Ambiguities in Direction-of-Arrival Estimation with Linear Arrays

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Abstract—In this paper, we present a novel approach to compute ambiguities in thinned uniform linear arrays, i.e., sparse non-uniform linear arrays, via a mixed-integer program. Ambiguities arise when there exists a set of distinct directionsof-arrival, for which the corresponding steering matrix is rank-deficient and are associated with nonunique parameter estimation. Our approach uses Young tableaux for which a submatrix of the steering matrix has a vanishing determinant, which can be expressed through vanishing sums of unit roots. Each of these vanishing sums then corresponds to an ambiguous set of directions-ofarrival. We derive a method to enumerate such ambiguous sets using a mixed-integer program and present results on several examples.

Index Terms—Ambiguities, uniqueness, identifiability, array processing, direction-of-arrival estimation, sparse linear arrays, thinned linear arrays, spark, Kruskal rank, Young tableaux.

I. Introduction

Sparse uniform linear arrays, obtained from thinning uniform linear arrays (ULAs), are widely used in sensor array processing due to their ability to maintain the aperture size of the corresponding full ULA while reducing the number of array elements. This is associated with as significant decrease of the array costs including power consumption, hardware and computational complexity [28] as well as a reduction in the mainbeam width and mutual coupling [18], [19]. A variety of array thinning techniques have been proposed in literature to control the side and grating lobe levels of the array. These techniques can be classified into three main areas [48]: i) analytic thinning, e.g., based in prime number selection, where the array is formed from a $\lambda/2$ ULA by selecting sensors at prime multiples of the baseline as, e.g., in nested arrays, coprime arrays, and minimum redundancy arrays [29], [49], [25], ii) statistical thinning techniques, where sensors are selected randomly [34], and iii) optimization based thinning in which an appropriate error function is minimized [50], [17], [13], [15], [33], [27].

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Low side and grating lobe levels are important indicators for the resolution performance for sparse linear arrays in beamforming and Direction-of-Arrival (DoA) estimation applications. However, a fundamental question in the context of multisource DoA estimation in thinned arrays is that of the maximum number of sources that can uniquely be determined from an array measurement and the characterization of ambiguities in the measurements. Ambiguities in the array manifold arise when there exists a set of distinct directions-of-arrivals in the field of view, for which the corresponding steering matrix is rank-deficient. In this case it is impossible to uniquely determine the parameters of interest from single snapshot measurements.

The concept and mathematical framework for the ambiguity problem in direction finding for linear arrays dates back to Schmidt [36], who introduced the rank of an ambiguity based on linear dependency in the array steering matrix and thus classified ambiguities based on this notion of rank. In the subsequent years, research focused onto specific array geometries which have desirable properties or are free of certain types of ambiguities [20], [44], [45], [46]. For the popular ULA geometry it is known that no ambiguities exist, if the intersensor spacing is at most half wavelength [22], [39], [43], [44]. A linear array that is not uniform is called a non-uniform linear array.

Manikas and Proukakis [22] use tools from differential geometry in order to analyze ambiguities for general linear arrays. They derive a sufficient condition for the presence of ambiguities in a linear array. This sufficient condition implies that every non-uniform linear array with integer positions suffers from ambiguities, see [1]. Moreover, Manikas and Proukakis derive a class of ambiguities that are present in every non-uniform integer linear array, see [22, Theorem 2.2]. We call these ambiguities uniform ambiguities, since they are derived from a uniform partitioning of the array manifold.

For symmetric linear arrays, which can be shifted globally to positions r_1, \ldots, r_M , such that $\sum_{i=1}^{M} (r_i)^n = 0$ holds for all odd $n \in \mathbb{N}$, Manikas and

Proukakis identify additional ambiguities, see [22, Theorem 3]. This ambiguity criterion for symmetric linear arrays is generalized in Dowlut [7, Theorem 3] and Manikas [21, Theorem 7.1] to a whole class of so-called *non-uniform* ambiguities that depend on a parameter $n \in \mathbb{N}$. Moreover, Manikas [21, Lemma 7.1] states that these non-uniform ambiguities converge to a uniform ambiguity for $n \to \infty$.

Wax and Ziskind [51] derive conditions for which the signal model (1) has a unique solution based on the number of sensors in the array and the spark of the steering matrix.

In a recent paper, Achanta et al. [2] investigate the spark of DFT matrices and use vanishing sums in order to find rows of a DFT matrix that induce a matrix with full spark. However, they do not draw the connection to generalized Vandermonde matrices and the Schur polynomial. Translated to our setting, the results of [2] can determine which linear arrays do not have any ambiguities within a fixed set of certain DoAs. In contrast, our paper determines ambiguities a fixed linear array.

Contribution: Which non-uniform integer linear array suffers from ambiguities, besides the ones already known in the literature? In order to answer this question, we need to find steering vectors $\mathbf{a}(\theta_i)$ (or more precisely DoAs θ_i) such that the steering matrix for the position vector \mathbf{r} has spark at most M. This means that the M sensors cannot uniquely localize the signal sources with directions θ_i . In the following, we present a novel approach for identifying ambiguities in (non-uniform) integer linear arrays. This approach can in theory be used to compute all ambiguities from which a given integer linear array suffers, see Section III-C. However, those computations are too expensive to be of practical use, even for small arrays. By making a (small) structural assumption, we are able to formulate a mixed-integer problem (MIP) that is capable of enumerating ambiguities, see Section IV. At least for non-symmetric integer linear arrays, we show that this approach can find all ambiguities already known in the literature (namely those from [22, Theorem 2.2]), see Proposition IV.5. Moreover, we demonstrate the usefulness of our approach in Section V by presenting examples of non-symmetric integer linear arrays for which we find many previously unknown ambiguities.

Remark I.1. Our notion of ambiguity and the failure to uniquely resolve source signals can also be used to characterize ambiguities in co-arrays corresponding to non-fully augmentable arrays [1].

In this paper, the following notations are used:

matrices are denoted by boldface uppercase letters **A**, vectors are denoted by boldface lowercase letters **a**, and scalars are denoted by regular letters a. Symbols $(\cdot)^{\top}$ and $(\cdot)^{-1}$ denote the transpose, and inverse of the (matrix) argument. For $n \in \mathbb{N}$, we define $[n] := \{1, \ldots, n\}$.

II. PROBLEM DESCRIPTION

Consider a sparse (thinned) uniform linear array composed of M sensors located on the x-axis in \mathbb{R}^2 at positions corresponding to integer multiples $r_1 < r_2 < \cdots < r_M$ of a common baseline $d \in [0,1]$ measured in half wavelength. Define $\mathbf{r} = (r_1,\ldots,r_M) \in \mathbb{Z}^M$. A superposition of L narrowband waveforms emitted from sources at azimuth angles $\mathbf{\Theta} = (\theta_1,\ldots,\theta_L)$ is impinging on the array. Throughout the paper, we denote the azimuth angles $\theta \in \mathbf{\Theta}$ as Directions-of-Arrival (DoA). The received signal \mathbf{y} in a noise-free setting is expressed as

$$\mathbf{y} = \mathbf{A}(\mathbf{\Theta}) \,\mathbf{x},\tag{1}$$

where $\mathbf{x} \in \mathbb{C}^L$ is the emitted signal and $\mathbf{A}(\mathbf{\Theta}) = [\mathbf{a}(\theta_1), \dots, \mathbf{a}(\theta_L)] \in \mathbb{C}^{M \times L}$ the array steering matrix with columns

$$\mathbf{a}(\theta) = [z^{r_1}, \dots, z^{r_M}]^\top, \tag{2}$$

for $\theta \in \mathbf{\Theta}$ and $z = e^{-j\pi d \cos(\theta)}$. By using the *electrical* angle $\Phi = -\pi d \cos(\theta)$, this simplifies to $z = e^{j\Phi}$. We then denote the array steering matrix with $\mathbf{A}(\mathbf{\Phi})$.

In the case of linear measurement systems, unique recovery is assessed from the spark of the measurement matrix \mathbf{A} , which is defined as the smallest number of linearly dependent columns in \mathbf{A} . If the spark is large enough, then it is possible to uniquely recover \mathbf{x} from its measurements $\mathbf{A}\mathbf{x}$, even if the linear system is underdetermined, see, e.g., the book [9].

Uniqueness of the measurement model (1) is of importance for DoA estimation as well and directly related to the number of sources that can be identified from the measurements [51]. In our setting, the steering matrix $\mathbf{A}(\mathbf{\Theta})$ depends on the unknown directions from which the signals in \mathbf{x} impinge on the linear array **r**. This leads to a generalized notion of the spark: The ability to uniquely recover a signal \mathbf{x} coming from directions Θ depends on the spark of the induced steering matrix $\mathbf{A}(\mathbf{\Theta})$. If the spark of $\mathbf{A}(\mathbf{\Theta})$ for a fixed linear array \mathbf{r} is not full, then there exists a subset of the columns of $A(\Theta)$, i.e., a subset of steering vectors, which are linearly dependent and thus induce a rank-deficient submatrix of $\mathbf{A}(\mathbf{\Theta})$. This is called an *ambiguity*, which can be formally defined as follows.

