Distance Distributions in Finite Uniformly Random Networks: Theory and Applications

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Abstract

In wireless networks, the knowledge of nodal distances is essential for several areas such as system configuration, performance analysis and protocol design. In order to evaluate distance distributions in random networks, the underlying nodal arrangement is almost universally taken to be an infinite Poisson point process. While this assumption is valid in some cases, there are also certain impracticalities to this model. For example, practical networks are non-stationary, and the number of nodes in disjoint areas are not independent. This paper considers a more realistic network model where a finite number of nodes are uniformly randomly distributed in a general d-dimensional ball of radius R and characterizes the distribution of Euclidean distances in the system. The key result is that the probability density function of the distance from the center of the network to its nth nearest neighbor follows a generalized beta distribution. This finding is applied to study network characteristics such as energy consumption, interference, outage and connectivity.

I. INTRODUCTION

A. Motivation

In wireless channels, the received signal strength falls off with distance according to a power law, at a rate termed the large scale path loss exponent (PLE) [1]. Given a transceiver distance d, the signal power at the receiver is attenuated by a factor of $d^{-\alpha}$, where α is the PLE. Consequently, in wireless networks, distances (or equivalently, the distribution of distances for random networks) between nodes strongly impact the signal-to-noise-and-interference ratios (SINRs), and therefore the link reliabilities as well. The knowledge of the nodal distances is therefore essential for several important areas such as the throughput and performance analysis and the design of protocols and algorithms.

In wireless networks, nodes are typically scattered randomly over an area or volume, and the distance distributions follow from the spatial stochastic process governing the locations of the nodes. For the sake of analytical convenience, the arrangement of nodes in a random network is commonly taken to be a homogeneous (or stationary) Poisson point process (PPP). For the so-called "Poisson network" of intensity λ , the number of nodes in any given set of Lebesgue measure V is Poisson with mean λV , and the number of nodes in disjoint sets are independent of each other. Even though the PPP assumption can lead to some insightful results, practical networks differ from Poisson networks in certain aspects. First, networks are formed by usually scattering a fixed (and finite) number of nodes in a given area (or very close to it). If nodes are uniformly randomly distributed, they form a *binomial point process* (BPP), which we describe shortly. Secondly, since the area or volume of deployment is necessarily finite, the point process formed is non-stationary and often non-isotropic, meaning that the network characteristics as seen from a node's perspective is not homogeneous for all nodes. For example, the mean farthest neighbor distances are larger for nodes lying near the boundary than for the ones closer to the center. Furthermore, the number of nodes in disjoint sets are not independent but governed by a

multinomial distribution. Fig. 1 shows a realization of the two processes with the same density and depicts that the PPP is clearly not a good model, since there may be more points in the realization than the number of nodes deployed. In particular when the number of nodes is small, the Poisson model is inaccurate. The main shortcoming of the Poisson assumption is, however, the independence of the number of nodes in disjoint areas. For example, if all users or nodes are located in a certain part of the network area, the remaining area is necessarily empty. This simple fact is not captured by the Poisson model. This motivates the need to study and accurately characterize finite uniformly random networks, in an attempt to extend the plethora of results for the PPP to the often more realistic case of the BPP. We call this new model a "binomial network".

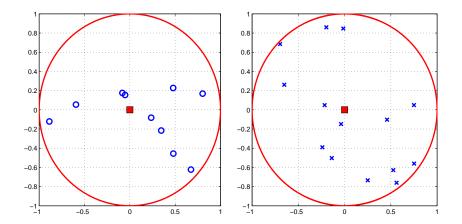


Fig. 1. (Left) A realization of 10 sensor nodes uniformly randomly distributed in a circular area of unit radius. (Right) This realization of the Poisson network with the same density ($\lambda = 3.18$) has 14 nodes. The shaded box at the origin represents the base station.

