# Blind Super-Resolution of Point Sources via Projected Gradient Descent 

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#### Abstract

Blind super-resolution can be cast as a low rank matrix recovery problem by exploiting the inherent simplicity of the signal and the low dimensional structure of point spread functions. In this paper, we develop a simple yet efficient non-convex projected gradient descent method for this problem based on the low rank structure of the vectorized Hankel matrix associated with the target matrix. Theoretical analysis indicates that the proposed method exactly converges to the target matrix with a linear convergence rate under the similar conditions as convex approaches. Numerical results show that our approach is competitive with existing convex approaches in terms of recovery ability and efficiency.


## 1 Introduction

Blind super-resolution is the problem of estimating high-resolution information of a signal from its lowresolution measurements when the point spread functions (PSFs) are unknown. Such problem arises in a wide variety of applications, including seismic data analysis [25], nuclear magnetic resonance spectroscopy [27], multi-user communication system [23], and 3D single-molecule microscopy [28]. In particular, when the knowledge of PSFs is available, blind super-resolution reduces to the super-resolution problem [7] [8].

Without any additional assumptions, blind super-resolution of point sources is an ill-posed problem. To alleviate this issue, it is common to assume that the PSFs belong to a known low-dimensional subspace. Under this assumption and utilizing the lift technique, blind super-resolution of point sources can be formulated as a matrix recovery problem. By exploiting low dimensional structures of the target matrix, a series of works [15, 37, 21, 12, 31 theoretically studied under which conditions the target matrix can be recovered. The author in 15 considered the setting where the PSF is shared among all point sources, and established the recovery guarantees for the atomic norm minimization (ANM) method. Yang et al. 37] further studied the same method, but with multiple unknown PSFs. Li et al. 21] provided robust analysis of blind 1D super-resolution and later the work in [31 generalized [21 to 2D case. Recently, Chen et al. [12] proposed a nuclear norm minimization method based on the vectorized Hankel lift framework, which also appears in [38, 40] but for matrix completion. Moreover, [12] established the corresponding exact recovery guarantees. While strong theoretical guarantees have been built for blind super-resolution based on convex methods, these approaches are computational inefficient for the high dimensional setting. Therefore it is necessary to design efficient and provable algorithms to deal with the large-scale regime.

In the past few years, substantial progress has been made on designing and analyzing provable fast algorithms for applications from science and engineering via non-convex optimization [20, 13, 16, including

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matrix completion [14, 42], phase retrieval [29], blind deconvolution [22], spectrally sparse signal recovery [5, 6], to name just a few. The goal of this work is to develop an efficient non-convex algorithm for blind super-resolution problem.

### 1.1 Comparisons with Related Work and Main Contributions

Our work is closely related to [12, [5, 43]. As already mentioned, 12 proposed a convex approach called Vectorized Hankel Lift (VHL) for blind super-resolution. Based on this framework, we develop an efficient and provable non-convex algorithm for blind super-resolution of point sources. More precisely, we parameter the vectorized Hankel matrix corresponding to a candidate solution in terms of the Burer-Monteiro factorization and develop a projected gradient descent method to directly recover the low-rank factors.

Our algorithm is inspired by the method in [5], where the projected gradient descent was developed for spectrally sparse signal recovery problem based on the low rank structure of the Hankel matrix corresponding to the target signal. Despite this, both the structures of sensing operator and target matrix in this paper are substantially different from that in [5. Therefore, the convergence analysis in [5] can not be easily extended to our model.

Recently, 43] follows our work and develops an iterative hard thresholding method based on the framework of vectorzied Hankel lift to solve blind super-resolution problem. It is worth pointing out that they directly apply our result to bound the initialization error of their method. Furthermore, their proof idea is inspired by the guarantee analysis of low rank matrix recovery over Riemannian manifold [6, 36]. Therefore the proof techniques are totally different with ours.

The main contributions of this work are summerized as follows. Firstly, we present a new non-convex algorithm called projected gradient descent via vectorized Hankel lift (PGD-VHL) for blind super-resolution. Numerical experiments show that PGD-VHL is competitive with convex recovery methods such as ANM and VHL in terms of recovery ability, but is much more efficient. Secondly, we establish the recovery performance of PGD-VHL. Our results show that PGD-VHL started from a spectral initialization converges linearly to the target matrix under the similar sample complexity as convex approaches. Lastly, it is worth mentioning that the theoretical guarantee of PGD-VHL requires a slightly milder assumption on the low-dimensional subspace than that for VHL in [12].

### 1.2 Organization and Notation

The rest of this paper is organized as follows. Section 2 gives the problem setup of blind super-resolution of point sources. Section 3 presents the PGD-VHL algorithm whose the exact recovery guarantee is provided in Section 4. Numerical evaluations are presented to illustrate the performance of PGD-VHL in Section 5 All proofs are deferred to Section 6. Finally, we conclude this paper and propose some future work in Section 7

Some notations used throughout this paper are presented as follows. Symbols for vectors, matrices and operators are in bold lowercase letters, bold uppercase letters and calligraphic letters, respectively. In this paper, vectors and matrices are indexed starting with zero. For a complex number $x$, its real part is denoted by $\Re(x)$. The transpose, complex conjugate, complex transpose, spectral norm and Frobenius norm of matrix $\boldsymbol{X}$ are denoted as $\boldsymbol{X}^{\top}, \overline{\boldsymbol{X}}, \boldsymbol{X}^{\mathrm{H}},\|\boldsymbol{X}\|$ and $\|\boldsymbol{X}\|_{\mathrm{F}}$, respectively. The inner product of two matrices $\boldsymbol{X}_{1}$ and $\boldsymbol{X}_{2}$ is defined as $\left\langle\boldsymbol{X}_{1}, \boldsymbol{X}_{2}\right\rangle=\operatorname{trace}\left(\boldsymbol{X}_{1}^{\mathrm{H}} \boldsymbol{X}_{2}\right)$. Moreover, we will refer to $\boldsymbol{A} \odot \boldsymbol{B}$ and $\boldsymbol{A} \otimes \boldsymbol{B}$ as the Hadamard product and Kroncker product, respectively. We use $\boldsymbol{x}[\ell]$ to denote the $\ell$-th entry of $\boldsymbol{x}$ and $\boldsymbol{X}(j,:)$ to denote the $j$ th row of $\boldsymbol{X}$. Moreover, we use the MATLAB notation $\boldsymbol{X}(i: j, k)$ to denote a vector of length $j-i+1$, with entries $\boldsymbol{X}_{i, k}, \cdots, \boldsymbol{X}_{j, k}$. The identity operator are denoted as $\mathcal{I}$. Let $\mathcal{H}$ be the vectorized Hankel lift operator which maps a matrix $\boldsymbol{X} \in \mathbb{C}^{s \times n}$ into an $s n_{1} \times n_{2}$ matrix,

$$
\mathcal{H}(\boldsymbol{X})=\left[\begin{array}{cccc}
\boldsymbol{x}_{0} & \boldsymbol{x}_{1} & \cdots & \boldsymbol{x}_{n_{2}-1}  \tag{1.1}\\
\boldsymbol{x}_{1} & \boldsymbol{x}_{2} & \cdots & \boldsymbol{x}_{n_{2}} \\
\vdots & \vdots & \ddots & \vdots \\
\boldsymbol{x}_{n_{1}-1} & \boldsymbol{x}_{n_{1}} & \cdots & \boldsymbol{x}_{n-1}
\end{array}\right] \in \mathbb{C}^{s n_{1} \times n_{2}}
$$

where $\boldsymbol{x}_{i} \in \mathbb{C}^{s}$ is the $i$-th column of $\boldsymbol{X}$ and $n_{1}+n_{2}=n+1$. We denote the adjoint of $\mathcal{H}$ by $\mathcal{H}^{*}$, which is a linear mapping from $\mathbb{C}^{s n_{1} \times n_{2}}$ to $\mathbb{C}^{s \times n}$. In particular, for any matrix $\boldsymbol{Z} \in \mathbb{C}^{s n_{1} \times n_{2}}$, the $i$-th column of $\mathcal{H}^{*}(\boldsymbol{Z})$ is given by

$$
\mathcal{H}^{*}(\boldsymbol{Z}) \boldsymbol{e}_{i}=\sum_{\substack{j+k=i \\ 0 \leq j \leq n_{1}-1,0 \leq k \leq n_{2}-1}} \boldsymbol{z}_{j, k},
$$

where $\boldsymbol{z}_{j, k}$ is the $(j, k)$-th block of $\boldsymbol{Z}$ such that $\boldsymbol{z}_{j, k}=\boldsymbol{Z}(j s:(j+1) s-1, k)$. Letting $\mathcal{D}^{2}=\mathcal{H}^{*} \mathcal{H}$, we have

$$
\mathcal{D}^{2}(\boldsymbol{X})=\left[\begin{array}{lll}
w_{0} \boldsymbol{x}_{0} & \cdots & w_{n-1} \boldsymbol{x}_{n-1}
\end{array}\right]
$$

where the scale $w_{i}$ is defined as

$$
w_{i}=\#\left\{(j, k) \mid j+k=i, 0 \leq j \leq n_{1}-1,0 \leq k \leq n_{2}-1\right\} .
$$

Moreover, we define $\mathcal{G}=\mathcal{H} \mathcal{D}^{-1}$. The adjoint of $\mathcal{G}$ denoted $\mathcal{G}^{*}$ is given by $\mathcal{G}^{*}=\mathcal{D}^{-1} \mathcal{H}^{*}$. Additionally, $\mathcal{G}$ and $\mathcal{G}^{*}$ satisfy

$$
\begin{equation*}
\mathcal{G}^{*} \mathcal{G}=\mathcal{I}, \quad\|\mathcal{G}\|=1, \quad \text { and }\left\|\mathcal{G}^{*}\right\| \leq 1 \tag{1.2}
\end{equation*}
$$

We use $\boldsymbol{G}_{i}$ to denote the matrix defined by

$$
\begin{equation*}
\boldsymbol{G}_{i}=\frac{1}{\sqrt{w_{i}}} \sum_{\substack{j+k=i \\ 0 \leq j \leq n_{1}-1,0 \leq k \leq n_{2}-1}} \boldsymbol{e}_{j} \boldsymbol{e}_{k}^{\top} \tag{1.3}
\end{equation*}
$$

Then one has

$$
\begin{equation*}
\mathcal{G}(\boldsymbol{X})=\sum_{i=0}^{n-1} \mathcal{G}\left(\boldsymbol{x}_{i} \boldsymbol{e}_{i}^{\boldsymbol{\top}}\right)=\sum_{i=0}^{n-1} \boldsymbol{G}_{i} \otimes \boldsymbol{x}_{i} \tag{1.4}
\end{equation*}
$$

where $\boldsymbol{G}_{i} \otimes \boldsymbol{x}_{i}$ denotes the Kronecker product between $\boldsymbol{G}_{i}$ and $\boldsymbol{x}_{i}$.
Throughout this paper, $c, c_{0}, c_{1}, \cdots$ denote absolute positive numerical constants whose values may vary from line to line. The notation $n=\mathcal{O}(m)$ means that there exists an absolute constant $c>0$ such that $n \leq c m$.

## 2 Problem formulation

The point source signal model can be represented as a superposition of $r$ spikes

$$
\begin{equation*}
x(t)=\sum_{k=1}^{r} d_{k} \delta\left(t-\tau_{k}\right) \tag{2.1}
\end{equation*}
$$

where $\delta(\cdot)$ is the Dirac function, $d_{k} \in \mathbb{C}$ and $\tau_{k} \in[0,1)$ are the amplitude and location of the $k$-th point source, respectively. Let $\left\{g_{k}(t)\right\}_{k=1}^{r}$ be the unknown point spread functions depending on the locations of point sources. The observation is a convolution between $x(t)$ and $\left\{g_{k}(t)\right\}_{k=1}^{r}$, that is,

$$
\begin{equation*}
y(t)=\sum_{k=1}^{r} d_{k} \delta\left(t-\tau_{k}\right) * g_{k}(t)=\sum_{k=1}^{r} d_{k} \cdot g_{k}\left(t-\tau_{k}\right) \tag{2.2}
\end{equation*}
$$

After taking the Fourier transform and sampling, we obtain the measurements as

$$
\begin{equation*}
\boldsymbol{y}[j]=\sum_{k=1}^{r} d_{k} e^{-2 \pi \imath \tau_{k} \cdot j} \hat{g}_{k}[j], \quad j=0, \cdots n-1 . \tag{2.3}
\end{equation*}
$$

Let $\boldsymbol{g}_{k}=\left[\begin{array}{lll}\hat{g}_{k}[0] & \cdots & \hat{g}_{k}[n-1\end{array}\right]^{\top}$ be a vector corresponding to the $k$-th unknown point spread function. The goal is to estimate $\left\{d_{k}, \tau_{k}\right\}_{k=1}^{r}$ as well as $\left\{\boldsymbol{g}_{k}\right\}_{k=1}^{r}$ from (2.3).

Obviously, the problem of blind super-resolution is ill-posed without any additional assumptions, because the number of unknowns in (2.3) is $\mathcal{O}(n r)$, which is larger than the number of samples $n$. To tackle this issue, we follow the same route as that in [1, 15, 37, 12] and assume that all the Fourier samples of the unknown PSFs $\left\{\boldsymbol{g}_{k}\right\}_{k=1}^{r}$ belong to a known low-dimensional subspace spanned by the columns of $\boldsymbol{B} \in \mathbb{C}^{n \times s}$ with $s<n$, i.e.,

$$
\begin{equation*}
\boldsymbol{g}_{k}=\boldsymbol{B} \boldsymbol{h}_{k} \tag{2.4}
\end{equation*}
$$

where $\boldsymbol{h}_{k} \in \mathbb{C}^{s}$ denotes the unknown directional vector of $\boldsymbol{g}_{k}$ in the subspace. According to the subspace assumption (2.4) and using the lift trick [1], we can easily rewrite (2.3) as a set of linear measurements with respect to the target matrix $\boldsymbol{X}^{\natural}=\sum_{k=1}^{r} d_{k} \boldsymbol{h}_{k} \boldsymbol{a}_{\tau_{k}}^{\top}$,

$$
\begin{equation*}
\boldsymbol{y}[j]=\left\langle\boldsymbol{b}_{j} \boldsymbol{e}_{j}^{\top}, \boldsymbol{X}^{\natural}\right\rangle, \quad j=0, \cdots, n-1, \tag{2.5}
\end{equation*}
$$

where $\boldsymbol{a}_{\tau_{k}}=\left[1, e^{-2 \pi \iota \tau_{k}}, \cdots, e^{-2 \pi \iota \tau_{k} \cdot(n-1)}\right]^{\top}, \boldsymbol{b}_{j} \in \mathbb{C}^{s}$ is the $j$ th column vector of $\boldsymbol{B}^{\mathrm{H}}, \boldsymbol{e}_{j}$ is the $j$-th standard basis of $\mathbb{R}^{n}$. The measurement model (2.5) can be rewritten succinctly as

$$
\begin{equation*}
\boldsymbol{y}=\mathcal{A}\left(\boldsymbol{X}^{\natural}\right) \tag{2.6}
\end{equation*}
$$

where $\mathcal{A}: \mathbb{C}^{s \times n} \rightarrow \mathbb{C}^{n}$ is the linear operator. Let $\mathcal{A}^{*}$ be the adjoint operator of $\mathcal{A}$ which is given by $\mathcal{A}^{*}(\boldsymbol{y})=\sum_{j=0}^{n-1} \boldsymbol{y}[j] \boldsymbol{b}_{j} \boldsymbol{e}_{j}^{\top}$. Furthermore, define $\boldsymbol{D}=\operatorname{diag}\left(\sqrt{w_{0}}, \cdots, \sqrt{w_{n-1}}\right)$. We have $\boldsymbol{D} \mathcal{A}(\boldsymbol{X})=\mathcal{A D}(\boldsymbol{X})$ for any $\boldsymbol{X}$. The measurements can be reformulated as

$$
\begin{equation*}
\boldsymbol{D} \boldsymbol{y}=\mathcal{A D}\left(\boldsymbol{X}^{\natural}\right) \tag{2.7}
\end{equation*}
$$

Note that once the data matrix $\boldsymbol{X}^{\natural}$ is reconstructed, the frequencies $\left\{\tau_{k}\right\}_{k=1}^{r}$ can be retrieved through spatial smoothing MUSIC [18, 17, [39, [12, and the amplitudes $\left\{d_{k}\right\}_{k=1}^{r}$ and coefficients $\left\{\boldsymbol{h}_{k}\right\}_{k=1}^{r}$ can be estimated by solving an over-determined linear system. Therefore in this work we focus on the problem of recovering $\boldsymbol{X}^{\natural}$ from its linear measurements (2.6).

It has been shown that $\mathcal{H}\left(\boldsymbol{X}^{\natural}\right)$ is a rank-r matrix [12] and thus the matrix $\mathcal{H}\left(\boldsymbol{X}^{\natural}\right)$ admits low rank structure when $r \ll \min \left(s n_{1}, n_{2}\right)$. Equipped with the low rank structure of $\mathcal{H}\left(\boldsymbol{X}^{\natural}\right)$, it is natural to recover $\boldsymbol{X}^{\natural}$ by solving the constrained least squares problem

$$
\begin{equation*}
\min _{\boldsymbol{X}} \frac{1}{2}\|\boldsymbol{D} \boldsymbol{y}-\mathcal{A D}(\boldsymbol{X})\|_{2}^{2} \quad \text { s.t. } \operatorname{rank}(\mathcal{H}(\boldsymbol{X}))=r \tag{2.8}
\end{equation*}
$$

Letting $\boldsymbol{Z}=\mathcal{H}(\boldsymbol{X})=\mathcal{G D}(\boldsymbol{X})$ for any $\boldsymbol{X}$, it can be verified that $\left(\mathcal{I}-\mathcal{G} \mathcal{G}^{*}\right)(\boldsymbol{Z})=\mathbf{0}$. To eliminate the rank constrain in (2.8), we apply the Burer-Monteiro factorization (4) to parameterize $\boldsymbol{Z}$ as $\boldsymbol{Z}=\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}$, where $\boldsymbol{L} \in \mathbb{C}^{s n_{1} \times r}$ and $\boldsymbol{R} \in \mathbb{C}^{n_{2} \times r}$ are two rank-r matrices. Therefore, the optimization problem (2.8) can be rewritten as

$$
\begin{equation*}
\min _{\boldsymbol{L}, \boldsymbol{R}} \frac{1}{2}\left\|\boldsymbol{D} \boldsymbol{y}-\mathcal{A G}^{*}\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}\right)\right\|_{2}^{2} \text { s.t. }\left(\mathcal{I}-\mathcal{G G}^{*}\right)\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}\right)=\mathbf{0} \tag{2.9}
\end{equation*}
$$

Before introducing our algorithm, we make an assumption that $\boldsymbol{Z}^{\natural}=\mathcal{H}\left(\boldsymbol{X}^{\natural}\right)$ is $\mu_{1}$-incoherent which is defined below.