Definition II.1 ([22]). Let $\mathbf{r} \in \mathbb{R}^M$ be a linear array with M sensors and let $\mathbf{\Theta} = [\theta_1, \dots, \theta_L]^{\top}$ be an ordered vector of DoAs with $L \leq M$. Then $\mathbf{\Theta}$ is called an *ambiguous vector of DoAs*, if

$$\operatorname{rank}(\mathbf{A}(\mathbf{\Theta})) = \operatorname{rank}([\mathbf{a}(\theta_1), \dots, \mathbf{a}(\theta_L)]) < L.$$

Furthermore, its rank of ambiguity is defined as $\rho_a = \operatorname{rank}(\mathbf{A}(\mathbf{\Theta})) \in \mathbb{N}$.

An important question is whether the steering matrix $\mathbf{A}(\boldsymbol{\Theta})$ for a fixed linear array and a vector of directions $\boldsymbol{\Theta}$ has full spark, i.e., if the spark is given by min $\{M, L\}$.

Note that more signal sources than sensor positions M always produce an ambiguity, since the rank of the corresponding submatrix can be at most M. Thus, we focus on the search of ambiguities with at most M signal sources.

In order to simplify the presentation, we make some assumptions without loss of generalization.

- (A1) We assume that the first sensor of a linear array $\mathbf{r} = (r_1, \dots, r_M)^{\top}$ is located in the origin, i.e., $r_1 = 0$, since a global shift of the array positions does not change the ambiguities.
- (A2) We assume that the electrical angles Φ lie in $\Omega = [-\pi d, \pi d]$, since $\mathbf{a}(\theta) = \mathbf{a}(2\pi \theta)$ for $\theta \in [0, 2\pi]$. For d = 1, we assume $\Phi \in [-\pi, \pi)$, since in this case $\mathbf{a}(0) = \mathbf{a}(\pi)$. This implies a one-to-one correspondence between electrical angles and DoAs, so that from now on, we will also use electrical angles.
- (A3) An appropriate global rotation of the DoAs amounts to multiplying the steering matrix $\mathbf{A}(\boldsymbol{\Theta})$ with a constant diagonal matrix with unit complex entries, which retains the ambiguity property. Thus, we assume without loss of generality that $\theta_1 = 0$ resulting in an electrical angle $\Phi_1 = -\pi d$.

Furthermore, we assume that $\mathbf{r} \in \mathbb{N}^M$. Recall that \mathbf{r} denotes the integer multiples of the common baseline $d \in [0, 1]$, measured in half wavelength. Such sparse uniform sampling is often used in practical systems due the simplicity of the hardware design and several attractive features associated with it, such as the existence of efficient search-free rooting-based DoA estimation techniques [12], [3], [32], [16], [41] and virtual signal decorrelation procedures [37], [47], [30].

III. A NOVEL APPROACH FOR DETECTING AMBIGUITIES IN LINEAR ARRAYS

Let Assumptions (A1)–(A3) hold and let $\mathbf{r} \in \mathbb{N}^M$. We first give a compact overview over our procedure to find ambiguities that is presented in the next sections.

- 1) The array steering matrix $\mathbf{A}(\mathbf{\Phi})$ for \mathbf{r} induces ambiguities if some of its $M \times M$ submatrices have a nonzero determinant. Therefore, we search for M electrical angles Φ_1, \ldots, Φ_M such that the determinant of their induced submatrix of $\mathbf{A}(\mathbf{\Phi})$ vanishes.
- 2) Every submatrix of $\mathbf{A}(\mathbf{\Phi})$ is a generalized Vandermonde determinant, see Section III-A. This determinant is divisible by the classical Vandermonde determinant, and this quotient is equal to the Schur polynomial $s_{\lambda}(\mathbf{z})$, see Equation (4). Thus, instead of searching for roots of the generalized Vandermonde determinant, we can search for roots of the Schur polynomial.
- 3) The Schur polynomial $s_{\lambda}(\mathbf{z})$ can be represented using semistandard Young tableaux (SSYTs), see Section III-B and Lemma III.1 therein.
- 4) In our case, $z_i = e^{j\Phi_i}$ for $i \in [M]$, and if $\Phi_i \in \{2\pi q : q \in \mathbb{Q}\}$, the Schur polynomial is a sum of unit roots. This means, we search unit roots such that their sum vanishes and the relationship to the SSYTs due to Lemma III.1 is satisfied, see Section III-D.
- 5) We construct these vanishing sums of unit roots by adding up rotated minimal vanishing sums. This can be formulated as a mixed-integer problem, see Section IV.

A. Generalized Vandermonde Matrix

Our goal is to determine whether there exist electrical angles $\Phi_1 < \cdots < \Phi_M$ and an $M \times M$ submatrix of the corresponding array steering matrix for an integer linear array \mathbf{r} with zero determinant. Every $M \times M$ submatrix of $\mathbf{A}(\mathbf{\Phi})$ is a generalized Vandermonde matrix of the form

$$\mathbf{B_r}(\mathbf{z}) = egin{pmatrix} z_1^{r_1} & \dots & z_M^{r_1} \ z_1^{r_2} & \dots & z_M^{r_2} \ \vdots & \ddots & \vdots \ z_1^{r_M} & \dots & z_M^{r_M} \end{pmatrix},$$

with $z_i = e^{j\Phi_i}$, $i \in [M]$ and $\Phi_1 < \cdots < \Phi_M$. We define the polynomial

$$V_{\mathbf{r}}: \mathbb{C}^M \to \mathbb{C}, \quad V_{\mathbf{r}}(\mathbf{z}) := \det(\mathbf{B}_{\mathbf{r}}(\mathbf{z}))$$
 (3)

as the generalized Vandermonde determinant. Every root of V_r induces an ambiguity.

If $\mathbf{r} = (0, 1, \dots, M-1)^{\top}$, then $V_{\mathbf{r}} : \mathbb{C}^{M} \to \mathbb{C}$ is the classical Vandermonde determinant

$$V(\mathbf{z}) = \prod_{1 \le i \le k \le M} (z_k - z_i).$$

In this case, we obtain that $V(\mathbf{z}) = 0$ if and only if $z_i = z_k$ for indices $i \neq k$. This means that ambiguities only arise if there are two equal electrical angles $\Phi_i = \Phi_k$ for $i \neq k$ [38].

For steering matrices with a non-uniform linear array, $\mathbf{B_r}(\mathbf{z})$ is a so-called generalized Vandermonde matrix, and its determinant $V_{\mathbf{r}}(\mathbf{z})$ is a generalized Vandermonde determinant. For more information on generalized Vandermonde matrices and determinants, see, e.g., [14], [4], [8], [35] as well as [24], [26] for results in case that $\mathbf{z} \in \mathbb{R}^M$ and that $\mathbf{V_r}$ is defined over a finite field, respectively.

In the present paper, we have $\mathbf{z} \in \mathbb{C}^M$. In this case, it is well known that $V_{\mathbf{r}}(\mathbf{z})$ is divisible by the (classical) Vandermonde determinant $V(\mathbf{z})$, see, e.g., [6]. The ratio

$$s_{\lambda}(\mathbf{z}) := \frac{V_{\mathbf{r}}(\mathbf{z})}{V(\mathbf{z})} \tag{4}$$

with $\lambda = \mathbf{r} - \boldsymbol{\delta}$, and $\boldsymbol{\delta} = (0, 1, \dots, M - 1)^{\top}$ is the so-called *Schur polynomial*. Here, λ is sorted non-decreasingly.

An ambiguous vector of electrical angles $[\Phi_1, \ldots, \Phi_M]^{\top}$ induces a generalized Vandermonde matrix with a zero determinant, and thus forms a root of the Schur polynomial $s_{\lambda}(\mathbf{z})$ with $z_i = e^{j\Phi_i}$, $i \in [M]$. In order to find such roots, we introduce Young tableaux in the next section, as they can be used to represent the Schur polynomial.

B. Young Tableaux

The notations, definitions and statements in this section are taken from [40].

Given $\lambda = (\lambda_1, \dots, \lambda_M)^{\top}$, $0 \le \lambda_1 \le \dots \le \lambda_M$, the Young diagram of shape λ is defined as a collection of boxes that are arranged in M left-justified rows such that the number of boxes in row M - i + 1 is λ_i . A semistandard Young tableau (SSYT) of shape λ is obtained by filling the Young diagram with positive integers $i \in [M]$ such that entries increase weakly along each row and increase strictly down each column¹.

For example, if $\lambda = (1,3,4)^{\top}$, we need to fill the Young diagram with integers from the set $\{1,2,3\}$ and one possible SSYT is the following:

For a given λ , we define \mathcal{T}_{λ} as the set of all SSYTs with entries in [M]. Each tableau $T \in \mathcal{T}_{\lambda}$ defines a weight vector $\alpha(T) = (\alpha_1(T), \dots, \alpha_M(T))^{\top}$, where $\alpha_i(T)$ is the number of times i appears. By

1	1	2	3
2	3	3	
3			

TABLE I: Number N of SSYTs for selected linear arrays.

