2}^{(m)}}\frac{1}{N}\hat{\omega}(z_2)\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_2)^2\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\chi(r)\phi_M\mathrm{d}z_2 \tag{131}
$$
$$
-\mathrm{j}r\sum_{q=1}^{L}w_q\frac{\sqrt{N}}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{z_2}^{(m)}}\frac{2}{N}\hat{\omega}(z_2)\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z_2)^3\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{x}_j}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\chi(r)\phi_M\mathrm{d}z_2
$$
$$
+\mathrm{j}r\sum_{q=1}^{L}w_q\frac{\sqrt{N}}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{z_2}^{(m)}}\frac{1}{N}\frac{\hat{\omega}(z_2)^2}{z_2}\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z_2)^2\right]\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\chi(r)\phi_M\mathrm{d}z_2
$$
$$
-\mathrm{j}r\sum_{q=1}^{L}w_q\frac{\sqrt{N}}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{z_2}^{(m)}}\frac{1}{N}\frac{\hat{\omega}(z_2)^2}{z_2}\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z_2)^2\right]\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\chi(r)\phi_M\mathrm{d}z_2
$$

Next, substituting (131) into (129) and computing the double sum we can equivalently express the expectation $\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\chi(r)\phi_M\right]$ as

$$
\begin{aligned}
\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\chi(r)\phi_M\right] =&-\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\chi(r)\phi_M\frac{1}{N}\mathrm{tr}\!\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]\right]+\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\chi(r)\phi_M\right]+\mathcal{O}\!\left(N^{-\mathbb{N}}\right)\\
&-\mathrm{j}r\sum_{q=1}^{L}w_q\frac{\sqrt{N}}{2\pi\mathrm{j}}\frac{1}{N}\oint_{\mathcal{C}_{z_2}^{(m)}}\mathbb{E}\!\left[\hat{h}(z_2)\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\chi(r)\phi_M\right]\mathrm{d}z_2\\
&+\mathrm{j}r\sum_{q=1}^{L}w_q\frac{\sqrt{N}}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{z_2}^{(m)}}\frac{1}{N^2}\mathbb{E}\!\left[\hat{\omega}(z_2)\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)^2\hat{\boldsymbol{R}}\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\chi(r)\phi_M\right]\mathrm{d}z_2\\
&-\mathrm{j}r\sum_{q=1}^{L}w_q\frac{\sqrt{N}}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{z_2}^{(m)}}\frac{2}{N^2}\mathbb{E}\!\left[\hat{\omega}(z_2)\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z_2)^3\hat{\boldsymbol{R}}\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\chi(r)\phi_M\right]\mathrm{d}z_2\\
&+\mathrm{j}r\sum_{q=1}^{L}w_q\frac{\sqrt{N}}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{z_2}^{(m)}}\frac{1}{N^2}\mathbb{E}\!\left[\frac{\hat{\omega}(z_2)^2}{z_2}\frac{1}{N}\mathrm{tr}\!\left[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z_2)^2\right]\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\chi(r)\phi_M\right]\mathrm{d}z_2\\
&-\mathrm{j}r\sum_{q=1}^{L}w_q\frac{\sqrt{N}}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{z_2}^{(m)}}\frac{1}{N^2}\mathbb{E}\!\left[\frac{\hat{\omega}(z_2)^2}{z_2}\frac{1}{N}\mathrm{tr}\!\left[\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z_2)^2\right]\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\chi(r)\phi_M\right]\mathrm{d}z_2.
\end{aligned}
\tag{132}
$$

In (132) it can be seen that due to the regularizer ϕ_M all expectations will converge to bounded quantities as M and N increase without bound at the same rate. Moreover, the last four terms on the right hand side in (132) decay one order faster than the third term on the right hand side. Therefore, we neglect the last four terms and add an error term $\mathcal{O}(N^{-3/2})$ such that

$$
\begin{aligned}
\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\chi(r)\phi_M\right] =&-\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\chi(r)\phi_M\frac{1}{N}\mathrm{tr}\!\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]\right]+\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\chi(r)\phi_M\right]+\mathcal{O}\!\left(N^{-3/2}\right)\\
&-\mathrm{j}r\sum_{q=1}^{L}w_q\frac{\sqrt{N}}{2\pi\mathrm{j}}\frac{1}{N}\oint_{\mathcal{C}_{z_2}^{(m)}}\mathbb{E}\!\left[\hat{h}(z_2)\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\chi(r)\phi_M\right]\mathrm{d}z_2.
\end{aligned}
\tag{133}
$$

In the following we add the quantity $\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\chi(r)\phi_M\right]\frac{1}{N}\mathrm{tr}\!\left[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\right]$ on both sides of (133). For the left hand side of (133) we obtain

$$
\begin{aligned}
&\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\chi(r)\phi_M\right]+\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\chi(r)\phi_M\right]\frac{1}{N}\mathrm{tr}\!\left[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\right]\\
=&\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\chi(r)\phi_M\right]\left(1+\frac{1}{N}\mathrm{tr}\!\left[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\right]\right) = \mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\chi(r)\phi_M\right]\frac{\omega(z_1)}{z_1},
\end{aligned}
\tag{134}
$$

where we have used $\left(1+\frac{1}{N}\mathrm{tr}\!\left[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\right]\right)=\frac{\omega(z_1)}{z_1}$ which follows from (86). For the right hand side of (133) we obtain

$$
\begin{aligned}
&-\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\chi(r)\phi_M\frac{1}{N}\mathrm{tr}\!\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]\right]+\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\chi(r)\phi_M\right]+\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\chi(r)\phi_M\right]\frac{1}{N}\mathrm{tr}\!\left[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\right]\\
&-\mathrm{j}r\sum_{q=1}^{L}\frac{w_q}{2\pi\mathrm{j}}\frac{1}{N}\sqrt{N}\oint_{\mathcal{C}_{z_2}^{(m)}}\mathbb{E}\!\left[\hat{h}(z_2)\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\chi(r)\phi_M\right]\mathrm{d}z_2+\mathcal{O}\!\left(N^{-3/2}\right)\\
=&-\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\chi(r)\phi_M\alpha(z_1)\right]+\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\chi(r)\phi_M\right]\\
&-\mathrm{j}r\sum_{q=1}^{L}\frac{w_q}{2\pi\mathrm{j}}\frac{1}{N}\sqrt{N}\oint_{\mathcal{C}_{z_2}^{(m)}}\mathbb{E}\!\left[\hat{h}(z_2)\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\chi(r)\phi_M\right]\mathrm{d}z_2+\mathcal{O}\!\left(N^{-3/2}\right),
\end{aligned}
\tag{135}
$$

where we have introduced

$$
\alpha(z_1) = \frac{1}{N}\mathrm{tr}\!\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\right]\phi_M-\frac{1}{N}\mathrm{tr}\!\left[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\right].
\tag{136}
$$

Equating (134) and (135) and multiplying by $\frac{z_1}{\omega(z_1)}$ on both sides yields

$$
\begin{aligned}
\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\chi(r)\phi_M\right] =&-\frac{z_1}{\omega(z_1)}\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\chi(r)\phi_M\alpha(z_1)\right]+\frac{z_1}{\omega(z_1)}\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\chi(r)\phi_M\right]+\mathcal{O}\!\left(N^{-3/2}\right)\\
&-\frac{z_1}{\omega(z_1)}\mathrm{j}r\sum_{q=1}^{L}\frac{w_q}{2\pi\mathrm{j}}\frac{1}{N}\sqrt{N}\oint_{\mathcal{C}_{z_2}^{(m)}}\mathbb{E}\!\left[\hat{h}(z_2)\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\chi(r)\phi_M\right]\mathrm{d}z_2.
\end{aligned}
\tag{137}
$$

In the following we apply the resolvent identity in (102) to the left hand side of (137) and subtract $\frac{z_1}{\omega(z_1)}\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\chi(r)\phi_M\right]$ on both sides

$$
\begin{aligned}
&z_1\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\chi(r)\phi_M\right]+\boldsymbol{I}_M\mathbb{E}[\chi(r)\phi_M]-\frac{z_1}{\omega(z_1)}\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\chi(r)\phi_M\right]\\
=&-\frac{z_1}{\omega(z_1)}\mathbb{E}\!\left[\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\chi(r)\phi_M\alpha(z_1)\right]-\mathrm{j}r\frac{z_1}{\omega(z_1)}\sum_{q=1}^{L}\frac{w_q}{2\pi\mathrm{j}}\frac{1}{N}\sqrt{N}\oint_{\mathcal{C}_{z_2}^{(m)}}\mathbb{E}\!\left[\hat{h}(z_2)\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\chi(r)\phi_M\right]\mathrm{d}z_2+\mathcal{O}\!\left(N^{-3/2}\right).
\end{aligned}
\tag{138}
$$

Next, we multiply both sides of (138) by $-\bar{\boldsymbol{Q}}(z_1)$

$$
\begin{aligned}
&-z_1\mathbb{E}\Big[\hat{\boldsymbol{Q}}(z_1)\bar{\boldsymbol{Q}}(z_1)\chi(r)\phi_M\Big]-\mathbb{E}\Big[\bar{\boldsymbol{Q}}(z_1)\chi(r)\phi_M\Big]+\frac{z_1}{\omega(z_1)}\mathbb{E}\Big[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\chi(r)\phi_M\Big] \\
&=\frac{z_1}{\omega(z_1)}\mathbb{E}\Big[\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\bar{\boldsymbol{Q}}(z_1)\chi(r)\phi_M\alpha(z_1)\Big] \\
&\quad+\mathrm{j}r\frac{z_1}{\omega(z_1)}\sum_{q=1}^{L}\frac{w_q}{2\pi\mathrm{j}}\frac{1}{N}\sqrt{N}\oint_{\mathcal{C}_{z_2}^{(m)}}\mathbb{E}\Big[\hat{h}(z_2)\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\bar{\boldsymbol{Q}}(z_1)\chi(r)\phi_M\Big]\mathrm{d}z_2+\mathcal{O}\Big(N^{-3/2}\Big).
\end{aligned}
\tag{139}
$$

Using $\boldsymbol{I}_M = -z_1\bar{\boldsymbol{Q}}(z_1)+\frac{z_1}{\omega(z_1)}\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)$ which immediately follows from the definition of $\bar{\boldsymbol{Q}}(z_1)$ in (85) we can develop

$$
\begin{aligned}
-z_1\mathbb{E}\Big[\hat{\boldsymbol{Q}}(z_1)\bar{\boldsymbol{Q}}(z_1)\chi(r)\phi_M\Big]+\frac{z_1}{\omega(z_1)}\mathbb{E}\Big[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\chi(r)\phi_M\Big] &=\mathbb{E}\Big[\hat{\boldsymbol{Q}}(z_1)\Big(-z_1\bar{\boldsymbol{Q}}(z_1)+\frac{z_1}{\omega(z_1)}\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\Big)\chi(r)\phi_M\Big] \\
&=\mathbb{E}\Big[\hat{\boldsymbol{Q}}(z_1)\chi(r)\phi_M\Big].
\end{aligned}
\tag{140}
$$

Substituting (140) into (139) and multiplying by $\sqrt{N}\bar{h}(z_1)\boldsymbol{v}_p^{\mathrm{H}}$ from the left hand side and by $\boldsymbol{v}_p$ from the right hand side yields

$$
\begin{aligned}
&\sqrt{N}\bar{h}(z_1)\mathbb{E}\Big[\boldsymbol{v}_p^{\mathrm{H}}\Big(\hat{\boldsymbol{Q}}(z_1)-\bar{\boldsymbol{Q}}(z_1)\Big)\boldsymbol{v}_p\chi(r)\phi_M\Big] \\
&=\sqrt{N}\bar{h}(z_1)\frac{z_1}{\omega(z_1)}\mathbb{E}\Big[\boldsymbol{v}_p^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p\chi(r)\phi_M\alpha(z_1)\Big] \\
&\quad+\mathrm{j}r\bar{h}(z_1)\frac{z_1}{\omega(z_1)}\sum_{q=1}^{L}\frac{w_q}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{z_2}^{(m)}}\mathbb{E}\Big[\hat{h}(z_2)\boldsymbol{v}_p^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p\chi(r)\phi_M\Big]\mathrm{d}z_2+\mathcal{O}\Big(N^{-3/2}\Big),
\end{aligned}
\tag{141}
$$

Please note that (141) is equivalent to the first quantity of interest in (106). Hence, it remains to study the asymptotic stochastic behavior of (141). We will do so by analyzing the asymptotic stochastic behavior of the first term on the right hand side of (141) first.

Lemma 11. *Let* $\zeta = \boldsymbol{v}_p^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p\phi_M$ *then*

$$
\sqrt{N}\mathbb{E}[\zeta\chi(r)\alpha(z_1)] = \mathcal{O}\Big(N^{-1/2}\Big).
$$

Proof. See Appendix G. ∎

The asymptotic stochastic behavior of the second term on the right hand side of (141) is analyzed next.