Assumption 2.1. Let $\boldsymbol{Z}^{\natural}=\boldsymbol{U} \boldsymbol{\Sigma} \boldsymbol{V}^{\mathrm{H}}$ be the singular value decomposition of $\boldsymbol{Z}^{\natural}$, where $\boldsymbol{U} \in \mathbb{C}^{s n_{1} \times r}, \boldsymbol{\Sigma} \in \mathbb{R}^{r \times r}$ and $\boldsymbol{V} \in \mathbb{C}^{n_{2} \times r}$. Denote $\boldsymbol{U}^{\mathrm{H}}=\left[\begin{array}{lll}\boldsymbol{U}_{0}^{\mathrm{H}} & \cdots & \boldsymbol{U}_{n_{1}-1}^{\mathrm{H}}\end{array}\right]^{\mathrm{H}}$, where $\boldsymbol{U}_{j}=\boldsymbol{U}[j s:(j+1) s-1,:]$ is the $j$-th block of $\boldsymbol{U}$ for $j=0, \cdots n_{1}-1$. The matrix $\boldsymbol{Z}^{\natural}$ is $\mu_{1}$-incoherent if $\boldsymbol{U}$ and $\boldsymbol{V}$ obey that

$$
\max _{0 \leq j \leq n_{1}-1}\left\|\boldsymbol{U}_{j}\right\|_{\mathrm{F}}^{2} \leq \frac{\mu_{1} r}{n} \text { and } \max _{0 \leq k \leq n_{2}-1}\left\|\boldsymbol{e}_{k}^{\top} \boldsymbol{V}\right\|_{2}^{2} \leq \frac{\mu_{1} r}{n}
$$

for some positive constant $\mu_{1}$.

Remark 2.1. Assumption 2.1] is the same as the one made in [11, 40] for low rank matrix recovery and is used in [12] for blind super-resolution. It has been established that Assumption 2.1 is obeyed when the minimum wrap-up distance between the locations of point sources is greater than about $2 / n$.

Let $\mu$ and $\sigma$ be two numerical constants such $\mu_{1} \leq \mu$ and $\sigma_{1} \leq \sigma$, and $\mathcal{M}$ be a convex set defined as follows

$$
\mathcal{M}=\left\{\left[\begin{array}{l}
\boldsymbol{L}  \tag{2.10}\\
\boldsymbol{R}
\end{array}\right]: \max _{0 \leq j \leq n_{1}-1}\left\|\boldsymbol{L}_{j}\right\|_{\mathrm{F}} \leq \sqrt{\frac{\mu r \sigma}{n}},\|\boldsymbol{R}\|_{2, \infty} \leq \sqrt{\frac{\mu r \sigma}{n}}\right\}
$$

where $\boldsymbol{L}_{j}$ is the $j$-th block of $\boldsymbol{L}$. Define

$$
\boldsymbol{M}^{\natural}=\left[\begin{array}{c}
\boldsymbol{L}^{\natural} \\
\boldsymbol{R}^{\natural}
\end{array}\right]=\left[\begin{array}{l}
\boldsymbol{U} \boldsymbol{\Sigma}^{1 / 2} \\
\boldsymbol{V} \boldsymbol{\Sigma}^{1 / 2}
\end{array}\right] .
$$

Since $\boldsymbol{Z}^{\natural}$ is $\mu_{1}$-incoherent, we have $\boldsymbol{M}^{\natural} \in \mathcal{M}$. Therefore, we consider a penalized version of (2.9) for recovering the factorized matrices:

$$
\begin{equation*}
\min _{\boldsymbol{M} \in \mathcal{M}}\left\{f(\boldsymbol{M}):=\frac{1}{2}\left\|\boldsymbol{D} \boldsymbol{y}-\mathcal{A \mathcal { G } ^ { * }}\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}\right)\right\|_{2}^{2}+\frac{1}{2}\left\|\left(\mathcal{I}-\mathcal{G} \mathcal{G}^{*}\right)\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}\right)\right\|_{\mathrm{F}}^{2}+\frac{1}{16}\left\|\boldsymbol{L}^{\mathrm{H}} \boldsymbol{L}-\boldsymbol{R}^{\mathrm{H}} \boldsymbol{R}\right\|_{\mathrm{F}}^{2}\right\} \tag{2.11}
\end{equation*}
$$

where $\boldsymbol{M}=\left[\begin{array}{ll}\boldsymbol{L}^{\mathrm{H}} & \boldsymbol{R}^{\mathrm{H}}\end{array}\right]^{\mathrm{H}} \in \mathbb{C}^{\left(s n_{1}+n_{2}\right) \times r}$, and the last term penalizes the mismatch between $\boldsymbol{L}$ and $\boldsymbol{R}$, which is widely used in rectangular low rank matrix recovery (34, 42, 16].

## 3 Algorithm: projected gradient descent

Inspired by [5, we design a projected gradient descent method for the problem (2.11), which is summarized in Algorithm 1 . The initialization involves two steps: (1) computes the best rank $r$ approximation of $\mathcal{H} \mathcal{A}^{*}(\boldsymbol{y})$

```
Algorithm 1 PGD-VHL
    Input: \(\mathcal{A}, \boldsymbol{y}, n, s, r\)
    Initialization:
        \(\widehat{\boldsymbol{Z}}_{0}=\widehat{\boldsymbol{U}}_{0} \widehat{\boldsymbol{\Sigma}}_{0} \widehat{\boldsymbol{V}}_{0}^{\mathrm{H}}=\mathcal{P}_{r} \mathcal{H} \mathcal{A}^{*}(\boldsymbol{y})\)
        \(\widehat{\boldsymbol{L}}_{0}=\widehat{\boldsymbol{U}}_{0} \widehat{\boldsymbol{\Sigma}}_{0}^{1 / 2}, \quad \widehat{\boldsymbol{R}}_{0}=\widehat{\boldsymbol{V}}_{0} \widehat{\boldsymbol{\Sigma}}_{0}^{1 / 2}\)
        \(\widehat{\boldsymbol{M}}_{0}=\left[\begin{array}{ll}\widehat{\boldsymbol{L}}_{0}^{\mathrm{H}} & \widehat{\boldsymbol{R}}_{0}^{\mathrm{H}}\end{array}\right]^{\mathrm{H}}\)
        \(\left(\boldsymbol{L}_{0}, \boldsymbol{R}_{0}\right)=\mathcal{P}_{\mathcal{M}}\left(\left(\widehat{\boldsymbol{L}}_{0}, \widehat{\boldsymbol{R}}_{0}\right)\right)\)
        \(\boldsymbol{M}_{0}=\left[\begin{array}{ll}\boldsymbol{L}_{0}^{\mathrm{H}} & \boldsymbol{R}_{0}^{\mathrm{H}}\end{array}\right]^{\mathrm{H}}\)
    while not convergence do
        \(\boldsymbol{M}_{t+1}=\mathcal{P}_{\mathcal{M}}\left(\boldsymbol{M}_{t}-\eta \nabla f\left(\boldsymbol{M}_{t}\right)\right)\).
    end while
```

via one step hard thresholding $\mathcal{P}_{r}(\cdot)$, where $\mathcal{A}^{*}$ is the adjoint of $\mathcal{A}$ and $\mathcal{P}_{r}(\boldsymbol{Z})$ is the best rank $r$ approximation of $\boldsymbol{Z} ;(2)$ projects the low-rank factors of best rank- $r$ approximated matrix onto the convex feasible set $\mathcal{M}$. Given a matrix $\boldsymbol{M}=\left[\begin{array}{ll}\boldsymbol{L}^{H} & \boldsymbol{R}^{H}\end{array}\right]^{H}$, the projection onto $\mathcal{M}$, denoted by $\left[\begin{array}{ll}\widehat{\boldsymbol{L}}^{H} & \widehat{\boldsymbol{R}}^{H}\end{array}\right]^{H}$, has a closed form solution:

$$
\widehat{\boldsymbol{L}}_{j}= \begin{cases}\boldsymbol{L}_{j} & \text { if }\left\|\boldsymbol{L}_{j}\right\|_{\mathrm{F}} \leq \sqrt{\frac{\mu r \sigma}{n}} \\ \frac{1}{\left\|\boldsymbol{L}_{j}\right\|_{\mathrm{F}}} \boldsymbol{L}_{j} \cdot \sqrt{\frac{\mu r \sigma}{n}} & \text { otherwise }\end{cases}
$$

for $0 \leq j \leq n_{1}-1$ and

$$
\boldsymbol{e}_{k}^{\top} \widehat{\boldsymbol{R}}= \begin{cases}\boldsymbol{e}_{k}^{\top} \boldsymbol{R} & \text { if }\left\|\boldsymbol{e}_{k}^{\top} \boldsymbol{R}\right\|_{2} \leq \sqrt{\frac{\mu r \sigma}{n}} \\ \frac{\boldsymbol{e}_{k}^{\top} \boldsymbol{R}}{\left\|\boldsymbol{e}_{k}^{\top} \boldsymbol{R}\right\|_{2}} \cdot \sqrt{\frac{\mu r \sigma}{n}} & \text { otherwise }\end{cases}
$$

for $0 \leq k \leq n_{2}-1$. Let $\boldsymbol{M}_{t}$ be the current estimator. The algorithm updates $\boldsymbol{M}_{t}$ along gradient descent direction $-\nabla f\left(\boldsymbol{M}_{t}\right)$ with step size $\eta$, followed by projection onto the set $\mathcal{M}$. The gradient of $f(\boldsymbol{M})$ is computed with respect to Wirtinger calculus given by $\nabla f=\left[\begin{array}{ll}\nabla_{\boldsymbol{L}}^{\mathrm{H}} f & \nabla_{\boldsymbol{R}}^{\mathrm{H}} f\end{array}\right]^{\mathrm{H}}$ where

$$
\begin{aligned}
& \nabla_{\boldsymbol{L}} f=\left(\mathcal{G} \mathcal{A}^{*}\left(\mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}\right)-\boldsymbol{D} \boldsymbol{y}\right)\right) \boldsymbol{R}+\left(\left(\mathcal{I}-\mathcal{G G}^{*}\right)\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}\right)\right) \boldsymbol{R}+\frac{1}{4} \boldsymbol{L}\left(\boldsymbol{L}^{\mathrm{H}} \boldsymbol{L}-\boldsymbol{R}^{\mathrm{H}} \boldsymbol{R}\right), \\
& \nabla_{\boldsymbol{R}} f=\left(\mathcal{G} \mathcal{A}^{*}\left(\mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}\right)-\boldsymbol{D} \boldsymbol{y}\right)\right)^{\mathrm{H}} \boldsymbol{L}+\left(\left(\mathcal{I}-\mathcal{G} \mathcal{G}^{*}\right)\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}\right)\right)^{\mathrm{H}} \boldsymbol{L}+\frac{1}{4} \boldsymbol{R}\left(\boldsymbol{R}^{\mathrm{H}} \boldsymbol{R}-\boldsymbol{L}^{\mathrm{H}} \boldsymbol{L}\right) .
\end{aligned}
$$

Indeed PGD-VHL algorithm can be efficiently implemented. To obtain the computational cost of $\nabla f$, we first introduce some notations. Let $\mathcal{H}_{v}$ be the Hankel operator which maps a vector $\boldsymbol{x} \in \mathbb{C}^{n}$ into an $n_{1} \times n_{2}$ matrix,

$$
\mathcal{H}_{v}(\boldsymbol{x})=\left[\begin{array}{ccc}
x_{0} & \cdots & x_{n_{2}-1} \\
\vdots & \ddots & \vdots \\
x_{n_{1}-1} & \cdots & x_{n-1}
\end{array}\right]
$$

where $x_{i}$ is the $i$-th entry of $\boldsymbol{x}$. The adjoint of $\mathcal{H}_{v}$, denoted by $\mathcal{H}_{v}^{*}$, is a linear mapping from $n_{1} \times n_{2}$ to $n$. It can be seen that $\mathcal{H}_{v}^{*}\left(\boldsymbol{L}_{v} \boldsymbol{R}_{v}^{\mathrm{H}}\right)$ can be computed via $r$ fast convolutions by noting that

$$
\begin{aligned}
{\left[\mathcal{H}_{v}^{*}\left(\boldsymbol{L}_{v} \boldsymbol{R}_{v}^{\mathrm{H}}\right)\right]_{i} } & =\left[\mathcal{H}_{v}^{*}\left(\sum_{s=1}^{r} \boldsymbol{L}_{v}[:, s] \overline{\boldsymbol{R}_{v}}[:, s]^{\top}\right)\right]_{i} \\
& =\sum_{s=1}^{r}\left[\mathcal{H}_{v}^{*}\left(\boldsymbol{L}_{j}[:, s] \overline{\boldsymbol{R}}[:, s]^{\top}\right)\right]_{i} \\
& =\sum_{s=1}^{r}\left(\boldsymbol{L}_{v}[:, s] * \overline{\boldsymbol{R}_{v}}[:, s]\right)[i]
\end{aligned}
$$

where $\boldsymbol{L}_{v} \in \mathbb{C}^{n_{1} \times r}$ and $\boldsymbol{R}_{v} \in \mathbb{C}^{n_{2} \times r}$. In addition, we can compute $\left(\mathcal{H}_{v}(\boldsymbol{x})\right) \boldsymbol{R}_{v}$ by $r$ fast Hankel matrix-vector multiplications, that is,

$$
\begin{aligned}
\left(\mathcal{H}_{v}(\boldsymbol{x}) \boldsymbol{R}_{v}\right)[j, s] & =\sum_{k=0}^{n_{2}-1} \boldsymbol{x}[j+k] \boldsymbol{R}_{v}[k, s] \\
& =\sum_{k=0}^{n_{2}-1} \tilde{\boldsymbol{x}}[n-1-j-k] \boldsymbol{R}_{v}[k, s] \\
& =\left(\tilde{\boldsymbol{x}} * \boldsymbol{R}_{v}[:, s]\right)[n-1-j],
\end{aligned}
$$

where $\tilde{\boldsymbol{x}}$ is a vector reversing the order of $\boldsymbol{x}$. Therefore the computational complexity of both $\mathcal{H}_{v}^{*}\left(\boldsymbol{L}_{v} \boldsymbol{R}_{v}^{\mathrm{H}}\right)$ and $\left(\mathcal{H}_{v}(\boldsymbol{x})\right) \boldsymbol{R}_{v}$ is $\mathcal{O}(r n \log n)$ flops. Moreover, the authors in 12 show that $\mathcal{H}(\boldsymbol{X})=\boldsymbol{P} \widetilde{\mathcal{H}}(\boldsymbol{X})$, where $\widetilde{\mathcal{H}}(\boldsymbol{X})$ is a matrix constructed by stacking all $\left\{\mathcal{H}_{v}\left(\boldsymbol{e}_{\ell}^{\top} \boldsymbol{X}\right)\right\}_{\ell=1}^{s}$ on top of one another, and $\boldsymbol{P}$ is a permutation matrix. Therefore we can compute $\mathcal{G}^{*}\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}\right)$ and $\mathcal{G} \mathcal{D}(\boldsymbol{X}) \boldsymbol{R}$ by using $\mathcal{O}(\operatorname{srn} \log n)$ flops. Thus the implementation of our algorithm is very efficient and the main computational complexity in each step is $\mathcal{O}\left(s r^{2} n+\operatorname{srn} \log n\right)$.

## 4 Main result

In this section, we provide a theoretical analysis of PGD-VHL under a random subspace model.
Assumption 4.1. The column vectors $\left\{\boldsymbol{b}_{i}\right\}_{i=0}^{n-1}$ of $\boldsymbol{B}^{\boldsymbol{H}}$ are independently and identically drawn from a distribution $F$ which satisfies the following conditions:

$$
\begin{align*}
& \mathbb{E}\left[\boldsymbol{b}_{i} \boldsymbol{b}_{i}^{\mathrm{H}}\right]=\boldsymbol{I}_{s}, \quad i=0, \cdots, n-1,  \tag{4.1}\\
& \max _{0 \leq \ell \leq s-1}\left|\boldsymbol{b}_{i}[\ell]\right|^{2} \leq \mu_{0}, \quad i=0, \cdots, n-1 . \tag{4.2}
\end{align*}
$$

Remark 4.1. Assumption 4.1 is a standard assumption in RIPless compressed sensing [10] and blind superresolution [15, 37, 21, [30, 12]. It implies the spectral flatness over point spread functions, which is satisfied in OFDM signals [2] and noisy radar waveforms [26]. Assumption 4.1] holds with $\mu_{0}=1$ by many common random ensembles, for instance, when the components of $\boldsymbol{b}$ are Rademacher random variables taking the values $\pm 1$ with equal probability or when $\boldsymbol{b}$ is uniformly sampled from the rows of a Discrete Fourier Transform (DFT) matrix.