Array N	Array	N	Array	N	Array	N
$\begin{array}{c cccc} \hline (0,2) & 2 \\ (0,3) & 3 \\ (0,4) & 4 \\ (0,5) & 5 \\ (0,1,3) & 3 \\ (0,1,4) & 6 \\ (0,1,5) & 10 \\ \hline \end{array}$	$ \begin{vmatrix} (0,2,3) \\ (0,2,4) \\ (0,2,5) \\ (0,3,4) \\ (0,3,5) \\ (0,4,5) \\ (0,1,2,4) \end{vmatrix} $	$ \begin{array}{c} 3 \\ 8 \\ 15 \\ 6 \\ 15 \\ 10 \\ 4 \end{array} $	$ \begin{array}{c} (0,1,2,5) \\ (0,1,3,4) \\ (0,1,3,5) \\ (0,1,4,5) \\ (0,2,3,4) \\ (0,2,3,5) \\ (0,2,4,5) \end{array} $	6 20 20 4 15	$ \begin{array}{c} (0,3,4,5) \\ (0,1,2,3,5) \\ (0,1,2,4,5) \\ (0,1,3,4,5) \\ (0,2,3,4,5) \end{array} $	5 10 10

definition, $\sum_{i=1}^{M} \alpha_i(T) = \sum_{i=1}^{M} \lambda_i$ holds. In the above example, $(\alpha_1(T), \alpha_2(T), \alpha_3(T))^{\top} = (2, 2, 4)^{\top}$.

SSYTs have the beautiful property that they can be used to represent the Schur polynomial. This is known in the literature under the term "Young's Rule", and it plays a key role in our procedure, since it yields a polynomial representation of the quotient of the generalized Vandermonde determinant and the classical Vandermonde determinant.

Lemma III.1 (Stanley [40]). Consider $\lambda = (\lambda_1, \dots, \lambda_M)^{\top}$ with $0 \le \lambda_1 \le \dots \le \lambda_M$. Then,

$$s_{\lambda}(\mathbf{z}) = \sum_{T \in \mathcal{T}_{\lambda}} \prod_{i=1}^{M} z_i^{\alpha_i(T)}.$$

Lemma III.1 is the reason why we only allowed integers $i \in [M]$ to appear in an SSYT in the definition at the beginning of this section. From Lemma III.1, we deduce that the degree of s_{λ} is $\sum_{i=1}^{M} \lambda_{i}$. Moreover, if $\lambda \neq 0$, then s_{λ} is homogeneous. For a formula of the total number of SSYTs of shape λ with entries in [M], see [10, Theorem 6.3], which is a simple consequence of [40, Lemma 7.21.1 and Theorem 7.21.2]. For all linear arrays with positions in $(0,1,2,3,4,5)^{\top}$, Table I lists the number N of corresponding SSYTs.

Example III.2. Consider the simple array geometry $\mathbf{r} = (1, 2, 5)^{\top}$ with baseline d = 1. With $\mathbf{r} = \lambda + \delta$ and $\delta = (0, 1, 2)^{\top}$, we obtain that $\lambda = (1, 1, 3)^{\top}$. Then the Young tableaux in \mathcal{T}_{λ} are the ones listed in Figure 1. The associated Schur polynomial is

$$s_{\lambda}(\mathbf{z}) = z_1 z_2 z_3 (z_1^2 + z_1 z_2 + z_2^2 + z_1 z_3 + z_2 z_3 + z_3^2),$$

whose roots are all $\mathbf{z} \in \mathbb{C}^3$ with $z_1 = \frac{1}{2}(-z_2 - z_3 \pm \sqrt{-3z_2^2 - 2z_3z_2 - 3z_3^2})$ or $z_1 = 0$ or $z_2 = 0$ or $z_3 = 0$.

¹Note that in general, a Young diagram can be filled with arbitrary positive integers $i \in \mathbb{N}$ to obtain an SSYT, but as it will become clear later in this section, in our setting, only integers $i \in [M]$ are allowed.

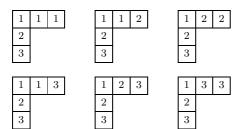


Fig. 1: Young tableaux for $\lambda = (1, 1, 3)^{\top}$

C. Algebraic root finding

Since the Schur polynomial $s_{\lambda}(\mathbf{z})$ is a complex polynomial in the variables z_i , we can search roots of the Schur polynomial algebraically by splitting each variable $z_i = x_i + j \cdot y_i$ with $x_i = \text{Re}(z_i)$ and $y_i = \text{Im}(z_i)$. Thus, $s_{\lambda}^{R}(\mathbf{x}, \mathbf{y}) \coloneqq \text{Re}(s_{\lambda}(\mathbf{z}))$ and $s_{\lambda}^{I}(\mathbf{x}, by) \coloneqq \text{Im}(s_{\lambda}(\mathbf{z}))$ are real polynomials in the variables x_i , y_i . This yields a polynomial equation system, which can be solved using, for instance, elimination theory or multivariate resultants. For more details, see, e.g., [42].

Thus, for very small examples, the roots of the Schur polynomial can be derived algebraically. However, if M or $|\mathcal{T}_{\lambda}|$ is large, then a numerical solution is usually difficult. Moreover, the algebraic approach becomes computationally extremely expensive. As an alternative, the idea in Section IV is to formulate a mixed-integer program (MIP) whose feasible solutions correspond to roots of the Schur polynomial. Before we state this result, we introduce the definition of vanishing sums of unit roots in the next section, as it turns out that these play an important role for the detection of roots of the Schur polynomial.

D. Vanishing Sums of Unit Roots

Let $\mathcal{T}_{\lambda} = \{T_1, \ldots, T_N\}$ be the set of all SSYTs of shape λ with cardinality $N := |\mathcal{T}_{\lambda}|$. In our setting, the variables z_i are given by $z_i = e^{j\Phi_i}$, $\Phi_i \in [-\pi d, \pi d]$, for $i \in [M]$. The Schur polynomial can be written as

$$s_{\lambda}(\mathbf{z}) = \sum_{\ell=1}^{N} \prod_{i=1}^{M} z_i^{\alpha_{i\ell}} = \sum_{\ell=1}^{N} \prod_{i=1}^{M} e^{j\alpha_{i\ell}\Phi_i}$$
$$= \sum_{\ell=1}^{N} e^{j\sum_{i=1}^{M} \alpha_{i\ell}\Phi_i}, \tag{5}$$

where $\boldsymbol{\alpha} = (\alpha_{1,\ell}, \dots, \alpha_{M,\ell})^{\top} \in \mathbb{Z}^{M \times N}$ is the vector $\boldsymbol{\alpha}(T_{\ell})$ defined in Section III-B. Recall that our goal is to search for roots of the Schur polynomial, which

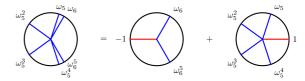


Fig. 2: Example of a nontrivial sum of unit roots

means to check whether there exists a vector $\mathbf{\Phi} \in [-\pi d, \pi d]^M$ such that

$$s_{\lambda}(\mathbf{z}) = \sum_{\ell=1}^{N} e^{j\sigma_{\ell}} = 0, \tag{6}$$

$$\sigma_{\ell} = \sum_{i=1}^{M} \alpha_{i\ell} \, \Phi_i \pmod{2\pi}, \quad \forall \ell \in [N].$$
 (7)

Since $e^{j\sigma_{\ell}} = e^{j(\sigma_{\ell} + k \cdot 2\pi)}$ for $k \in \mathbb{N}$, Equation (7) only needs to hold modulo 2π .

For $m \in \mathbb{N}$, define $\omega_m = \mathrm{e}^{j2\pi/m}$. Then, ω_m^v is called an m-th unit root for $v \in \{0, 1, \ldots, m-1\}$. If all $\sigma_\ell/2\pi$ are rational, then the sum $\sum_{\ell=1}^N \mathrm{e}^{j\sigma_\ell} = 0$ is called a vanishing sum of unit roots.

The most important special case of a vanishing sum of unit roots is the sum of all m-th unit roots, that is, $\omega_m + \omega_m^2 + \cdots + \omega_m^m = 0$ holds for an integer m > 1, which can be seen using a geometric sum. In the literature these sums are often denoted as *trivial*. An example of a nontrivial vanishing sum of unit roots is

$$\omega_6 + \omega_6^5 + \omega_5 + \omega_5^2 + \omega_5^3 + \omega_5^4 = 0.$$
 (8)

This sum can be written as the sum of the trivial sum $\sum_{v=1}^5 \omega_5^v = 0$ and the rotated trivial sum $\omega_6 + \omega_6^3 + \omega_6^5 = \omega_6^1 \cdot (\sum_{v=1}^3 \omega_3^v) = 0$, see Figure 2.

Remark III.3. An obvious way to find unit roots $e^{j\sigma_{\ell}}$ and electrical angles Φ_i corresponding to an ambiguity of a linear array would be a feasibility problem with Constraints (6) and (7). However, the real-valued reformulation of $\sum_{\ell} e^{j\sigma_{\ell}} = 0$, with $\sigma_{\ell} \in [0, 2\pi)$ is $\sum_{\ell} \cos(\sigma_{\ell}) = 0$ and $\sum_{\ell} \sin(\sigma_{\ell}) = 0$, which is a trigonometric constraint that is hard to handle in most solvers. The same problem emerges, if $e^{j\sigma_{\ell}}$ is replaced by a complex variable ω_{ℓ} for all $\ell \in [M]$. This would yield $\sum_{\ell} \text{Re}(\omega_{\ell}) = \sum_{\ell} \text{Im}(\omega_{\ell}) = 0$ and $\text{Re}(\omega_{\ell})^2 + \text{Im}(\omega_{\ell})^2 = 1$. In this case, we need to control the argument of the complex variables ω_{ℓ} to model Equation (7), which is nonlinear.

In order to avoid the issue in Remark III.3, we make the following additional assumption.

(A4) For every root of the Schur polynomial, the corresponding exponential sum $\sum_{\ell=1}^{N} e^{j\sigma_{\ell}}$ is a

vanishing sum of roots of unity, possibly rotated by a complex number on the unit circle.