Lemma 12. *Let* $\zeta_1 = \boldsymbol{v}_p^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\phi_M$ *and* $\zeta_2 = \boldsymbol{v}_q^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p\phi_M$ *then the expectation* $\mathbb{E}\Big[\hat{h}(z_2)\zeta_1\zeta_2\chi(r)\Big]$ *can be approximated by*

$$
\mathbb{E}\Big[\hat{h}(z_2)\zeta_1\zeta_2\chi(r)\Big] =\bar{h}(z_2)\mathbb{E}[\zeta_1]\mathbb{E}[\zeta_2]\mathbb{E}[\chi(r)]+\mathcal{O}\Big(N^{-1/2}\Big).
$$

Proof. See Appendix H. ∎

Using Lemmas 11 and 12 we can approximate the quantity on the right hand side of (141) by

$$
\sqrt{N}\bar{h}(z_1)\mathbb{E}\Big[\boldsymbol{v}_p^{\mathrm{H}}\Big(\hat{\boldsymbol{Q}}(z_1)-\bar{\boldsymbol{Q}}(z_1)\Big)\boldsymbol{v}_p\chi(r)\phi_M\Big] = \mathrm{j}r\bar{h}(z_1)\frac{z_1}{\omega(z_1)}\sum_{q=1}^{L}\frac{w_q}{2\pi\mathrm{j}}\oint_{\mathcal{C}_{z_2}^{(m)}}\bar{h}(z_2)\mathbb{E}[\zeta_1]\mathbb{E}[\zeta_2]\mathbb{E}[\chi(r)]\mathrm{d}z_2+\mathcal{O}\Big(N^{-1/2}\Big),
\tag{142}
$$

With the help of Lemma 17 and Lemma 16 we can compute the expectations $\mathbb{E}[\zeta_1]$ and $\mathbb{E}[\zeta_2]$

$$
\mathbb{E}[\zeta_1] =\boldsymbol{v}_p^{\mathrm{H}}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{v}_q\Big(\frac{1}{1-\gamma(z_1,z_2)}\Big)+\mathcal{O}\big(N^{-1}\big)
\tag{143}
$$

$$
\mathbb{E}[\zeta_2] =\frac{z_2}{\omega(z_2)}\boldsymbol{v}_q^{\mathrm{H}}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{v}_p+\mathcal{O}\big(N^{-1}\big),
\tag{144}
$$

where

$$
\gamma(z_1,z_2) = \frac{z_1}{\omega(z_1)}\frac{z_2}{\omega(z_2)}\frac{1}{N}\mathrm{tr}\big[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\big].
\tag{145}
$$

Substituting the expressions in (143) and (144) into (142) yields the expression in (109).

APPENDIX G
PROOF OF LEMMA 11

Using the Cauchy-Schwarz inequality we can upper-bound

$$|\mathbb{E}[\zeta\chi(r)\alpha(z_1)]|^2 \leq \mathbb{E}[|\zeta\chi(r)|^2]\mathbb{E}[|\alpha(z_1)|^2] = \mathbb{E}[|\zeta|^2|\chi(r)|^2]\mathbb{E}[|\alpha(z_1)|^2] = \mathbb{E}[|\zeta|^2]\mathbb{E}[|\alpha(z_1)|^2],$$

where we have used $|\chi(r)|^2 = 1$. According to Lemma 13, the expectation and the variance of $\alpha(z_1)$ in (136) decay with $\mathbb{E}[\alpha(z_1)] = \mathcal{O}(N^{-1})$ and $\mathrm{var}[\alpha(z_1)] = \mathcal{O}(N^{-2})$, respectively. Furthermore, it can be seen in Lemma 16 that the expectation $\mathbb{E}[\zeta]$ converges to something deterministic and the variance decays with $\mathrm{var}[\zeta] = \mathcal{O}(N^{-1})$. Hence, $\mathbb{E}[|\alpha(z_1)|^2] = \mathrm{var}[\alpha(z_1)] + \mathbb{E}[\alpha(z_1)]^2 = \mathcal{O}(N^{-2})$ and $\mathbb{E}[|\zeta|^2] = \mathrm{var}[\zeta] + \mathbb{E}[\zeta]^2 = \mathcal{O}(1)$ and the quantity of interest can be upper-bounded by

$$\sqrt{N}\mathbb{E}[\zeta\chi(r)\alpha(z_1)] \leq \sqrt{N}\sqrt{\mathbb{E}[|\zeta|^2]\mathbb{E}[|\alpha(z_1)|^2]} = \mathcal{O}\left(\frac{\sqrt{N}}{N}\right) = \mathcal{O}\left(N^{-1/2}\right).$$

APPENDIX H
PROOF OF LEMMA 12

Let us use the following notation $(\zeta_1\zeta_2)^{(\circ)} = (\zeta_1\zeta_2) - \mathbb{E}[(\zeta_1\zeta_2)]$ such that $\mathbb{E}\left[(\zeta_1\zeta_2)^{(\circ)}\chi(r)\right] = \mathbb{E}[((\zeta_1\zeta_2) - \mathbb{E}[\zeta_1\zeta_2])\chi(r)] = \mathbb{E}[\zeta_1\zeta_2\chi(r)] - \mathbb{E}[\zeta_1\zeta_2]\mathbb{E}[\chi(r)]$. Hence, the quantity of interest can equivalently be expressed as

$$\mathbb{E}[\zeta_1\zeta_2\chi(r)] = \mathbb{E}\left[(\zeta_1\zeta_2)^{(\circ)}\chi(r)\right] + \mathbb{E}[\zeta_1\zeta_2]\mathbb{E}[\chi(r)]. \tag{146}$$

Next, we apply the Cauchy-Schwarz inequality to upper bound the expectation $\mathbb{E}\left[(\zeta_1\zeta_2)^{(\circ)}\chi(r)\right]$ according to

$$\left|\mathbb{E}\left[(\zeta_1\zeta_2)^{(\circ)}\chi(r)\right]\right|^2 \leq \mathbb{E}\left[\left|(\zeta_1\zeta_2)^{(\circ)}\right|^2\right]\mathbb{E}[|\chi(r)|^2] = \mathrm{var}[\zeta_1\zeta_2], \tag{147}$$

where we have used $|\chi(r)| = 1$ and $\mathbb{E}\left[|(\zeta_1\zeta_2)^{(\circ)}|^2\right] = \mathbb{E}[|\zeta_1\zeta_2 - \mathbb{E}[\zeta_1\zeta_2]|^2] = \mathrm{var}[\zeta_1\zeta_2]$. Using the Nash-Poincaré inequality allows to upper bound the variance

$$\mathrm{var}[\zeta_1\zeta_2] \leq \sum_{i=1}^{M}\sum_{j=1}^{N}\left(\mathbb{E}\left[\left|\frac{\partial(\zeta_1\zeta_2)}{\partial X_{ij}}\right|^2\right] + \mathbb{E}\left[\left|\frac{\partial(\zeta_1\zeta_2)}{\partial X_{ij}^*}\right|^2\right]\right) = \mathcal{O}(N^{-1}), \tag{148}$$

where both derivatives can be upper bounded itself by applying Jensen's inequality

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\frac{\partial(\zeta_1\zeta_2)}{\partial X_{ij}}\right|^2\right] = \sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\left(\frac{\partial\zeta_1}{\partial X_{ij}}\right)\zeta_2 + \zeta_1\left(\frac{\partial\zeta_2}{\partial X_{ij}}\right)\right|^2\right] \leq 2\sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\left(\frac{\partial\zeta_1}{\partial X_{ij}}\right)\zeta_2\right|^2\right] + 2\sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\zeta_1\left(\frac{\partial\zeta_2}{\partial X_{ij}}\right)\right|^2\right], \tag{149}$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\frac{\partial(\zeta_1\zeta_2)}{\partial X_{ij}^*}\right|^2\right] = \sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\left(\frac{\partial\zeta_1}{\partial X_{ij}^*}\right)\zeta_2 + \zeta_1\left(\frac{\partial\zeta_2}{\partial X_{ij}^*}\right)\right|^2\right] \leq 2\sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\left(\frac{\partial\zeta_1}{\partial X_{ij}^*}\right)\zeta_2\right|^2\right] + 2\sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\zeta_1\left(\frac{\partial\zeta_2}{\partial X_{ij}^*}\right)\right|^2\right]. \tag{150}$$

The first order derivatives of ζ_1 with respect to X_{ij} and X_{ij}^* are given in (257) and (258) respectively, where $s_1 = v_p$ and $s_2 = v_q$. Furthermore, the first order derivatives of ζ_2 with respect to X_{ij} and X_{ij}^* are given in (243) and (244) where $s_1 = v_q$ and $s_2 = \bar{Q}(z_1)v_p$. Substituting the derivatives into (149) and (150) and applying Jensen's inequality a second time it can be seen that

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\frac{\partial(\zeta_1\zeta_2)}{\partial X_{ij}}\right|^2\right] \leq \mathcal{O}(N^{-1}), \qquad \sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\frac{\partial(\zeta_1\zeta_2)}{\partial X_{ij}^*}\right|^2\right] \leq \mathcal{O}(N^{-1}),$$

since

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\left(\frac{\partial\zeta_1}{\partial X_{ij}}\right)\zeta_2\right|^2\right] \leq \mathcal{O}(N^{-1}), \qquad \sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\zeta_1\left(\frac{\partial\zeta_2}{\partial X_{ij}}\right)\right|^2\right] \leq \mathcal{O}(N^{-1}),$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\left(\frac{\partial\zeta_1}{\partial X_{ij}^*}\right)\zeta_2\right|^2\right] \leq \mathcal{O}(N^{-1}), \qquad \sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\zeta_1\left(\frac{\partial\zeta_2}{\partial X_{ij}^*}\right)\right|^2\right] \leq \mathcal{O}(N^{-1}).$$

Hence, the variance in (148) decays with $\mathrm{var}[\zeta_1\zeta_2] = \mathcal{O}(N^{-1})$ and as a direct consequence the expectation $\mathbb{E}\left[(\zeta_1\zeta_2)^{(\circ)}\chi(r)\right]$ decays with $\mathbb{E}\left[(\zeta_1\zeta_2)^{(\circ)}\chi(r)\right] \leq \mathcal{O}(N^{-1/2})$ which follows from (147). With this, we can approximate the quantity of interest in (146) by

$$\mathbb{E}[\zeta_1\zeta_2\chi(r)] = \mathbb{E}[\zeta_1\zeta_2]\mathbb{E}[\chi(r)] + \mathcal{O}\left(N^{-1/2}\right), \tag{151}$$

and it remains to study the asymptotic stochastic behavior of $\mathbb{E}[\zeta_1\zeta_2]$. Introducing the covariance

$$\mathrm{cov}[\zeta_1, \zeta_2] = \mathbb{E}\left[\zeta_1^{(\circ)}\zeta_2^{(\circ)}\right] = \mathbb{E}[(\zeta_1 - \mathbb{E}[\zeta_1])(\zeta_2 - \mathbb{E}[\zeta_2])] = \mathbb{E}[\zeta_1\zeta_2] - \mathbb{E}[\zeta_1]\mathbb{E}[\zeta_2],$$

allows us to equivalently express $\mathbb{E}[\zeta_1\zeta_2]$ as

$$\mathbb{E}[\zeta_1\zeta_2] = \mathrm{cov}[\zeta_1,\,\zeta_2] + \mathbb{E}[\zeta_1]\mathbb{E}[\zeta_2]. \tag{152}$$

The covariance $\mathrm{cov}[\zeta_1,\,\zeta_2]$ can be upper-bounded using the Cauchy-Schwarz inequality

$$|\mathrm{cov}[\zeta_1,\,\zeta_2]|^2 = |\mathbb{E}[(\zeta_1 - \mathbb{E}[\zeta_1])(\zeta_2 - \mathbb{E}[\zeta_2])]|^2 \le \mathbb{E}[|\zeta_1 - \mathbb{E}[\zeta_1]|^2]\mathbb{E}[|\zeta_2 - \mathbb{E}[\zeta_2]|^2] = \mathrm{var}[\zeta_1]\mathrm{var}[\zeta_2] = \mathcal{O}(N^{-2}), \tag{153}$$

such that $\mathrm{cov}[\zeta_1,\,\zeta_2] \le \sqrt{\mathrm{var}[\zeta_1]\mathrm{var}[\zeta_2]} = \mathcal{O}(N^{-1})$ where we have used $\mathrm{var}[\zeta_1] = \mathcal{O}(N^{-1})$ and $\mathrm{var}[\zeta_2] = \mathcal{O}(N^{-1})$ which follows from Lemmas 16 and 17. Substituting the square root of (153) into (152) we have successfully decorrelated $\mathbb{E}[\zeta_1\zeta_2] = \mathbb{E}[\zeta_1]\mathbb{E}[\zeta_2] + \mathcal{O}(N^{-1})$. Inserting $\mathbb{E}[\zeta_1\zeta_2] = \mathbb{E}[\zeta_1]\mathbb{E}[\zeta_2] + \mathcal{O}(N^{-1})$ into the expression in (151) we obtain

$$\begin{aligned}\mathbb{E}[\zeta_1\zeta_2\chi(r)] &= \mathbb{E}[\zeta_1\zeta_2]\mathbb{E}[\chi(r)] + \mathcal{O}\left(N^{-1/2}\right) = (\mathrm{cov}[\zeta_1,\,\zeta_2] + \mathbb{E}[\zeta_1]\mathbb{E}[\zeta_2])\mathbb{E}[\chi(r)] + \mathcal{O}\left(N^{-1/2}\right) \\ &= (\mathcal{O}(N^{-1}) + \mathbb{E}[\zeta_1]\mathbb{E}[\zeta_2])\mathbb{E}[\chi(r)] + \mathcal{O}\left(N^{-1/2}\right) = \mathbb{E}[\zeta_1]\mathbb{E}[\zeta_2]\mathbb{E}[\chi(r)] + \mathcal{O}\left(N^{-1/2}\right).\end{aligned} \tag{154}$$

By adding and subtracting $\bar{h}(z_2)$ inside the expectation of the quantity of interest $\mathbb{E}\left[\hat{h}(z_2)\zeta_1\zeta_2\chi(r)\right]$ we can write

$$\begin{aligned}\mathbb{E}\left[\hat{h}(z_2)\zeta_1\zeta_2\chi(r)\right] &= \mathbb{E}\left[\left(\hat{h}(z_2) - \bar{h}(z_2) + \bar{h}(z_2)\right)\zeta_1\zeta_2\chi(r)\right] = \mathbb{E}\left[\left(\hat{h}(z_2) - \bar{h}(z_2)\right)\zeta_1\zeta_2\chi(r)\right] + \mathbb{E}\left[\bar{h}(z_2)\zeta_1\zeta_2\chi(r)\right] \\ &= \mathbb{E}\left[\left(\hat{h}(z_2) - \bar{h}(z_2)\right)\zeta_1\zeta_2\chi(r)\right] + \bar{h}(z_2)\mathbb{E}[\zeta_1\zeta_2\chi(r)].\end{aligned} \tag{155}$$

Applying the Cauchy-Schwarz inequality to the first term on the right hand side of (155) we can upper-bound

$$\left|\mathbb{E}\left[\left(\hat{h}(z_2) - \bar{h}(z_2)\right)\zeta_1\zeta_2\chi(r)\right]\right|^2 \le \mathbb{E}\left[\left|\hat{h}(z_2) - \bar{h}(z_2)\right|^2\right]\mathbb{E}[|\zeta_1\zeta_2\chi(r)|^2]. \tag{156}$$