Now we present the main result, whose proofs are deferred to Section 6
Theorem 4.1. Let $\mu \geq \mu_{1}$ and $\sigma=\sigma_{1}\left(\widehat{\boldsymbol{\Sigma}}_{0}\right) /(1-\varepsilon)$ for $0 \leq \varepsilon \leq 1 / 3$. Let $\eta \leq \frac{\sigma_{r}}{9000\left(\mu_{0} \mu s r \sigma_{1}\right)^{2}}, \beta=\frac{\sigma_{r}}{72}$, and $\boldsymbol{M}^{\natural}=\left[\begin{array}{ll}\boldsymbol{L}^{\natural^{\mathrm{H}}} & \boldsymbol{R}^{\natural^{\mathrm{H}}}\end{array}\right]^{\mathrm{H}}$. Suppose $\boldsymbol{X}^{\natural}$ obeys the Assumption 2.1 and the subspace $\boldsymbol{B}$ satisfies the Assumption 4.1 . If

$$
n \geq c_{0} \varepsilon^{-2} \mu_{0}^{2} \mu s^{2} r^{2} \kappa^{2} \log ^{2}(s n)
$$

with probability at least $1-c_{1}(s n)^{-c_{2}}$, the sequence $\left\{\boldsymbol{M}_{t}\right\}$ returned by Algorithm 1 satisfies

$$
\begin{equation*}
\operatorname{dist}^{2}\left(\boldsymbol{M}_{t}, \boldsymbol{M}^{\natural}\right) \leq(1-\eta \beta)^{t} \cdot \frac{\varepsilon^{2} \sigma_{r}}{\mu_{0} s} \tag{4.3}
\end{equation*}
$$

where $c_{0}, c_{1}, c_{2}$ are absolute constants, $\sigma_{1}=\sigma_{1}\left(\boldsymbol{Z}^{\natural}\right), \sigma_{r}=\sigma_{r}\left(\boldsymbol{Z}^{\natural}\right), \kappa$ is the condition number of $\boldsymbol{Z}^{\natural}$, and the distance $\operatorname{dist}\left(\boldsymbol{M}, \boldsymbol{M}^{\natural}\right)$ is defined as

$$
\operatorname{dist}\left(\boldsymbol{M}, \boldsymbol{M}^{\natural}\right)=\min _{\boldsymbol{Q} \boldsymbol{Q}^{\mathrm{H}}=\boldsymbol{Q}^{\mathrm{H}} \boldsymbol{Q}=\boldsymbol{I}_{r}}\left\|\boldsymbol{M}-\boldsymbol{M}^{\natural} \boldsymbol{Q}\right\|_{\mathrm{F}} .
$$

Remark 4.2. It is worth noting that our results require slightly milder assumptions than that in [12]. The theoretical performance in [12] is established based on an additional assumption, which requires a lower bound of $\ell_{2}$ norm of the row vector of $\boldsymbol{B}$. However, the performance guarantee of $P G D-V H L$ is independent of this assumption.

Remark 4.3. Compared with the sample complexity established in [12 for the nuclear norm minimization method, which is $n \geq c \mu_{0} \mu_{1} \cdot s r \log ^{4}(s n)$, Theorem 4.1 implies that PGD-VHL is sub-optimal in terms of $s$ and $r$. The extra $r$ factor is caused by the technical derivation. Since the convergence is established based on the Frobenius norm and the Frobenius norm of the initial error is bounded by its spectral norm. Details can be found in Section 6.1. The extra s factor is introduced to ensure the initial error to be sufficiently small, i.e., on the order of $1 /\left(\mu_{0} s\right)$, which helps to derive the linear convergence rate of $P G D-V H L$. We admit that it is an artifact of our proof because subsequent numerical experiments indicate that there approximately exists a linear relationship between $n$ and $s$ or $n$ and $r$.

Remark 4.4. Theorem 4.1 implies that $P G D-V H L$ converges to $M^{\natural}$ with a linear rate. Therefore, after $T=\mathcal{O}\left(\left(\mu_{0} \mu s r \kappa\right)^{2} \log (1 / \epsilon)\right)$ iterations, we have $\operatorname{dist}^{2}\left(\boldsymbol{M}_{T}, \boldsymbol{M}^{\natural}\right) \leq \epsilon \cdot \operatorname{dist}^{2}\left(\boldsymbol{M}_{0}, \boldsymbol{M}^{\natural}\right)$. Given the iterates $\boldsymbol{M}_{T}$ returned by $P G D-V H L$, we can estimate $\boldsymbol{X}_{T}$ by $\mathcal{D}^{-1} \mathcal{G}^{*}\left(\boldsymbol{L}_{T} \boldsymbol{R}_{T}^{\mathrm{H}}\right)$.

Remark 4.5. As already mentioned, once the data matrix $\boldsymbol{X}^{\natural}$ is recovered, the locations $\left\{\tau_{k}\right\}_{k=1}^{r}$ can be computed from it by spatial smoothing MUSIC algorithm [18, 17, 39, 12] and the weights $\left\{d_{k}, \boldsymbol{h}_{k}\right\}_{k=1}^{r}$ can be estimated by solving an overdetermined linear system [12].

Remark 4.6. Though we have mainly focused on the one-dimensional (1D) blind super-resolution problem, the model and analysis are also applicable for higher dimensional case. Due to space limitations, we omit the theoretical results but provide numerical simulations for two-dimensional (2D) blind super-resolution problem in Section V.B.


Figure 1: The phase transitions of VHL, ANM and PGD-VHL when $n=64$. Top: frequencies are randomly generated; Bottom: frequencies obey the separation condition $\Delta:=\min _{k \neq j}\left|\tau_{k}-\tau_{j}\right| \geq 1 / n$. The red curve plots the hyperbola curve $r s=20$.

## 5 Numerical simulations

In this section, a series of numerical results are provided to illustrate the performance of PGD-VHL. We conduct simulations on 1D signals and 2D signals separately. Moreover, we implement our algorithm with 2D MUSIC [3, 41] to solve the joint delay-Doppler estimation problem, which is an important issue arising in orthogonal frequency-division multiplexing (OFDM) signals. The numerical simulations are executed from MATLAB R2021b on a macOS machine with multi-core Intel CPU at 2.3 GHz CPU and 16 GB RAM. Our code is available at "https://github.com/jcchen2017/PGDVHL".

### 5.1 Simulation for 1D Signals

We begin by providing the numerical results for 1 D signals. The data matrix $\boldsymbol{X}^{\natural} \in \mathbb{C}^{s \times n}$ is generated by $\sum_{k=1}^{r} d_{k} \boldsymbol{h}_{k} \boldsymbol{a}_{\tau_{k}}^{\top}$. Here the locations $\left\{\tau_{k}\right\}_{k=1}^{r}$ of the point sources are randomly generated from $[0,1)$, the coefficients $\left\{\boldsymbol{h}_{k}\right\}_{k=1}^{r}$ are i.i.d. sampled from standard Gaussian with normalization, and the amplitudes $\left\{d_{k}\right\}_{k=1}^{r}$ are selected to be $d_{k}=\left(1+10^{c_{k}}\right) e^{-\imath \phi_{k}}$, where $c_{k}$ is uniformly sampled from $[0,1]$ and $\phi_{k}$ is uniformly sampled from $[0,2 \pi)$. Moreover, the columns of $\boldsymbol{B}$ are uniformly sampled from the DFT matrix. The stepsize of PGD-VHL is chosen via backtracking line search.

The first experiment studies the recovery ability of PGD-VHL through the framework of phase transition and we compare it with two convex recovery methods: VHL [12] and ANM [37]. Both VHL and ANM are solved by CVX [19. PGD-VHL will be terminated if $\left\|y-\mathcal{A}\left(\boldsymbol{X}_{t}\right)\right\|_{2} \leq 10^{-5}$ or a maximum number of
iterations is reached. The tests are conducted with $n=64$ and the varied $s$ and $r$. We repeat 20 random trials and record the probability of successful recovery in our tests. A trial is declared to be successful if $\left\|\boldsymbol{X}_{t}-\boldsymbol{X}^{\natural}\right\|_{\mathrm{F}} /\left\|\boldsymbol{X}^{\natural}\right\|_{\mathrm{F}} \leq 10^{-3}$. Figure $\prod_{\text {(a) }}$, (b) and (c) show the phase transitions of VHL, ANM and PGDVHL when the locations of point sources are randomly generated, and Figure (d), 1(e) and (f) illustrate the phase transitions of VHL, ANM and PGD-VHL when the separation condition $\Delta:=\min _{j \neq k}\left|\tau_{j}-\tau_{k}\right| \geq 1 / n$ is imposed. In this figure, white color means successful recovery while black color indicates failure. It is interesting to observe that PGD-VHL has a higher phase transition curve than VHL whether the separation condition is satisfied or not. Moreover, by comparing Figure 1 (b) with (c), we observe that PGD-VHL is less sensitive to the separation condition than ANM.

In the second experiment, we study the phase transition of PGD-VHL when one of $r$ and $s$ is fixed. Note that in this test, the separation condition is not imposed for PGD-VHL. Figure 2(a) indicates an approximately linear relationship between $s$ and $n$ for the successful recovery when the number of point sources is fixed to be $r=4$. The same linear relationship between $r$ and $n$ can be observed when the dimension of the subspace is fixed to be $s=4$, see Figure 2(b). Therefore there exists a gap between our theory and empirical observation and we leave it as future work.


Figure 2: (a) The phase transition of PGD-VHL for varying $n$ and $s$ when $r=4$. The red line plots the straight line $n=2.5 s$. (b) The phase transition of PGD-VHL for varying $n$ and $r$ when $s=4$. The red line plots the straight line $n=2.5 r$.

In the third simulation, we investigate the convergence rate of PGD-VHL for $n=1024$ with fixed $s$ or $r$. The results are shown in Figure 3. The $y$-axis denotes $\log \left(\left\|\boldsymbol{X}_{t}-\boldsymbol{X}^{\natural}\right\|_{\mathrm{F}} /\left\|\boldsymbol{X}^{\natural}\right\|_{\mathrm{F}}\right)$ and the $x$-axis represents the iteration number. It can be clearly seen that PGD-VHL converges linearly as shown in our main theorem. Also it is worth pointing out that PGD-VHL can be implemented in high dimensional regimes, where we take $n=1024$.


Figure 3: (a) Convergence of PGD-VHL for varying $s=4,6,8$ when $n=1024$ and $r=4$. (b) Convergence of PGD-VHL for varying $r=4,6,8$ when $n=1024$ and $s=4$.

In the fourth simulation, we conduct the tests to demonstrate the robustness of PGD-VHL to additive noise. More specifically, we collect the measurements corrupted by the noise vector $\boldsymbol{e}=\sigma_{\boldsymbol{e}} \cdot\|\boldsymbol{y}\|_{2} \cdot \boldsymbol{w} /\|\boldsymbol{w}\|_{2}$, where $\boldsymbol{y}$ is the uncontaminated observations, $\boldsymbol{w}$ is the standard Gaussian vector with i.i.d. entries and $\sigma_{e}$ denotes the noise level. In the tests, the noise level $\sigma_{e}$ is taken from $10^{-3}$ to 1 , corresponding to the signal-to-noise ratio (SNR) from 60 to 0 dB . For each $\sigma_{e}, 10$ random trials are conducted with $s=r=4$. As for the number of measurements, we choose $n=64$ and $n=128$ for comparison. PGD-VHL is set to be terminated when $\left\|\boldsymbol{X}^{t+1}-\boldsymbol{X}^{t}\right\|_{\mathrm{F}} /\left\|\boldsymbol{X}^{t}\right\|_{\mathrm{F}} \leq 10^{-7}$. In Figure 4 , the average relative reconstruction error is plotted with SNR. It can be clearly seen that the relationship between the relative reconstruction error and the noise level is linear for PGD-VHL. Moreover, the relative reconstruction error decreases with the increase of the number of measurements.

We finally compare the running time for ANM, VHL and PGD-VHL when the number of measurements is varied, the number of spikes $r$ is fixed to be 3 and the dimension of subspace $s$ is also fixed to be 3 . Note that both VHL and ANM are solved by SDPT3 [32] based on CVX [19]. We repeat 10 random trials for each test. The average computational time for each tested algorithms are shown in Table I. The symbol " - " indicates that the algorithm was terminated due to the lack of memory. It can be seen that PGD-VHL significantly improve the running time compared with ANM and VHL when $n$ is large.

Table 1: Running time comparison for 1D signals when $s=r=3$.

| Methods | $n=64$ | $n=128$ | $n=256$ | $n=512$ |
| :---: | :---: | :---: | :---: | :---: |
| ANM | 1.6278 s | 7.2992 s | 67.6007 s | - |
| VHL | 62.2369 s | 748.3695 s | - | - |
| PGD-VHL | 1.3254 s | 4.4417 s | 19.3004 s | 57.4518 s |



Figure 4: Performance of PGD-VHL under different noise levels

### 5.2 Simulation for 2D Signals

In this part, we evaluate the performance of our algorithm for 2 D signals. The data matrix is given by $\boldsymbol{X}^{\natural}=\sum_{k=1}^{r} d_{k} \boldsymbol{h}_{k}\left(\boldsymbol{a}_{\tau_{2 k}} \otimes \boldsymbol{a}_{\tau_{1 k}}\right)^{\top} \in \mathbb{C}^{s \times n_{1} n_{2}}$ and the samples are generated by

$$
\begin{equation*}
\boldsymbol{y}[j]=\left\langle\boldsymbol{b}_{j} \boldsymbol{e}_{j}^{\top}, \boldsymbol{X}^{\natural}\right\rangle, \quad j=0, \cdots, n_{1} n_{2}-1 . \tag{5.1}
\end{equation*}
$$

Here the two dimensional locations $\tau_{k}:=\left(\tau_{1 k}, \tau_{2 k}\right)$ are uniformly sampled from $[0,1) \times[0,1)$, the amplitudes $\left\{d_{k}\right\}_{k=1}^{r}$, the coefficients $\left\{\boldsymbol{h}_{k}\right\}_{k=1}^{r}$ and the subspace columns $\left\{\boldsymbol{b}_{j}\right\}_{j=0}^{n_{1} n_{2}-1}$ are generated by the same way as the 1D case. Let $\boldsymbol{X}_{\ell}^{\natural}=\sum_{k=1}^{r} d_{k} e^{-2 \imath \pi \tau_{2 k} \cdot \ell}\left(\boldsymbol{h}_{k} \boldsymbol{a}_{\tau_{1 k}}^{\top}\right)$ be an $s \times n_{1}$ matrix, where $\ell=0, \cdots, n_{2}-1$. Let $\mathcal{H}\left(\boldsymbol{X}^{\natural}\right)$
be the two-fold vectorized Hankel matrix of $\boldsymbol{X}^{\natural}$ defined as follows:

$$
\mathcal{H}\left(\boldsymbol{X}^{\natural}\right)=\left[\begin{array}{ccc}
\mathcal{H}\left(\boldsymbol{X}_{0}^{\natural}\right) & \cdots & \mathcal{H}\left(\boldsymbol{X}_{K_{2}-1}^{\natural}\right) \\
\vdots & \ddots & \vdots \\
\mathcal{H}\left(\boldsymbol{X}_{K_{1}-1}^{\natural}\right) & \cdots & \mathcal{H}\left(\boldsymbol{X}_{n_{2}}^{\natural}\right)
\end{array}\right] \in \mathbb{C}^{s L_{1} K_{1} \times L_{2} K_{2}}
$$

where $\mathcal{H}\left(\boldsymbol{X}_{\ell}^{\natural}\right) \in \mathbb{C}^{s L_{1} \times L_{2}}$ is the vectorized Hankel matrix defined in (1.1). Here $L_{1}+L_{2}=n_{1}+1$ and $K_{1}+K_{2}=n_{2}+1$. It has been shown in 12 that $\mathcal{H}\left(\boldsymbol{X}^{\natural}\right)$ is a rank- $r$ matrix. Therefore, we can naturally generalize our algorithm to the 2D case, and then use the two-dimensional PGD-VHL (PGD-VHL (2D)) to recover $\boldsymbol{X}^{\natural}$ from (5.1). The phase transition is shown in Figure 5 and the convergence rate of PGD-VHL (2D) for different $n_{1}$ and $n_{2}$ is shown in Figure 6. Overall, the performance of PGD-VHL (2D) exhibits a similar phenomenon to the 1D case.


Figure 5: The phase transition of PGD-VHL (2D) for varying $s$ and $r$ when $n_{1}=13$ and $n_{2}=9$. The locations are randomly generated. The red line plots the straight line $s r=25$.


Figure 6: Convergence of PGD-VHL (2D) for 2D signals with $\left(n_{1}, n_{2}\right) \in\{(23,19),(33,29),(53,49)\}$. Here we fix $s=3$ and $r=3$.

### 5.3 Simulation for joint delay-Doppler estimation from OFDM signals

Furthermore, we evaluate the performance of PGD-VHL (2D) for the problem of joint delay-Doppler estimation from OFDM signals. In this problem, the transmitted signal is divided into $M$ blocks and $N$ orthogonal subcarriers are used in each block. Then the received samples in the $n$-th subcarrier and $m$-th block can be formulated as 41]

$$
\begin{equation*}
\boldsymbol{y}_{m}[n]=\sum_{k=1}^{r} d_{k} e^{-2 \imath \pi\left(n \cdot \Delta f \tau_{k}+m \cdot \bar{T} f_{k}\right)} \boldsymbol{g}_{m}[n], \tag{5.2}
\end{equation*}
$$

where $r$ is the number of propagation paths for communication channel, $\left\{\tau_{k}, f_{k}\right\}_{k=1}^{r}$ are the delays and Doppler frequencies, $\left\{d_{k}\right\}_{k=1}^{r} \subset \mathbb{C}$ denote the channel coefficients, $\Delta f$ is the frequency spacing of adjacent subcarriers, $\bar{T}$ is the duaration of each transmission block with $\bar{T} f_{k} \ll 1$ as stated in 41, and $\left\{\boldsymbol{g}_{m}[n]\right\}_{n=0, \cdots, N-1}$ denote data symbols in the $m$-th block. For sake of simplicity, we define $\phi_{k}=\Delta f \tau_{k} \in$ $[0,1), \psi_{k}=\bar{T} f_{k} \in[0,1)$. Concatenating $\boldsymbol{y}_{m}[n]$ in the vector $\boldsymbol{y}_{m} \in \mathbb{C}^{N}$ and stacking $\boldsymbol{y}_{m}$ yield that

$$
\boldsymbol{y}=\left(\sum_{k=0}^{r} d_{k} \boldsymbol{a}_{\psi_{k}} \otimes \boldsymbol{a}_{\phi_{k}}\right) \odot \boldsymbol{g} \in \mathbb{C}^{M N},
$$

where $\boldsymbol{g}=\left[\begin{array}{lll}\boldsymbol{g}_{1} & \cdots & \boldsymbol{g}_{M}\end{array}\right]^{\top} \in \mathbb{C}^{M N}$ and $\boldsymbol{g}_{m}=\left[\begin{array}{lll}\boldsymbol{g}_{m}[0] & \cdots \boldsymbol{g}_{m}[N-1]\end{array}\right]^{\top} \in \mathbb{C}^{N}$. As pointed in [2, 35], since the waveforms in OFDM have a flat spectrum, $\boldsymbol{g}$ can be approximately represented as $\boldsymbol{g}=\boldsymbol{B} \boldsymbol{h}$, where $\boldsymbol{B}$ should obey the isotropy and incoherence properties in Assumption 4.1. Then the received samples can be rewritten as

$$
\begin{equation*}
\boldsymbol{y}[j]=\left\langle\boldsymbol{b}_{j} \boldsymbol{e}_{j}^{\top}, \sum_{k=1}^{r} d_{k} \boldsymbol{h}\left(\boldsymbol{a}_{\psi_{k}} \otimes \boldsymbol{a}_{\phi_{k}}\right)^{\top}\right\rangle \tag{5.3}
\end{equation*}
$$

for $j=0, \cdots, M N-1$. It can be seen that (5.3) is a special case of (5.1) where $\boldsymbol{h}$ is independent to the two dimensional locations $\left\{\left(\psi_{k}, \phi_{k}\right)\right\}_{k=1}^{r}$.