Formally speaking, a d-dimensional BPP is formed as a result of distributing N points independently uniformly in a compact set $W \subset \mathbb{R}^d$ [3]. The intensity of this process is defined to be $N/\nu_d(W)$. For any Borel subset V of W, the number of points in V, $\Phi(V)$, is binomial(n, p)with parameters n = N and $p = \nu_d(V)/\nu_d(W)$, where $\nu_d()$ is the standard d-dimensional Lebesgue measure. Accordingly,

$$\Pr(\Phi(V) = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

By this property, the number of nodes in disjoint sets are joint via a multinomial distribution. This network model applies to mobile ad hoc networks and wireless networks with infrastructure, such as cellular telephony networks or sensor networks.

In this paper, we study the Euclidean distance properties in a general *d*-dimensional binomial network. We also provide results on network characteristics such as energy consumption, interference, outage and connectivity based on the distance distributions.

B. Related Work

Even though the knowledge of node locations in wireless networks is important for their analysis and design, relatively few results are available in the literature in this area. Moreover, much of existing work deals only with moments of the distances (means and variances) or characterizes the exact distribution only for very specific system models.

In [4], the probability density function (pdf) and cumulative distribution function (cdf) of distances between nodes are derived for networks with uniformly random and Gaussian distributed nodes over a rectangular area. [5] studies mean internodal distance properties for several kinds of multihop systems such as ring networks, Manhattan street networks, hypercubes and shufflenets. [6] provides closed-form expressions for the distributions in *d*-dimensional homogeneous PPPs and describes several applications of the results for large networks. [7] derives the joint distribution of distances of nodes from a common reference point for planar networks with a finite number of nodes randomly distributed on a square. [8] considers square random networks and determines the pdf and cdf of nearest neighbor and internodal distances. [9] investigates one-dimensional multihop networks with randomly located nodes and analyzes the distributions of single-hop and multiple-hop distances.

II. DISTANCE DISTRIBUTION TO THE NEIGHBORS

In this section, we determine the distribution of the Euclidean distance to the n^{th} nearest point from the origin for a general *d*-dimensional isotropic BPP. It will be established that this random variable follows a generalized beta distribution. We also comment on the distances to the nearest and farthest neighbors and the void probabilities.

Theorem 2.1: In a point process consisting of N nodes uniformly randomly distributed in a d-dimensional ball of radius R centered at the origin, the Euclidean distance R_n from the origin to its nth nearest neighbor is distributed as a generalized beta distribution i.e.,

$$f_{R_n}(r) = \frac{d}{R} \frac{B(n-1/d+1, N-n+1)}{B(N-n+1, n)} \beta\left(\left(\frac{r}{R}\right)^d; n-\frac{1}{d}+1, N-n+1\right), \quad r \in [0, R],$$

where $\beta(x; a, b)$ is the beta density function¹ defined as $\beta(x; a, b) = \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}$. B(a, b) denotes the beta function and is expressible in terms of gamma functions as $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$.

Proof: Consider the BPP with N points uniformly randomly distributed in a d-dimensional ball W of radius R centered at the origin o, i.e., $W = \mathbb{B}_d(o, R)$. The volume of this ball $\nu_d(W)$ is equal to $c_d R^d$ [3], where

$$c_d = \frac{\pi^{d/2}}{\Gamma(1+d/2)}$$

is the volume of the unit ball in \mathbb{R}^d . The important cases include $c_1 = 2$, $c_2 = \pi$ and $c_3 = 4\pi/3$. The density of this process is equal to $N/(c_d R^d)$.

The complementary cumulative distribution function (ccdf) of R_n is the probability that there are less than n points in the ball $\mathbb{B}_d(o, r)$:

$$\bar{F}_{R_n}(r) = \sum_{k=0}^{n-1} \binom{N}{k} p^k (1-p)^{N-k}, \quad 0 \le r \le R,$$
(1)

¹Mathematica: PDF[BetaDistribution[a, b],x].

where $p = c_d r^d / c_d R^d = (r/R)^d$.