The subsequent Example III.7 demonstrates that there exist linear arrays with ambiguities that violate this assumption. However, the Assumption (A4) will allow us to use a purely linear procedure to find many ambiguities that are present in a linear array. This procedure relies on the following theorem, which reduces the case of general vanishing sums to minimal vanishing sums, i.e., vanishing sums, such that no proper subsum also vanishes.

Theorem III.4 (Mann [23], Corollary 1.1). Let $\sum_{i=1}^{k} a_i \eta_i = 0$ be a vanishing sum with $a_i \in \mathbb{Z} \setminus \{0\}$, $i \in [k]$, and unit roots η_i , $i \in [k]$. Then this sum can be written as

$$\sum_{i=1}^{k} a_i \, \eta_i = \zeta_1 \sum_{i=1}^{k_1} a_i \, \nu_i + \dots + \zeta_u \sum_{i=k_{u-1}+1}^{k_u} a_i \, \nu_i,$$

with unit roots ζ_1, \ldots, ζ_u , $0 =: k_0 < k_1 < \cdots < k_u := k$ and all ν_i are $(p_1 p_2 \cdots p_s)$ -th unit roots with $0 < p_t \le \max\{k_j - k_{j-1} : j \in [u]\}$, for prime numbers p_t , $t \in [s]$. Moreover, each vanishing sum $\sum_{i=k_j+1}^{k_{j+1}} a_i \nu_i = 0$, $j = 0, \ldots, u-1$ is minimal.

Note that the upper bound for p_t in Theorem III.4 does not depend on t. Moreover, the representation in Theorem III.4 is clearly not unique.

Because of Theorem III.4, we only need to consider minimal vanishing sums, as they can be rotated and linearly combined to obtain general rotated vanishing sums and thus ambiguities as roots of the Schur polynomial. If we allow the linear coefficients ζ_i in Theorem III.4 to be complex numbers on the unit circle, i.e, $\zeta_i = \mathrm{e}^{j2\pi v_i}$, $v_i \in [0,1)$, instead of roots of unity, we are also able to find slightly more general exponential sums that sum to zero and thus induce ambiguities.

Remark III.5. There are no (nontrivial) vanishing sums of length 1. Furthermore, every vanishing sum of length 4 can be written as the sum of two (rotated) minimal vanishing sums of length 2, and thus, there are no minimal vanishing sums of length 4, which can be deduced from [31, Lemma 3.2 and Lemma 3.3].

Remark III.6 (Limitation of the approach). At least for symmetric linear arrays there exist ambiguities which cannot be represented as a linear combination of minimal vanishing sums with rotation factors of the form $e^{j2\pi v}$, $v \in [0,1)$ and thus violate Assumption (A4). In this case, the approach of this section is not able to detect these ambiguities. The

following example shows such an ambiguity, which can be obtained by using [21, Theorem 7.1].

Example III.7. Consider the linear array $\mathbf{r} = (0, 1, 3, 4)^{\top}$ with baseline d = 1 and $\boldsymbol{\lambda} = (0, 0, 1, 1)^{\top}$. The Schur polynomial for \mathbf{r} is given by

$$s_{\lambda} = e^{j(\Phi_1 + \Phi_2)} + e^{j(\Phi_1 + \Phi_3)} + e^{j(\Phi_1 + \Phi_4)} + e^{j(\Phi_2 + \Phi_3)} + e^{j(\Phi_2 + \Phi_4)} + e^{j(\Phi_3 + \Phi_4)}$$

The ambiguity

$$\mathbf{\Phi} = \begin{bmatrix} 2\arctan(\Delta_1) - \pi, \ 2\arctan(\Delta_2) - \pi, \\ \pi - 2\arctan(\Delta_2), \ \pi - 2\arctan(\Delta_1) \end{bmatrix}^{\top}$$
$$\approx (43.6^{\circ}, 77.35^{\circ}, 102.7^{\circ}, 136.4^{\circ})^{\top},$$

with
$$\Delta_1 = \sqrt{\frac{1}{3}(12 - \sqrt{129})}$$
, $\Delta_2 = \sqrt{\frac{1}{3}(12 + \sqrt{129})}$,

yields $\Phi_1 + \Phi_4 = \Phi_2 + \Phi_3$, whereas all other exponents of s_{λ} are pairwise different. This means, in order to encode the solution Φ by summing rotated minimal vanishing sums of roots of unity, two of the six resulting roots of unity must be equal (after a possible rotation). Since there are N = 6 SSYTs for the linear array r, there are three ways of summing rotated minimal vanishing sums in order to obtain a vanishing sum of length 6, namely, 6 = 3 + 3 = 2 + 2 + 2. Note that there are no minimal vanishing sums of lengths 1 and 4, see Remark III.5. However, it is easy to see that for all three cases, it is not possible to find rotation factors such that two roots of unity are equal and five are pairwise different. Thus, the ambiguity given by Φ cannot be represented using sums of rotated minimal vanishing sums, so that Assumption (A4) is violated.

IV. ENUMERATION USING MINIMAL VANISHING SUMS

Recall that finding ambiguous DoAs for the linear array $\mathbf{r} = \boldsymbol{\lambda} + \boldsymbol{\delta}$ corresponds to finding roots of the Schur polynomial $s_{\boldsymbol{\lambda}}$ in (5). The Schur polynomial is given by a sum of exponentials, where the exponents need to fulfill a linear relation modulo 2π , which originates from the SSYTs of shape $\boldsymbol{\lambda}$ for the linear array \mathbf{r} , see (6) and (7). We assumed that $s_{\boldsymbol{\lambda}}$ can be represented using linear combinations of minimal vanishing sums with coefficients $\zeta = \mathrm{e}^{j2\pi v}$, $v \in [0,1)$. By that, we can find a fairly large subset of roots of the Schur polynomial. If N is the number of SSYTs of shape $\boldsymbol{\lambda}$, then the following procedure can be used to find ambiguities in \mathbf{r} .

1) Choose a partition $N = p_1 + \cdots + p_k$, $k \in [N]$, with $p_i \in [N] \setminus \{1, 4\}$, i.e., without using the elements 1

- and 4, since there are no minimal vanishing sums of unit roots of these lengths, see Remark III.5.
- 2) For each partition element p_i choose a minimal vanishing sum S_i of length p_i .
- 3) Choose an assignment of the roots of unity appearing in S_i to the variables σ_{ℓ} , $\ell \in [N]$.
- 4) Check if there exists a solution to the linear equation system (7), where each S_i can be rotated by an arbitrary $e^{j2\pi v_i}$, with $v_i \in [0,1)$, c.f. Theorem III.4 and the discussion thereafter.

Clearly, for this approach to work, we need to know all possible minimal vanishing sums that can be used to build the desired vanishing sum. For $N \leq 12$, all minimal vanishing sums (up to rotations) of length N are characterized by Poonen and Rubinstein [31, Table 1 and Theorem 3]², and their construction can be easily extended to N > 12.

A. The MIP-Formulation

The approach described above to find ambiguities for a given linear array \mathbf{r} can be formulated as a mixed-integer (linear) program (MIP). Every feasible solution of the MIP then corresponds to an ambiguity for \mathbf{r} , such that by enumerating feasible solutions of the MIP, it is possible to obtain many ambiguities for \mathbf{r} . The only requirement is that all minimal vanishing sums with length up to N need to be known in advance. The following parameters are used in the MIP-formulation (9) in Figure 3:

M := Number of sensors in the linear array $\mathbf{r} = \boldsymbol{\lambda} + \boldsymbol{\delta}, \text{ i.e., } \mathbf{r} \in \mathbb{Z}^M,$

 $d := \text{common baseline of the array positions } \mathbf{r},$

 $N := \text{number of SSYTs of shape } \lambda$,

$$\mathbf{P} \coloneqq [\ \underbrace{\frac{2,\dots,2}{N}}_{\lfloor \frac{N}{2} \rfloor \ \mathrm{many}} \ \lfloor \frac{3,\dots,3}{3} \ , \ \underbrace{\frac{5,\dots,5}{5}}_{\lfloor \frac{N}{5} \rfloor \ \mathrm{many}} \ \lfloor \frac{N}{6} \rfloor \ \mathrm{many}}_{\lfloor \frac{N}{6} \rfloor \ \mathrm{many}}],$$

 $m_i := \#$ minimal vanishing sums of length P_i

 $u_{t,k}^{(i)} := k$ -th root of unity of t-th minimal vanishing sum of length P_i , $i \in [|P|]$, $t \in [m_i]$, $k \in [P_i]$, $I := \{(i, t, k) : i \in [|\mathbf{P}|], t \in [m_i], k \in [P_i]\}.$

The vector \mathbf{P} is needed to model all partitions of N that can be used in the first step of the procedure above, where $|\mathbf{P}|$ denotes its number of entries.

The variables are:

• $\mathbf{q}^{(i)} \in \{0,1\}^{m_i}$ with $q_t^{(i)} = 1$ if and only if P_i is part of the chosen partition of N and the t-th minimal vanishing sum is selected for this P_i .

 2 Mann [23] already characterized all minimal vanishing sums up to N=7, and Conway and Jones [5] characterized all minimal vanishing sums up to N=9.