From Lemma 10 it follows that

$$\mathbb{E}\left[\left|\hat{h}(z_2) - \bar{h}(z_2)\right|^2\right] = \mathrm{var}\left[\hat{h}(z_2) - \bar{h}(z_2)\right] + \mathbb{E}\left[\hat{h}(z_2) - \bar{h}(z_2)\right]^2 = \mathcal{O}(N^{-2}),$$

and from our previous analysis we know that

$$\mathbb{E}[|\zeta_1\zeta_2\chi(r)|^2] = \mathbb{E}[|\zeta_1\zeta_2|^2|\chi(r)|^2] = \mathbb{E}[|\zeta_1\zeta_2|^2] = \mathrm{var}[\zeta_1\zeta_2] + \mathbb{E}[\zeta_1\zeta_2]^2 = \mathrm{var}[\zeta_1\zeta_2] + (\mathrm{cov}[\zeta_1,\,\zeta_2] + \mathbb{E}[\zeta_1]\mathbb{E}[\zeta_2])^2 = \mathcal{O}(1),$$

where we have used (148) and (152). Correspondingly, the first term on the right hand side of (155) decays with

$$\mathbb{E}\left[\left(\hat{h}(z_2) - \bar{h}(z_2)\right)\zeta_1\zeta_2\chi(r)\right] \le \sqrt{\mathbb{E}\left[\left|\hat{h}(z_2) - \bar{h}(z_2)\right|^2\right]\mathbb{E}[|\zeta_1\zeta_2\chi(r)|^2]} = \mathcal{O}(N^{-1})$$

and therefore

$$\mathbb{E}\left[\hat{h}(z_2)\zeta_1\zeta_2\chi(r)\right] = \bar{h}(z_2)\mathbb{E}[\zeta_1\zeta_2\chi(r)] + \mathcal{O}(N^{-1}). \tag{157}$$

Finally, substituting (154) into (157) we obtain the result of Lemma 12

$$\mathbb{E}\left[\hat{h}(z_2)\zeta_1\zeta_2\chi(r)\right] = \bar{h}(z_2)\mathbb{E}[\zeta_1]\mathbb{E}[\zeta_2]\mathbb{E}[\chi(r)] + \mathcal{O}\left(N^{-1/2}\right).$$

APPENDIX I
AUXILIARY LEMMAS

In this appendix, we provide some bounds on expectations and variances of different random functions of complex variables. The notation $\mathcal{O}(N^{-l})$ should be understood as a deterministic term whose magnitude is upper bounded by a quantity of the form $\varepsilon(z_1,z_2)N^{-l}$, where $\varepsilon(z_1,z_2)$ is a bivariate real-valued positive function independent of N such that $\sup_{(z_1,z_2)\in\mathcal{C}_{z_1}^{(m)}\times\mathcal{C}_{z_2}^{(m)}}\varepsilon(z_1,z_2) < \infty$. Further insights are provided in [19, Remark 1, supplementary material].

Lemma 13. *Let B denote an $M\times M$ deterministic matrix with bounded spectral norm. Then, as $M,N \to \infty$ at the same rate*

$$\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[B\hat{Q}(z_1)\phi_M\right]\right] = \frac{1}{N}\mathrm{tr}\left[B\bar{Q}(z_1)\right] + \mathcal{O}(N^{-1}) \tag{158}$$

$$\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[B\hat{Q}(z_1)R^{1/2}\frac{XX^{\mathrm{H}}}{N}\phi_M\right]\right] = \frac{z_1}{\omega(z_1)}\frac{1}{N}\mathrm{tr}\left[B\bar{Q}(z_1)R^{1/2}\right] + \mathcal{O}(N^{-1}) \tag{159}$$

and also

$$\mathrm{var}\left[\frac{1}{N}\mathrm{tr}\left[B\hat{Q}(z_1)\phi_M\right]\right] = \mathcal{O}(N^{-2}) \tag{160}$$

$$\mathrm{var}\left[\frac{1}{N}\mathrm{tr}\left[B\hat{Q}(z_1)R^{1/2}\frac{XX^{\mathrm{H}}}{N}\phi_M\right]\right] = \mathcal{O}(N^{-2}). \tag{161}$$

Proof. See Appendix J and [19, Lemma 8, supplementary material]. ∎

Lemma 14. *Let B denote an $M \times M$ deterministic matrix with bounded spectral norm. Then, as $M, N \to \infty$ at the same rate*

$$\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[B\hat{Q}(z_1)R\hat{Q}(z_2)\phi_M\right]\right] = \frac{1}{1-\gamma(z_1,z_2)}\frac{1}{N}\mathrm{tr}\left[B\bar{Q}(z_1)R\bar{Q}(z_2)\right] + \mathcal{O}(N^{-1}) \tag{162}$$

$$\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{XX^{\mathrm{H}}}{N}\phi_M\right]\right] = \frac{z_2}{\omega(z_2)}\frac{1}{1-\gamma(z_1,z_2)}\frac{1}{N}\mathrm{tr}\left[B\bar{Q}(z_1)R\bar{Q}(z_2)R^{1/2}\right]$$
$$- \frac{\gamma(z_1,z_2)}{1-\gamma(z_1,z_2)}\frac{1}{N}\mathrm{tr}\left[B\bar{Q}(z_1)R^{1/2}\right] + \mathcal{O}(N^{-1}) \tag{163}$$

where

$$\gamma(z_1,z_2) = \frac{z_1}{\omega(z_1)}\frac{z_2}{\omega(z_2)}\frac{1}{N}\mathrm{tr}\left[R\bar{Q}(z_1)R\bar{Q}(z_2)\right] \tag{164}$$

and also

$$\mathrm{var}\left[\frac{1}{N}\mathrm{tr}\left[B\hat{Q}(z_1)R\hat{Q}(z_2)\phi_M\right]\right] = \mathcal{O}(N^{-2}) \tag{165}$$

$$\mathrm{var}\left[\frac{1}{N}\mathrm{tr}\left[B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{XX^{\mathrm{H}}}{N}\phi_M\right]\right] = \mathcal{O}(N^{-2}). \tag{166}$$

Proof. See Appendix K and [19, Lemma 9, supplementary material]. ∎

Lemma 15. *As $M, N \to \infty$ at the same rate*

$$\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\hat{Q}(z_1)\hat{R}\hat{Q}(z_1)\phi_M\right]\right] = \frac{z_1^2}{\omega(z_1)^2}\frac{1}{N}\mathrm{tr}\left[R\bar{Q}(z_1)^2\right]\frac{\partial\omega(z_1)}{\partial z_1} + \mathcal{O}(N^{-1})$$

and the variance decays with

$$\mathrm{var}\left[\frac{1}{N}\mathrm{tr}\left[\hat{Q}(z_1)\hat{R}\hat{Q}(z_1)\phi_M\right]\right] = \mathcal{O}(N^{-2}).$$

Proof. See Appendix L. ∎

Lemma 16. *Let s_1, s_2 denote two generic $M \times 1$ deterministic column vectors. Then, as $M, N \to \infty$ at the same rate*

$$\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\right] = \frac{z_2}{\omega(z_2)}s_1^{\mathrm{H}}\bar{Q}(z_2)Rs_2 + \mathcal{O}(N^{-1}) \tag{167}$$

and also

$$\mathrm{var}\left[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\right] = \mathcal{O}(N^{-1}) \tag{168}$$

Proof. See Appendix M. ∎

Lemma 17. *Let s_1, s_2 denote two generic $M \times 1$ deterministic column vectors. Then, as $M, N \to \infty$ at the same rate*

$$\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)s_2\phi_M\right] = s_1^{\mathrm{H}}\bar{Q}(z_1)R\bar{Q}(z_2)s_2\left(\frac{1}{1-\gamma(z_1,z_2)}\right) + \mathcal{O}(N^{-1}) \tag{169}$$

and also

$$\mathrm{var}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)s_2\phi_M\right] = \mathcal{O}(N^{-1}) \tag{170}$$

where

$$\gamma(z_1,z_2) = \frac{z_1}{\omega(z_1)}\frac{z_2}{\omega(z_2)}\frac{1}{N}\mathrm{tr}\left[R\bar{Q}(z_1)R\bar{Q}(z_2)\right] \tag{171}$$

Proof. See Appendix N. ∎

APPENDIX J
PROOF OF LEMMA 13

1) Variances: We start by upper-bounding the variance in (160) by applying the Nash-Poincaré inequality to

$$\mathrm{var}\left[\mathrm{tr}\left[B\hat{Q}(z_1)\phi_M\right]\right] \leq \sum_{i=1}^{M}\sum_{j=1}^{N}\left(\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}}\mathrm{tr}\left[B\hat{Q}(z_1)\phi_M\right]\right|^2\right] + \mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}^*}\mathrm{tr}\left[B\hat{Q}(z_1)\phi_M\right]\right|^2\right]\right), \tag{172}$$

where

$$\frac{\partial}{\partial X_{ij}}\mathrm{tr}\left[B\hat{Q}(z_1)\phi_M\right] = \sum_{m=1}^{M}e_m^{\mathrm{T}}B\left(\frac{\partial}{\partial X_{ij}}\hat{Q}(z_1)\right)\phi_M e_m = -\sum_{m=1}^{M}e_m^{\mathrm{T}}B\hat{Q}(z_1)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_1)\phi_M e_m + \mathcal{O}\left(N^{-\mathbb{N}}\right), \tag{173}$$

$$\frac{\partial}{\partial X_{ij}^*}\mathrm{tr}\left[B\hat{Q}(z_1)\phi_M\right] = \sum_{m=1}^{M}e_m^{\mathrm{T}}B\left(\frac{\partial}{\partial X_{ij}^*}\hat{Q}(z_1)\right)\phi_M e_m = -\sum_{m=1}^{M}e_m^{\mathrm{T}}B\hat{Q}(z_1)R^{1/2}\frac{x_j e_i^{\mathrm{T}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_1)\phi_M e_m + \mathcal{O}\left(N^{-\mathbb{N}}\right). \tag{174}$$

It follows from (173) and (174) that

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}}\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\phi_M\right]\right|^2\right]=\mathcal{O}\left(\frac{M}{N}\right),\qquad\sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}^*}\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\phi_M\right]\right|^2\right]=\mathcal{O}\left(\frac{M}{N}\right),$$

and consequently the variance in (172) is upper-bounded by $\text{var}\left[\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\right]\right]\leq\mathcal{O}(M/N)$. Hence, the variance in (160) decays with

$$\text{var}\left[\frac{1}{N}\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\phi_M\right]\right]=\frac{1}{N^2}\text{var}\left[\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\phi_M\right]\right]\leq\frac{1}{N^2}\mathcal{O}\left(\frac{M}{N}\right)=\mathcal{O}(N^{-2}).$$

Next, we analyze the variance in (161). Therefore, we use the Nash-Poincaré inequality to upper-bound

$$\text{var}\left[\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\phi_M\right]\right]\leq\sum_{i=1}^{M}\sum_{j=1}^{N}\left(\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}}\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\phi_M\right]\right|^2\right]+\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}^*}\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\phi_M\right]\right|^2\right]\right),$$
(175)

where

$$\frac{\partial}{\partial X_{ij}}\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\phi_M\right]=-\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\text{H}}}{N}\boldsymbol{R}^{\text{H}/2}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\boldsymbol{e}_m\phi_M$$
$$+\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\text{H}}}{N}\boldsymbol{e}_m\phi_M+\mathcal{O}\left(N^{-\mathbb{N}}\right),$$
(176)

$$\frac{\partial}{\partial X_{ij}^*}\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\phi_M\right]=-\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\text{T}}}{N}\boldsymbol{R}^{\text{H}/2}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\boldsymbol{e}_m\phi_M$$
$$+\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\text{T}}}{N}\boldsymbol{e}_m\phi_M+\mathcal{O}\left(N^{-\mathbb{N}}\right).$$
(177)

Using Jensen's inequality we can upper-bound the squared absolute value of the two derivatives in (176) and (177) by

$$\left|\frac{\partial}{\partial X_{ij}}\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\phi_M\right]\right|^2\leq2\left|\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\text{H}}}{N}\boldsymbol{R}^{\text{H}/2}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\boldsymbol{e}_m\phi_M\right|^2$$
$$+2\left|\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\text{H}}}{N}\boldsymbol{e}_m\phi_M\right|^2+\mathcal{O}\left(N^{-\mathbb{N}}\right),$$
(178)

and

$$\left|\frac{\partial}{\partial X_{ij}^*}\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\phi_M\right]\right|^2\leq2\left|\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\text{T}}}{N}\boldsymbol{R}^{\text{H}/2}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\boldsymbol{e}_m\phi_M\right|^2$$
$$+2\left|\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\text{T}}}{N}\boldsymbol{e}_m\phi_M\right|^2+\mathcal{O}\left(N^{-\mathbb{N}}\right).$$
(179)

It follows from (178) and (179) that

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}}\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\phi_M\right]\right|^2\right]=\mathcal{O}\left(\frac{M}{N}\right),\qquad\sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}^*}\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\phi_M\right]\right|^2\right]=\mathcal{O}\left(\frac{M}{N}\right),$$

and the variance in (175) is upper bounded by $\text{var}\left[\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\right]\right]\leq\mathcal{O}(M/N)$. As a direct consequence the variance in (161) decays with

$$\text{var}\left[\frac{1}{N}\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\phi_M\right]\right]=\frac{1}{N^2}\text{var}\left[\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\phi_M\right]\right]\leq\frac{1}{N^2}\mathcal{O}\left(\frac{M}{N}\right)=\mathcal{O}(N^{-2}).$$

2) Expectations: To derive the expectation in (158) we express the trace as sum

$$\frac{1}{N}\mathbb{E}\left[\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\phi_M\right]\right]=\frac{1}{N}\sum_{m=1}^{M}\mathbb{E}\left[\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{e}_m\phi_M\right]=\frac{1}{N}\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\mathbb{E}\left[\hat{\boldsymbol{Q}}(z_1)\phi_M\right]\boldsymbol{e}_m.$$
(180)

By replacing the random resolvent $\hat{\boldsymbol{Q}}(z_1)$ with its asymptotic deterministic equivalent $\bar{\boldsymbol{Q}}(z_1)$ in (85) we obtain the expectation in (158) in Lemma 13

$$\frac{1}{N}\mathbb{E}\left[\text{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\phi_M\right]\right]=\frac{1}{N}\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{e}_m+\mathcal{O}(N^{-1})=\frac{1}{N}\text{tr}\left[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\right]+\mathcal{O}(N^{-1}),$$
(181)

where we have used $\mathbb{E}\left[\hat{\boldsymbol{Q}}(z)\phi_M\right]=\bar{\boldsymbol{Q}}(z)+\mathcal{O}(N^{-1})$.