In our numerical simulation, We set $N=13, M=9$ and $s=r=4$. Each row of $\boldsymbol{B}$ is generated from the following distribution described in [15, 35], i.e.,

$$
\boldsymbol{b}_{k}=\left[\begin{array}{llll}
1 & e^{2 \pi \imath f_{k}} & \cdots & e^{2 \pi \imath(s-1) f_{k}}
\end{array}\right]^{\top}
$$

for $k=0, \ldots, M N-1$, where $f_{k}$ is chosen uniformly at random in $[0,1]$. The locations $\left\{\left(\psi_{k}, \phi_{k}\right)\right\}_{k=1}^{r}$ are chosen uniformly at random from $[0,1) \times[0,1)$, and the coefficient vector $\boldsymbol{h}$ is generated from standard Gaussian with normalization. The data matrix $\sum_{k=1}^{r} d_{k} \boldsymbol{h}\left(\boldsymbol{a}_{\psi_{k}} \otimes \boldsymbol{a}_{\phi_{k}}\right)^{\top}$ can be firstly recovered via PGD-VHL (2D), then the locations $\left\{\left(\psi_{k}, \phi_{k}\right)\right\}_{k=1}^{r}$ are retrieved by 2D MUSIC and the channel coefficients $\left\{d_{k}\right\}_{k=1}^{r}$ are estimated by solving an overdetermined linear system [12]. The results are presented in Figure[7] It is shown that implementing PGD-VHL (2D) with 2D MUSIC can exactly recover delays and Doppler frequencies.


Figure 7: Performance of PGD-VHL (2D) for joint delay-Doppler estimation problem.

## 6 Proof of Theorem 4.1

The proof follows a well established route that has been widely used in non-convex optimization for low rank matrix recovery [9, 42, 5]. In a nutshell, the initialization provided in Algorithm 1 will be shown to lie in a basin of attraction where the sequence returned by Algorithm 1 converges linearly to the true solution. Despite this, the proof details are quite involved and substantially different. We first list two useful lemmas, whose proofs are deferred to Section 6.1 and 6.2

Lemma 6.1. Suppose $\boldsymbol{Z}^{\natural}$ is $\mu_{1}$-incoherent and $n \geq c_{0} \varepsilon^{-2} \kappa^{2} \mu_{0}^{2} \mu s^{2} r^{2} \log ^{2}(s n)$. Then one has

$$
\operatorname{dist}^{2}\left(\boldsymbol{M}_{0}, \boldsymbol{M}^{\natural}\right) \leq \frac{\varepsilon^{2} \sigma_{r}}{\mu_{0} s}
$$

with probability at least $1-(s n)^{-c_{1}}$.
Lemma 6.2. Let $\left\{\boldsymbol{M}_{t}\right\}$ be the sequence returned by Algorithm 1. Denote $\boldsymbol{\Delta}_{t}:=\boldsymbol{M}_{t}-\boldsymbol{M}^{\natural} \boldsymbol{Q}_{t}$, where

$$
Q_{t}=\arg \min _{Q Q^{H}=Q^{H} Q=I_{r}}\left\|M_{t}-M^{\natural} Q\right\|_{\mathrm{F}} .
$$

Let $\eta \leq \frac{\sigma_{r}}{9000\left(\mu_{0} \mu s r \sigma_{1}\right)^{2}}, \beta=\frac{\sigma_{r}}{72}$. Then with probability $1-(s n)^{-c}$, one has

$$
\operatorname{dist}^{2}\left(\boldsymbol{M}_{t}, \boldsymbol{M}^{\natural}\right) \leq(1-\eta \beta)^{t} \operatorname{dist}^{2}\left(\boldsymbol{M}_{0}, \boldsymbol{M}^{\natural}\right)
$$

for $t=1,2, \cdots$.
Combining Lemma 6.1 and Lemma 6.2 together, we complete the proof.

### 6.1 Proof of Lemma 6.1

We begin our presentation of the proof with a useful lemma whose proof is provided in Section 6.3
Lemma 6.3. Suppose that $Z^{\natural}$ is $\mu_{1}$-incoherent. Then with probability at least $1-(s n)^{-c_{1}}$, the matrix $\widehat{\boldsymbol{Z}}_{0}=\mathcal{P}_{r}\left(\mathcal{G D} \mathcal{A}^{*}(\boldsymbol{y})\right)$ obeys

$$
\begin{equation*}
\left\|\widehat{Z}_{0}-Z^{\natural}\right\| \leq c_{0} \sigma_{1} \sqrt{\frac{\mu_{0} \mu_{1} s r \log ^{2}(s n)}{n}} \tag{6.1}
\end{equation*}
$$

where $c_{0}$ and $c_{1}$ are absolute constants.
By Lemma 6.3 and the assumption that $n$ should be larger than $c_{0} \varepsilon^{-2} \kappa^{2} \mu_{0}^{2} \mu s^{2} r^{2} \log ^{2}(s n)$, the event

$$
\left\|\widehat{\boldsymbol{Z}}_{0}-\boldsymbol{Z}^{\natural}\right\| \leq c_{1} \sqrt{\frac{\mu_{0} \mu_{1} s r \log ^{2}(s n)}{n}} \sigma_{1}\left(\boldsymbol{Z}^{\natural}\right) \leq \varepsilon \sigma_{1}\left(\boldsymbol{Z}^{\natural}\right)
$$

occurs with probability at least $1-(s n)^{-c_{1}}$, which implies that

$$
\sigma_{1}\left(\boldsymbol{Z}^{\natural}\right) \leq \frac{\sigma_{1}\left(\widehat{\boldsymbol{Z}}_{0}\right)}{1-\varepsilon}=: \sigma \text { and } \sigma_{1}\left(\widehat{\boldsymbol{Z}}_{0}\right) \leq(1+\varepsilon) \sigma_{1} \leq 2 \sigma .
$$

By the definition of $\mathcal{M}$ in (2.10), it can be seen that $\boldsymbol{M}^{\natural} \widehat{\boldsymbol{Q}}_{0} \in \mathcal{M}$, where $\widehat{\boldsymbol{Q}}_{0}=\arg \min _{Q}\left\|\widehat{\boldsymbol{M}}_{0}-\boldsymbol{M}^{\natural} \boldsymbol{Q}\right\|_{F}$. Thus we have

$$
\begin{align*}
\operatorname{dist}\left(\boldsymbol{M}_{0}, \boldsymbol{M}^{\natural}\right) & =\min _{Q}\left\|\boldsymbol{M}_{0}-\boldsymbol{M}^{\natural} \boldsymbol{Q}\right\|_{F} \\
& \leq\left\|\boldsymbol{M}_{0}-\boldsymbol{M}^{\natural} \widehat{\boldsymbol{Q}}_{0}\right\|_{F} \\
& =\left\|\mathcal{P}_{\mathcal{M}}\left(\widehat{\boldsymbol{M}}_{0}-\boldsymbol{M}^{\natural} \widehat{\boldsymbol{Q}}_{0}\right)\right\|_{F} \\
& \leq\left\|\widehat{\boldsymbol{M}}_{0}-\boldsymbol{M}^{\natural} \widehat{\boldsymbol{Q}}_{0}\right\|_{F} \\
& =\operatorname{dist}\left(\widehat{\boldsymbol{M}}_{0}, \boldsymbol{M}^{\natural}\right) . \tag{6.2}
\end{align*}
$$

To complete the proof, it suffices to control $\operatorname{dist}\left(\widehat{\boldsymbol{M}}_{0}, \boldsymbol{M}^{\natural}\right)$. A straightforward computation yields that

$$
\begin{aligned}
\operatorname{dist}^{2}\left(\widehat{\boldsymbol{M}}_{0}, \boldsymbol{M}^{\natural}\right) & \stackrel{(a)}{\leq} \frac{1}{2(\sqrt{2}-1) \sigma_{r}^{2}\left(\boldsymbol{M}^{\natural}\right)}\left\|\widehat{\boldsymbol{M}}_{0} \widehat{\boldsymbol{M}}_{0}^{\mathrm{H}}-\boldsymbol{M}^{\natural} \boldsymbol{M}^{\natural^{\mathrm{H}}}\right\|_{\mathrm{F}}^{2} \\
& \stackrel{(b)}{=} \frac{1}{4(\sqrt{2}-1) \sigma_{r}}\left\|\widehat{\boldsymbol{M}}_{0} \widehat{\boldsymbol{M}}_{0}^{\mathrm{H}}-\boldsymbol{M}^{\natural} \boldsymbol{M}^{\natural}\right\|_{\mathrm{F}}^{\mathrm{H}} \|^{2} \\
& \stackrel{(c)}{\leq} \frac{1}{4(\sqrt{2}-1) \sigma_{r}} \cdot 4\left\|\widehat{\boldsymbol{Z}}_{0}-\boldsymbol{Z}^{\natural}\right\|_{\mathrm{F}}^{2} \\
& \leq \frac{1}{(\sqrt{2}-1) \sigma_{r}} \cdot 2 r\left\|\widehat{\boldsymbol{Z}}_{0}-\boldsymbol{Z}^{\natural}\right\|^{2} \\
& \leq \frac{2 r}{(\sqrt{2}-1) \sigma_{r}} \cdot \frac{\mu_{0} \mu s r \log ^{2}(s n)}{n} \sigma_{1}^{2} \\
& \leq \frac{\varepsilon^{2} \sigma_{r}}{\mu_{0} s}
\end{aligned}
$$

provided that $n \geq c_{0} \varepsilon^{-2} \kappa^{2} \mu_{0}^{2} \mu s^{2} r^{2} \log ^{2}(s n)$, where step (a) is due to Lemma 5.4 in [34], step (b) has used the fact $\sigma_{r}\left(\boldsymbol{M}^{\natural}\right)=\sqrt{2} \sigma_{r}$, and the step (c) can be derived as follows.

Let $\boldsymbol{A}_{1}, \boldsymbol{B}_{1} \in \mathbb{C}^{s n_{1} \times r}$ and $\boldsymbol{A}_{2}, \boldsymbol{B}_{2} \in \mathbb{C}^{n_{2} \times r}$ be four complex matrices. It is direct to obtain that

$$
\begin{align*}
\left\langle\boldsymbol{A}_{1} \boldsymbol{A}_{1}^{\mathrm{H}}, \boldsymbol{B}_{1} \boldsymbol{B}_{1}^{\mathrm{H}}\right\rangle+\left\langle\boldsymbol{A}_{2} \boldsymbol{A}_{2}^{\mathrm{H}}, \boldsymbol{B}_{2} \boldsymbol{B}_{2}^{\mathrm{H}}\right\rangle & =\left\langle\boldsymbol{A}_{1}^{\mathrm{H}} \boldsymbol{B}_{1}, \boldsymbol{A}_{1}^{\mathrm{H}} \boldsymbol{B}_{1}\right\rangle+\left\langle\boldsymbol{A}_{2}^{\mathrm{H}} \boldsymbol{B}_{2}, \boldsymbol{A}_{2}^{\mathrm{H}} \boldsymbol{B}_{2}\right\rangle \\
& \geq 2 \Re\left(\left\langle\boldsymbol{A}_{1} \boldsymbol{A}_{2}^{\mathrm{H}}, \boldsymbol{B}_{1} \boldsymbol{B}_{2}^{\mathrm{H}}\right\rangle\right) . \tag{6.3}
\end{align*}
$$

Then a simple calculation yields that

$$
\left\|\widehat{\boldsymbol{M}}_{0} \widehat{\boldsymbol{M}}_{0}^{\mathrm{H}}-\boldsymbol{M}^{\natural} \boldsymbol{M}^{\mathrm{h}}\right\|_{\mathrm{F}}^{2}=2\left\|\widehat{\boldsymbol{U}}_{0} \widehat{\boldsymbol{\Sigma}}_{0} \widehat{\boldsymbol{V}}_{0}^{\mathrm{H}}-\boldsymbol{U} \boldsymbol{\Sigma} \boldsymbol{V}^{\mathrm{H}}\right\|_{\mathrm{F}}^{2}+\left\|\widehat{\boldsymbol{U}}_{0} \widehat{\boldsymbol{\Sigma}}_{0} \widehat{\boldsymbol{U}}_{0}^{\mathrm{H}}-\boldsymbol{U} \boldsymbol{\Sigma} \boldsymbol{U}^{\mathrm{H}}\right\|_{\mathrm{F}}^{2}+\left\|\widehat{\boldsymbol{V}}_{0} \widehat{\boldsymbol{\Sigma}}_{0} \widehat{\boldsymbol{V}}_{0}^{\mathrm{H}}-\boldsymbol{V} \boldsymbol{\Sigma} \boldsymbol{V}^{\mathrm{H}}\right\|_{\mathrm{F}}^{2} .
$$

Denote $\boldsymbol{A}_{1}=\widehat{\boldsymbol{U}}_{0} \widehat{\boldsymbol{\Sigma}}_{0}^{1 / 2}, \boldsymbol{B}_{1}=\boldsymbol{U} \boldsymbol{\Sigma}^{1 / 2}, \boldsymbol{A}_{2}=\widehat{\boldsymbol{V}}_{0} \widehat{\boldsymbol{\Sigma}}_{0}^{1 / 2}$ and $\boldsymbol{B}_{2}=\boldsymbol{V} \boldsymbol{\Sigma}^{1 / 2}$. Applying (6.3) can be easily verified that

$$
\left\|\widehat{\boldsymbol{U}}_{0} \widehat{\boldsymbol{\Sigma}}_{0} \widehat{\boldsymbol{U}}_{0}^{\mathrm{H}}-\boldsymbol{U} \boldsymbol{\Sigma} \boldsymbol{U}^{\mathrm{H}}\right\|_{\mathrm{F}}^{2}+\left\|\widehat{\boldsymbol{V}}_{0} \widehat{\boldsymbol{\Sigma}}_{0} \widehat{\boldsymbol{V}}_{0}^{\mathrm{H}}-\boldsymbol{V} \boldsymbol{\Sigma} \boldsymbol{V}^{\mathrm{H}}\right\|_{\mathrm{F}}^{2} \leq 2\left\|\widehat{\boldsymbol{U}}_{0} \widehat{\boldsymbol{\Sigma}}_{0} \widehat{\boldsymbol{V}}_{0}^{\mathrm{H}}-\boldsymbol{U} \boldsymbol{\Sigma} \boldsymbol{V}^{\mathrm{H}}\right\|_{\mathrm{F}}^{2} .
$$

Hence

$$
\begin{aligned}
\left\|\widehat{\boldsymbol{M}}_{0} \widehat{\boldsymbol{M}}_{0}^{\mathrm{H}}-\boldsymbol{M}^{\natural} \boldsymbol{M}^{\natural^{\mathrm{H}}}\right\|_{\mathrm{F}}^{2} & \leq 4\left\|\widehat{\boldsymbol{U}}_{0} \widehat{\boldsymbol{\Sigma}}_{0} \widehat{\boldsymbol{V}}_{0}^{\mathrm{H}}-\boldsymbol{U} \boldsymbol{\Sigma} \boldsymbol{V}^{\mathrm{H}}\right\|_{\mathrm{F}}^{2} \\
& =4\left\|\widehat{\boldsymbol{Z}}_{0}-\boldsymbol{Z}^{\natural}\right\|_{\mathrm{F}}^{2} .
\end{aligned}
$$

Thus we complete the proof of Lemma 6.1.

### 6.2 Proof of Lemma 6.2

We first establish a regularity condition in the following lemma whose proof is provided in Section 6.4
Lemma 6.4. Let $\boldsymbol{\Delta}=\boldsymbol{M}-\boldsymbol{M}^{\natural} \boldsymbol{Q}$, where $\boldsymbol{Q}=\arg \min _{\boldsymbol{Q} \boldsymbol{Q}^{H}=\boldsymbol{Q}^{\dagger} \boldsymbol{Q}=\boldsymbol{I}_{r}}\left\|\boldsymbol{M}-\boldsymbol{M}^{\natural} \boldsymbol{Q}\right\|_{\mathrm{F}}$. Then one has

$$
\begin{equation*}
\Re(\langle\nabla f(\boldsymbol{M}), \boldsymbol{\Delta}\rangle) \geq \frac{\eta}{2}\|\nabla f(\boldsymbol{M})\|_{\mathrm{F}}^{2}+\frac{\beta}{2}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2} \tag{6.4}
\end{equation*}
$$

happens with high probability for all $\boldsymbol{M}$ such that $\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2} \leq \frac{\varepsilon^{2} \sigma_{r}}{\mu_{0} s}$, where $\eta \leq \frac{\sigma_{r}}{9000\left(\mu_{0} \mu s r \sigma_{1}\right)^{2}}$, and $\beta=\frac{\sigma_{r}}{72}$.