- $\mathbf{b}^{(i)} \in \{0,1\}^{m_i \times P_i \times N}$. We have $b_{t,k,\ell}^{(i)} = 1$ if and only if σ_{ℓ} is assigned to the k-th root of unity in the t-th minimal vanishing sum chosen for the i-th element of the partition.
- $\mathbf{v}^{(i)} \in [0, 2\pi)^{m_i}$: rotation factors for the minimal vanishing sums of length P_i .
- $\mathbf{w}^{(i)} \in [0, 2\pi)^{m_i \times P_i \times N}$: auxiliary variables for linearization: $w_{t,k,\ell}^{(i)} = v_t^{(i)} \cdot b_{t,k,\ell}^{(i)}$.
- $\mathbf{z} \in \mathbb{Z}^N$: models that a rotation is always applied modulo 2π . Since the rotation values lie in $[0, 2\pi)$, all z_{ℓ} can be assumed to be binary.
- $\sigma \in [0, 2\pi)^N$, $\Phi \in [-\pi d, \pi d]^M$, $\mathbf{x} \in \mathbb{Z}^N$: model Equations (6) and (7), where x is needed for the modulo operation. Constraint (9i) implies that each x_ℓ is bounded by $-\frac{1}{2}d\sum_{m=1}^M \alpha_{m,\ell} 1 \le x_\ell \le \frac{1}{2}d(\sum_{m=2}^M \alpha_{m,\ell} \alpha_{1,\ell})$, for $\ell \in [N]$, see Constraint (9k).

Constraints (9a) and (9b) ensure that a valid partition of N together with corresponding minimal vanishing sums is chosen. The subsequent Constraints (9c)–(9h) model the assignment of the chosen roots of unity to the variables σ_{ℓ} , using a linearization of the bilinear constraint

$$\sum_{i=1}^{|P|} \sum_{t=1}^{m_i} \sum_{k=1}^{P_i} b_{t,k,\ell}^{(i)} \left(v_t^{(i)} + u_{t,k}^{(i)} \right) - 2\pi z_{\ell} = \sigma_{\ell}, \ \forall \ell \in [N].$$

Finally, Constraint (9i) models the linear equations (7). Because of Assumption (A3), we can fix the first electrical angle $\Phi_1 = -\pi d$ in Constraint (9j). In terms of vanishing sums, this amounts to a global rotation of all appearing roots of unity which does not destroy the sum being 0, as can be seen by

$$\sum_{\ell=1}^{N} e^{j(2\pi v + \mu)} = \sum_{\ell=1}^{N} e^{j2\pi v} \cdot e^{j\mu}.$$
 (10)

In order to remove some symmetric solutions, the variables Φ are be ordered increasingly in Constraint (9j). This is justified by Theorem IV.1 in the next section. Additionally, the strict inequalities prevent trivial ambiguities consisting of two or more equal electrical angles. These strict inequalities together with the upper bounds for $\mathbf{v}^{(i)}$, $\mathbf{w}^{(i)}$, Φ and σ are modeled using non-strict inequalities with a small $\epsilon = 0.001$, so that (9) is indeed a MIP.

Note that there could exist different feasible solutions that correspond to the same ambiguous vector of electrical angles. There are different reasons for this behavior. First, the decomposition of a vanishing sum of unit roots into minimal vanishing sums is not unique, such that different sums of (rotated) minimal vanishing sums of unit roots can lead to the

$$\sum_{i=1}^{|P|} P_i \cdot \left(\sum_{t=1}^{m_i} q_t^{(i)}\right) = N, \tag{9a}$$

$$\sum_{t=1}^{m_i} q_t^{(i)} \le 1 \qquad \forall i \in [|P|], \tag{9b}$$

$$b_{t,k,\ell}^{(i)} \le q_t^{(i)} \qquad \forall (i,t,k,\ell) \in I \times [N], \tag{9c}$$

$$\sum_{i=1}^{|P|} \sum_{t=1}^{m_i} \sum_{k=1}^{P_i} b_{t,k,\ell}^{(i)} = 1, \qquad \forall \ell \in [N],$$
 (9d)

$$\sum_{\ell=1}^{N} b_{t,k,\ell}^{(i)} \le 1, \qquad \forall (i,t,k) \in I, \tag{9e}$$

$$\sum_{i=1}^{|P|} \sum_{t=1}^{m_i} \sum_{k=1}^{P_i} w_{t,k,\ell}^{(i)} + b_{t,k,\ell}^{(i)} u_{t,k}^{(i)} - 2\pi z_{\ell} = \sigma_{\ell}, \qquad \forall \ell \in [N],$$
(9f)

$$w_{t,k,\ell}^{(i)} \le b_{t,k,\ell}^{(i)} \cdot 2\pi, \qquad \forall (i,t,k,\ell) \in I \times [N], \tag{9g}$$

$$2\pi(-1 + b_{t,k,\ell}^{(i)}) + w_{t,k,\ell}^{(i)} \le 2\pi(1 - b_{t,k,\ell}^{(i)}) + w_{t,k,\ell}^{(i)}, \qquad \forall (i,t,k,\ell) \in I \times [N],$$
(9h)

$$\sum_{m=1}^{M} \alpha_{m,\ell} \, \Phi_m - 2\pi \, x_\ell = \sigma_\ell, \qquad \forall \, \ell \in [N], \tag{9i}$$

$$-\pi d = \Phi_1 < \Phi_2 < \dots < \Phi_M \le \pi d,\tag{9j}$$

$$-\frac{1}{2}d\sum_{m=1}^{M}\alpha_{m,\ell} - 1 \le x_{\ell} \le \frac{1}{2}d\left(\sum_{m=2}^{M}\alpha_{m,\ell} - \alpha_{1,\ell}\right) \qquad \forall \ell \in [N], \tag{9k}$$

$$\mathbf{q}^{(i)} \in \{0,1\}^{m_i}, \ \mathbf{b}^{(i)} \in \{0,1\}^{m_i \times P_i \times N}, \ \mathbf{z} \in \{0,1\}^N, \ \mathbf{x} \in \mathbb{Z}^N, \ \mathbf{w}^{(i)} \in [0,2\pi)^{m_i \times P_i \times N}$$
$$\mathbf{v}^{(i)} \in [0,2\pi)^{m_i}, \ \mathbf{\Phi} \in [-\pi d, \pi d]^M, \ \boldsymbol{\sigma} \in [0,2\pi)^N.$$

Fig. 3: Feasibility MIP for enumerating all ambiguities.

same vanishing sum and thus, to the same ambiguity. Second, there can be different assignments of the chosen unit roots to the variables σ_{ℓ} that lead to the same electrical angles Φ_m .

B. Analysis of the MIP formulation

Theorem IV.1. Consider a feasible solution $\mathbf{X} = (\mathbf{b}^{(i)}, \mathbf{q}^{(i)}, \mathbf{x}, \mathbf{z}, \mathbf{v}^{(i)}, \mathbf{w}^{(i)}, \mathbf{\Phi}, \boldsymbol{\sigma})$ for (9) and let $\tau \in S_M$ be an arbitrary permutation of [M]. Then there exists a feasible solution $\tilde{\mathbf{X}} = (\tilde{\mathbf{b}}^{(i)}, \tilde{\mathbf{q}}^{(i)}, \tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \tilde{\mathbf{v}}^{(i)}, \tilde{\mathbf{w}}^{(i)}, \tilde{\mathbf{\Phi}}, \tilde{\boldsymbol{\sigma}})$ with $\tilde{\mathbf{\Phi}} = (\Phi_{\tau(1)}, \dots, \Phi_{\tau(M)}) \in [-\pi d, \pi d]^M$.

Note that the statement is not trivial: If the generalized Vandermonde determinant is 0, permuting Φ does not change this, and it does not change the Schur polynomial. However, it is not clear whether there still exists a sum of rotated minimal vanishing sums and assignment of the appearing roots of unity so that Constraint (9i) is satisfied.

Proof. Let $\mathbf{r} = \boldsymbol{\lambda} + \boldsymbol{\delta}$ be a linear array, and let $\mathcal{T}_{\boldsymbol{\lambda}}$ be the set of all SSYTs of shape $\boldsymbol{\lambda}$. We first prove the assertion that if there exists an SSYT $T_{\ell} \in \mathcal{T}_{\boldsymbol{\lambda}}$ with weight vector $\boldsymbol{\alpha}(T_{\ell}) = (\alpha_{1,\ell}, \dots, \alpha_{M,\ell})$, then for all permutations $\tau \in S_M$ there also exists an SSYT $T_k \in \mathcal{T}_{\boldsymbol{\lambda}}$ with $\boldsymbol{\alpha}(T_k) = (\alpha_{1,k}, \dots, \alpha_{M,k}) = (\alpha_{\tau^{-1}(1),\ell}, \dots, \alpha_{\tau^{-1}(M),\ell}) =: \tau(\boldsymbol{\alpha}(T_{\ell}))$.

Let β , β' be two possible weight vectors of an SSYT of shape λ so that β and β' only differ by swapping two consecutive entries. Then there exists a bijection between the SSYTs of shape λ with weight vector β and β' : Consider the entries β_i and β_{i+1} of the weight vector β , and let T be an SSYT of shape λ with weight vector β . Select all columns of T that contain exactly one entry equal to i or i+1. All other columns contain either no or two such entries. In each row of T replace each i appearing in these columns by i+1 and vice versa. After reordering the rows so that

these are again sorted nondecreasingly, we obtain an SSYT of shape λ with weight vector β' . This yields the desired bijection, see also [40, Proof of Theorem 7.10.2]. Thus, for a given weight vector α and a given permutation $\tau \in S_M$ we can find an SSYT with weight vector $\tau(\alpha)$ by decomposing τ into a sequence of transpositions of consecutive entries and using the compositions of the respective bijections.