To derive the expectation in (159) we express the trace as sum and the sample covariance matrix $\hat{\boldsymbol{R}}$ as shown in (103) and apply the integration by parts formula

$$
\begin{aligned}
\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right] &=\frac{1}{N}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{m=1}^{M}\mathbb{E}\left[X_{ij}\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{e}_m\phi_M\right] \\
&=\frac{1}{N}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{m=1}^{M}\mathbb{E}\left[\frac{\partial}{\partial X_{ij}^*}\left(\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{e}_m\phi_M\right)\right],
\end{aligned}
\tag{182}
$$

where

$$
\begin{aligned}
\frac{\partial}{\partial X_{ij}^*}\left(\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{e}_m\phi_M\right) &=-\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{e}_m\phi_M \\
&\quad+\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{e}_m\phi_M+\mathcal{O}\left(N^{-\mathbb{N}}\right).
\end{aligned}
\tag{183}
$$

Substituting the derivative in (183) into (182) and computing the triple sum yields

$$
\begin{aligned}
\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right] &=-\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\right]\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]\phi_M\right] \\
&\quad+\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\phi_M\right]\right]+\mathcal{O}\left(N^{-\mathbb{N}}\right).
\end{aligned}
\tag{184}
$$

Next, we decorrelate the first term on the right hand side of (184). Therefore, we express the covariance as

$$
\begin{aligned}
&\mathrm{cov}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right],\ \frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\phi_M\right]\right] \\
&=\mathbb{E}\left[\left(\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right)^{(\circ)}\left(\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\phi_M\right]\right)^{(\circ)}\right] \\
&=\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\right]\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]\phi_M\right]-\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right]\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\phi_M\right]\right],
\end{aligned}
\tag{185}
$$

Afterwards, we apply the Cauchy-Schwarz inequality to upper-bound the covariance

$$
\begin{aligned}
&\left|\mathrm{cov}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right],\ \frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\phi_M\right]\right]\right|^2 \\
&\leq\mathbb{E}\left[\left|\left(\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right)^{(\circ)}\right|^2\right]\mathbb{E}\left[\left|\left(\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\phi_M\right]\right)^{(\circ)}\right|^2\right] \\
&=\mathrm{var}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right]\mathrm{var}\left[\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\phi_M\right]\right].
\end{aligned}
\tag{186}
$$

Using the previously derived variances in (160) and (161) we know that

$$
\mathrm{var}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right]=\mathcal{O}\left(N^{-2}\right),\qquad \mathrm{var}\left[\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\phi_M\right]\right]=\mathcal{O}\left(N^{-2}\right),
$$

and therefore the covariance in (185) decays with

$$
\mathrm{cov}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right],\ \frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\phi_M\right]\right]\leq\mathcal{O}\left(N^{-2}\right).
$$

This allows to decorrelate the first term on the right hand side of (184) as follows

$$
\begin{aligned}
&\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\right]\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]\phi_M\right] \\
&=\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right]\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\phi_M\right]\right]+\mathrm{cov}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right],\ \frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\phi_M\right]\right] \\
&=\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right]\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\phi_M\right]\right]+\mathcal{O}\left(N^{-2}\right).
\end{aligned}
\tag{187}
$$

Substituting the decorrelated expression in (187) into (184) yields

$$
\begin{aligned}
\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right] &=\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\phi_M\right]\right] \\
&\quad-\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right]\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\phi_M\right]\right]+\mathcal{O}\left(N^{-2}\right),
\end{aligned}
$$

which can be rearranged as follows

$$\frac{1}{N}\mathbb{E}\Big[\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\phi_M\Big]\Big]\Big(1+\frac{1}{N}\mathbb{E}\Big[\text{tr}\Big[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\phi_M\Big]\Big]\Big)=\frac{1}{N}\mathbb{E}\Big[\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\phi_M\Big]\Big]+\mathcal{O}(N^{-2}). \tag{188}$$

Using the previously derived expectation in (158) we know that (188) simplifies to

$$\frac{1}{N}\mathbb{E}\Big[\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\phi_M\Big]\Big]\Big(1+\frac{1}{N}\text{tr}\Big[\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\Big]\Big)=\frac{1}{N}\text{tr}\Big[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\Big]+\mathcal{O}(N^{-1}). \tag{189}$$

Using $\frac{\omega(z_1)}{z_1}=(1+\frac{1}{N}\text{tr}\big[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\big])$ which immediately follows from (86) and multiplying (189) on both sides with $\frac{z_1}{\omega(z_1)}$ we obtain (159).

APPENDIX K
PROOF OF LEMMA 14

1) Variances: In order to upper-bound the covariance in (165) we apply the Nash-Poincaré inequality

$$\text{var}\Big[\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]\leq\sum_{i=1}^{M}\sum_{j=1}^{N}\Big(\mathbb{E}\Big[\Big|\frac{\partial}{\partial X_{ij}}\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big|^2\Big]+\mathbb{E}\Big[\Big|\frac{\partial}{\partial X_{ij}^*}\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big|^2\Big]\Big), \tag{190}$$

where the first order derivative with respect to X_{ij} and X_{ij}^* are given by

$$\frac{\partial}{\partial X_{ij}}\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]=-\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\text{H}}}{N}\boldsymbol{R}^{\text{H}/2}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{e}_m\phi_M$$
$$-\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\text{H}}}{N}\boldsymbol{R}^{\text{H}/2}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{e}_m\phi_M+\mathcal{O}\Big(N^{-\mathbb{N}}\Big), \tag{191}$$

and

$$\frac{\partial}{\partial X_{ij}^*}\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]=-\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\text{T}}}{N}\boldsymbol{R}^{\text{H}/2}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{e}_m\phi_M$$
$$-\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\text{T}}}{N}\boldsymbol{R}^{\text{H}/2}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{e}_m\phi_M+\mathcal{O}\Big(N^{-\mathbb{N}}\Big). \tag{192}$$

Using Jensen's inequality we can upper bound the squared absolute value of the first order derivative with respect to X_{ij} in (191)

$$\Big|\frac{\partial}{\partial X_{ij}}\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big|^2\leq 2\Big|\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\text{H}}}{N}\boldsymbol{R}^{\text{H}/2}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{e}_m\phi_M\Big|^2$$
$$+2\Big|\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\text{H}}}{N}\boldsymbol{R}^{\text{H}/2}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{e}_m\phi_M\Big|^2+\mathcal{O}\Big(N^{-\mathbb{N}}\Big) \tag{193}$$

and the first order derivative with respect to X_{ij}^* in (192)

$$\Big|\frac{\partial}{\partial X_{ij}^*}\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big|^2\leq 2\Big|\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\text{T}}}{N}\boldsymbol{R}^{\text{H}/2}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{e}_m\phi_M\Big|^2$$
$$+2\Big|\sum_{m=1}^{M}\boldsymbol{e}_m^{\text{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\text{T}}}{N}\boldsymbol{R}^{\text{H}/2}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{e}_m\phi_M\Big|^2+\mathcal{O}\Big(N^{-\mathbb{N}}\Big) \tag{194}$$

Furthermore, it can be observed in (193) and (194) that

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\Big[\Big|\frac{\partial}{\partial X_{ij}}\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big|^2\Big]=\mathcal{O}\Big(\frac{M}{N}\Big), \qquad \sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\Big[\Big|\frac{\partial}{\partial X_{ij}^*}\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big|^2\Big]=\mathcal{O}\Big(\frac{M}{N}\Big).$$

Consequently, the variance in (190) is upper-bounded by $\text{var}\Big[\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]\leq\mathcal{O}(M/N)$ and the variance in (165) decays with

$$\text{var}\Big[\frac{1}{N}\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]=\frac{1}{N^2}\text{var}\Big[\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]\leq\frac{1}{N^2}\mathcal{O}\Big(\frac{M}{N}\Big)=\mathcal{O}(N^{-2}).$$

In the following we upper-bound the variance in (166) by using the Nash-Poincaré inequality

$$\text{var}\Big[\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\phi_M\Big]\Big]\leq\sum_{i=1}^{M}\sum_{j=1}^{N}\Big(\mathbb{E}\Big[\Big|\frac{\partial}{\partial X_{ij}}\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\phi_M\Big]\Big|^2\Big]$$
$$+\mathbb{E}\Big[\Big|\frac{\partial}{\partial X_{ij}^*}\text{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\text{H}}}{N}\phi_M\Big]\Big|^2\Big]\Big), \tag{195}$$

where the first order derivative with respect to X_{ij} is given by

$$\frac{\partial}{\partial X_{ij}}\mathrm{tr}\left[B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{XX^{\mathrm{H}}}{N}\phi_M\right] = -\sum_{m=1}^{M}e_m^{\mathrm{T}}B\hat{Q}(z_1)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_1)R\hat{Q}(z_2)\frac{XX^{\mathrm{H}}}{N}e_m\phi_M$$
$$-\sum_{m=1}^{M}e_m^{\mathrm{T}}B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_2)\frac{XX^{\mathrm{H}}}{N}e_m\phi_M \tag{196}$$
$$+\sum_{m=1}^{M}e_m^{\mathrm{T}}B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}e_m\phi_M+\mathcal{O}\left(N^{-\mathbb{N}}\right),$$

and the first order derivative with respect to X_{ij} is given by

$$\frac{\partial}{\partial X_{ij}^{*}}\mathrm{tr}\left[B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{XX^{\mathrm{H}}}{N}\phi_M\right] = -\sum_{m=1}^{M}e_m^{\mathrm{T}}B\hat{Q}(z_1)R^{1/2}\frac{x_j e_i^{\mathrm{T}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_1)R\hat{Q}(z_2)\frac{XX^{\mathrm{H}}}{N}e_m\phi_M$$
$$-\sum_{m=1}^{M}e_m^{\mathrm{T}}B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{x_j e_i^{\mathrm{T}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_2)\frac{XX^{\mathrm{H}}}{N}e_m\phi_M \tag{197}$$
$$+\sum_{m=1}^{M}e_m^{\mathrm{T}}B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{x_j e_i^{\mathrm{T}}}{N}e_m\phi_M+\mathcal{O}\left(N^{-\mathbb{N}}\right).$$

Using Jensen's inequality we can upper-bound the absolute value squared of the first order derivative in (196) by

$$\left|\frac{\partial}{\partial X_{ij}}\mathrm{tr}\left[B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{XX^{\mathrm{H}}}{N}\phi_M\right]\right|^2 \leq 3\left|\sum_{m=1}^{M}e_m^{\mathrm{T}}B\hat{Q}(z_1)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_1)R\hat{Q}(z_2)\frac{XX^{\mathrm{H}}}{N}e_m\phi_M\right|^2$$
$$+3\left|\sum_{m=1}^{M}e_m^{\mathrm{T}}B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_2)\frac{XX^{\mathrm{H}}}{N}e_m\phi_M\right|^2 \tag{198}$$
$$+3\left|\sum_{m=1}^{M}e_m^{\mathrm{T}}B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}e_m\phi_M\right|^2+\mathcal{O}\left(N^{-\mathbb{N}}\right)$$

as well as the absolute value squared of the first order derivative in (197)

$$\left|\frac{\partial}{\partial X_{ij}^{*}}\mathrm{tr}\left[B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{XX^{\mathrm{H}}}{N}\phi_M\right]\right|^2 \leq 3\left|\sum_{m=1}^{M}e_m^{\mathrm{T}}B\hat{Q}(z_1)R^{1/2}\frac{x_j e_i^{\mathrm{T}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_1)R\hat{Q}(z_2)\frac{XX^{\mathrm{H}}}{N}e_m\phi_M\right|^2$$
$$+3\left|\sum_{m=1}^{M}e_m^{\mathrm{T}}B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{x_j e_i^{\mathrm{T}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_2)\frac{XX^{\mathrm{H}}}{N}e_m\phi_M\right|^2 \tag{199}$$
$$+3\left|\sum_{m=1}^{M}e_m^{\mathrm{T}}B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{x_j e_i^{\mathrm{T}}}{N}e_m\phi_M\right|^2+\mathcal{O}\left(N^{-\mathbb{N}}\right).$$

From (198) and (199) it follows that

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}}\mathrm{tr}\left[B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{XX^{\mathrm{H}}}{N}\phi_M\right]\right|^2\right] = \mathcal{O}\left(\frac{M}{N}\right),$$
$$\sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}^{*}}\mathrm{tr}\left[B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{XX^{\mathrm{H}}}{N}\phi_M\right]\right|^2\right] = \mathcal{O}\left(\frac{M}{N}\right),$$

and consequently the variance in (195) $\mathrm{var}\left[\mathrm{tr}\left[B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{XX^{\mathrm{H}}}{N}\phi_M\right]\right] = \mathcal{O}(M/N)$. Hence the variance in (166) decays with

$$\mathrm{var}\left[\frac{1}{N}\mathrm{tr}\left[B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{XX^{\mathrm{H}}}{N}\phi_M\right]\right] = \frac{1}{N^2}\mathrm{var}\left[\mathrm{tr}\left[B\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{XX^{\mathrm{H}}}{N}\phi_M\right]\right] \leq \frac{1}{N^2}\mathcal{O}\left(\frac{M}{N}\right) = \mathcal{O}(N^{-2}).$$