Let $\widehat{\boldsymbol{M}}_{t+1}=\boldsymbol{M}_{t}-\eta \nabla f\left(\boldsymbol{M}_{t}\right)$. Under the condition of (6.4), one has

$$
\begin{aligned}
\operatorname{dist}^{2}\left(\boldsymbol{M}_{t+1}, \boldsymbol{M}^{\natural}\right) & \stackrel{(a)}{\leq} \operatorname{dist}^{2}\left(\widehat{\boldsymbol{M}}_{t+1}, \boldsymbol{M}^{\natural}\right) \\
& \leq\left\|\widehat{\boldsymbol{M}}_{t+1}-\boldsymbol{M}^{\natural} \boldsymbol{Q}_{t}\right\|_{\mathrm{F}}^{2} \\
& =\left\|\boldsymbol{\Delta}_{t}-\eta \nabla f\left(\boldsymbol{M}_{t}\right)\right\|_{\mathrm{F}}^{2} \\
& =\left\|\boldsymbol{\Delta}_{t}\right\|_{\mathrm{F}}^{2}+\eta^{2}\left\|\nabla f\left(\boldsymbol{M}_{t}\right)\right\|_{\mathrm{F}}^{2}-2 \eta \Re\left(\left\langle\boldsymbol{\Delta}_{t}, \nabla f\left(\boldsymbol{M}_{t}\right)\right\rangle\right) \\
& \stackrel{(b)}{\leq}\left\|\boldsymbol{\Delta}_{t}\right\|_{\mathrm{F}}^{2}+\eta^{2}\left\|\nabla f\left(\boldsymbol{M}_{t}\right)\right\|_{\mathrm{F}}^{2}-\eta^{2}\left\|\nabla f\left(\boldsymbol{M}_{t}\right)\right\|_{\mathrm{F}}^{2}-\beta \eta\left\|\boldsymbol{\Delta}_{t}\right\|_{\mathrm{F}}^{2} \\
& =(1-\eta \beta)\left\|\boldsymbol{\Delta}_{t}\right\|_{\mathrm{F}}^{2},
\end{aligned}
$$

where step (a) follows the same argument as (6.2) and step (b) is due to (6.4). Then a little algebra yields that

$$
\begin{aligned}
\operatorname{dist}^{2}\left(\boldsymbol{M}_{t+1}, \boldsymbol{M}^{\natural}\right) & \leq(1-\eta \beta)^{t+1} \operatorname{dist}^{2}\left(\boldsymbol{M}_{0}, \boldsymbol{M}^{\natural}\right) \\
& \leq(1-\eta \beta)^{t+1} \frac{\varepsilon^{2} \sigma_{r}}{\mu_{0} s}
\end{aligned}
$$

which completes the proof of Lemma 6.2,

### 6.3 Proof of Lemma 6.3

Notice that $\mathcal{G D} \mathcal{A}^{*}(\boldsymbol{y})=\mathcal{G} \mathcal{A}^{*} \boldsymbol{D}(\boldsymbol{y})=\mathcal{G} \mathcal{A}^{*} \mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right)$. A simple computation yields that

$$
\begin{aligned}
\mathbb{E}\left[\mathcal{G A}^{*} \mathcal{A G}^{*}\left(\boldsymbol{Z}^{\natural}\right)\right] & =\mathbb{E}\left[\mathcal{G}\left(\sum_{i=0}^{n-1}\left\langle\boldsymbol{b}_{i} \boldsymbol{e}_{i}^{\top}, \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right)\right\rangle \boldsymbol{b}_{i} \boldsymbol{e}_{i}^{\top}\right)\right] \\
& =\mathcal{G}\left(\sum_{i=0}^{n-1} \mathbb{E}\left[\boldsymbol{b}_{i} \boldsymbol{b}_{i}^{\mathrm{H}} \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right) \boldsymbol{e}_{i} \boldsymbol{e}_{i}^{\top}\right]\right) \\
& =\mathcal{G} \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right)=\boldsymbol{Z}^{\natural},
\end{aligned}
$$

where the third equality is due to the isotropy property of $\left\{\boldsymbol{b}_{i}\right\}_{i=0}^{n-1}$. Let us first bound

$$
\left\|\mathcal{G} \mathcal{A}^{*} \mathcal{A G}^{*}\left(\boldsymbol{Z}^{\natural}\right)-\mathbb{E}\left[\mathcal{G} \mathcal{A}^{*} \mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right)\right]\right\|
$$

by the matrix Bernstein inequality (6.29). The matrix $\mathcal{G} \mathcal{A}^{*} \mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right)-\mathbb{E}\left[\mathcal{G} \mathcal{A}^{*} \mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right)\right]$ can be rewritten as

$$
\begin{aligned}
\mathcal{G A}^{*} \mathcal{A \mathcal { G }}^{*}\left(\boldsymbol{Z}^{\natural}\right)-\mathbb{E}\left[\mathcal{G \mathcal { A }}^{*} \mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right)\right] & =\sum_{i=0}^{n-1} \mathcal{G}\left(\left(\boldsymbol{b}_{i} \boldsymbol{b}_{i}^{\mathrm{H}}-\boldsymbol{I}_{s}\right) \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right) \boldsymbol{e}_{i} \boldsymbol{e}_{i}^{\top}\right) \\
& =\sum_{i=0}^{n-1} \boldsymbol{G}_{i} \otimes\left(\left(\boldsymbol{b}_{i} \boldsymbol{b}_{i}^{\mathrm{H}}-\boldsymbol{I}_{s}\right) \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right) \boldsymbol{e}_{i}\right) \\
& =: \sum_{i=0}^{n-1} \boldsymbol{Y}_{i} .
\end{aligned}
$$

Notice that $\left\{\boldsymbol{Y}_{i}\right\}_{i=0}^{n-1}$ are independent mean-zero random matrices with

$$
\begin{aligned}
\left\|\boldsymbol{Y}_{i}\right\| & =\left\|\boldsymbol{G}_{i} \otimes\left(\left(\boldsymbol{b}_{i} b_{i}^{\mathrm{H}}-\boldsymbol{I}_{s}\right) \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right) \boldsymbol{e}_{i}\right)\right\| \\
& \leq\left\|\boldsymbol{G}_{i}\right\| \cdot\left\|\left(\boldsymbol{b}_{i} b_{i}^{\mathrm{H}}-\boldsymbol{I}_{s}\right) \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right) \boldsymbol{e}_{i}\right\|
\end{aligned}
$$

$$
\begin{aligned}
& \leq \frac{1}{\sqrt{\omega_{i}}} \cdot \max \left\{\left\|\boldsymbol{b}_{i}\right\|_{2}^{2}, 1\right\} \cdot\left\|\mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right) \boldsymbol{e}_{i}\right\|_{2} \\
& \stackrel{(a)}{\leq} s \mu_{0} \cdot \max _{i} \frac{1}{\sqrt{\omega_{i}}}\left\|\mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right) \boldsymbol{e}_{i}\right\|_{2} \\
& \stackrel{(b)}{\leq} s \mu_{0} \cdot \frac{\mu_{1} r}{n} \sigma_{1},
\end{aligned}
$$

where step (a) follows from (4.2) and step (b) is due to Lemma 6.6. Moreover, letting $\boldsymbol{w}_{i}:=\left(\boldsymbol{b}_{i} \boldsymbol{b}_{i}^{\mathrm{H}}-\right.$ $\left.\boldsymbol{I}_{s}\right) \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right) \boldsymbol{e}_{i}$, we have

$$
\begin{aligned}
\left\|\mathbb{E}\left[\sum_{i=0}^{n-1} \boldsymbol{Y}_{i}^{\mathrm{H}} \boldsymbol{Y}_{i}\right]\right\| & =\left\|\sum_{i=0}^{n-1} \mathbb{E}\left[\left(\boldsymbol{G}_{i} \otimes \boldsymbol{w}_{i}\right)^{\mathrm{H}}\left(\boldsymbol{G}_{i} \otimes \boldsymbol{w}_{i}\right)\right]\right\| \\
& =\left\|\sum_{i=0}^{n-1}\left(\boldsymbol{G}_{i}^{\mathrm{H}} \boldsymbol{G}_{i}\right) \cdot \mathbb{E}\left[\left\|\boldsymbol{w}_{i}\right\|_{2}^{2}\right]\right\| \\
& \leq \sum_{i=0}^{n-1}\left\|\boldsymbol{G}_{i}^{\mathrm{H}} \boldsymbol{G}_{i}\right\| \cdot \mathbb{E}\left[\left\|\boldsymbol{w}_{i}\right\|_{2}^{2}\right] \\
& \leq(a) \\
& \sum_{i=0}^{n-1} \frac{1}{w_{i}} \cdot s \mu_{0}\left\|\mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right) \boldsymbol{e}_{i}\right\|_{2}^{2} \\
& \leq s \mu_{0} \cdot \sqrt{\frac{\mu_{1} r \log (s n)}{n}} \sigma_{1},
\end{aligned}
$$

where step (a) follows from $\left\|\boldsymbol{G}_{i}\right\| \leq 1 / \sqrt{w_{i}}$ and the fact $\mathbb{E}\left[\left\|\boldsymbol{w}_{i}\right\|_{2}^{2}\right] \leq s \mu_{0} \cdot\left\|\mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right) \boldsymbol{e}_{i}\right\|_{2}^{2}$, and step (b) is due to Lemma 6.6. Moreover, the fact used in step (a) can be proved as follows:

$$
\begin{aligned}
\mathbb{E}\left[\left\|\boldsymbol{w}_{i}\right\|_{2}^{2}\right] & =\mathbb{E}\left[\boldsymbol{e}_{i}^{\mathrm{T}}\left(\mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right)\right)^{\mathrm{H}}\left(\boldsymbol{b}_{i} \boldsymbol{b}_{i}^{\mathrm{H}}-\boldsymbol{I}_{s}\right)^{2} \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right) \boldsymbol{e}_{i}\right] \\
& =\boldsymbol{e}_{i}^{\top}\left(\mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right)\right)^{\mathrm{H}} \mathbb{E}\left[\left(\boldsymbol{b}_{i} \boldsymbol{b}_{i}^{\mathrm{H}}-\boldsymbol{I}_{s}\right)^{2}\right] \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right) \boldsymbol{e}_{i} \\
& =\boldsymbol{e}_{i}^{\top}\left(\mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right)\right)^{\mathrm{H}} \mathbb{E}\left[\left\|\boldsymbol{b}_{i}\right\|_{2}^{2} \boldsymbol{b}_{i} \boldsymbol{b}_{i}^{\mathrm{H}}-\boldsymbol{I}_{s}\right] \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right) \boldsymbol{e}_{i} \\
& \leq\left(\mu_{0} s-1\right) \cdot\left\|\mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right) \boldsymbol{e}_{i}\right\|_{2}^{2} \\
& \leq \mu_{0} s\left\|\mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right) \boldsymbol{e}_{i}\right\|_{2}^{2},
\end{aligned}
$$

where the third line is due to (4.1). Similarly, one has

$$
\begin{aligned}
\left\|\mathbb{E}\left[\sum_{i=0}^{n-1} \boldsymbol{Y}_{i} \boldsymbol{Y}_{i}^{\mathrm{H}}\right]\right\| & \leq \sum_{i=0}^{n-1}\left\|\boldsymbol{G}_{i}\right\|^{2} \cdot\left\|\mathbb{E}\left[\boldsymbol{w}_{i} \boldsymbol{w}_{i}^{\mathrm{H}}\right]\right\| \\
& \leq \sum_{i=0}^{n-1}\left\|\boldsymbol{G}_{i}\right\|^{2} \cdot \mathbb{E}\left[\left\|\boldsymbol{w}_{i}\right\|_{2}^{2}\right] \\
& \leq s \mu_{0} \cdot \sqrt{\frac{\mu_{1} r \log (s n)}{n}} \sigma_{1}
\end{aligned}
$$

Applying the matrix Bernstein inequality (6.29) shows that, with probability greater than $1-(s n)^{-c_{1}}$,

$$
\left\|\mathcal{G} \mathcal{A}^{*} \mathcal{A G}^{*}\left(\boldsymbol{Z}^{\natural}\right)-\mathbb{E}\left[\mathcal{G} \mathcal{A}^{*} \mathcal{A G}^{*}\left(\boldsymbol{Z}^{\natural}\right)\right]\right\| \leq c_{1} \sqrt{s \mu_{0} \log (s n)} \cdot \sqrt{\frac{\mu_{1} r \log (s n)}{n}} \sigma_{1}+c_{2} s \mu_{0} \log (s n) \cdot \frac{\mu_{1} r}{n} \sigma_{1}
$$

$$
\leq c_{3} \sqrt{\frac{\mu_{0} \mu_{1} s r \log ^{2}(s n)}{n}} \sigma_{1}
$$

provided $n \geq c_{0} \mu_{0} \mu s r \log ^{2}(s n)$. Therefore, the event

$$
\begin{aligned}
&\left\|\widehat{\boldsymbol{Z}}_{0}-\boldsymbol{Z}^{\natural}\right\|=\left\|\mathcal{P}_{r}\left(\mathcal{G} \mathcal{A} \mathcal{A}^{*} \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right)\right)-\boldsymbol{Z}^{\natural}\right\| \\
& \leq\left\|\mathcal{P}_{r}\left(\mathcal{G} \mathcal{A} \mathcal{A}^{*} \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right)\right)-\mathcal{G} \mathcal{A}^{*} \mathcal{A \mathcal { G }}^{*}\left(\boldsymbol{Z}^{\natural}\right)\right\| \\
& \quad+\left\|\mathcal{G} \mathcal{A}^{*} \mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right)-\boldsymbol{Z}^{\natural}\right\| \\
& \leq 2\left\|\mathcal{G} \mathcal{A}^{*} \mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{Z}^{\natural}\right)-\boldsymbol{Z}^{\natural}\right\| \\
& \leq c_{0} \sqrt{\frac{\mu_{0} \mu s r \log ^{2}(s n)}{n}} \sigma_{1} .
\end{aligned}
$$

occurs with probability at least $1-(s n)^{-c_{1}}$. Finally we complete the proof.

### 6.4 Proof of Lemma 6.4

The proof includes two parts. We will show that

$$
\begin{equation*}
\Re(\langle\nabla f(\boldsymbol{M}), \boldsymbol{\Delta}\rangle) \geq \frac{1}{72} \sigma_{r}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+\frac{1}{8}\left\|\boldsymbol{M}^{\mathrm{h}^{\mathrm{H}}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2} \tag{6.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\|\nabla f(\boldsymbol{M})\|_{\mathrm{F}}^{2} \leq 125\left(\mu_{0} \mu s r \sigma_{1}\right)^{2}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+\frac{1}{2} \sigma_{1}\left\|\boldsymbol{M}^{\natural^{\mathrm{H}}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2} \tag{6.6}
\end{equation*}
$$

provided $\varepsilon \leq \frac{1}{3}$. By the assumption

$$
\eta \leq \frac{\sigma_{r}}{9000\left(\mu_{0} \mu s r \sigma_{1}\right)^{2}} \leq \frac{1}{2 \sigma_{1}} \text { and } \beta=\frac{\sigma_{r}}{72}
$$

we have

$$
\begin{aligned}
\frac{\eta}{2}\|\nabla f(\boldsymbol{M})\|_{\mathrm{F}}^{2}+\frac{\beta}{2}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2} & \leq\left(\frac{1}{2} \frac{\sigma_{r}}{9000\left(\mu_{0} \mu s r \sigma_{1}\right)^{2}} \cdot 125\left(\mu_{0} \mu s r \sigma_{1}\right)^{2}+\frac{1}{2} \frac{\sigma_{r}}{72}\right)\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+\frac{1}{2} \cdot \frac{1}{2 \sigma_{1}} \cdot \frac{1}{2} \sigma_{1}\left\|\boldsymbol{M}^{\natural^{\mathrm{H}}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2} \\
& \leq \frac{\sigma_{r}}{72}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+\frac{1}{8}\left\|\boldsymbol{M}^{\mathrm{\natural}}{ }^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2} \\
& \leq \Re(\langle\nabla f(\boldsymbol{M}), \boldsymbol{\Delta}\rangle) .
\end{aligned}
$$

### 6.4.1 Proof of (6.5)

Let $\nabla f_{1}, \nabla f_{2}$ and $\nabla f_{3}$ be the matrices given by

$$
\begin{aligned}
\nabla f_{1}= & {\left[\begin{array}{l}
\left(\mathcal{G} \mathcal{A}^{*}\left(\mathcal{A G}^{*}\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}\right)-\boldsymbol{D} \boldsymbol{y}\right)\right) \boldsymbol{R} \\
\left(\mathcal{G A}^{*}\left(\mathcal{A G}^{*}\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}\right)-\boldsymbol{D} \boldsymbol{y}\right)\right)^{\mathrm{H}} \boldsymbol{L}
\end{array}\right], } \\
\nabla f_{2}= & =\left[\begin{array}{l}
\left(\left(\mathcal{I}-\mathcal{G} \mathcal{G}^{*}\right)\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}\right)\right) \boldsymbol{R} \\
\left(\left(\boldsymbol{\mathcal { I }}-\mathcal{G \mathcal { G }}^{*}\right)\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}\right)\right)^{\mathrm{H}} \boldsymbol{L}
\end{array}\right], \\
\nabla f_{3}= & \frac{1}{4}\left[\begin{array}{l}
\boldsymbol{L}\left(\boldsymbol{L}^{\mathrm{H}} \boldsymbol{L}-\boldsymbol{R}^{\mathrm{H}} \boldsymbol{R}\right) \\
\boldsymbol{R}\left(\boldsymbol{R}^{\mathrm{H}} \boldsymbol{R}-\boldsymbol{L}^{\mathrm{H}} \boldsymbol{L}\right)
\end{array}\right] .
\end{aligned}
$$

A straightforward computation yields that

$$
\Re(\langle\nabla f(\boldsymbol{M}), \boldsymbol{\Delta}\rangle)=\Re\left(\left\langle\nabla f_{1}, \boldsymbol{\Delta}\right\rangle\right)+\Re\left(\left\langle\nabla f_{2}, \boldsymbol{\Delta}\right\rangle\right)+\Re\left(\left\langle\nabla f_{3}, \boldsymbol{\Delta}\right\rangle\right)
$$

We will bound these three terms separately. For the sake of simplification, let $\boldsymbol{\Delta}=\left[\begin{array}{ll}\boldsymbol{\Delta}_{\boldsymbol{L}}^{H} & \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}}\end{array}\right]^{\mathrm{H}}$ where $\boldsymbol{\Delta}_{\boldsymbol{L}}=\boldsymbol{L}-\boldsymbol{L}^{\natural} \boldsymbol{Q} \in \mathbb{C}^{s n_{1} \times r}$ and $\boldsymbol{\Delta}_{\boldsymbol{R}}=\boldsymbol{R}-\boldsymbol{R}^{\natural} \boldsymbol{Q} \in \mathbb{C}^{n_{2} \times r}$.