Consider the solution X and permutation τ . In order to prove the existence of solution \tilde{X} with $\tilde{\Phi}_m = \Phi_{\tau(m)}, m \in [M]$, we show that there exists a permutation $\gamma \in S_N$ with $\tilde{\sigma}_\ell = \sigma_{\gamma(\ell)}$:

$$\begin{split} \tilde{\sigma}_{\ell} &= \alpha_{1,\ell} \, \tilde{\Phi}_1 + \dots + \alpha_{M,\ell} \, \tilde{\Phi}_M - 2 \, \tilde{x}_{\ell} \\ &= \alpha_{1,\ell} \, \Phi_{\tau(1)} + \dots + \alpha_{M,\ell} \, \Phi_{\tau(M)} - 2 \, \tilde{x}_{\ell} \\ &= \alpha_{\tau^{-1}(1),\ell} \, \Phi_1 + \dots + \alpha_{\tau^{-1}(M),\ell} \, \Phi_M - 2 \, \tilde{x}_{\ell}. \end{split}$$

By the assertion, there exists an SSYT T with weight vector $\tau(\boldsymbol{\alpha}(T_{\ell}))$. Defining the permutation γ such that $\alpha_{m,\gamma(\ell)} = \alpha_{\tau^{-1}(m),\ell}$ for all $m \in [M]$, i.e., such that the SSYT $T_{\gamma}(\ell)$ has weight vector $\tau(\boldsymbol{\alpha}(T_{\ell}))$, and setting

$$\begin{split} \tilde{\sigma}_{\ell} &= \sigma_{\gamma(\ell)}, \ \tilde{x}_{\ell} = x_{\gamma(\ell)}, \ \tilde{z}_{\ell} = z_{\gamma(\ell)}, \ \tilde{b}_{t,k,\ell}^{(i)} = b_{t,k,\gamma(\ell)}^{(i)}, \\ \tilde{w}_{t,k,\ell}^{(i)} &= w_{t,k,\gamma(\ell)}^{(i)}, \ \tilde{q}_{t}^{(i)} = q_{t}^{(i)}, \ \tilde{v}_{t}^{(i)} = v_{t}^{(i)}, \end{split}$$

for all $(i, t, k, \ell) \in I \times [N]$ yields the desired feasible solution $\tilde{\mathbf{X}}$ with $\tilde{\mathbf{\Phi}} \in [-\pi d, \pi d]^M$ and $\tilde{\Phi}_m = \Phi_{\tau(m)}$ for all $m \in [M]$.

The following two Lemmas state that for a linear array with integer positions, the feasibility problem (9) finds all ambiguities in the array that can be represented using a linear combination of minimal vanishing sums with coefficients on the complex unit circle.

Lemma IV.2. For a linear array with positions corresponding to integer multiples $\mathbf{r} \in \mathbb{Z}^M$ of a common baseline $d \leq 1$ measured in half wavelength, each feasible solution of the feasibility-MIP (9) corresponds to an ambiguous vector of electrical angles.

Proof. Let $\mathbf{X} = (\mathbf{b}^{(i)}, \mathbf{q}^{(i)}, \mathbf{x}, \mathbf{z}, \mathbf{v}^{(i)}, \mathbf{w}^{(i)}, \mathbf{\Phi}, \boldsymbol{\sigma})$ be a feasible solution for (9). It is clear by construction that the Schur polynomial $s_{\lambda}(\mathbf{z})$ with $z_i = e^{j\Phi_i}$ satisfies $s_{\lambda}(\mathbf{z}) = 0$. By definition, this implies that the generalized Vandermonde determinant $V_{\mathbf{r}}(\mathbf{z})$ vanishes. Thus, the array steering matrix $\mathbf{A}(\mathbf{\Phi})$ is rank-deficient, i.e., Φ_1, \ldots, Φ_M are ambiguous.

The next Lemma is an immediate consequence of the definition of the Schur polynomial and the arguments above.

Lemma IV.3. Let $\mathbf{r} \in \mathbb{Z}^M$ be an arbitrary integer linear array where the positions are multiples of a common baseline $d \leq 1$ measured in half wavelength. Each ambiguous vector of electrical angles that forms a root of the Schur polynomial and that can be represented using sums of rotated minimal vanishing sums, corresponds to at least one feasible solution of the feasibility-MIP (9).

Let us now relate our approach to the *uniform* ambiguities from [22, Theorem 2.2]. For the ease of presentation, we restate this result in terms of electrical angles.

Theorem IV.4 ([22]). Let $\mathbf{r} = (r_1, \dots, r_M)^{\top} \in \mathbb{R}^M$ be an arbitrary linear array with positions measured in half wavelength and baseline d = 1. Define the vector $\Phi_{i,j}$ of electrical angles as

$$\Phi_{i,j} := \left[-\pi, -\pi \left(1 - \frac{2}{|r_i - r_j|} \right), -\pi \left(1 - \frac{4}{|r_i - r_j|} \right), \\
\dots, -\pi \left(1 - \frac{2c}{|r_i - r_j|} \right) \right]^\top,$$

where $i \neq j \in [M]$ and $c \in \mathbb{N}$ is the largest integer satisfying $c < |r_i - r_j|$. Then, if $\Phi_{i,j}$ contains at least M elements, any subvector of M elements from $\Phi_{i,j}$ is an ambiguous vector.

Note that these are all ambiguities that are currently known in the literature for non-symmetric integer linear arrays.

We can now prove our main result, namely, that for an integer linear array, our approach is able to identify all *uniform ambiguities* from Theorem IV.4. This directly implies we can find all all ambiguities previously known in the literature for *non-symmetric* integer linear arrays.

Proposition IV.5. Let $\mathbf{r} \in \mathbb{Z}^M$ be an arbitrary integer linear array where the positions are multiples of a common baseline $d \leq 1$ measured in half wavelength. Let $\tilde{\mathbf{r}} = d \cdot \mathbf{r} \in \mathbb{R}^M$, be the array \mathbf{r} rescaled to a baseline $\tilde{d} = 1$, such that \mathbf{r} and $\tilde{\mathbf{r}}$ are in fact two representations (with different baselines) of the same linear array. Then any ambiguity of the form $\Phi_{i,j}$ as stated in Theorem IV.4 (for the representation $\tilde{\mathbf{r}}$) corresponds to at least one solution of the feasibility-MIP (9) (for the representation \mathbf{r}).

Proof. Let $\Phi \in \mathbb{R}^M$ be an ambiguity for $\tilde{\mathbf{r}}$ in the form of Theorem IV.4, and assume w.l.o.g. that the first electrical angle is $-\pi d$, i.e., $\Phi_1 = -\pi d$ and for $m = \{2, \ldots, M\}$:

$$\Phi_m = -\pi d \left(1 - \frac{2c_m}{|\tilde{r}_i - \tilde{r}_j|} \right) = -\pi d \left(1 - \frac{2c_m}{d|r_i - r_j|} \right),$$

with $i \neq j \in [M]$, $c_m \in \mathbb{N}$ and $c_{m_1} \neq c_{m_2}$ for all $m_1 \neq m_2 \in [M]$. The variables σ_{ℓ} in (9) are then given by

$$\sigma_{\ell} = \sum_{m=1}^{M} \alpha_{m,\ell} \Phi_m \mod 2\pi,$$

for $m \in [M]$ and $\ell \in [N]$. Thus,

$$\sum_{\ell=1}^{N} e^{j\sigma_{\ell}} = \sum_{\ell=1}^{N} \exp\left(j \sum_{m=1}^{M} \alpha_{m,\ell} (\Phi_{m} + \pi d - \pi d)\right)$$
$$= e^{-j\pi dK} \sum_{\ell=1}^{N} \exp\left(j \sum_{m=1}^{M} \alpha_{m,\ell} (\Phi_{m} + \pi d)\right), \quad (11)$$

where $K = \sum_{m=1}^{M} \alpha_{m,\ell}$ is the same constant for all $\ell \in [N]$. Since $\Phi_m + \pi d \in \{2\pi w : w \in \mathbb{Q}\}$ for all $m \in [M]$ it holds that (11) is a sum of roots of unity, rotated by a complex number on the complex unit circle.

Since Φ forms an ambiguous vector of electrical angles for the linear array \mathbf{r} , $z_m = \exp(j\Phi_m)$ form a root of the generalized Vandermonde determinant $V_{\mathbf{r}}(\mathbf{z})$ and thus of the Schur polynomial $s_{\lambda}(\mathbf{z})$, where $\lambda = \mathbf{r} - (0, 1, \dots, M-1)$. Equations (5), (6) and (7) imply $0 = s_{\lambda}(\mathbf{z}) = \sum_{\ell=1}^{N} \exp(j\sigma_{\ell})$, and thus (11) is a rotated vanishing sum of roots of unity. Theorem III.4 implies that (11) can be written as linear combination of minimal vanishing sums with coefficients on the complex unit circle. By Lemma IV.3, there is at least one feasible solution of the MIP (9) that corresponds to the ambiguity Φ , which finishes the proof.