 \bar{F}_{R_n} can be written in terms of the regularized incomplete beta function as

$$\bar{F}_{R_n}(r) = I_{1-p}(N-n+1,n), \quad 0 \le r \le R,$$
(2)

where

$$I_x(a,b) = \frac{\int_0^x t^{a-1} (1-t)^{b-1} \mathrm{d}t}{B(a,b)}.$$

The pdf of the distance function is $f_{R_n} = -d\bar{F}_{R_n}/dr$ and we have

$$f_{R_n}(r) = -\frac{d}{dr} I_{1-p}(N-n+1,n)$$

$$= -\left(\frac{d(1-p)}{dr}\right) \frac{(1-p)^{N-n} p^{n-1}}{B(N-n+1,n)}$$

$$= \frac{d}{R} \left(\frac{r}{R}\right)^{d-1} \frac{(1-p)^{N-n} p^{n-1}}{B(N-n+1,n)}$$

$$= \frac{d}{R} \frac{(1-p)^{N-n} p^{n-1/d}}{B(N-n+1,n)}$$

$$= \frac{d}{R} \frac{B(n-1/d+1,N-n+1)}{B(N-n+1,n)} \beta\left(\left(\frac{r}{R}\right)^d; n-\frac{1}{d}+1,N-n+1\right)$$
(3)

for $0 \le r \le R$. The final equality casts R_n as a generalized beta-distributed variable. \Box

Corollary 2.2: For the practical cases of d = 1 and d = 2, we have

$$f_{R_n}(r) = \frac{1}{R}\beta\left(\frac{r}{R}; n, N - n + 1\right)$$

and

$$f_{R_n}(r) = \frac{2}{R} \frac{\Gamma(n+\frac{1}{2})\Gamma(N+1)}{\Gamma(n)\Gamma(N+\frac{3}{2})} \beta\left(\frac{r^2}{R^2}; n+\frac{1}{2}, N-n+1\right)$$

respectively.

Fig. 2 plots the distance pdfs for the cases of d = 1 and d = 2.

Remarks:

 The void probability p⁰_B of the point process is defined as the probability of there being no point of the process in the test set B ⊆ W [3]. For a BPP with N points distributed over a set W, it is easy to see that

$$p_B^0 = (1 - \nu(B)/\nu(W))^N.$$
(4)

For the isotropic BPP considered above, when the test set is $B = \mathbb{B}_d(0, r)$, we have $p_B^0 = I_{1-p}(N, 1) = (1-p)^N.$

Of interest in particular are the nearest and farthest neighbor distances. The nearest neighbor distance pdf is given by

$$f_{R_1}(r) = \frac{dN}{r} \left(1 - \left(\frac{r}{R}\right)^d\right)^{N-1} \left(\frac{r}{R}\right)^d,\tag{5}$$

and the distance to the farthest point from the origin is distributed as

$$f_{R_N}(r) = \frac{dN}{r} \left(\frac{r}{R}\right)^{Nd}, \quad 0 \le r \le R.$$
(6)

Both these are distributed as the generalized Kumaraswamy distributions [10].

3) For a one-dimensional BPP, $f_{R_n}(r) = f_{R_{N-n+1}}(R-r)$, and therefore knowledge of the distance pdfs for the nearest $\lceil N/2 \rceil$ neighbors gives complete information on the distance distributions to the other nodes.

We wish to compare the distance distributions for a BPP and a PPP. However, note that in general, the PPP may have fewer points than the number dropped. In order to make a fair comparison, we condition on the fact that there are at least N points present in the PPP model. The following corollary establishes the distance pdfs for such a conditioned PPP. Also note that conditioned on there being exactly N points present, the PPP is equivalent to a BPP [3].