2) Expectations: In the following we derive the two expectations in (162) and (163). Using the resolvent identity $z_2\hat{Q}(z_2) = \hat{Q}(z_2)\hat{R}-I_M$ in (102) we can express the expectation in (162) as

$$\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[B\hat{Q}(z_1)R\hat{Q}(z_2)\phi_M\right]\right] = z_2^{-1}\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[B\hat{Q}(z_1)R\hat{Q}(z_2)\hat{R}\phi_M\right]\right] - z_2^{-1}\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[B\hat{Q}(z_1)R\phi_M\right]\right]. \tag{200}$$

Next, we develop the first term on the right hand side of (200) by rewriting the trace as sum and the sample covariance $\hat{R}$ as shown in (103) and apply the integration by parts formula

$$
\begin{aligned}
\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\phi_M\Big]\Big] =& \frac{1}{N}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{m=1}^{M}\mathbb{E}\Big[X_{ij}\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{e}_m\phi_M\Big] \\
=& \frac{1}{N}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{m=1}^{M}\mathbb{E}\Big[\frac{\partial}{\partial X_{ij}^*}\Big(\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{e}_m\phi_M\Big)\Big],
\end{aligned}
\tag{201}
$$

where the first order derivative with respect to X_{ij}^* is given by

$$
\begin{aligned}
\frac{\partial}{\partial X_{ij}^*}\Big(\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{e}_m\phi_M\Big) =& \boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{e}_m\phi_M+\mathcal{O}\Big(N^{-\mathbb{N}}\Big) \\
& -\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{e}_m\phi_M \\
& -\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{e}_m\phi_M.
\end{aligned}
\tag{202}
$$

Substituting (202) into (201) and computing the triple sum we can equivalently express the first term on the right hand side of (200) as

$$
\begin{aligned}
\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\phi_M\Big]\Big] =& \frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\Big]\Big]-\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\Big]\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\Big]\phi_M\Big] \\
& -\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\Big]\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\Big]\phi_M\Big]+\mathcal{O}\Big(N^{-\mathbb{N}}\Big).
\end{aligned}
\tag{203}
$$

Inserting (203) into (200) yields

$$
\begin{aligned}
\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] =& z_2^{-1}\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\Big]\Big]-z_2^{-1}\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\Big]\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\Big]\phi_M\Big] \\
& -z_2^{-1}\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\Big]\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\Big]\phi_M\Big]-z_2^{-1}\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\phi_M\Big]\Big]+\mathcal{O}\Big(N^{-\mathbb{N}}\Big).
\end{aligned}
\tag{204}
$$

Next, we need to decorrelate the second and third term on the right hand side of (204). We begin with the second term on the right hand side of (204) by analyzing the following covariance

$$
\begin{aligned}
\mathrm{cov}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\phi_M\Big],\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] =& \mathbb{E}\Big[\Big(\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\phi_M\Big]\Big)^{(\circ)}\Big(\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big)^{(\circ)}\Big] \\
=& \mathbb{E}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\Big]\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\Big]\phi_M\Big] \\
& -\mathbb{E}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\phi_M\Big]\Big]\mathbb{E}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big].
\end{aligned}
\tag{205}
$$

Applying the Cauchy-Schwarz inequality to (205) yields

$$
\begin{aligned}
& \Big|\mathrm{cov}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\phi_M\Big],\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]\Big|^2 \\
\leq & \mathbb{E}\Big[\Big|\Big(\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\phi_M\Big]\Big)^{(\circ)}\Big|^2\Big]\mathbb{E}\Big[\Big|\Big(\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big)^{(\circ)}\Big|^2\Big] \\
= & \mathrm{var}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\phi_M\Big]\Big]\mathrm{var}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] = \mathcal{O}\big(N^{-4}\big),
\end{aligned}
\tag{206}
$$

where we have used (161) in Lemma 13 and (165) in Lemma 14 to obtain

$$
\mathrm{var}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\phi_M\Big]\Big] = \mathcal{O}\big(N^{-2}\big), \qquad \mathrm{var}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] = \mathcal{O}\big(N^{-2}\big).
$$

It follows from (206) that the covariance in (205) decays with

$$
\mathrm{cov}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\phi_M\Big],\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] = \mathcal{O}\big(N^{-2}\big).
\tag{207}
$$

Hence, we can decorrelate the second quantity on the right hand side of (204) as

$$
\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\right]\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\right]\phi_M\right]=\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\phi_M\right]\right]\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]
$$
$$
+\mathrm{cov}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\phi_M\right],\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right] \tag{208}
$$
$$
=\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\phi}_M\right]\right]\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]+\mathcal{O}(N^{-2}).
$$

Let us now decorrelate the third quantity on the right hand side of (204) by applying the same technique. We start by analyzing the following covariance

$$
\mathrm{cov}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\phi_M\right],\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]=\mathbb{E}\left[\left(\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\phi_M\right]\right)^{(\circ)}\left(\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right)^{(\circ)}\right]
$$
$$
=\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\right]\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\right]\phi_M\right] \tag{209}
$$
$$
-\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\phi_M\right]\right]\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right].
$$

Applying the Cauchy Schwarz inequality to the covariance in (209) yields

$$
\left|\mathrm{cov}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\phi_M\right],\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]\right|^2
$$
$$
\leq\mathbb{E}\left[\left|\left(\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\phi_M\right]\right)^{(\circ)}\right|^2\right]\mathbb{E}\left[\left|\left(\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right)^{(\circ)}\right|^2\right] \tag{210}
$$
$$
=\mathrm{var}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\phi_M\right]\right]\mathrm{var}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]=\mathcal{O}(N^{-4}),
$$

where we have used (166) in Lemma 14 and (160) in Lemma 13 to obtain

$$
\mathrm{var}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\phi_M\right]\right]=\mathcal{O}(N^{-2}),\qquad\mathrm{var}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]=\mathcal{O}(N^{-2}).
$$

From (210) it follows that the covariance in (209) decays with

$$
\mathrm{cov}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\phi_M\right],\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]=\mathcal{O}(N^{-2}).
$$

Using (209) we can decorrelated the third term on the right hand side of (204) as

$$
\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\right]\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\right]\phi_M\right]=\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\phi_M\right]\right]\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]
$$
$$
+\mathrm{cov}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\phi_M\right],\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right] \tag{211}
$$
$$
=\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\phi_M\right]\right]\mathbb{E}\left[\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]+\mathcal{O}(N^{-2}).
$$

Substituting the two decorrelated terms in (208) and (211) into (204) yields

$$
\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]=z_2^{-1}\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\right]\right]
$$
$$
-z_2^{-1}\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\phi_M\right]\right]\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]
$$
$$
-z_2^{-1}\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\phi_M\right]\right]\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right] \tag{212}
$$
$$
-z_2^{-1}\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\phi_M\right]\right]+\mathcal{O}(N^{-2}).
$$

Using the previously derived expectation in (158) in Lemma 13 we can equivalently express (212) as

$$
\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]=z_2^{-1}\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\right]\right]
$$
$$
-z_2^{-1}\frac{z_1}{\omega(z_1)}\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]
$$
$$
-z_2^{-1}\frac{1}{N}\mathbb{E}\left[\mathrm{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\phi_M\right]\right]\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\right] \tag{213}
$$
$$
-z_2^{-1}\frac{1}{N}\mathrm{tr}\left[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]+\mathcal{O}(N^{-1}).
$$

In order to derive the second expectation in (163) we express the sample covariance matrix $\hat{\boldsymbol{R}}$ as shown in (103) and rewrite the trace as sum to apply the integration by parts formula

$$\frac{1}{N}\mathbb{E}\left[\operatorname{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right] = \frac{1}{N}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{m=1}^{M}\mathbb{E}\left[X_{ij}\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{e}_m\phi_M\right]$$
$$= \frac{1}{N}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{m=1}^{M}\mathbb{E}\left[\frac{\partial}{\partial X_{ij}^*}\left(\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{e}_m\phi_M\right)\right],$$

(214)

where the first order derivative with respect to X_{ij}^* is given by

$$\frac{\partial}{\partial X_{ij}^*}\left(\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{e}_m\phi_M\right) = -\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{e}_m\phi_M$$
$$-\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{e}_m\phi_M$$
$$+\boldsymbol{e}_m^{\mathrm{T}}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{e}_m\phi_M+\mathcal{O}\left(N^{-\mathbb{N}}\right).$$

(215)

Substituting the derivative in (215) into (214) and computing the triple sum yields

$$\frac{1}{N}\mathbb{E}\left[\operatorname{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right] = -\frac{1}{N}\mathbb{E}\left[\operatorname{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\right]\frac{1}{N}\operatorname{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\right]\phi_M\right]$$
$$-\frac{1}{N}\mathbb{E}\left[\operatorname{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\right]\frac{1}{N}\operatorname{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\right]\phi_M\right]$$
$$+\frac{1}{N}\mathbb{E}\left[\operatorname{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\phi_M\right]\right]+\mathcal{O}\left(N^{-\mathbb{N}}\right).$$

(216)

In the following we decorrelate the first and second term on the right hand side of (216). We start with the first term on the right hand side of (216) by analyzing the covariance

$$\operatorname{cov}\left[\frac{1}{N}\operatorname{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right], \frac{1}{N}\operatorname{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]$$
$$= \mathbb{E}\left[\left(\frac{1}{N}\operatorname{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right)^{(\circ)}\left(\frac{1}{N}\operatorname{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right)^{(\circ)}\right]$$
$$= \frac{1}{N}\mathbb{E}\left[\operatorname{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\right]\frac{1}{N}\operatorname{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\right]\phi_M\right]$$
$$- \mathbb{E}\left[\frac{1}{N}\operatorname{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right]\mathbb{E}\left[\frac{1}{N}\operatorname{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right].$$

(217)

Using the Cauchy-Schwarz inequality we can upper-bound (217) by

$$\left|\operatorname{cov}\left[\frac{1}{N}\operatorname{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right], \frac{1}{N}\operatorname{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]\right|^2$$
$$\leq \mathbb{E}\left[\left|\left(\frac{1}{N}\operatorname{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right)^{(\circ)}\right|^2\right]\mathbb{E}\left[\left|\left(\frac{1}{N}\operatorname{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right)^{(\circ)}\right|^2\right]$$
$$= \operatorname{var}\left[\frac{1}{N}\operatorname{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right]\operatorname{var}\left[\frac{1}{N}\operatorname{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right] = \mathcal{O}(N^{-4}),$$

(218)

where we have used (161) in Lemma 13 and (165) in Lemma 14 to obtain

$$\operatorname{var}\left[\frac{1}{N}\operatorname{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right] = \mathcal{O}(N^{-2}), \qquad \operatorname{var}\left[\frac{1}{N}\operatorname{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right] = \mathcal{O}(N^{-2}).$$

From (218) it follows that the covariance in (217) decays with

$$\operatorname{cov}\left[\frac{1}{N}\operatorname{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right], \frac{1}{N}\operatorname{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right] = \mathcal{O}(N^{-2}).$$

With this we have decorrelated the first term on the right hand side of (216) as

$$\frac{1}{N}\mathbb{E}\left[\operatorname{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\right]\frac{1}{N}\operatorname{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\right]\phi_M\right]$$
$$= \frac{1}{N}\mathbb{E}\left[\operatorname{tr}\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right]\frac{1}{N}\mathbb{E}\left[\operatorname{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]+\mathcal{O}(N^{-2}),$$

(219)

which follows from (217). In order to decorrelate the second term on the right hand side of (216) we analyze the covariance

$$
\mathrm{cov}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\Big],\ \frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]
$$

$$
=\mathbb{E}\Big[\Big(\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\Big]\Big)^{(\circ)}\Big(\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big)^{(\circ)}\Big]
$$

$$
=\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\Big]\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\Big]\phi_M\Big]
$$

$$
-\mathbb{E}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\Big]\Big]\mathbb{E}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big].
$$

(220)

Next, we apply the Cauchy-Schwarz inequality to (220)

$$
\Big|\mathrm{cov}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\Big],\ \frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]\Big|^2
$$

$$
\leq\mathbb{E}\Big[\Big|\Big(\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\Big]\Big)^{(\circ)}\Big|^2\Big]\mathbb{E}\Big[\Big|\Big(\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big)^{(\circ)}\Big|^2\Big]
$$

$$
=\mathrm{var}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\Big]\Big]\mathrm{var}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]=\mathcal{O}(N^{-4}),
$$

(221)

where we have used (166) in Lemma 14 and (160) in Lemma 13 to obtain

$$
\mathrm{var}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\Big]\Big]=\mathcal{O}(N^{-2}),\qquad \mathrm{var}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]=\mathcal{O}(N^{-2}).
$$

As a direct consequence of (221) the covariance in (220) decays with

$$
\mathrm{cov}\Big[\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\Big],\ \frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]=\mathcal{O}(N^{-2}),
$$

and we can decorrelated the second term on the right hand side of (216) as

$$
\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\Big]\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\Big]\phi_M\Big]
$$

$$
=\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\Big]\Big]\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]+\mathcal{O}(N^{-2}).
$$

(222)

Substituting the two decorrelated terms in (219) and (222) into (216) yields

$$
\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\Big]\Big]=-\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\Big]\Big]\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]
$$

$$
-\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\Big]\Big]\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]
$$

$$
+\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\phi_M\Big]\Big]+\mathcal{O}(N^{-2}).
$$

(223)

Using the previously derived expectation in (158) and adding $\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\Big]\Big]\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\Big]$ on both sides of (223) yields

$$
\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\Big]\Big]\Big(1+\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\Big]\Big)
$$

$$
=-\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\Big]\Big]\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]
$$

$$
+\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\phi_M\Big]\Big]+\mathcal{O}(N^{-1}).
$$

(224)

Multiplying (224) by $\frac{z_2}{\omega(z_2)}$ on both sides where $1+\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\Big]=\frac{\omega(z_2)}{z_2}$ which follows from (86) and using the expectation in (159) in Lemma 13 we obtain

$$
\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\Big]\Big]=-\frac{1}{N}\frac{z_1}{\omega(z_1)}\frac{z_2}{\omega(z_2)}\mathrm{tr}\Big[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\Big]\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]
$$

$$
+\frac{1}{N}\frac{z_2}{\omega(z_2)}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\phi_M\Big]\Big]+\mathcal{O}(N^{-1}).
$$