Bounding $\Re\left(\left\langle\nabla f_{1}, \boldsymbol{\Delta}\right\rangle\right) \quad$ By applying $\boldsymbol{D} \boldsymbol{y}=\mathcal{A G}^{*}\left(\boldsymbol{Z}^{\natural}\right)$, we can rewrite $\Re\left(\left\langle\nabla f_{1}, \boldsymbol{\Delta}\right\rangle\right)$ as

$$
\begin{equation*}
\Re\left(\left\langle\nabla f_{1}, \boldsymbol{\Delta}\right\rangle\right)=\Re\left(\left\langle\mathcal{G} \mathcal{A}^{*} \mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}-\boldsymbol{L}^{\natural} \boldsymbol{R}^{\natural} \mathrm{H}\right), \boldsymbol{\Delta}_{\boldsymbol{L}} \boldsymbol{R}^{\mathrm{H}}+\boldsymbol{L} \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}}\right\rangle\right) . \tag{6.7}
\end{equation*}
$$

Notice that

$$
\begin{align*}
\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}-\boldsymbol{L}^{\natural} \boldsymbol{R}^{\natural}{ }^{\mathrm{H}} & =\left(\boldsymbol{\Delta}_{\boldsymbol{L}}+\boldsymbol{L}^{\natural} \boldsymbol{Q}\right)\left(\boldsymbol{\Delta}_{\boldsymbol{R}}+\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}}-\boldsymbol{L}^{\natural} \boldsymbol{R}^{\natural^{\mathrm{H}}} \\
& =\underbrace{\boldsymbol{\Delta}_{\boldsymbol{L}} \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}}}_{:=\Psi}+\underbrace{\boldsymbol{\Delta}_{\boldsymbol{L}}\left(\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}}+\left(\boldsymbol{L}^{\natural} \boldsymbol{Q}\right) \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}}}_{:=\boldsymbol{\Phi}} . \tag{6.8}
\end{align*}
$$

and

$$
\begin{align*}
\boldsymbol{\Delta}_{\boldsymbol{L}} \boldsymbol{R}^{\mathrm{H}}+\boldsymbol{L} \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}} & =\boldsymbol{\Delta}_{\boldsymbol{L}}\left(\boldsymbol{\Delta}_{\boldsymbol{R}}+\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}}+\left(\boldsymbol{\Delta}_{\boldsymbol{L}}+\boldsymbol{L}^{\natural} \boldsymbol{Q}\right) \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}} \\
& =2 \boldsymbol{\Delta}_{\boldsymbol{L}} \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}}+\boldsymbol{\Delta}_{\boldsymbol{L}}\left(\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}}+\left(\boldsymbol{L}^{\natural} \boldsymbol{Q}\right) \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}} \\
& =2 \boldsymbol{\Psi}+\boldsymbol{\Phi} . \tag{6.9}
\end{align*}
$$

Then $\Re\left(\left\langle\nabla f_{1}, \boldsymbol{\Delta}\right\rangle\right)$ can be bounded as follows:

$$
\begin{align*}
& \Re\left(\left\langle\nabla f_{1}, \boldsymbol{\Delta}\right\rangle\right)=\Re\left(\left\langle\mathcal{A G}^{*}(\mathbf{\Phi}+\mathbf{\Psi}), \mathcal{A G}^{*}(\mathbf{\Phi}+2 \boldsymbol{\Psi})\right\rangle\right) \\
& =\left\|\mathcal{A G}^{*}(\mathbf{\Phi})\right\|_{2}^{2}+2\left\|\mathcal{A} \mathcal{G}^{*}(\mathbf{\Psi})\right\|_{2}^{2}+3 \Re\left(\left\langle\mathcal{A G}^{*}(\mathbf{\Phi}), \mathcal{A G}^{*}(\mathbf{\Psi})\right\rangle\right) \\
& \geq\left\|\mathcal{A G}^{*}(\mathbf{\Phi})\right\|_{2}^{2}+2\left\|\mathcal{A \mathcal { G } ^ { * }}(\mathbf{\Psi})\right\|_{2}^{2}-3\left|\left\langle\mathcal{A G}^{*}(\mathbf{\Phi}), \mathcal{A G}^{*}(\mathbf{\Psi})\right\rangle\right| \\
& \geq\left\|\mathcal{A G}^{*}(\mathbf{\Phi})\right\|_{2}^{2}+2\left\|\mathcal{A} \mathcal{G}^{*}(\mathbf{\Psi})\right\|_{2}^{2}-3\left\|\mathcal{A G}^{*}(\mathbf{\Phi})\right\|_{2} \cdot\left\|\mathcal{A} \mathcal{G}^{*}(\mathbf{\Psi})\right\|_{2} \\
& \geq \frac{3}{4}\left\|\mathcal{A G}^{*}(\mathbf{\Phi})\right\|_{2}^{2}-7\left\|\mathcal{A} \mathcal{G}^{*}(\mathbf{\Psi})\right\|_{2}^{2} \\
& \geq \frac{3}{4}\left\|\mathcal{A G}^{*}(\boldsymbol{\Phi})\right\|_{2}^{2}-\frac{7 \varepsilon^{2}}{4} \sigma_{r}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}, \tag{6.10}
\end{align*}
$$

where the second line is due to $\langle\boldsymbol{x}, \boldsymbol{x}\rangle=\|\boldsymbol{x}\|_{2}^{2}$ and $\Re(\langle\boldsymbol{x}, \boldsymbol{y}\rangle)=\Re(\langle\boldsymbol{y}, \boldsymbol{x}\rangle)$ for any vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^{n}$, the third line is due to $\Re(\langle\boldsymbol{x}, \boldsymbol{y}\rangle) \leq|\langle\boldsymbol{x}, \boldsymbol{y}\rangle|$, and the last second line is due to $a+2 b-3 \sqrt{a b} \geq \frac{3}{4} a-7 b$ for any $a, b \geq 0$ and the last line is due to the fact $\left\|\mathcal{A} \mathcal{G}^{*}(\boldsymbol{\Psi})\right\|_{2}^{2} \leq \frac{\varepsilon^{2}}{4} \sigma_{r}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}$. Moreover the fact used in the last line can be proved as follows:

$$
\begin{align*}
\left\|\mathcal{A G}^{*}(\boldsymbol{\Psi})\right\|_{2}^{2} & =\left\|\mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{\Delta}_{\boldsymbol{L}} \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}}\right)\right\|_{2}^{2} \\
& \leq\left(\|\mathcal{A}\|\left\|\mathcal{G}^{*}\right\| \cdot\left\|\boldsymbol{\Delta}_{\boldsymbol{L}} \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}}\right\|_{\mathrm{F}}\right)^{2} \\
& \leq \mu_{0} s \cdot \frac{1}{4}\left(\left\|\boldsymbol{\Delta}_{\boldsymbol{L}}\right\|_{\mathrm{F}}^{2}+\left\|\boldsymbol{\Delta}_{\boldsymbol{R}}\right\|_{\mathrm{F}}^{2}\right)^{2} \\
& =\frac{\mu_{0} s}{4}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{4} \\
& \leq \frac{\varepsilon^{2}}{4} \sigma_{r}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2} \tag{6.11}
\end{align*}
$$

where the second line is due to $\|\mathcal{A}\| \leq \sqrt{\mu_{0} s},\left\|\mathcal{G}^{*}\right\| \leq 1$ and the last line is due to $\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2} \leq \frac{\varepsilon^{2} \sigma_{r}}{\mu_{0} s}$. Plugging (6.10) into (6.7) reveals that

$$
\begin{equation*}
\Re\left(\left\langle\nabla f_{1}, \boldsymbol{\Delta}\right\rangle\right) \geq \frac{3}{4}\left\|\mathcal{A} \mathcal{G}^{*}(\boldsymbol{\Phi})\right\|_{2}^{2}-\frac{7 \varepsilon^{2}}{4} \sigma_{r}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2} \tag{6.12}
\end{equation*}
$$

Bounding $\Re\left(\left\langle\nabla f_{2}, \boldsymbol{\Delta}\right\rangle\right)$ This term can be bounded as follows:

$$
\Re\left(\left\langle\nabla f_{2}, \boldsymbol{\Delta}\right\rangle\right)=\Re\left(\left\langle\left(\mathcal{I}-\mathcal{G} \mathcal{G}^{*}\right)\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}\right), \boldsymbol{\Delta}_{\boldsymbol{L}} \boldsymbol{R}^{\mathrm{H}}+\boldsymbol{L} \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}}\right\rangle\right)
$$

$$
\begin{align*}
= & \Re\left(\left\langle\left(\mathcal{I}-\mathcal{G G}^{*}\right)\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}-\boldsymbol{L}^{\natural} \boldsymbol{R}^{\mathrm{\natural}^{\mathrm{H}}}\right), \boldsymbol{\Delta}_{\boldsymbol{L}} \boldsymbol{R}^{\mathrm{H}}+\boldsymbol{L} \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}}\right\rangle\right) \\
= & \Re\left(\left\langle\left(\mathcal{I}-\mathcal{G G}^{*}\right)(\boldsymbol{\Phi}+\boldsymbol{\Psi}),(\boldsymbol{\Phi}+2 \boldsymbol{\Psi})\right\rangle\right) \\
\geq & \left\|\left(\mathcal{I}-\mathcal{G G}^{*}\right)(\boldsymbol{\Phi})\right\|_{\mathrm{F}}^{2}+2\left\|\left(\mathcal{I}-\mathcal{G} \mathcal{G}^{*}\right)(\boldsymbol{\Psi})\right\|_{\mathrm{F}}^{2} \\
& \quad-3\left\|\left(\mathcal{I}-\mathcal{G} \mathcal{G}^{*}\right)(\boldsymbol{\Phi})\right\|_{\mathrm{F}} \cdot\left\|\left(\mathcal{I}-\mathcal{G} \mathcal{G}^{*}\right)(\boldsymbol{\Psi})\right\|_{\mathrm{F}} \\
\geq & \frac{3}{4}\left\|\left(\mathcal{I}-\mathcal{G G}^{*}\right)(\boldsymbol{\Phi})\right\|_{\mathrm{F}}^{2}-7\left\|\left(\mathcal{I}-\mathcal{G} \mathcal{G}^{*}\right)(\boldsymbol{\Psi})\right\|_{\mathrm{F}}^{2} \tag{6.13}
\end{align*}
$$

where the second line is due to $\left(\mathcal{I}-\mathcal{G} \mathcal{G}^{*}\right)\left(\boldsymbol{L}^{\natural} \boldsymbol{R}^{\natural^{\mathrm{H}}}\right)=\mathbf{0}$ and the last line follows from the fact that $a+2 b-$ $3 \sqrt{a b} \geq \frac{3}{4} a-7 b$ for any $a, b \geq 0$. Combining (6.12) and (6.13) together, one has

$$
\begin{align*}
\Re\left(\left\langle\nabla f_{1}, \boldsymbol{\Delta}\right\rangle\right)+\Re\left(\left\langle\nabla f_{2}, \boldsymbol{\Delta}\right\rangle\right) & \geq \frac{3}{4}\left\|\mathcal{A G}^{*}(\boldsymbol{\Phi})\right\|_{2}^{2}-\frac{7 \varepsilon^{2}}{4} \sigma_{r}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+\frac{3}{4}\left\|\left(\mathcal{I}-\mathcal{G G}^{*}\right)(\boldsymbol{\Phi})\right\|_{\mathrm{F}}^{2}-7\left\|\left(\mathcal{I}-\mathcal{G} \mathcal{G}^{*}\right)(\boldsymbol{\Psi})\right\|_{\mathrm{F}}^{2} \\
& \geq \frac{3}{4}\left(\left\|\mathcal{A} \mathcal{G}^{*}(\boldsymbol{\Phi})\right\|_{2}^{2}+\left\|\left(\mathcal{I}-\mathcal{G G}^{*}\right)(\boldsymbol{\Phi})\right\|_{\mathrm{F}}^{2}\right)-\frac{7 \varepsilon^{2}}{4} \sigma_{r}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}-7\|\boldsymbol{\Psi}\|_{\mathrm{F}}^{2} \tag{6.14}
\end{align*}
$$

where the last line follows from the fact that $\mathcal{I}-\mathcal{G G}^{*}$ is a projection operator. Let $T$ be the tangent space at $\boldsymbol{Z}^{\natural}$ defined as follows

$$
T:=\left\{\boldsymbol{U} \boldsymbol{J}^{\mathrm{H}}+\boldsymbol{K} \boldsymbol{V}^{\mathrm{H}}: \boldsymbol{J} \in \mathbb{C}^{n_{2} \times r}, \boldsymbol{K} \in \mathbb{C}^{s n_{1} \times r}\right\}
$$

It can be seen that $\boldsymbol{\Phi} \in T$. Therefore a simple calculation yields that

$$
\begin{align*}
\left\|\mathcal{A G}^{*}(\boldsymbol{\Phi})\right\|_{2}^{2}+\left\|\left(\mathcal{I}-\mathcal{G G}^{*}\right)(\boldsymbol{\Phi})\right\|_{\mathrm{F}}^{2} & =\left\langle\mathcal{G} \mathcal{A}^{*} \mathcal{A G}^{*}(\boldsymbol{\Phi}), \boldsymbol{\Phi}\right\rangle+\langle\boldsymbol{\Phi}, \boldsymbol{\Phi}\rangle-\left\langle\mathcal{G} \mathcal{G}^{*}(\boldsymbol{\Phi}), \boldsymbol{\Phi}\right\rangle \\
& \geq\|\boldsymbol{\Phi}\|_{\mathrm{F}}^{2}-\left|\left\langle\mathcal{G}\left(\mathcal{I}-\mathcal{A}^{*} \mathcal{A}\right) \mathcal{G}^{*}(\boldsymbol{\Phi}), \boldsymbol{\Phi}\right\rangle\right| \\
& =\|\boldsymbol{\Phi}\|_{\mathrm{F}}^{2}-\left|\left\langle\mathcal{P}_{T} \mathcal{G}\left(\mathcal{I}-\mathcal{A}^{*} \mathcal{A}\right) \mathcal{G}^{*} \mathcal{P}_{T}(\boldsymbol{\Phi}), \boldsymbol{\Phi}\right\rangle\right| \\
& \geq(1-\varepsilon)\|\boldsymbol{\Phi}\|_{\mathrm{F}}^{2} \tag{6.15}
\end{align*}
$$

where the last line follows from Lemma 6.7 and the assumption on $n$. In addition, we have

$$
\begin{align*}
\|\boldsymbol{\Psi}\|_{\mathrm{F}}^{2} & \leq\left(\left\|\boldsymbol{\Delta}_{\boldsymbol{L}}\right\|_{\mathrm{F}}\left\|\boldsymbol{\Delta}_{\boldsymbol{R}}\right\|_{\mathrm{F}}\right)^{2} \\
& \leq \frac{1}{4}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{4} \\
& \leq \frac{\varepsilon^{2} \sigma_{r}}{4}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2} \tag{6.16}
\end{align*}
$$

where we use $\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2} \leq \frac{\varepsilon^{2} \sigma_{r}\left(\boldsymbol{Z}^{\natural}\right)}{s \mu_{0}} \leq \varepsilon^{2} \sigma_{r}\left(\boldsymbol{Z}^{\natural}\right)$. Hence plugging (6.15) and (6.16) into (6.14) yields that

$$
\begin{align*}
\Re\left(\left\langle\nabla f_{1}, \boldsymbol{\Delta}\right\rangle\right)+\Re\left(\left\langle\nabla f_{2}, \boldsymbol{\Delta}\right\rangle\right) \geq & \frac{3}{4}(1-\varepsilon)\|\boldsymbol{\Phi}\|_{\mathrm{F}}^{2}-\frac{7 \varepsilon^{2}}{2} \sigma_{r}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2} \\
\geq & \frac{1}{2}\|\boldsymbol{\Phi}\|_{\mathrm{F}}^{2}-\frac{7 \varepsilon^{2}}{2} \sigma_{r}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2} \\
= & \frac{1}{2}\left(\left\|\boldsymbol{\Delta}_{\boldsymbol{L}}\left(\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}}\right\|_{\mathrm{F}}^{2}+\left\|\boldsymbol{L}^{\natural} \boldsymbol{Q} \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}}\right\|_{\mathrm{F}}^{2}\right)-\frac{7 \varepsilon^{2}}{2} \sigma_{r}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2} \\
& \quad \Re \Re\left(\left\langle\boldsymbol{\Delta}_{\boldsymbol{L}}\left(\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}},\left(\boldsymbol{L}^{\natural} \boldsymbol{Q}\right) \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}}\right\rangle\right) \\
\geq & \left(\frac{1}{2}-\frac{7 \varepsilon^{2}}{2}\right) \sigma_{r}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+\Re\left(\left\langle\boldsymbol{\Delta}_{\boldsymbol{L}}^{\mathrm{H}}\left(\boldsymbol{L}^{\natural} \boldsymbol{Q}\right),\left(\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{\Delta}_{\boldsymbol{R}}\right\rangle\right) \\
\geq & \frac{1}{9} \sigma_{r}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+\Re\left(\left\langle\boldsymbol{\Delta}_{\boldsymbol{L}}^{\mathrm{H}}\left(\boldsymbol{L}^{\natural} \boldsymbol{Q}\right),\left(\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{\Delta}_{\boldsymbol{R}}\right\rangle\right) \tag{6.17}
\end{align*}
$$

where the last line is due to $\varepsilon \leq \frac{1}{3}$.