Corollary 2.3: Consider a PPP of density λ over a finite volume $\mathbb{B}_d(o, R)$. Conditioned on

there being at least N points in the ball, the distance distribution from the origin to the n^{th} nearest neighbor ($n \leq N$) is given by

$$f_{R_n}'(r) = \frac{\lambda dc_d r^{d-1} \left(A_{n-1}(r) \left(1 - \sum_{k=0}^{N-n-1} B_k(r) \right) \right)}{1 - \sum_{k=0}^{N-1} e^{-\lambda c_d R^d} (\lambda c_d R^d)^k / k!}, \quad r \in [0, R],$$
(7)

where $A_k(r) := e^{-\lambda c_d r^d} \left(\lambda c_d r^d\right)^k / k!$ and $B_k(r) := e^{-\lambda c_d \left(R^d - r^d\right)} \left(\lambda c_d \left(R^d - r^d\right)\right)^k / k!.$

Proof: The complimentary conditional cdf of R_n is given by

$$\bar{F'}_{R_n}(r) = \Pr\left(\Phi\left(\mathbb{B}_d(o,r)\right) < n | \Phi\left(\mathbb{B}_d(o,R)\right) \ge N\right) \\
= \frac{\Pr\left(\Phi\left(\mathbb{B}_d(o,r)\right) < n, \Phi\left(\mathbb{B}_d(o,R)\right) \ge N\right)}{\Pr\left(\Phi\left(\mathbb{B}_d(o,R)\right) \ge N\right)} \\
\overset{(a)}{=} \frac{\sum_{k=0}^{n-1} \Pr\left(\Phi\left(\mathbb{B}_d(o,r)\right) = k\right) \Pr\left(\Phi\left(\mathbb{B}_d(o,R) \setminus \mathbb{B}_d(o,r)\right) \ge N - k\right)}{\Pr\left(\Phi\left(\mathbb{B}_d(o,R)\right) \ge N\right)} \\
= \frac{\sum_{k=0}^{n-1} A_k(r) \left(1 - \sum_{l=0}^{N-k-1} B_l(r)\right)}{1 - \sum_{k=0}^{N-1} e^{-\lambda c_d R^d} (\lambda c_d R^d)^k / k!},$$
(8)

where (a) is obtained from the property that the number of points in disjoint sets are independent of each other. It is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}r}A_{k}(r) = \begin{cases} \lambda dc_{d}r^{d-1} \left(A_{k-1}(r) - A_{k}(r)\right) & k > 0\\ -\lambda dc_{d}r^{d-1}A_{0}(r) & k = 0 \end{cases}$$
(i)

and

$$\frac{\mathrm{d}}{\mathrm{d}r}B_{l}(r) = \begin{cases} \lambda dc_{d}r^{d-1} \left(B_{l}(r) - B_{l-1}(r)\right) & l > 0\\ \lambda dc_{d}r^{d-1}B_{0}(r) & l = 0. \end{cases}$$
(ii)

Therefore, we have

$$\frac{d}{dr} \sum_{l=0}^{N-k-1} B_l(r) = \lambda dc_d r^{d-1} B_{N-k-1}(r).$$
(iii)

The details of the remainder of the proof are straightforward but tedious and are omitted here. Since the pdf of the conditional distance distribution is $f'_{R_n} = -d\bar{F'}_{R_n}/dr$, one basically has to differentiate the numerator in (8), and after some simplifications using (i)-(iii), it will be seen that the conditional distance pdf is identical to (7).

Fig. 2 depicts the conditional pdfs of the distances for one- and two-dimensional Poisson processes and compares it with the distance distributions for a BPP.

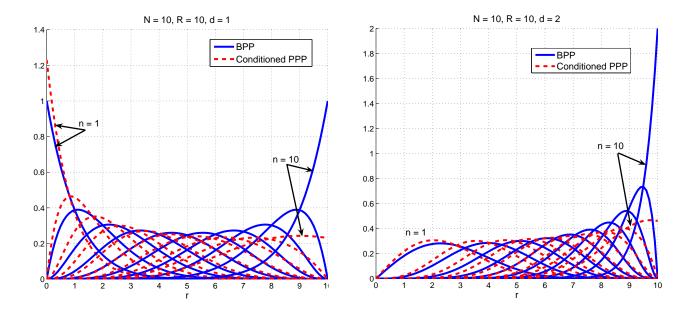


Fig. 2. Distance pdfs for each of the neighbors for one- and two- dimensional binomial and conditioned Poisson networks.