(225)

Next, we replace $\boldsymbol{B}$ by $\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{B}$ in (225) such that

$$
\begin{aligned}
\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\right]\right] =&\;\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\phi_M\right]\right]\\
=&-\frac{1}{N}\frac{z_1}{\omega(z_1)}\frac{z_2}{\omega(z_2)}\mathrm{tr}\!\left[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]\\
&+\frac{1}{N}\frac{z_2}{\omega(z_2)}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\right]\right]+\mathcal{O}(N^{-1}).
\end{aligned}
\tag{226}
$$

Inserting the previous expression in (226) into (213) yields

$$
\begin{aligned}
\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right] =&-z_2^{-1}\frac{z_1}{\omega(z_1)}\frac{1}{N}\mathrm{tr}\!\left[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\right]\right]\left(1-\frac{z_2}{\omega(z_2)}\frac{1}{N}\mathrm{tr}\!\left[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\right]\right)\\
&+z_2^{-1}\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\right]\right]\left(1-\frac{z_2}{\omega(z_2)}\frac{1}{N}\mathrm{tr}\!\left[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\right]\right)\\
&-z_2^{-1}\frac{1}{N}\mathrm{tr}\!\left[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]+\mathcal{O}(N^{-1}).
\end{aligned}
\tag{227}
$$

Now using $\frac{z_2}{\omega(z_2)}=1-\frac{z_2}{\omega(z_2)}\frac{1}{N}\mathrm{tr}\!\left[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\right]$ which follows from (86) we can simplify the expression in (227) as

$$
\begin{aligned}
\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right] =&-\frac{z_1}{\omega(z_1)\omega(z_2)}\frac{1}{N}\mathrm{tr}\!\left[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\right]\right]\\
&+\frac{1}{\omega(z_2)}\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\right]\right]-z_2^{-1}\frac{1}{N}\mathrm{tr}\!\left[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]+\mathcal{O}(N^{-1}).
\end{aligned}
\tag{228}
$$

Replacing $\boldsymbol{B}$ in (228) by $\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}$ we obtain

$$
\begin{aligned}
\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right] =&-\frac{z_1}{\omega(z_1)\omega(z_2)}\frac{1}{N}\mathrm{tr}\!\left[\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\right]\right]\\
&+\frac{1}{\omega(z_2)}\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\right]\right]-z_2^{-1}\frac{1}{N}\mathrm{tr}\!\left[\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]+\mathcal{O}(N^{-1}).
\end{aligned}
\tag{229}
$$

By subtracting and adding $\omega(z_2)\boldsymbol{I}_M$ inside the trace of the second term on the right hand side of (229), we can equivalently express $\frac{1}{\omega(z_2)}\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\right]\right]$ as

$$
\begin{aligned}
\frac{1}{\omega(z_2)}\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\right]\right] =&\frac{1}{\omega(z_2)}\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)(\boldsymbol{R}-\omega(z_2)\boldsymbol{I}_M+\omega(z_2)\boldsymbol{I}_M)\bar{\boldsymbol{Q}}(z_2)\phi_M\right]\right]\\
=&\frac{1}{\omega(z_2)}\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)(\boldsymbol{R}-\omega(z_2)\boldsymbol{I}_M)\bar{\boldsymbol{Q}}(z_2)\phi_M\right]\right]\\
&+\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\bar{\boldsymbol{Q}}(z_2)\phi_M\right]\right]\\
=&\frac{1}{\omega(z_2)}\frac{\omega(z_2)}{z_2}\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]+\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\bar{\boldsymbol{Q}}(z_2)\phi_M\right]\right]\\
=&\frac{1}{z_2}\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]+\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\bar{\boldsymbol{Q}}(z_2)\phi_M\right]\right],
\end{aligned}
\tag{230}
$$

where we have used the resolvent $\bar{\boldsymbol{Q}}(z_2)=\frac{\omega(z_2)}{z_2}(\boldsymbol{R}-\omega(z_2)\boldsymbol{I}_M)^{-1}$ in (85). Rearranging the expression in (229) according to

$$
\begin{aligned}
&\frac{z_1}{\omega(z_1)\omega(z_2)}\frac{1}{N}\mathrm{tr}\!\left[\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\right]\right]\\
=&-\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]\\
&+\frac{1}{\omega(z_2)}\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\right]\right]-z_2^{-1}\frac{1}{N}\mathrm{tr}\!\left[\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]+\mathcal{O}(N^{-1}),
\end{aligned}
\tag{231}
$$

and substituting (230) into (231) and multiplying by z_2 on both sides yields

$$
\begin{aligned}
&\frac{z_1 z_2}{\omega(z_1)\omega(z_2)}\frac{1}{N}\mathrm{tr}\!\left[\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\right]\right]\\
=&\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]-\frac{1}{N}\mathrm{tr}\!\left[\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\right]+\mathcal{O}(N^{-1}),
\end{aligned}
$$

which can equivalently be expressed as

$$
\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]\left(\frac{z_1 z_2}{\omega(z_1)\omega(z_2)}\frac{1}{N}\mathrm{tr}\!\left[\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\right]-1\right)=-\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\phi_M\right]\right]+\mathcal{O}(N^{-1}).
\tag{232}
$$

Using the previously derived expectation in (158) in Lemma 13 we obtain

$$
\frac{1}{N}\mathbb{E}\!\left[\mathrm{tr}\!\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\right]\right]=\frac{1}{1-\gamma(z_1,z_2)}\frac{1}{N}\mathrm{tr}\!\left[\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\right]+\mathcal{O}(N^{-1}),
\tag{233}
$$

where

$$\gamma(z_1,z_2) = \frac{z_1 z_2}{\omega(z_1)\omega(z_2)}\frac{1}{N}\mathrm{tr}\big[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\big]. \tag{234}$$

Replacing $\boldsymbol{B}$ by $\bar{\boldsymbol{Q}}(z_2)\boldsymbol{B}$ in (228) yields

$$\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\bar{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] = -\frac{z_1}{\omega(z_1)\omega(z_2)}\frac{1}{N}\mathrm{tr}\big[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\big]\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\Big]\Big]$$
$$+\frac{1}{\omega(z_2)}\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] - \frac{1}{z_2}\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] + \mathcal{O}(N^{-1}). \tag{235}$$

Substituting (233) into (235) we obtain

$$\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\bar{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] = -\frac{z_1}{\omega(z_1)\omega(z_2)}\frac{1}{N}\mathrm{tr}\big[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\big]\frac{1}{1-\gamma(z_1,z_2)}\frac{1}{N}\mathrm{tr}\big[\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\big]$$
$$+\frac{1}{\omega(z_2)}\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] - \frac{1}{z_2}\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] + \mathcal{O}(N^{-1}). \tag{236}$$

By adding and subtracting $\omega(z_2)\boldsymbol{I}_M$ inside the trace of the second term on the right hand side of (236) and using $\bar{\boldsymbol{Q}}(z_2) = \frac{\omega(z_2)}{z_2}(\boldsymbol{R}-\omega(z_2)\boldsymbol{I}_M)^{-1}$ in (85), we can equivalently express $\frac{1}{\omega(z_2)}\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]$ as

$$\frac{1}{\omega(z_2)}\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] = \frac{1}{\omega(z_2)}\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)(\boldsymbol{R}-\omega(z_2)\boldsymbol{I}_M+\omega(z_2)\boldsymbol{I}_M)\bar{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]$$
$$= \frac{1}{z_2}\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] + \frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\bar{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big]. \tag{237}$$

Substituting (237) into (236) yields

$$\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\bar{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] = -\frac{z_1}{\omega(z_1)\omega(z_2)}\frac{1}{N}\mathrm{tr}\big[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\big]\frac{1}{1-\gamma(z_1,z_2)}\frac{1}{N}\mathrm{tr}\big[\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\big]$$
$$+\frac{1}{z_2}\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] + \frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\bar{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] \tag{238}$$
$$-\frac{1}{z_2}\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] + \mathcal{O}(N^{-1}).$$

Rearranging (238) and using the previously derived expectation in (158) in Lemma 13 we obtain the expectation in (162) of Lemma 14

$$\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] = \frac{z_1 z_2}{\omega(z_1)\omega(z_2)}\frac{1}{N}\mathrm{tr}\big[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\big]\frac{1}{1-\gamma(z_1,z_2)}\frac{1}{N}\mathrm{tr}\big[\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\big]$$
$$+\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\phi_M\Big]\Big] + \mathcal{O}(N^{-1}) \tag{239}$$
$$= \frac{1}{1-\gamma(z_1,z_2)}\frac{1}{N}\mathrm{tr}\big[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\big] + \mathcal{O}(N^{-1}),$$

where we have used $\gamma(z_1,z_2)$ in (234). Using the previously derived expectation in (162) we can develop

$$\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\phi_M\Big]\Big] = \frac{1}{1-\gamma(z_1,z_2)}\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\Big] + \mathcal{O}(N^{-1}), \tag{240}$$

$$\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\phi_M\Big]\Big] = \frac{1}{1-\gamma(z_1,z_2)}\frac{1}{N}\mathrm{tr}\big[\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\big] + \mathcal{O}(N^{-1}) \tag{241}$$

Substituting (240) and (241) into (225) we obtain the expectation in (163) in Lemma 14

$$\frac{1}{N}\mathbb{E}\Big[\mathrm{tr}\Big[\boldsymbol{B}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{N}\phi_M\Big]\Big] = -\frac{z_1}{\omega(z_1)}\frac{z_2}{\omega(z_2)}\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\Big]\frac{1}{1-\gamma(z_1,z_2)}\frac{1}{N}\mathrm{tr}\big[\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}\big]$$
$$+\frac{z_2}{\omega(z_2)}\frac{1}{1-\gamma(z_1,z_2)}\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\Big] + \mathcal{O}(N^{-1})$$
$$= \frac{z_2}{\omega(z_2)}\frac{1}{1-\gamma(z_1,z_2)}\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\Big]$$
$$-\frac{\gamma(z_1,z_2)}{1-\gamma(z_1,z_2)}\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{B}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\Big] + \mathcal{O}(N^{-1}).$$

APPENDIX L
PROOF OF LEMMA L

A. Variance

Using the Nash-Poincaré inequality we can upper-bound the variance of

$$\mathrm{var}\Big[\frac{1}{N}\mathrm{tr}\Big[\hat{\boldsymbol{Q}}(z)^2\hat{\boldsymbol{R}}\phi_M\Big]\Big] \leq \frac{1}{N^2}\sum_{i=1}^{M}\sum_{j=1}^{N}\Bigg(\mathbb{E}\Bigg[\Big|\frac{\partial}{\partial X_{ij}}\mathrm{tr}\Big[\hat{\boldsymbol{Q}}(z)^2\hat{\boldsymbol{R}}\phi_M\Big]\Big|^2\Bigg] + \mathbb{E}\Bigg[\Big|\frac{\partial}{\partial X_{ij}^*}\mathrm{tr}\Big[\hat{\boldsymbol{Q}}(z)^2\hat{\boldsymbol{R}}\phi_M\Big]\Big|^2\Bigg]\Bigg),$$

where

$$\frac{\partial}{\partial X_{ij}}\mathrm{tr}\Big[\hat{\boldsymbol{Q}}(z)^2\hat{\boldsymbol{R}}\phi_M\Big]=\frac{\partial}{\partial X_{ij}}\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l$$

$$=\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\Big(\frac{\partial}{\partial X_{ij}}\hat{\boldsymbol{Q}}(z)\Big)\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l\phi_M+\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\Big(\frac{\partial}{\partial X_{ij}}\hat{\boldsymbol{R}}\Big)\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l\phi_M+\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\hat{\boldsymbol{R}}\Big(\frac{\partial}{\partial X_{ij}}\hat{\boldsymbol{Q}}(z)\Big)\boldsymbol{e}_l\phi_M$$

$$=-\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z)\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l\phi_M+\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l\phi_M$$

$$-\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l\phi_M+\mathcal{O}\Big(N^{-\mathbb{N}}\Big)$$

and

$$\frac{\partial}{\partial X_{ij}^*}\mathrm{tr}\Big[\hat{\boldsymbol{Q}}(z)^2\hat{\boldsymbol{R}}\phi_M\Big]=\frac{\partial}{\partial X_{ij}}\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l\phi_M$$

$$=\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\Big(\frac{\partial}{\partial X_{ij}^*}\hat{\boldsymbol{Q}}(z)\Big)\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l\phi_M+\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\Big(\frac{\partial}{\partial X_{ij}^*}\hat{\boldsymbol{R}}\Big)\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l\phi_M+\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\hat{\boldsymbol{R}}\Big(\frac{\partial}{\partial X_{ij}^*}\hat{\boldsymbol{Q}}(z)\Big)\boldsymbol{e}_l\phi_M$$

$$=-\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z)\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l\phi_M+\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l\phi_M$$

$$-\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l\phi_M+\mathcal{O}\Big(N^{-\mathbb{N}}\Big).$$