Moreover, the computational results in Section V below show that for non-symmetric integer linear arrays, our approach finds many more ambiguities than previously known. In general, it remains an open question, whether there also exist ambiguities for non-symmetric integer linear arrays which cannot be expressed as linear combination of minimal vanishing sums with coefficients on the complex unit circle, or if our approach indeed finds *all* ambiguities that are present in such an array.

Before presenting computational results, let us shortly discuss some details of enumerating the feasible solutions with our approach in the next section.

C. Enumerating all Feasible Solutions of the MIP

Since it is possible that the MIP (9) has infinitely many feasible solutions, we do not count all feasible solutions, but only all configurations of the integer and binary variables \mathbf{b} , \mathbf{q} , \mathbf{x} , \mathbf{z} such that after fixing all these variables the remaining problem has at least

one feasible solution. Because all integer and binary variables are bounded, there are only finitely many configurations that need to be checked.

After all feasible configurations of the integer and binary variables have been found, we apply a post-processing step in order to find the ambiguities corresponding to each configuration. Due to the rotations $v_t^{(i)}$ of the used minimal vanishing sums, there can exist configurations of the integer (and binary) variables with infinitely many feasible solutions for the continuous variables. These solutions form a whole class of ambiguities with a number of parameters, see Example V.1. Altogether, we end up with either finitely many ambiguities, or finitely many classes of ambiguities, each of them depending on a number of parameters. The ambiguities can be converted into DoAs by using $-\pi d \cos(\theta_m) = \Phi_m$.

Remark IV.6. In order to reduce the computational effort of enumerating all feasible integer solutions of (9), the problem can be divided into smaller subproblems, one for each possible partition. In each of the smaller MIPs, most of the variables $\mathbf{q}^{(i)} \in \{0,1\}^{m_i}$ can be fixed, according to the corresponding partition. Moreover, this eliminates many symmetric solutions in terms of the variables $q_i^{(i)}$.

V. Computational Results

In this section, we use the approach of obtaining ambiguities by enumerating the feasible solutions of (9) to identify ambiguities for some exemplary integer linear arrays. To enumerate all feasible solutions we use the counting feature of SCIP 6.0.0 [11]. We use CPLEX 12.7.1.0 as LP solver. The first Example V.1 was enumerated using the full Problem (9), whereas for Example V.2, we divided Problem (9) into smaller ones, one for each possible partition as described in Remark IV.6. In all examples, the baseline is d=1. The used model files can be obtained via the website of the last author.

The computation for Example V.1 was performed on a Linux desktop with 3.6 GHz Intel Core i7-7700 Quad-Core CPUs having 16 GB main memory and 8 MB cache, whereas the computations for Example V.2 were done on on a Linux cluster with 3.5 GHz Intel Xeon E5-1620 Quad-Core CPUs, having 32 GB main memory and 10 MB cache. All computations were performed single-threaded and, in the case of Example V.2 with a timelimit of 450 000 s. For this Example, we also display the time needed to enumerate all feasible solutions, the total number of solutions that were enumerated, as well as the number of processed nodes in the enumeration process.

Example V.1 shows a linear array with an infinite number of ambiguities, even after fixing the first electrical angle to $-\pi d$.

Example V.1. Consider the linear array $\mathbf{r} = \boldsymbol{\lambda} + \boldsymbol{\delta} = (0, 1, 3, 4)^{\top}$, where $\boldsymbol{\delta} = (0, 1, 2, 3)^{\top}$ and $\boldsymbol{\lambda} = (0, 0, 1, 1)^{\top}$. The possible Young tableaux are shown in Figure 4.

1	1	1	2	2	3	
2	3	4	3	4	4	

Fig. 4: Young tableaux for $\lambda = (0, 0, 1, 1)^{\top}$

This results in the Schur polynomial

$$s_{\lambda}(\mathbf{z}) = z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4$$

Since **r** has N=6 corresponding SSYTs, there are three possible partitions: $6=2+2+2=3+3=6.^3$ For the partitions 6=2+2+2 and 6=3+3, there exist infinitely many valid solutions for the feasibility problem (9), and thus infinitely many ambiguities, which can be classified into three classes.

The partition 6 = 2 + 2 + 2 yields the two infinite classes of electrical angles

$$-\pi \cdot \begin{bmatrix} 1, v, 0, -v \end{bmatrix}^{\top}, \quad -\pi \cdot \begin{bmatrix} 1, 1-v, 1-2v, -v \end{bmatrix}^{\top},$$

for all $v \in (0,1)$. The partition 6 = 3 + 3 yields the following class of electrical angles for $v \in (-1, 1)$ and $\gamma = \arccos(-\pi v)$:

$$-\pi \cdot \left[1, \frac{1}{3}, -\frac{1}{3}, v\right]^{\top} \approx \left[0^{\circ}, 70.53^{\circ}, 109.47^{\circ}, \gamma\right]^{\top}.$$

Since already the steering vectors corresponding to the electrical angles $-\pi \cdot [1, \frac{1}{3}, -\frac{1}{3}]^{\top}$ are linearly dependent and thus induce a rank-deficient steering matrix, an arbitrary electrical angle $v \in (-\pi, \pi)$ can be added to the three electrical angles in order to obtain an ambiguity with four electrical angles.

For the third partition 6 = 6, there are finitely many corresponding solutions, namely the eight ambiguities in Table II. Note that all three ambiguities that were already found by using the methods from Proukakis and Manikas [22], are contained in one of the three classes.

Some of the non-uniform ambiguities obtained with the methods for symmetric linear arrays in [21] are contained in $-\pi \cdot [1, v, 0, -v]^{\top}$, but there also exist non-uniform ambiguities which cannot be found with our approach, see Example III.7.

Altogether, our approach finds two new classes of ambiguities for this particular linear array that

TABLE II: Unique ambiguities for Partition 6 = 6 in the linear array $\mathbf{r} = (0, 1, 3, 4)$.

Electrical Angle	Degree
$\begin{array}{c} -\pi \left[1,\ 14/15,8/15,\ -3/15\right] \\ -\pi \left[1,\ 14/15,2/15,\ -9/15\right] \\ -\pi \left[1,\ 9/15,8/15,\ -4/15\right] \\ -\pi \left[1,\ 9/15,\ -2/15,\ -14/15\right] \\ -\pi \left[1,\ 4/15,\ -2/15,\ -3/15\right] \\ -\pi \left[1,\ 4/15,\ -8/15,\ -9/15\right] \\ -\pi \left[1,\ 3/15,\ 2/15,\ -4/15\right] \\ -\pi \left[1,\ 3/15,\ -8/15,\ -14/15\right] \end{array}$	[0°, 21.04°, 57.77°, 101.54°] [0°, 21.04°, 82.34°, 126.87°] [0°, 53.13°, 57.77°, 105.47°] [0°, 53.13°, 97.66°, 158.96°] [0°, 74.53°, 97.66°, 101.54°] [0°, 74.53°, 122.23°, 126.87°] [0°, 78.46°, 82.34°, 105.47°] [0°, 78.46°, 122.23°, 158.96°]

TABLE III: Definition of $\mathbf{f}_j(v)$ and $\mathbf{f}_6^{(k)}(v)$ using electrical angles.

```
 \begin{split} \mathbf{f}_2 &= \pi \left[ v - 1, \, v \right], \\ \mathbf{f}_3 &= \pi \left[ v - 1, \, v - \frac{1}{3}, \, v + \frac{1}{3} \right], \\ \mathbf{f}_5 &= \pi \left[ v - 1, \, v - \frac{1}{3}, \, v + \frac{1}{3} \right], \\ \mathbf{f}_6 &= \pi \left[ v - 1, \, v - \frac{1}{5}, \, v + \frac{1}{5}, \, v + \frac{3}{5} \right], \\ \mathbf{f}_6 &= \pi \left[ v - 1, \, v - \frac{14}{15}, \, v - \frac{4}{15}, \, v + \frac{1}{5}, \, v + \frac{3}{5} \right], \\ \mathbf{f}_6 &= \pi \left[ v - 1, \, v - \frac{14}{15}, \, v - \frac{4}{15}, \, v - \frac{1}{15}, \, v + \frac{3}{15}, \, v + \frac{3}{5} \right], \\ \mathbf{f}_6 &= \pi \left[ v - 1, \, v - \frac{3}{5}, \, v - \frac{1}{15}, \, v + \frac{1}{15}, \, v + \frac{3}{5} \right], \\ \mathbf{f}_6 &= \pi \left[ v - 1, \, v - \frac{3}{5}, \, v - \frac{1}{15}, \, v - \frac{1}{15}, \, v + \frac{3}{15}, \, v + \frac{3}{15}, \\ \mathbf{f}_6 &= \pi \left[ v - 1, \, v - \frac{3}{5}, \, v - \frac{4}{15}, \, v + \frac{1}{15}, \, v + \frac{1}{16} \right], \\ \mathbf{f}_6 &= \pi \left[ v - 1, \, v - \frac{3}{5}, \, v - \frac{4}{15}, \, v + \frac{2}{15}, \, v + \frac{4}{15}, \, v + \frac{1}{15} \right], \\ \mathbf{f}_6 &= \pi \left[ v - 1, \, v - \frac{14}{15}, \, v - \frac{8}{15}, \, v - \frac{1}{15}, \, v + \frac{4}{15}, \, v + \frac{1}{3} \right], \\ \mathbf{f}_6 &= \pi \left[ v - 1, \, v - \frac{14}{15}, \, v - \frac{8}{15}, \, v - \frac{1}{15}, \, v + \frac{4}{15}, \, v + \frac{1}{3} \right], \\ \mathbf{f}_6 &= \pi \left[ v - 1, \, v - \frac{14}{15}, \, v - \frac{8}{15}, \, v - \frac{1}{15}, \, v + \frac{4}{15}, \, v + \frac{1}{3} \right], \\ \mathbf{f}_7 &= \pi \left[ v - \frac{1}{15}, \, v - \frac{1}{15}, \, v - \frac{1}{15}, \, v + \frac{4}{15}, \, v + \frac{1}{3} \right], \\ \mathbf{f}_7 &= \pi \left[ v - \frac{1}{15}, \, v - \frac{1}{15}, \, v - \frac{1}{15}, \, v - \frac{1}{15}, \, v + \frac{1}{15}, \, v + \frac{1}{3} \right], \\ \mathbf{f}_8 &= \pi \left[ v - 1, \, v - \frac{14}{15}, \, v - \frac{8}{15}, \, v - \frac{1}{15}, \, v + \frac{4}{15}, \, v + \frac{1}{3}, \, v + \frac{1}{3} \right], \\ \mathbf{f}_7 &= \pi \left[ v - \frac{1}{15}, \, v - \frac{1}{1
```

have not been known in the literature. Additionally, all previously known ambiguities which can be expressed as sums of rotated minimal vanishing sums are found by our approach as well.