When a large number of nodes are distributed randomly over a large area, their arrangement can be well approximated by an infinite homogeneous PPP. The PPP model for the nodal distribution is ubiquitously used for wireless networks and may be justified by claiming that nodes are dropped from an aircraft in large numbers; for mobile ad hoc networks, it may be argued that terminals move independently of each other. We now present the following corollary to the earlier theorem, which also reproduces a result from [6].

Corollary 2.4: In an infinite PPP with intensity λ on \mathbb{R}^d , the distance R_n , between a point and its n^{th} neighbor is distributed according to the generalized Gamma distribution.

$$f_{R_n}(r) = e^{-\lambda c_d r^d} \frac{d(\lambda c_d r^d)^n}{r\Gamma(n)}, \quad r \in \mathbb{R}.$$
(9)

Proof: If the total number of points N tends to infinity in such a way that the density $\lambda = N/(c_d R^d)$ remains a constant, then the BPP asymptotically (as $R \to \infty$) behaves as a PPP [3]. Taking $R = \sqrt[d]{N/c_d \lambda}$ and applying the limit as $N \to \infty$, we obtain for a PPP,

$$f_{R_n}(r) = \lim_{N \to \infty} \frac{d}{R} \frac{(1-p)^{N-n} p^{n-1/d} \Gamma(N+1)}{\Gamma(N-n+1)\Gamma(n)}$$

$$= \frac{d}{r\Gamma(n)} (\lambda c_d r^d)^n \lim_{N \to \infty} \left(1 - \frac{\lambda c_d r^d}{N}\right)^N \frac{N(N-1)\dots(N-n+1)}{N^n}$$

$$= e^{-\lambda c_d r^d} \frac{d(\lambda c_d r^d)^n}{r\Gamma(n)}.$$

for $r \in \mathbb{R}$. This is an alternate proof to the one provided in [6].

For a PPP, we can also specify how the internodal distances are distributed. By the stationarity of the PPP and Slivnyak's theorem [3], the distance between the origin and its n^{th} nearest neighbor is the same as the distance between an arbitrary point and its n^{th} nearest neighbor. [6] studies distance distributions in a PPP in more detail and shows that the nearest neighbor distribution for any node is exponential and Rayleigh-distributed for the one-dimensional and two-dimensional PPP respectively. However, since the BPP is non-stationary, it is not straightforward to specify how the internodal distances are distributed in that case.

III. Moments of R_n

We now use the distance pdf to compute its moments. The γ^{th} moment of R_n is calculated as follows².

$$\mathbb{E}[R_n^{\gamma}] = \frac{d}{R} \frac{1}{B(N-n+1,n)} \int_0^R \left[r^{\gamma} \left(\frac{r}{R}\right)^{nd-1} \left(1 - \left(\frac{r}{R}\right)^d\right)^{N-n} \right] dr.$$

$$\stackrel{(a)}{=} \frac{R^{\gamma}}{B(N-n+1,n)} \int_0^1 t^{n+\gamma/d-1} (1-t)^{N-n} dt$$

$$= \frac{R^{\gamma}}{B(N-n+1,n)} B_x(n+\gamma/d, N-n+1)|_0^1$$

$$\stackrel{(b)}{=} \begin{cases} \frac{R^{\gamma} \Gamma(N+1) \Gamma(\gamma/d+n)}{\Gamma(n) \Gamma(\gamma/d+N+1)} & \text{if } n+\gamma/d > 0 \\ \infty & \text{otherwise} \end{cases}$$

$$= \begin{cases} R^{\gamma} n^{[\gamma/d]} / (N+1)^{[\gamma/d]} & \text{if } n+\gamma/d > 0 \\ \infty & \text{otherwise}, \end{cases}$$
(10)

where $B_x[a, b]$ is the incomplete beta function³ and $x^{[n]} = \Gamma(x+n)/\Gamma(x)$ denotes the rising Pochhammer symbol notation. Here, (a) is obtained on making the substitution $r = Rt^{1/d}$ and (b) using the following identities:

$$B_0(a,b) = \begin{cases} 0 & \mathcal{R}e(a) > 0 \\ -\infty & \mathcal{R}e(a) \le 0, \end{cases}$$

and $B_1(a,b) = B(a,b)$ if $\mathcal{R}e(b) > 0$.