Using Jensen's inequality we can upper-bound both derivatives by

$$\Big|\frac{\partial}{\partial X_{ij}}\mathrm{tr}\Big[\hat{\boldsymbol{Q}}(z)^2\hat{\boldsymbol{R}}\phi_M\Big]\Big|^2\leq 3\Big|\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z)\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l\phi_M\Big|^2$$

$$+3\Big|\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l\phi_M\Big|^2$$

$$+3\Big|\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l\phi_M\Big|^2+\mathcal{O}\Big(N^{-\mathbb{N}}\Big),$$

and

$$\Big|\frac{\partial}{\partial X_{ij}^*}\mathrm{tr}\Big[\hat{\boldsymbol{Q}}(z)^2\hat{\boldsymbol{R}}\phi_M\Big]\Big|^2\leq 3\Big|\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z)\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l\phi_M\Big|^2$$

$$+3\Big|\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l\phi_M\Big|^2$$

$$+3\Big|\sum_{l=1}^{M}\boldsymbol{e}_l^{\mathrm{H}}\hat{\boldsymbol{Q}}(z)\hat{\boldsymbol{R}}\hat{\boldsymbol{Q}}(z)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z)\boldsymbol{e}_l\phi_M\Big|^2+\mathcal{O}\Big(N^{-\mathbb{N}}\Big),$$

such that

$$\mathbb{E}\Big[\Big|\frac{\partial}{\partial X_{ij}}\mathrm{tr}\Big[\hat{\boldsymbol{Q}}(z)^2\hat{\boldsymbol{R}}\phi_M\Big]\Big|^2\Big]=\mathcal{O}\Big(\frac{M}{N}\Big)\qquad\mathbb{E}\Big[\Big|\frac{\partial}{\partial X_{ij}^*}\mathrm{tr}\Big[\hat{\boldsymbol{Q}}(z)^2\hat{\boldsymbol{R}}\phi_M\Big]\Big|^2\Big]=\mathcal{O}\Big(\frac{M}{N}\Big)$$

and consequently the variance of the quantity of interest decays with

$$\mathrm{var}\Big[\frac{1}{N}\mathrm{tr}\Big[\hat{\boldsymbol{Q}}(z)^2\hat{\boldsymbol{R}}\phi_M\Big]\Big]=\mathcal{O}\big(N^{-2}\big).$$

B. Expectation

The expectation can be derived by taking the derivative of (159) with respect to z for $\boldsymbol{B}=\boldsymbol{I}_M$ which yields

$$\frac{\partial}{\partial z}\frac{1}{N}\mathbb{E}\Big[\hat{\boldsymbol{Q}}(z)\hat{\boldsymbol{R}}\phi_M\Big]=\frac{1}{N}\frac{\partial\omega(z)}{\partial z}\mathrm{tr}\Big[\boldsymbol{R}(\boldsymbol{R}-\omega(z)\boldsymbol{I}_M)^{-2}\phi_M\Big]+\mathcal{O}\big(N^{-1}\big)$$

$$=\frac{1}{N}\omega'(z)\mathrm{tr}\Big[\boldsymbol{R}(\boldsymbol{R}-\omega(z)\boldsymbol{I}_M)^{-2}\phi_M\Big]+\mathcal{O}\big(N^{-1}\big)$$

$$=\frac{z^2}{\omega(z)^2}\frac{1}{N}\mathrm{tr}\Big[\boldsymbol{R}\bar{\boldsymbol{Q}}(z)^2\Big]\omega'(z)+\mathcal{O}\big(N^{-1}\big).$$

APPENDIX M

PROOF OF LEMMA 16

A. Variance

The variance $\mathrm{var}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big]$ can be upper-bounded using the Nash-Poincaré inequality which states that

$$\mathrm{var}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big] \leq \sum_{i=1}^{M}\sum_{j=1}^{N}\left(\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}}s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\right|^2\right]+\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}^*}s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\right|^2\right]\right), \tag{242}$$

with first order derivative with respect to X_{ij}

$$\frac{\partial}{\partial X_{ij}}\Big(s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big) = s_1^{\mathrm{H}}\hat{Q}(z_2)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}R^{\mathrm{H}/2}s_2\phi_M - s_1^{\mathrm{H}}\hat{Q}(z_2)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_2)\hat{R}s_2\phi_M + \mathcal{O}\Big(N^{-\mathbb{N}}\Big), \tag{243}$$

and first order derivative with respect to X_{ij}^*

$$\frac{\partial}{\partial X_{ij}^*}\Big(s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big) = s_1^{\mathrm{H}}\hat{Q}(z_2)R^{1/2}\frac{x_j e_i^{\mathrm{T}}}{N}R^{\mathrm{H}/2}s_2\phi_M - s_1^{\mathrm{H}}\hat{Q}(z_2)R^{1/2}\frac{x_j e_i^{\mathrm{T}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_2)\hat{R}s_2\phi_M + \mathcal{O}\Big(N^{-\mathbb{N}}\Big). \tag{244}$$

Jensen's inequality allows to upper-bound the absolute value squared of the first order derivative with respect to X_{ij} in (243) by

$$\left|\frac{\partial}{\partial X_{ij}}\Big(s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big)\right|^2 \leq 2\left|s_1^{\mathrm{H}}\hat{Q}(z_2)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}R^{\mathrm{H}/2}s_2\phi_M\right|^2 + 2\left|s_1^{\mathrm{H}}\hat{Q}(z_2)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_2)\hat{R}s_2\phi_M\right|^2 + \mathcal{O}\Big(N^{-\mathbb{N}}\Big), \tag{245}$$

as well as the first order derivative with respect to X_{ij}^* in (244) by

$$\left|\frac{\partial}{\partial X_{ij}^*}\Big(s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big)\right|^2 \leq 2\left|s_1^{\mathrm{H}}\hat{Q}(z_2)R^{1/2}\frac{x_j e_i^{\mathrm{T}}}{N}R^{\mathrm{H}/2}s_2\phi_M\right|^2 + 2\left|s_1^{\mathrm{H}}\hat{Q}(z_2)R^{1/2}\frac{x_j e_i^{\mathrm{T}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_2)\hat{R}s_2\phi_M\right|^2 + \mathcal{O}\Big(N^{-\mathbb{N}}\Big). \tag{246}$$

From (245) and (246) it follows that

$$\sum_{i=1}^{M}\sum_{j=1}^{M}\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}}\Big(s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big)\right|^2\right] \leq \mathcal{O}\big(N^{-1}\big), \qquad \sum_{i=1}^{M}\sum_{j=1}^{M}\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}^*}\Big(s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big)\right|^2\right] \leq \mathcal{O}\big(N^{-1}\big). \tag{247}$$

Substituting (247) into (242) we obtain the expression in Lemma 16 in (168)

$$\mathrm{var}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big] \leq \sum_{i=1}^{M}\sum_{j=1}^{N}\left(\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}}s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\right|^2\right]+\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}^*}s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\right|^2\right]\right) \leq \mathcal{O}\big(N^{-1}\big).$$

B. Expectation

In order to derive the expectation in (167) we use the expression of the sample covariance matrix in (103) and apply the integration by parts formula to develop

$$\mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big] = \sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[X_{ij}s_1^{\mathrm{H}}\hat{Q}(z_2)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}R^{\mathrm{H}/2}s_2\phi_M\right]$$
$$= \sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\frac{\partial}{\partial X_{ij}^*}\left(s_1^{\mathrm{H}}\hat{Q}(z_2)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}R^{\mathrm{H}/2}s_2\phi_M\right)\right], \tag{248}$$

where the first order derivative with respect to X_{ij}^* is given by

$$\frac{\partial}{\partial X_{ij}^*}\left(s_1^{\mathrm{H}}\hat{Q}(z_2)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}R^{\mathrm{H}/2}s_2\phi_M\right) = s_1^{\mathrm{H}}\hat{Q}(z_2)R^{1/2}\frac{e_i e_i^{\mathrm{T}}}{N}R^{\mathrm{H}/2}s_2\phi_M$$
$$- s_1^{\mathrm{H}}\hat{Q}(z_2)R^{1/2}\frac{x_j e_i^{\mathrm{H}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_2)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}R^{\mathrm{H}/2}s_2\phi_M + \mathcal{O}\Big(N^{-\mathbb{N}}\Big). \tag{249}$$

Substituting (249) into (248) yields

$$\mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big] = \mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)Rs_2\phi_M\Big] - \mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\frac{1}{N}\mathrm{tr}\Big[\hat{Q}(z_2)R\Big]\Big] + \mathcal{O}\Big(N^{-\mathbb{N}}\Big). \tag{250}$$

Furthermore, we can use the definition of $\alpha(z_2)$ in (136) to express

$$\phi_M\frac{1}{N}\mathrm{tr}\Big[\hat{Q}(z_2)R\Big] = \alpha(z_2) + \frac{1}{N}\mathrm{tr}\Big[R\bar{Q}(z_2)\Big]. \tag{251}$$

Replacing $\frac{1}{N}\text{tr}\big[\hat{Q}(z_2)R\big]$ in the second term on the right hand side of (250) by (251) leads to

$$\mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big] = \mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)Rs_2\phi_M\Big] - \mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\alpha(z_2)\Big] - \mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big]\frac{1}{N}\text{tr}\big[R\bar{Q}(z_2)\big] + \mathcal{O}\Big(N^{-\mathbb{N}}\Big). \quad (252)$$

By bringing the third term of the right hand side of (252) to the left hand side and using $\frac{\omega(z_2)}{z_2} = 1 + \frac{1}{N}\text{tr}\big[R\bar{Q}(z_2)\big]$ which follows from (86) we obtain

$$\mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big]\frac{\omega(z_2)}{z_2} = \mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)Rs_2\phi_M\Big] - \mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\alpha(z_2)\Big] + \mathcal{O}\Big(N^{-\mathbb{N}}\Big). \quad (253)$$

Multiplying by $\frac{z_2}{\omega(z_2)}$ on both sides of (253) yields

$$\mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big] = \frac{z_2}{\omega(z_2)}\mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)Rs_2\phi_M\Big] - \frac{z_2}{\omega(z_2)}\mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\alpha(z_2)\Big] + \mathcal{O}\Big(N^{-\mathbb{N}}\Big). \quad (254)$$

According to Lemma 13 the expectation and the variance of $\alpha(z_2)$ in (136) decay with $\mathbb{E}[\alpha(z_2)] = \mathcal{O}(N^{-1})$ and $\text{var}[\alpha(z_2)] = \mathcal{O}(N^{-1})$. Hence, $\mathbb{E}\big[|\alpha(z_2)|^2\big] = \text{var}[\alpha(z_2)] + \mathbb{E}[\alpha(z_2)]^2 = \mathcal{O}(N^{-2})$ and we can upper-bound the second term on the right hand side of (254) using the Cauchy-Schwarz inequality

$$\mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\alpha(z_2)\Big] \leq \sqrt{\mathbb{E}\Big[\Big|s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big|^2\Big]\mathbb{E}\big[|\alpha(z_2)|^2\big]} = \mathcal{O}\big(N^{-1}\big),$$

and (254) simplifies to

$$\mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big] = \frac{z_2}{\omega(z_2)}\mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)Rs_2\phi_M\Big] + \mathcal{O}\big(N^{-1}\big). \quad (255)$$

By definition the resolvent $\hat{Q}(z_2)$ converges to its deterministic equivalent $\bar{Q}(z_2)$ in (85). Hence, the expectation in (167) in Lemma 16 is obtained by replacing $\hat{Q}(z_2)$ with $\bar{Q}(z_2)$ in (255) according to

$$\mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)\hat{R}s_2\phi_M\Big] = \frac{z_2}{\omega(z_2)}s_1^{\mathrm{H}}\bar{Q}(z_2)Rs_2 + \mathcal{O}\big(N^{-1}\big),$$

since $\mathbb{E}\Big[s_1^{\mathrm{H}}\hat{Q}(z_2)Rs_2\phi_M\Big] = s_1^{\mathrm{H}}\mathbb{E}\Big[\hat{Q}(z_2)\phi_M\Big]Rs_2$.

APPENDIX N
PROOF OF LEMMA 17

A. Variance

The variance $\text{var}\Big[s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)s_2\phi_M\Big]$ can be upper-bounded by applying the Nash-Poincaré inequality

$$\text{var}\Big[s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)s_2\phi_M\Big] \leq \sum_{i=1}^{M}\sum_{j=1}^{N}\left(\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}}s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)s_2\phi_M\right|^2\right] + \mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}^*}s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)s_2\phi_M\right|^2\right]\right), \quad (256)$$

where the first order derivative with respect to X_{ij} is given by

$$\frac{\partial}{\partial X_{ij}}s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)s_2\phi_M = -s_1^{\mathrm{H}}\hat{Q}(z_1)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_1)R\hat{Q}(z_2)s_2\phi_M$$
$$-s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_2)s_2\phi_M + \mathcal{O}\Big(N^{-\mathbb{N}}\Big), \quad (257)$$

and the first order derivative with respect to X_{ij}^* is given by

$$\frac{\partial}{\partial X_{ij}^*}s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)s_2\phi_M = -s_1^{\mathrm{H}}\hat{Q}(z_1)R^{1/2}\frac{x_j e_i^{\mathrm{T}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_1)R\hat{Q}(z_2)s_2\phi_M$$
$$-s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{x_j e_i^{\mathrm{T}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_2)s_2\phi_M + \mathcal{O}\Big(N^{-\mathbb{N}}\Big). \quad (258)$$

Using Jensen's inequality we can upper-bound the squared absolute value of the first order derivative with respect to X_{ij} in (257) by

$$\left|\frac{\partial}{\partial X_{ij}}s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)s_2\phi_M\right|^2 \leq 2\left|s_1^{\mathrm{H}}\hat{Q}(z_1)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_1)R\hat{Q}(z_2)s_2\phi_M\right|^2$$
$$+2\left|s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)R^{1/2}\frac{e_i x_j^{\mathrm{H}}}{N}R^{\mathrm{H}/2}\hat{Q}(z_2)s_2\phi_M\right|^2 + \mathcal{O}\Big(N^{-\mathbb{N}}\Big) \quad (259)$$

as well as the squared absolute value of the first oder derivative with respect to X_{ij}^* in (258) by

$$\left|\frac{\partial}{\partial X_{ij}^*}\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{s}_2\phi_M\right|^2 \leq 2\left|\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{s}_2\phi_M\right|^2$$
$$+2\left|\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{s}_2\phi_M\right|^2+\mathcal{O}\left(N^{-\mathbb{N}}\right). \tag{260}$$

It can be observed in (259) and (260) that

$$\sum_{i=1}^{M}\sum_{j=1}^{M}\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}}\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{s}_2\phi_M\right|^2\right] \leq \mathcal{O}(N^{-1}), \qquad \sum_{i=1}^{M}\sum_{j=1}^{M}\mathbb{E}\left[\left|\frac{\partial}{\partial X_{ij}^*}\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{s}_2\phi_M\right|^2\right] \leq \mathcal{O}(N^{-1}),$$

and as a direct consequence the variance in (256) decays with $\mathrm{var}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{s}_2\phi_M\right] \leq \mathcal{O}(N^{-1})$ which coincides with the variance in (170) in Lemma 17.