Example V.2. Consider the linear array $\mathbf{r} = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12)$, which has N = 12 SSYTs. There are 14 partitions of N = 12 not using the numbers 1 and 4, namely

```
2+2+2+2+2+2=2+2+2+3+3=3+3+3+3
=2+2+3+5=2+5+5=2+2+2+6=3+3+6
=6+6=2+3+7=5+7=2+2+8=3+9=2+10=12.
```

The partitions 2+3+7 and 12 reached the timelimit of $450\,000$ seconds, so that for these partitions we possibly have enumerated only a subset of all feasible solutions. For a subset of the remaining partitions all ambiguities found by enumerating feasible solutions of the corresponding smaller MIP are displayed in Table IV, expressed as electrical angles. Here, $\mathbf{f}_j(v)$ and $\mathbf{f}_6^{(k)}(v)$ are vectors of electrical angles depending on a parameter, which are defined in Table III. Thus, the (infinite) classes of ambiguities are given by combining the specified vectors of electrical angles into one large vector of 12 electrical angles depending on a number of parameters.

In Table V the number of feasible solutions, the time needed for enumerating the feasible solutions and the number of processed nodes in the enumeration process are displayed for each partition.

³Note that there are no minimal vanishing sums of length 1 or 4, see Remark III.5.

TABLE IV: All found ambiguities in the linear array $\mathbf{r} = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12)$, for a subset of all possible partitions, expressed in electrical angles.

 $\begin{array}{ll} [\mathbf{f}_2(0),\ \mathbf{f}_2(v_1),\ \mathbf{f}_2(v_2),\ \mathbf{f}_3(w_1),\ \mathbf{f}_3(w_2)], \\ & \text{with } 0 < v_1 < v_2 < \frac{1}{2}, \text{and } 0 < w_1 < w_2 < \frac{1}{3}, \\ [\mathbf{f}_3(0),\ \mathbf{f}_2(v_1),\ \mathbf{f}_2(v_2),\ \mathbf{f}_2(v_3),\ \mathbf{f}_3(w_1)], \\ & \text{with } 0 < v_1 < v_2 < v_3 < \frac{1}{2}, \text{ and } 0 < w_1 < \frac{1}{3}. \end{array}$

Partition 3+3+3+3 — one class of infinitely many ambiguities

 $[\mathbf{f}_3(0), \ \mathbf{f}_3(v_1), \ \mathbf{f}_3(v_2), \ \mathbf{f}_3(v_3)], \ \text{with} \ 0 < v_1 < v_2 < v_3 < \frac{1}{3}.$

Partition 2 + 2 + 3 + 5 — three classes of infinitely many ambiguities

$$\begin{split} & [\mathbf{f}_3(0),\ \mathbf{f}_2(v_1),\ \mathbf{f}_2(v_2),\ \mathbf{f}_5(w)], \ \text{with} \ 0 < v_1 < v_2 < \frac{1}{2} \ \text{and} \ 0 < w < \frac{1}{5}. \\ & [\mathbf{f}_2(0),\ \mathbf{f}_2(v),\ \mathbf{f}_3(w),\ \mathbf{f}_5(u)], \ \text{with} \ 0 < v < \frac{1}{2}, \ 0 < w < \frac{1}{3} \ \text{and} \ 0 < w < \frac{1}{3}. \\ & [\mathbf{f}_5(0),\ \mathbf{f}_2(v_1),\ \mathbf{f}_2(v_2),\ \mathbf{f}_3(w)], \ \text{with} \ 0 < v_1 < v_2 < \frac{1}{2} \ \text{and} \ 0 < w < \frac{1}{3}. \end{split}$$

Partition 2+5+5 — two classes of infinitely many ambiguities

 $\begin{array}{l} [\mathbf{f}_{2}(0),\ \mathbf{f}_{5}(v_{1}),\ \mathbf{f}_{5}(v_{2})],\ \text{with}\ 0 < v_{1} < v_{2} < \frac{1}{5}. \\ [\mathbf{f}_{5}(0),\ \mathbf{f}_{2}(v),\ \mathbf{f}_{5}(w)],\ \text{with}\ 0 < v < \frac{1}{2}\ \text{and}\ 0 < w < \frac{1}{5}. \end{array}$

Partition 2+2+2+6 — twelve classes of infinitely many ambiguities

$$\begin{split} &[\mathbf{f}_2(0),\,\mathbf{f}_2(v_1),\,\mathbf{f}_2(v_2),\,\mathbf{f}_6^{(k)}(w)],\,\text{with }0 < v_1 < v_2 < \frac{1}{2},\,k \in [6].\\ &[\mathbf{f}_6^{(k)}(0),\,\mathbf{f}_2(v_1),\,\mathbf{f}_2(v_2),\,\mathbf{f}_2(v_3)],\,\text{with }0 < v_1 < v_2 < v_3 < \frac{1}{2},\,k \in [6]. \end{split}$$

Partition 3+3+6 — twelve classes of infinitely many ambiguities

$$\begin{split} & [\mathbf{f}_3(0),\ \mathbf{f}_3(v),\ \mathbf{f}_6^{\left(k\right)}(w)], \, \text{with} \, 0 < v < \frac{1}{2}, \, k \in [6]. \\ & [\mathbf{f}_6^{\left(k\right)}(0),\ \mathbf{f}_3(v_1),\ \mathbf{f}_3(v_2)], \, \text{with} \, 0 < v_1 < v_2 < \frac{1}{3}, \, k \in [6]. \end{split}$$

Partition 6 + 6 — thirty-six classes of infinitely many ambiguities

 $[\mathbf{f}_{6}^{k_{1}}(0), \ \mathbf{f}_{6}^{(k_{2})}(w)], \text{ with } k_{1}, k_{2} \in [6].$

TABLE V: Number of solutions, solution times and number of processed nodes for the MIPs for each partition for the linear array $\mathbf{r} = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12).$

Partition	# Solutions	Time (s)	# Nodes
2+2+2+2+2+2	160	719.43	2 204 798
2+2+2+3+3	26664	189691.62	74705045
3+3+3+3	405	5790.55	17067233
2+2+3+5	42336	300709.50	102050134
2+5+5	1680	20150.43	45847950
2+2+2+6	9968	85418.02	60992369
3+3+6	2736	26180.16	56723306
6+6	130	2495.70	9615581
2+3+7	13318	> 450 000.00	> 157 647 076
5+7	1289	37766.13	68727902
2+2+8	8046	260789.23	144652756
3+9	1188	44698.63	60887227
2+10	1830	75348.62	83449067
12	315	> 450 000.00	> 70 499 108

Theorem IV.4 yields one uniform ambiguity, namely

$$-\pi \cdot \big[\tfrac{5}{6}, \tfrac{4}{6}, \tfrac{3}{6}, \tfrac{2}{6}, \tfrac{1}{6}, \tfrac{0}{6}, -\tfrac{1}{6}, -\tfrac{2}{6}, -\tfrac{3}{6}, -\tfrac{4}{6}, -\tfrac{5}{6}\big]^\top$$

which is contained in the class

$$[\mathbf{f}_2(0), \, \mathbf{f}_2(v_1), \, \mathbf{f}_2(v_2), \, \mathbf{f}_2(v_3), \, \mathbf{f}_2(v_4), \, \mathbf{f}_2(v_5)]$$

of partition 2 + 2 + 2 + 2 + 2 + 2 + 2.

VI. CONCLUSION

We demonstrated that for several integer linear arrays our method is able to find more ambiguities than were known previously by the methods from Manikas and Proukakis [22] (general case) and the methods from Dowlut [7] and Manikas [21] (symmetric case). It turns out that arrays with a small number of SSYTs also have a small number of (infinite) classes of ambiguities and the number of SSYTs increases with the number of holes in the corresponding linear array.

Example V.1 shows that at least for symmetric linear arrays there exist non-uniform ambiguities that cannot be found with our approach described in Section IV. Therefore, the interesting question arises, whether there can exist ambiguities in non-symmetric linear arrays that cannot be represented using sums of rotated minimal vanishing sums (c.f. Example III.7 and the discussion thereafter).

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