Bounding $\Re\left(\left\langle\nabla f_{3}, \boldsymbol{\Delta}\right\rangle\right)$ Denote

$$
\boldsymbol{S}=\left[\begin{array}{ll}
\boldsymbol{I}_{S n_{1}} & \\
& -\boldsymbol{I}_{n_{2}}
\end{array}\right]
$$

We can bound $\Re\left(\left\langle\nabla f_{3}, \boldsymbol{\Delta}\right\rangle\right)$ as follows:

$$
\begin{align*}
4 \Re\left(\left\langle\nabla f_{3}, \boldsymbol{\Delta}\right\rangle\right)= & \Re\left(\left\langle\boldsymbol{S} \boldsymbol{M}^{\mathrm{H}} \boldsymbol{S} \boldsymbol{M}, \boldsymbol{\Delta}\right\rangle\right) \\
= & \Re\left(\left\langle\boldsymbol{M}^{\mathrm{H}} \boldsymbol{S} \boldsymbol{M}, \boldsymbol{M}^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}\right\rangle\right) \\
= & \Re\left(\left\langle\left(\boldsymbol{\Delta}+\boldsymbol{M}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{S}\left(\boldsymbol{\Delta}+\boldsymbol{M}^{\natural} \boldsymbol{Q}\right),\left(\boldsymbol{\Delta}+\boldsymbol{M}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}\right\rangle\right) \\
= & \left\|\boldsymbol{\Delta}^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2}+\left\|\left(\boldsymbol{M}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2}+3 \Re\left(\left\langle\left(\boldsymbol{M}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}, \boldsymbol{\Delta}^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}\right\rangle\right)+\Re\left(\left\langle\left(\boldsymbol{\boldsymbol { M } ^ { \natural } \boldsymbol { Q } ) ^ { \mathrm { H } } \boldsymbol { S } \boldsymbol { \Delta } , \boldsymbol { \Delta } ^ { \mathrm { H } } \boldsymbol { S } ( \boldsymbol { M } ^ { \natural } \boldsymbol { Q } ) \rangle )}\right.\right.\right. \\
= & \frac{1}{2}\left\|\left(\boldsymbol{M}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2}+\frac{1}{2}\left\|\left(\boldsymbol{M}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}+3 \boldsymbol{\Delta}^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2}-\frac{7}{2}\left\|\boldsymbol{\Delta}^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2}+\Re\left\langle\left(\boldsymbol{M}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}, \boldsymbol{\Delta}^{\mathrm{H}} \boldsymbol{S}\left(\boldsymbol{M}^{\natural} \boldsymbol{Q}\right)\right\rangle \\
= & \frac{1}{2}\left\|\left(\boldsymbol{M}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2}+\frac{1}{2}\left\|\left(\boldsymbol{M}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}+3 \boldsymbol{\Delta}^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2}-\frac{7}{2}\left\|\boldsymbol{\Delta}^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2} \\
& +\Re\left(\left\langle\left(\boldsymbol{M}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{\Delta}, \boldsymbol{\Delta}^{\mathrm{H}}\left(\boldsymbol{M}^{\natural} \boldsymbol{Q}\right)\right\rangle\right)-4 \Re\left(\left\langle\boldsymbol{\Delta}_{\boldsymbol{L}}^{\mathrm{H}}\left(\boldsymbol{L}^{\natural} \boldsymbol{Q}\right),\left(\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{\Delta}_{\boldsymbol{R}}\right\rangle\right) \\
\geq & \frac{1}{2}\left\|\left(\boldsymbol{M}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2}-\frac{7}{2}\left\|\boldsymbol{\Delta}^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2}-4 \Re\left(\left\langle\boldsymbol{\Delta}_{\boldsymbol{L}}^{\mathrm{H}}\left(\boldsymbol{L}^{\natural} \boldsymbol{Q}\right),\left(\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{\Delta}_{\boldsymbol{R}}\right\rangle\right) . \tag{6.18}
\end{align*}
$$

The fourth equality is due to $\left(\boldsymbol{M}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{S}\left(\boldsymbol{M}^{\natural} \boldsymbol{Q}\right)=\mathbf{0}$. The last equality follows from

$$
\begin{aligned}
\Re\left(\left\langle\boldsymbol{\Delta}_{\boldsymbol{L}}^{\mathrm{H}}\left(\boldsymbol{L}^{\natural} \boldsymbol{Q}\right),\left(\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{\Delta}_{\boldsymbol{R}}\right\rangle\right) & =\Re\left(\left\langle\left(\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{\Delta}_{\boldsymbol{R}}, \boldsymbol{\Delta}_{\boldsymbol{L}}^{\mathrm{H}}\left(\boldsymbol{L}^{\natural} \boldsymbol{Q}\right)\right\rangle\right) \\
& =\Re\left(\left\langle\left(\boldsymbol{L}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}} \boldsymbol{\Delta}_{\boldsymbol{L}}, \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}}\left(\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)\right\rangle\right) .
\end{aligned}
$$

Finally, combining (6.18) with (6.17) yields that

$$
\begin{align*}
\Re(\langle\nabla f(\boldsymbol{M}), \boldsymbol{\Delta}\rangle) & \geq \frac{1}{9} \sigma_{r}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+\frac{1}{8}\left\|\boldsymbol{M}^{\mathrm{\natural}^{\mathrm{H}}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2}-\frac{7}{8}\left\|\boldsymbol{\Delta}^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2} \\
& \geq \frac{1}{72} \sigma_{r}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+\frac{1}{8}\left\|\boldsymbol{M}^{\mathrm{b}^{\mathrm{H}}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2} \tag{6.19}
\end{align*}
$$

where the last line follows from that $\left\|\boldsymbol{\Delta}^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2} \leq\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{4} \leq \frac{\varepsilon^{2} \sigma_{r}}{\mu_{0} s}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}$ and $\varepsilon \leq \frac{1}{3}$.

### 6.4.2 Proof of (6.6)

Applying simple triangular inequality yields that

$$
\begin{align*}
\|\nabla f(\boldsymbol{M})\|_{\mathrm{F}}^{2} & =\left\|\nabla f_{1}+\nabla f_{2}+\nabla f_{3}\right\|_{\mathrm{F}}^{2} \\
& \leq 2\left\|\nabla f_{1}+\nabla f_{2}\right\|_{\mathrm{F}}^{2}+2\left\|\nabla f_{3}\right\|_{\mathrm{F}}^{2} \\
& \leq 4\left\|\nabla f_{1}\right\|_{\mathrm{F}}^{2}+4\left\|\nabla f_{2}\right\|_{\mathrm{F}}^{2}+2\left\|\nabla f_{3}\right\|_{\mathrm{F}}^{2} . \tag{6.20}
\end{align*}
$$

We will bound these three terms separately.
Bounding $\left\|\nabla f_{1}\right\|_{\mathrm{F}}^{2} \quad$ For any

$$
\boldsymbol{X}=\left[\begin{array}{ll}
\boldsymbol{X}_{\boldsymbol{L}}^{\mathrm{H}} & \boldsymbol{X}_{\boldsymbol{R}}^{\mathrm{H}}
\end{array}\right]^{\mathrm{H}} \in \mathbb{C}^{\left(s n_{1}+n_{2}\right) \times r}
$$

such that $\|\boldsymbol{X}\|_{\mathrm{F}}=1$, we have

$$
\begin{aligned}
\left|\left\langle\nabla f_{1}, \boldsymbol{X}\right\rangle\right|^{2} & =\left|\left\langle\mathcal{G} \mathcal{A}^{*}\left(\mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}\right)-\boldsymbol{D} \boldsymbol{y}\right), \boldsymbol{X}_{\boldsymbol{L}} \boldsymbol{R}^{\mathrm{H}}+\boldsymbol{L} \boldsymbol{X}_{\boldsymbol{R}}^{\mathrm{H}}\right\rangle\right|^{2} \\
& =\left|\left\langle\mathcal{G} \mathcal{A}^{*} \mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}-\boldsymbol{L}^{\natural} \boldsymbol{R}^{\mathrm{\natural}}\right), \boldsymbol{X}_{L} \boldsymbol{R}^{\mathrm{H}}+\boldsymbol{L} \boldsymbol{X}_{R}^{\mathrm{H}}\right\rangle\right|^{2}
\end{aligned}
$$

$$
\begin{align*}
\leq & \left(\left|\left\langle\mathcal{G} \mathcal{A}^{*} \mathcal{A \mathcal { G }}^{*}\left(\boldsymbol{L} \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}}\right), \boldsymbol{X}_{\boldsymbol{L}} \boldsymbol{R}^{\mathrm{H}}\right\rangle\right|\right. \\
& +\left|\left\langle\mathcal{G} \mathcal{A}^{*} \mathcal{A \mathcal { G }}^{*}\left(\boldsymbol{\Delta}_{\boldsymbol{L}}\left(\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}}\right), \boldsymbol{X}_{\boldsymbol{L}} \boldsymbol{R}^{\mathrm{H}}\right\rangle\right| \\
& +\left|\left\langle\mathcal{G} \mathcal{A}^{*} \mathcal{A \mathcal { G }}^{*}\left(\boldsymbol{L} \boldsymbol{\Delta}_{\boldsymbol{R}}^{H}\right), \boldsymbol{L} \boldsymbol{X}_{\boldsymbol{L}}^{\mathrm{H}}\right\rangle\right| \\
& \left.+\left|\left\langle\mathcal{G} \mathcal{A}^{*} \mathcal{A \mathcal { G }}^{*}\left(\boldsymbol{\Delta}_{\boldsymbol{L}}\left(\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}}\right), \boldsymbol{L} \boldsymbol{X}_{\boldsymbol{R}}^{\mathrm{H}}\right\rangle\right|\right)^{2} . \tag{6.21}
\end{align*}
$$

where we have used the fact $\boldsymbol{L} \boldsymbol{R}^{H}-\boldsymbol{L}^{\natural} \boldsymbol{R}^{\natural}{ }^{H}=\boldsymbol{L} \boldsymbol{\Delta}_{\boldsymbol{R}}^{H}+\boldsymbol{\Delta}_{\boldsymbol{L}}\left(\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}}$. Apparently, the upper bounds of the above four terms can be established similarly. We focus on the first term $\left|\left\langle\mathcal{G} \mathcal{A}^{*} \mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{L} \boldsymbol{\Delta}_{R}^{\mathrm{H}}\right), \boldsymbol{X}_{L} \boldsymbol{R}^{\mathrm{H}}\right\rangle\right|$ and its upper bound can be obtained as follows:

$$
\begin{align*}
\left|\left\langle\mathcal{G} \mathcal{A}^{*} \mathcal{A G}^{*}\left(\boldsymbol{L} \boldsymbol{\Delta}_{R}^{\mathrm{H}}\right), \boldsymbol{X}_{L} \boldsymbol{R}^{\mathrm{H}}\right\rangle\right| & =\left|\left\langle\mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{L} \boldsymbol{\Delta}_{R}^{\mathrm{H}}\right), \mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{X}_{L} \boldsymbol{R}^{\mathrm{H}}\right)\right\rangle\right| \\
& \leq \sum_{i=0}^{n-1}\left|\left\langle\boldsymbol{b}_{i} \boldsymbol{e}_{i}^{\top}, \mathcal{G}^{*}\left(\boldsymbol{L} \boldsymbol{\Delta}_{R}^{\mathrm{H}}\right)\right\rangle\right| \cdot\left|\left\langle\boldsymbol{b}_{i} \boldsymbol{e}_{i}^{\top}, \mathcal{G}^{*}\left(\boldsymbol{X}_{L} \boldsymbol{R}^{\mathrm{H}}\right)\right\rangle\right| \\
& =\sum_{i=0}^{n-1}\left|\left\langle\mathcal{G}\left(\boldsymbol{b}_{i} \boldsymbol{e}_{i}^{\top}\right), \boldsymbol{L} \boldsymbol{\Delta}_{R}^{\mathrm{H}}\right\rangle\right| \cdot\left|\left\langle\mathcal{G}\left(\boldsymbol{b}_{i} e_{i}^{\top}\right), \boldsymbol{X}_{L} \boldsymbol{R}^{\mathrm{H}}\right\rangle\right| \\
& =\sum_{i=0}^{n-1}\left|\left\langle\boldsymbol{G}_{i} \otimes \boldsymbol{b}_{i}, \boldsymbol{L} \boldsymbol{\Delta}_{R}^{\mathrm{H}}\right\rangle\right| \cdot\left|\left\langle\boldsymbol{G}_{i} \otimes \boldsymbol{b}_{i}, \boldsymbol{X}_{L} \boldsymbol{R}^{\mathrm{H}}\right\rangle\right| . \tag{6.22}
\end{align*}
$$

To complete the proof, it suffers to control $\left|\left\langle\boldsymbol{G}_{i} \otimes \boldsymbol{b}_{i}, \boldsymbol{L} \boldsymbol{\Delta}_{R}^{\mathrm{H}}\right\rangle\right|$ and $\left|\left\langle\boldsymbol{G}_{i} \otimes \boldsymbol{b}_{i}, \boldsymbol{X}_{L} \boldsymbol{R}^{\mathrm{H}}\right\rangle\right|$. Notice that

$$
\begin{align*}
\left|\left\langle\boldsymbol{G}_{i} \otimes \boldsymbol{b}_{i}, \boldsymbol{L} \boldsymbol{\Delta}_{R}^{\mathrm{H}}\right\rangle\right| & =\left|\frac{1}{\sqrt{w_{i}}} \sum_{j+k=i}\left\langle\left(\boldsymbol{e}_{j} \otimes \boldsymbol{b}_{i}\right) \boldsymbol{e}_{k}^{\top}, \boldsymbol{L} \boldsymbol{\Delta}_{R}^{\mathrm{H}}\right\rangle\right| \\
& \leq \frac{1}{\sqrt{w_{i}}} \sum_{j+k=i}\left\|\left(\boldsymbol{e}_{j}^{\top} \otimes \boldsymbol{b}_{i}^{\mathrm{H}}\right) \boldsymbol{L}\right\|_{2}\left\|\boldsymbol{e}_{k}^{\top} \boldsymbol{\Delta}_{R}\right\|_{2} \\
& \leq \sqrt{\sum_{j+k=i}\left\|\left(\boldsymbol{e}_{j}^{\top} \otimes \boldsymbol{b}_{i}^{\mathrm{H}}\right) \boldsymbol{L}\right\|_{2}^{2}\left\|\boldsymbol{e}_{k}^{\top} \boldsymbol{\Delta}_{R}\right\|_{2}^{2}} \\
& =\sqrt{\sum_{j+k=i}\left\|\boldsymbol{b}_{i}^{\mathrm{H}} \boldsymbol{L}_{j}\right\|_{2}^{2}\left\|\boldsymbol{e}_{k}^{\top} \boldsymbol{\Delta}_{R}\right\|_{2}^{2}} \\
& \leq \sqrt{\sum_{j+k=i}\left\|\boldsymbol{b}_{i}\right\|_{2}^{2}\left\|\boldsymbol{L}_{j}\right\|_{\mathrm{F}}^{2}\left\|\boldsymbol{e}_{k}^{\top} \boldsymbol{\Delta}_{R}\right\|_{2}^{2}} \\
& \leq \sqrt{\mu_{0} s} \cdot \sqrt{\sum_{j+k=i}\left\|\boldsymbol{L}_{j}\right\|_{\mathrm{F}}^{2}\left\|\boldsymbol{e}_{k}^{\top} \boldsymbol{\Delta}_{R}\right\|_{2}^{2}} \\
& \leq \sqrt{\mu_{0} s} \cdot \sqrt{\frac{\mu r \sigma}{n}} \cdot \sqrt{\sum_{j+k=i}\left\|\boldsymbol{e}_{k}^{\top} \boldsymbol{\Delta}_{R}\right\|_{2}^{2}} \tag{6.23}
\end{align*}
$$

Similarly one has

$$
\begin{align*}
\left|\left\langle\boldsymbol{G}_{i} \otimes \boldsymbol{b}_{i}, \boldsymbol{X}_{L} \boldsymbol{R}^{\mathrm{H}}\right\rangle\right| & \leq \sqrt{\mu_{0} s} \cdot \sqrt{\sum_{j+k=i}\left\|\left[\boldsymbol{X}_{\boldsymbol{L}}\right]_{j}\right\|_{\mathrm{F}}^{2}\left\|\boldsymbol{e}_{k}^{\top} \boldsymbol{R}\right\|_{2}^{2}} \\
& \leq \sqrt{\mu_{0} s} \cdot \sqrt{\frac{\mu r \sigma}{n}} \cdot \sqrt{\sum_{j+k=i}\left\|\left[\boldsymbol{X}_{L}\right]_{j}\right\|_{\mathrm{F}}^{2}} \tag{6.24}
\end{align*}
$$

Plugging (6.23) and (6.24) into (6.22) yields that

$$
\begin{aligned}
\left|\left\langle\mathcal{G} \mathcal{A}^{*} \mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{L} \boldsymbol{\Delta}_{R}^{\mathrm{H}}\right), \boldsymbol{X}_{L} \boldsymbol{R}^{\mathrm{H}}\right\rangle\right| & \leq \frac{\mu_{0} s \cdot \mu r \sigma}{n} \sum_{i=0}^{n-1} \sqrt{\sum_{j+k=i}\left\|e_{k}^{\top} \boldsymbol{\Delta}_{R}\right\|_{2}^{2}} \cdot \sqrt{\sum_{j+k=i}\left\|\left[\boldsymbol{X}_{\boldsymbol{L}}\right]_{j}\right\|_{\mathrm{F}}^{2}} \\
& \leq \mu_{0} s \cdot \mu r \sigma \cdot\left\|\boldsymbol{\Delta}_{\boldsymbol{R}}\right\|_{\mathrm{F}} \cdot\left\|\boldsymbol{X}_{\boldsymbol{L}}\right\|_{\mathrm{F}} .
\end{aligned}
$$

Using the same argument, one has

$$
\begin{aligned}
\left|\left\langle\mathcal{G} \mathcal{A}^{*} \mathcal{A G}^{*}\left(\boldsymbol{\Delta}_{L}\left(\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}}\right), \boldsymbol{X}_{\boldsymbol{L}} \boldsymbol{R}^{\mathrm{H}}\right\rangle\right| & \leq \mu_{0} s \cdot \mu r \sigma \cdot\left\|\boldsymbol{\Delta}_{\boldsymbol{L}}\right\|_{\mathrm{F}} \cdot\left\|\boldsymbol{X}_{\boldsymbol{L}}\right\|_{\mathrm{F}}, \\
\left|\left\langle\mathcal{G} \mathcal{A}^{*} \mathcal{A} \mathcal{G}^{*}\left(\boldsymbol{L} \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}}\right), \boldsymbol{L} \boldsymbol{X}_{\boldsymbol{L}}^{\mathrm{H}}\right\rangle\right| & \leq \mu_{0} s \cdot \mu r \sigma \cdot\left\|\boldsymbol{\Delta}_{\boldsymbol{R}}\right\|_{F} \cdot\left\|\boldsymbol{X}_{\boldsymbol{L}}\right\|_{\mathrm{F}}, \\
\left|\left\langle\mathcal{G} \mathcal{A}^{*} \mathcal{A G}^{*}\left(\boldsymbol{\Delta}_{\boldsymbol{L}}\left(\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}}\right), \boldsymbol{L} \boldsymbol{X}_{\boldsymbol{R}}^{\mathrm{H}}\right\rangle\right| & \leq \mu_{0} s \cdot \mu r \sigma \cdot\left\|\boldsymbol{\Delta}_{\boldsymbol{L}}\right\|_{\mathrm{F}} \cdot\left\|\boldsymbol{X}_{\boldsymbol{R}}\right\|_{\mathrm{F}} .
\end{aligned}
$$

Hence we can obtain

$$
\begin{aligned}
\left|\left\langle\nabla f_{1}, \boldsymbol{X}\right\rangle\right|^{2} & \leq\left(\mu_{0} \mu s r \sigma\right)^{2} \cdot\left(\left\|\boldsymbol{\Delta}_{R}\right\|_{\mathrm{F}}\left\|\boldsymbol{X}_{L}\right\|_{\mathrm{F}}+\left\|\boldsymbol{\Delta}_{L}\right\|_{\mathrm{F}}\left\|\boldsymbol{X}_{L}\right\|_{\mathrm{F}}+\left\|\boldsymbol{\Delta}_{R}\right\|_{\mathrm{F}}\left\|\boldsymbol{X}_{R}\right\|_{\mathrm{F}}+\left\|\boldsymbol{\Delta}_{L}\right\|_{\mathrm{F}}\left\|\boldsymbol{X}_{R}\right\|_{\mathrm{F}}\right)^{2} \\
& =\left(\mu_{0} \mu s r \sigma\right)^{2} \cdot\left(\left\|\boldsymbol{\Delta}_{L}\right\|_{\mathrm{F}}+\left\|\boldsymbol{\Delta}_{R}\right\|_{\mathrm{F}}\right)^{2} \cdot\left(\left\|\boldsymbol{X}_{L}\right\|_{\mathrm{F}}+\left\|\boldsymbol{X}_{R}\right\|_{\mathrm{F}}\right)^{2} \\
& \leq 4\left(\mu_{0} \mu s r \sigma\right)^{2}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}
\end{aligned}
$$

where the last line is due to $\|\boldsymbol{X}\|_{\mathrm{F}}^{2}=\left\|\boldsymbol{X}_{L}\right\|_{\mathrm{F}}^{2}+\left\|\boldsymbol{X}_{R}\right\|_{\mathrm{F}}^{2}=1$. Thus we have

$$
\begin{align*}
\left\|\nabla f_{1}\right\|_{\mathrm{F}}^{2} & =\sup _{\|\boldsymbol{X}\|_{\mathrm{F}}=1}\left|\left\langle\nabla f_{1}, \boldsymbol{X}\right\rangle\right|^{2} \\
& \leq 4\left(\mu_{0} \mu s r \sigma\right)^{2}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2} \tag{6.25}
\end{align*}
$$