B. Expectations

In order to derive the expectation in (169) we use the expression of the sample covariance matrix $\hat{\boldsymbol{R}}$ in (103) and apply the integration by parts formula

$$\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\boldsymbol{s}_2\phi_M\right] = \sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[X_{ij}\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{s}_2\phi_M\right]$$
$$= \sum_{i=1}^{M}\sum_{j=1}^{N}\mathbb{E}\left[\frac{\partial}{\partial X_{ij}^*}\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{s}_2\phi_M\right], \tag{261}$$

where the first order derivative with respect to X_{ij}^* is given by

$$\frac{\partial}{\partial X_{ij}^*}\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{s}_2\phi_M = -\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{s}_2\phi_M$$
$$-\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{x}_j\boldsymbol{e}_i^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{x}_j^{\mathrm{H}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{s}_2\phi_M$$
$$+\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}^{1/2}\frac{\boldsymbol{e}_i\boldsymbol{e}_i^{\mathrm{T}}}{N}\boldsymbol{R}^{\mathrm{H}/2}\boldsymbol{s}_2\phi_M+\mathcal{O}\left(N^{-\mathbb{N}}\right). \tag{262}$$

Substituting (262) into (261) and computing the double sum yields

$$\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\boldsymbol{s}_2\phi_M\right] = -\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\boldsymbol{s}_2\phi_M\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\right]\right] - \mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\boldsymbol{s}_2\phi_M\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\right]\right]$$
$$+\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\boldsymbol{s}_2\phi_M\right]+\mathcal{O}\left(N^{-\mathbb{N}}\right). \tag{263}$$

Using the definition of $\alpha(z_2)$ in (136) we know that

$$\phi_M\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\right] = \alpha(z_2)+\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\right],$$

which allows to express (263) as

$$\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\boldsymbol{s}_2\phi_M\right] = -\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\boldsymbol{s}_2\phi_M\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\right]\right] - \mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\boldsymbol{s}_2\phi_M\alpha(z_2)\right]$$
$$-\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\boldsymbol{s}_2\phi_M\right]\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\right]+\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\boldsymbol{s}_2\phi_M\right]+\mathcal{O}\left(N^{-\mathbb{N}}\right). \tag{264}$$

Bringing the third term on the right hand side of (264) to the other side and using $1+\frac{1}{N}\mathrm{tr}\left[\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\right] = \frac{\omega(z_2)}{z_2}$ which follows from (86) we obtain

$$\frac{\omega(z_2)}{z_2}\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\boldsymbol{s}_2\phi_M\right] = -\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\hat{\boldsymbol{R}}\boldsymbol{s}_2\phi_M\frac{1}{N}\mathrm{tr}\left[\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\right]\right] - \mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\boldsymbol{s}_2\phi_M\alpha(z_2)\right]$$
$$+\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{R}\boldsymbol{s}_2\phi_M\right]+\mathcal{O}\left(N^{-\mathbb{N}}\right). \tag{265}$$

By adding and subtracting $z_2\boldsymbol{I}_M$ to $\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\boldsymbol{s}_2\phi_M\right]$ and using the definition of the resolvent $\hat{\boldsymbol{Q}}(z_2)$ in (85) we can equivalently express the left hand side of (265) as

$$\frac{\omega(z_2)}{z_2}\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\boldsymbol{s}_2\phi_M\right] = \frac{\omega(z_2)}{z_2}\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\left(\hat{\boldsymbol{R}}-z_2\boldsymbol{I}_M+z_2\boldsymbol{I}_M\right)\boldsymbol{s}_2\phi_M\right]$$
$$= \frac{\omega(z_2)}{z_2}\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\left(\hat{\boldsymbol{R}}-z_2\boldsymbol{I}_M\right)\boldsymbol{s}_2\phi_M\right]+\frac{\omega(z_2)}{z_2}z_2\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{s}_2\phi_M\right] \tag{266}$$
$$= \frac{\omega(z_2)}{z_2}\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\boldsymbol{s}_2\phi_M\right]+\omega(z_2)\mathbb{E}\left[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{s}_2\phi_M\right].$$

Replacing the left hand side of (265) with the expression obtained in (266) and subtracting $\omega(z_2)\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)s_2\phi_M\right]$ on both sides yields

$$
\begin{aligned}
\frac{\omega(z_2)}{z_2}\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)Rs_2\phi_M\right] =& -\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)\hat{R}s_2\phi_M\frac{1}{N}\mathrm{tr}\left[\hat{Q}(z_1)R\hat{Q}(z_2)R\right]\right] \\
& -\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)\hat{R}s_2\phi_M\alpha(z_2)\right]+\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)(R-\omega(z_2)I_M)s_2\phi_M\right]+\mathcal{O}\left(N^{-\mathbb{N}}\right).
\end{aligned}
\tag{267}
$$

Next, we decorrelate the first term on the right hand side of (267) by analyzing the covariance

$$
\begin{aligned}
\mathrm{cov}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)\hat{R}s_2\phi_M,\ \phi_M\frac{1}{N}\mathrm{tr}\left[\hat{Q}(z_1)R\hat{Q}(z_2)R\right]\right] =& \mathbb{E}\left[\left(s_1^{\mathrm{H}}\hat{Q}(z_1)\hat{R}s_2\phi_M\right)^{(\circ)}\left(\phi_M\frac{1}{N}\mathrm{tr}\left[\hat{Q}(z_1)R\hat{Q}(z_2)R\right]\right)^{(\circ)}\right] \\
=& \mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)\hat{R}s_2\phi_M\frac{1}{N}\mathrm{tr}\left[\hat{Q}(z_1)R\hat{Q}(z_2)R\right]\right] \\
& -\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)\hat{R}s_2\phi_M\right]\mathbb{E}\left[\phi_M\frac{1}{N}\mathrm{tr}\left[\hat{Q}(z_1)R\hat{Q}(z_2)R\right]\right].
\end{aligned}
\tag{268}
$$

Using the Cauchy-Schwarz inequality we can upper-bound

$$
\begin{aligned}
& \left|\mathbb{E}\left[\left(s_1^{\mathrm{H}}\hat{Q}(z_1)\hat{R}s_2\phi_M\right)^{(\circ)}\left(\phi_M\frac{1}{N}\mathrm{tr}\left[\hat{Q}(z_1)R\hat{Q}(z_2)R\right]\right)^{(\circ)}\right]\right|^2 \\
\leq& \mathbb{E}\left[\left|\left(s_1^{\mathrm{H}}\hat{Q}(z_1)\hat{R}s_2\phi_M\right)^{(\circ)}\right|^2\right]\mathbb{E}\left[\left|\left(\phi_M\frac{1}{N}\mathrm{tr}\left[\hat{Q}(z_1)R\hat{Q}(z_2)R\right]\right)^{(\circ)}\right|^2\right] \\
=& \mathrm{var}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)\hat{R}s_2\phi_M\right]\mathrm{var}\left[\phi_M\frac{1}{N}\mathrm{tr}\left[\hat{Q}(z_1)R\hat{Q}(z_2)R\right]\right] = \mathcal{O}(N^{-3}),
\end{aligned}
\tag{269}
$$

where we have used (168) in Lemma 16 and (165) in Lemma 14 to obtain

$$
\mathrm{var}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)\hat{R}s_2\phi_M\right] = \mathcal{O}(N^{-1}), \qquad \mathrm{var}\left[\phi_M\frac{1}{N}\mathrm{tr}\left[\hat{Q}(z_1)R\hat{Q}(z_2)R\right]\right] = \mathcal{O}(N^{-2}).
$$

From (269) it follows that the covariance in (268) decays with

$$
\mathrm{cov}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)\hat{R}s_2\phi_M,\ \phi_M\frac{1}{N}\mathrm{tr}\left[\hat{Q}(z_1)R\hat{Q}(z_2)R\right]\right] \leq \mathcal{O}\left(N^{-3/2}\right),
$$

and we can express the first term on the right hand side of (267) as

$$
\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)\hat{R}s_2\phi_M\frac{1}{N}\mathrm{tr}\left[\hat{Q}(z_1)R\hat{Q}(z_2)R\right]\right] = \mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)\hat{R}s_2\phi_M\right]\mathbb{E}\left[\phi_M\frac{1}{N}\mathrm{tr}\left[\hat{Q}(z_1)R\hat{Q}(z_2)R\right]\right]+\mathcal{O}\left(N^{-3/2}\right).
\tag{270}
$$

Using the previously derived expectations in (167) in Lemma 16 and in (162) in Lemma 14 we can further develop (270) as

$$
\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)\hat{R}s_2\phi_M\frac{1}{N}\mathrm{tr}\left[\hat{Q}(z_1)R\hat{Q}(z_2)R\right]\right] = \left(\frac{z_1}{\omega(z_1)}s_1^{\mathrm{H}}\bar{Q}(z_1)Rs_2\right)\left(\frac{1}{1-\gamma(z_1,z_2)}\frac{1}{N}\mathrm{tr}\left[\bar{Q}(z_1)R\bar{Q}(z_2)R\right]\right)+\mathcal{O}(N^{-1}),
\tag{271}
$$

where $\gamma(z_1,z_2)$ is defined in (164). Substituting (271) into (267) and multiplying with $\frac{z_2}{\omega(z_2)}$ on both sides yields

$$
\begin{aligned}
\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)Rs_2\phi_M\right] =& -\frac{z_1}{\omega(z_1)}\frac{z_2}{\omega(z_2)}s_1^{\mathrm{H}}\bar{Q}(z_1)Rs_2\left(\frac{1}{1-\gamma(z_1,z_2)}\frac{1}{N}\mathrm{tr}\left[\bar{Q}(z_1)R\bar{Q}(z_2)R\right]\right) -\frac{z_2}{\omega(z_2)}\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)\hat{R}s_2\phi_M\alpha(z_2)\right] \\
& +\frac{z_2}{\omega(z_2)}\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)(R-\omega(z_2)I_M)s_2\phi_M\right]+\mathcal{O}\left(N^{-1}\right).
\end{aligned}
\tag{272}
$$

Inserting $\bar{Q}(z_2)s_2$ for s_2 and using $\bar{Q}(z_2) = \frac{\omega(z_2)}{z_2}(R-\omega(z_2)I_M)^{-1}$ we obtain

$$
\begin{aligned}
\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)R\bar{Q}(z_2)s_2\phi_M\right] =& -\frac{z_1}{\omega(z_1)}\frac{z_2}{\omega(z_2)}s_1^{\mathrm{H}}\bar{Q}(z_1)R\bar{Q}(z_2)s_2\left(\frac{1}{1-\gamma(z_1,z_2)}\frac{1}{N}\mathrm{tr}\left[\bar{Q}(z_1)R\bar{Q}(z_2)R\right]\right) \\
& -\frac{z_2}{\omega(z_2)}\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)\hat{R}\bar{Q}(z_2)s_2\phi_M\alpha(z_2)\right]+\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)s_2\phi_M\right]+\mathcal{O}\left(N^{-1}\right).
\end{aligned}
\tag{273}
$$

By rearranging (273) the quantity of interest in (169) can be expressed as

$$
\begin{aligned}
\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)s_2\phi_M\right] =& \mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)R\bar{Q}(z_2)s_2\phi_M\right]+s_1^{\mathrm{H}}\bar{Q}(z_1)R\bar{Q}(z_2)s_2\frac{\gamma(z_1,z_2)}{1-\gamma(z_1,z_2)} \\
& +\frac{z_2}{\omega(z_2)}\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)\hat{R}\bar{Q}(z_2)s_2\phi_M\alpha(z_2)\right]+\mathcal{O}\left(N^{-1}\right),
\end{aligned}
\tag{274}
$$

where we have used $\gamma(z_1,\gamma_2)$ in (171). In the following we upper-bound the third expression on the right hand side of (274). Using Cauchy-Schwarz inequality it can be seen that

$$
\left|\mathbb{E}\left[s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)\hat{R}\bar{Q}(z_2)s_2\phi_M\alpha(z_2)\right]\right|^2 \leq \mathbb{E}\left[\left|s_1^{\mathrm{H}}\hat{Q}(z_1)R\hat{Q}(z_2)\hat{R}\bar{Q}(z_2)s_2\phi_M\right|^2\right]\mathbb{E}\left[|\alpha(z_2)|^2\right],
$$

where $\mathbb{E}\big[|\alpha(z_2)|^2\big] = \text{var}[\alpha(z_2)] + \mathbb{E}[\alpha(z_2)]^2 = \mathcal{O}(N^{-2})$ which follows from Lemma 13 and the definition of $\alpha(z_2)$ in (136). Consequently, the third term on the right hand side of (274) decays as

$$\frac{z_2}{\omega(z_2)}\mathbb{E}\Big[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\hat{\boldsymbol{R}}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{s}_2\phi_M\alpha(z_2)\Big] \leq \mathcal{O}(N^{-1}),$$

and can therefore be neglected. Using the deterministic equivalent $\bar{\boldsymbol{Q}}(z_2)$ in (85) of the resolvent $\hat{\boldsymbol{Q}}(z_2)$ and $\mathbb{E}\Big[\hat{\boldsymbol{Q}}(z)\phi_M\Big] = \bar{\boldsymbol{Q}}(z)+\mathcal{O}(N^{-1})$ we obtain the expectation in (169) in Lemma 17

$$\mathbb{E}\Big[\boldsymbol{s}_1^{\mathrm{H}}\hat{\boldsymbol{Q}}(z_1)\boldsymbol{R}\hat{\boldsymbol{Q}}(z_2)\boldsymbol{s}_2\phi_M\Big] = \boldsymbol{s}_1^{\mathrm{H}}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{s}_2 + \boldsymbol{s}_1^{\mathrm{H}}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{s}_2\frac{\gamma(z_1,z_2)}{1-\gamma(z_1,z_2)}+\mathcal{O}(N^{-1})$$

$$= \left(\frac{1}{1-\gamma(z_1,z_2)}\right)\boldsymbol{s}_1^{\mathrm{H}}\bar{\boldsymbol{Q}}(z_1)\boldsymbol{R}\bar{\boldsymbol{Q}}(z_2)\boldsymbol{s}_2+\mathcal{O}(N^{-1}).$$

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