Bounding $\left\|\nabla f_{2}\right\|_{\mathrm{F}}^{2} \quad$ For any $\boldsymbol{X} \in \mathbb{C}^{\left(s n_{1}+n_{2}\right) \times r}$ such that $\|\boldsymbol{X}\|_{\mathrm{F}}=1$, one has

$$
\begin{aligned}
\left|\left\langle\nabla f_{2}, \boldsymbol{X}\right\rangle\right| & \leq\left|\left\langle\left(\mathcal{I}-\mathcal{G} \mathcal{G}^{*}\right)\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}\right), \boldsymbol{X}_{\boldsymbol{L}} \boldsymbol{R}^{\mathrm{H}}+\boldsymbol{L} \boldsymbol{X}_{\boldsymbol{R}}^{\mathrm{H}}\right\rangle\right| \\
& =\left|\left\langle\left(\mathcal{I}-\mathcal{G} \mathcal{G}^{*}\right)\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}-\boldsymbol{L}^{\natural} \boldsymbol{R}^{\natural}{ }^{\mathrm{H}}\right), \boldsymbol{X}_{\boldsymbol{L}} \boldsymbol{R}^{\mathrm{H}}+\boldsymbol{L} \boldsymbol{X}_{\boldsymbol{R}}^{\mathrm{H}}\right\rangle\right| \\
& \leq\left\|\left(\mathcal{I}-\mathcal{G} \mathcal{G}^{*}\right)\left(\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}-\boldsymbol{L}^{\natural} \boldsymbol{R}^{\natural}{ }^{\mathrm{H}}\right)\right\|_{\mathrm{F}} \cdot\left\|\boldsymbol{X}_{\boldsymbol{L}} \boldsymbol{R}^{\mathrm{H}}+\boldsymbol{L} \boldsymbol{X}_{\boldsymbol{R}}^{\mathrm{H}}\right\|_{\mathrm{F}} \\
& \leq\left\|\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}-\boldsymbol{L}^{\natural} \boldsymbol{R}^{\natural^{\mathrm{H}}}\right\|_{\mathrm{F}} \cdot\left\|\boldsymbol{X}_{\boldsymbol{L}} \boldsymbol{R}^{\mathrm{H}}+\boldsymbol{L} \boldsymbol{X}_{\boldsymbol{R}}^{\mathrm{H}}\right\|_{\mathrm{F}} .
\end{aligned}
$$

We bound both terms separately.

- Bounding $\left\|\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}-\boldsymbol{L}^{\natural} \boldsymbol{R}^{\mathrm{\natural}^{\mathrm{H}}}\right\|_{\mathrm{F}}$. It can be bounded as follows:

$$
\begin{aligned}
\left\|\boldsymbol{L} \boldsymbol{R}^{\mathrm{H}}-\boldsymbol{L}^{\natural} \boldsymbol{R}^{\natural}\right\|_{\mathrm{F}}^{\mathrm{H}} & =\left\|\boldsymbol{L} \boldsymbol{\Delta}_{\boldsymbol{R}}^{\mathrm{H}}+\boldsymbol{\Delta}_{\boldsymbol{L}}\left(\boldsymbol{R}^{\natural} \boldsymbol{Q}\right)^{\mathrm{H}}\right\|_{\mathrm{F}} \\
& \leq\|\boldsymbol{L}\|\left\|\boldsymbol{\Delta}_{\boldsymbol{R}}\right\|_{\mathrm{F}}+\left\|\boldsymbol{R}^{\natural}\right\|\left\|\boldsymbol{\Delta}_{\boldsymbol{L}}\right\|_{\mathrm{F}} \\
& \leq(1+\varepsilon) \sqrt{\sigma_{1}}\left\|\boldsymbol{\Delta}_{\boldsymbol{R}}\right\|_{\mathrm{F}}+\sqrt{\sigma_{1}}\left\|\boldsymbol{\Delta}_{L}\right\|_{\mathrm{F}} \\
& \leq(1+\varepsilon) \sqrt{\sigma_{1}} \cdot \sqrt{2}\|\boldsymbol{\Delta}\|_{\mathrm{F}},
\end{aligned}
$$

where we have used the fact $\|\boldsymbol{L}\| \leq\left\|\boldsymbol{\Delta}_{\boldsymbol{L}}\right\|+\left\|\boldsymbol{L}^{\text {घ }}\right\| \leq\left\|\boldsymbol{\Delta}_{\boldsymbol{L}}\right\|_{\mathrm{F}}+\sqrt{\sigma_{1}} \leq\|\boldsymbol{\Delta}\|_{\mathrm{F}}+\sqrt{\sigma_{1}} \leq(1+\varepsilon) \sqrt{\sigma_{1}}$.

- Bounding $\left\|\boldsymbol{X}_{\boldsymbol{L}} \boldsymbol{R}^{\mathrm{H}}+\boldsymbol{L} \boldsymbol{X}_{\boldsymbol{R}}^{\mathrm{H}}\right\|_{\mathrm{F}}$. It can be bounded as follows:

$$
\begin{aligned}
\left\|\boldsymbol{X}_{\boldsymbol{L}} \boldsymbol{R}^{\mathrm{H}}+\boldsymbol{L} \boldsymbol{X}_{\boldsymbol{R}}^{\mathrm{H}}\right\|_{\mathrm{F}} & \leq\|\boldsymbol{R}\|\left\|\boldsymbol{X}_{\boldsymbol{L}}\right\|_{\mathrm{F}}+\|\boldsymbol{L}\|\left\|\boldsymbol{X}_{\boldsymbol{R}}\right\|_{\mathrm{F}} \\
& \leq(1+\varepsilon) \sqrt{\sigma_{1}}\left(\left\|\boldsymbol{X}_{\boldsymbol{L}}\right\|_{\mathrm{F}}+\left\|\boldsymbol{X}_{\boldsymbol{R}}\right\|_{\mathrm{F}}\right) \\
& \leq \sqrt{2}(1+\varepsilon) \sqrt{\sigma_{1}} .
\end{aligned}
$$

Combining together and simple computation yields that

$$
\begin{align*}
\left\|\nabla f_{2}\right\|_{\mathrm{F}}^{2} & =\sup _{\|\boldsymbol{X}\|_{\mathrm{F}}=1}\left|\left\langle\nabla f_{3}, \boldsymbol{X}\right\rangle\right|^{2} \\
& \leq 4(1+\varepsilon)^{4} \sigma_{1}^{2} \cdot\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2} . \tag{6.26}
\end{align*}
$$

Bounding $\left\|\nabla f_{3}\right\|_{\mathrm{F}}^{2} \quad$ Recall that

$$
\boldsymbol{S}=\left[\begin{array}{ll}
\boldsymbol{I}_{s n_{1}} & \\
& -\boldsymbol{I}_{n_{2}}
\end{array}\right] .
$$

A straightforward computation yields that

$$
\begin{aligned}
& \left\|\nabla f_{3}\right\|_{\mathrm{F}}^{2}=\frac{1}{16}\left(\left\|\boldsymbol{L}\left(\boldsymbol{L}^{\mathrm{H}} \boldsymbol{L}-\boldsymbol{R}^{\mathrm{H}} \boldsymbol{R}\right)\right\|_{\mathrm{F}}^{2}+\left\|\boldsymbol{R}\left(\boldsymbol{R}^{\mathrm{H}} \boldsymbol{R} \boldsymbol{R}^{\mathrm{H}} \boldsymbol{R}\right)\right\|_{\mathrm{F}}^{2}\right) \\
& =\frac{1}{16}\left\|\boldsymbol{S M} \boldsymbol{M}^{\mathrm{H}} \boldsymbol{S} \boldsymbol{M}\right\|_{\mathrm{F}}^{2} \\
& =\frac{1}{16}\left\|\boldsymbol{S}\left(\boldsymbol{M} \boldsymbol{M}^{\mathrm{H}}-\boldsymbol{M}^{\natural} \boldsymbol{M}^{\text {n }^{\mathrm{H}}}\right) \boldsymbol{S} \boldsymbol{M}+\boldsymbol{S} \boldsymbol{M}^{\natural} \boldsymbol{M}^{\mathrm{q}^{\mathrm{H}}} \boldsymbol{S} \boldsymbol{M}\right\|_{\mathrm{F}}^{2} \\
& \leq \frac{1}{8}\left\|\boldsymbol{S}\left(\boldsymbol{M} \boldsymbol{M}^{\mathrm{H}}-\boldsymbol{M}^{\natural} \boldsymbol{M}^{\natural^{\mathrm{H}}}\right) \boldsymbol{S} \boldsymbol{M}\right\|_{F}^{2}+\frac{1}{8}\left\|\boldsymbol{S} \boldsymbol{M}^{\natural} \boldsymbol{M}^{\natural}{ }^{\mathrm{H}} \boldsymbol{S} \boldsymbol{M}\right\|_{F}^{2} \\
& \leq \frac{1}{8}\|\boldsymbol{M}\|^{2} \cdot\left\|M M^{\mathrm{H}}-\boldsymbol{M}^{\natural} \boldsymbol{M}^{\natural^{\mathrm{H}}}\right\|_{F}^{2}+\frac{1}{8}\left\|\boldsymbol{M}^{\natural}\right\|^{2} \cdot\left\|\boldsymbol{M}^{\natural^{\mathrm{H}}} \boldsymbol{S}\left(\boldsymbol{M}^{\natural}+\boldsymbol{\Delta}\right)\right\|_{F}^{2} \\
& \stackrel{(a)}{=} \frac{1}{8}\|\boldsymbol{M}\|^{2} \cdot\left\|\boldsymbol{M}^{\natural} \boldsymbol{\Delta}^{\mathrm{H}}+\boldsymbol{\Delta} \boldsymbol{M}^{\mathfrak{\natural}^{\mathrm{H}}}+\boldsymbol{\Delta} \boldsymbol{\Delta}^{\mathrm{H}}\right\|_{\mathrm{F}}^{2}+\frac{1}{8}\left\|\boldsymbol{M}^{\natural}\right\|^{2} \cdot\left\|\boldsymbol{M}^{\natural^{\mathrm{H}}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2} \\
& \leq \frac{3}{8}\|\boldsymbol{M}\|^{2} \cdot\left(2\left\|\boldsymbol{M}^{\natural}\right\|^{2}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{4}\right)+\frac{1}{8}\left\|\boldsymbol{M}^{\natural}\right\|^{2} \cdot\left\|\boldsymbol{M}^{\mathrm{G}^{\mathrm{H}}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2},
\end{aligned}
$$

where step (a) follows from $\boldsymbol{M}^{\natural^{\mathrm{H}}} \boldsymbol{S} \boldsymbol{M}^{\natural}=\mathbf{0}$. Notice that $\left\|\boldsymbol{M}^{\natural}\right\| \leq \sqrt{2 \sigma_{1}\left(\boldsymbol{Z}^{\natural}\right)}$ and $\|\boldsymbol{M}\| \leq\left\|\boldsymbol{M}-\boldsymbol{M}^{\natural} \boldsymbol{Q}\right\|_{\mathrm{F}}+$ $\left\|\boldsymbol{M}^{\natural}\right\| \leq \sqrt{\frac{\varepsilon^{2} \sigma_{r}}{\mu_{0} s}}+\sqrt{2 \sigma_{1}}$. Thus we have

$$
\begin{align*}
\left\|\nabla f_{3}\right\|_{\mathrm{F}}^{2} & \leq \frac{3}{8}\left(\sqrt{\frac{\varepsilon^{2} \sigma_{r}}{\mu_{0} s}}+\sqrt{2 \sigma_{1}}\right)^{2} \cdot\left(4 \sigma_{1}+\frac{\varepsilon^{2} \sigma_{r}}{\mu_{0} s}\right)\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+\frac{1}{4} \sigma_{1}\left\|\boldsymbol{M}^{\mathfrak{\natural}^{\mathrm{H}}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2} \\
& \leq \frac{3}{8} \cdot 2\left(\varepsilon^{2}+2\right) \sigma_{1} \cdot\left(4+\varepsilon^{2}\right) \sigma_{1} \cdot\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+\frac{1}{4} \sigma_{1}\left\|\boldsymbol{M}^{\mathrm{h}^{\mathrm{H}}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2} \\
& \leq \frac{19}{4} \sigma_{1}^{2} \cdot\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+\frac{1}{4} \sigma_{1}\left\|\boldsymbol{M}^{\natural^{\mathrm{H}}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2}, \tag{6.27}
\end{align*}
$$

where the last line is due to $\varepsilon \leq \frac{1}{3}$.
Finally, plugging (6.25), (6.26) and (6.27) into (6.20), we obtain that

$$
\begin{align*}
\|\nabla f(\boldsymbol{M})\|_{\mathrm{F}}^{2} & \leq 4 \cdot 4\left(\mu_{0} \mu s r \sigma\right)^{2}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+4 \cdot 4(1+\varepsilon)^{4} \sigma_{1}^{2} \cdot\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+2\left(\frac{19}{4} \sigma_{1}^{2} \cdot\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+\frac{1}{4} \sigma_{1}\left\|\boldsymbol{M}^{\mathrm{\natural}^{\mathrm{H}}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2}\right) \\
& \leq\left(64\left(\mu_{0} \mu s r\right)^{2}+16(1+\varepsilon)^{4}+\frac{19}{2}\right) \sigma_{1}^{2}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+\frac{1}{2} \sigma_{1} \| \boldsymbol{M}^{\mathrm{\natural}^{\mathrm{H}} \boldsymbol{S} \boldsymbol{\Delta} \|_{\mathrm{F}}^{2}} \\
& \leq 125\left(\mu_{0} \mu s r \sigma_{1}\right)^{2}\|\boldsymbol{\Delta}\|_{\mathrm{F}}^{2}+\frac{1}{2} \sigma_{1}\left\|\boldsymbol{M}^{\mathrm{\natural}} \boldsymbol{S} \boldsymbol{\Delta}\right\|_{\mathrm{F}}^{2} \tag{6.28}
\end{align*}
$$

where the second line is due to $\sigma \leq \frac{(1+\varepsilon) \sigma_{1}}{1-\varepsilon} \leq 2 \sigma_{1}$ and the last line follows from $\varepsilon \leq \frac{1}{3}<1$.

### 6.5 Auxiliary Lemmas

Lemma 6.5 (33, Theorem 6.1). Assume $\left\{\boldsymbol{X}_{i}\right\}_{i=1}^{n}$ are independent random matrices of dimension $n_{1} \times n_{2}$ and obey $\mathbb{E}\left[\boldsymbol{X}_{i}\right]=0$ and $\left\|\boldsymbol{X}_{i}\right\| \leq B$. Define

$$
\sigma^{2}:=\max \left\{\left\|\mathbb{E}\left[\sum_{i=1}^{n} \boldsymbol{X}_{i} \boldsymbol{X}_{i}^{\mathrm{H}}\right]\right\|,\left\|\mathbb{E}\left[\sum_{i=1}^{n} \boldsymbol{X}_{i}^{\mathrm{H}} \boldsymbol{X}_{i}\right]\right\|\right\} .
$$

Then the event

$$
\begin{equation*}
\left\|\sum_{i=1}^{n} \boldsymbol{X}_{i}\right\| \leq c\left(\sqrt{\sigma^{2} \log \left(n_{1}+n_{2}\right)}+B \log \left(n_{1}+n_{2}\right)\right) \tag{6.29}
\end{equation*}
$$

holds with probability at least $1-\left(n_{1}+n_{2}\right)^{-c_{1}}$, where $c, c_{1}>0$ are absolute constants.
Lemma 6.6 ([12], Lemma III.13). Suppose $\boldsymbol{Z}^{\natural}$ is $\mu_{1}$-incoherent. Then one has

$$
\begin{aligned}
& \sqrt{\sum_{i=0}^{n-1} \frac{\left\|\mathcal{G}^{*}(\boldsymbol{Z}) \boldsymbol{e}_{i}\right\|_{2}^{2}}{\omega_{i}}} \leq c_{1} \sqrt{\frac{\mu_{1} r \log (s n)}{n}} \sigma_{1}\left(\boldsymbol{Z}^{\natural}\right), \\
& \max _{0 \leq i \leq n-1} \frac{\left\|\mathcal{G}^{*}(\boldsymbol{Z}) \boldsymbol{e}_{i}\right\|_{2}}{\sqrt{\omega_{i}}} \leq \frac{\mu_{1} r}{n} \sigma_{1}\left(\boldsymbol{Z}^{\natural}\right) .
\end{aligned}
$$

Lemma 6.7 ([12], Corollary III.9). Under Assumption 1 and Assumption 2, let $T$ be the tangent space of $Z^{\natural}$, then the event

$$
\left\|\mathcal{P}_{T} \mathcal{G}\left(\mathcal{A}^{*} \mathcal{A}-\mathcal{I}\right) \mathcal{G}^{*} \mathcal{P}_{T}\right\| \leq c \sqrt{\frac{\mu_{0} \mu_{1} s r \log (s n)}{n}}
$$

occurs with probability at least $1-(s n)^{-c}$ for some universal constant $c>0$.

## 7 Conclusion

In this paper, we propose a non-convex method called PGD-VHL for low rank vectorized Hankel matrix recovery problem in blind super-resolution of point sources. Our theoretical analysis shows that PGD-VHL converges to the target matrix linearly when the number of samples is larger than $\mathcal{O}\left(s^{2} r^{2} \log ^{2}(s n)\right)$. The performance of PGD-VHL has also been demonstrated by our numerical simulations. For future work, it is interesting to study the recovery performance of PGD-VHL in the presence of noise and the behavior of vanilla gradient descent method for blind super-resolution.

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