The expected distance to the n^{th} nearest neighbor is thus

$$\mathbb{E}(R_n) = \frac{Rn^{[1/d]}}{(N+1)^{[1/d]}},\tag{11}$$

 $^{^2 \}mathrm{Note}$ that $\gamma \in \mathbb{R}$ in general, and is not restricted to being an integer.

³Mathematica: Beta[x, a, b].

and the variance of R_n is easily calculated as

$$\operatorname{Var}[R_n] = \frac{R^2 n^{[2/d]}}{(N+1)^{[2/d]}} - \left(\frac{R n^{[1/d]}}{(N+1)^{[1/d]}}\right)^2.$$
 (12)

Remarks:

- For one-dimensional networks, E[R_n] = Rn/(N + 1). Thus, on an average, it is as if the points are arranged regular lattice. In particular, when N is odd, the middle point is located exactly at the center on average.
- On the other hand, as d → ∞, E[R_n] → R and it is as if all the points are equidistant at maximum distance R from the origin.
- 3) In the general case, the mean distance of the n^{th} neighbor varies as $n^{1/d}$ for large n. This follows from the series expansion of the Pochhammer sequence [11]

$$n^{[q]} = n^q \left(1 - \mathcal{O}\left(1/n\right)\right)$$

Also, the variance $\rightarrow 0$ as *n* increases for d > 2. This is also observed in the case of a Poisson network [6].

 The mean internodal distance between the *i*th and *j*th nearest neighbors from the origin is simply given by (assuming *i* < *j*)

$$\mathbb{E}[d_{ij}] = \frac{R\left(j^{[1/d]} - i^{[1/d]}\right)}{(N+1)^{[1/d]}}$$

5) For the special case of $\gamma/d \in \mathbb{Z}$, we obtain

$$\mathbb{E}[R_n^{\gamma}] = R^{\gamma} \binom{n + \gamma/d - 1}{\gamma/d} \bigg/ \binom{N + \gamma/d}{\gamma/d},$$

IV. APPLICATIONS TO WIRELESS NETWORKS

We now apply the results obtained in the previous section to binomial networks. For the system model, we assume a d-dimensional network over a ball $\mathbb{B}_d(o, R)$, where N nodes are

uniformly randomly distributed. Nodes are assumed to communicate with a base station (BS) positioned at the origin. The attenuation in the channel is modeled by the large scale path loss function g with PLE α , i.e., $g(x) = ||x||^{-\alpha}$. The channel access scheme is taken to be the slotted ALOHA with contention parameter p.

A. Energy Consumption

Since battery life in wireless nodes are limited, energy consumption is a key design issue. The energy that is required to successfully deliver a packet over a distance r in a medium with PLE α is proportional to r^{α} . Therefore, the average energy required to deliver a packet from the n^{th} nearest neighbor to the BS is given by (10), with $\gamma = \alpha$. This approximately scales as $n^{\alpha/d}$ when the routing is taken over single hops. When $\alpha < d$, its more energy-efficient to use longer hops than when the PLE is greater than the number of dimensions.

B. Localization

In this section, we investigate conditional distance distributions and study their usefulness to localization algorithms, where it is beneficial to use as few beacons as possible for the estimation process.

Assume that the nodes in the network can talk to each other and determine how many other nodes are closer to the BS than they are, based on the strength of the received signal from the BS. However, the channel may be time-varying and hence, it might not be possible for all the nodes to determine their exact locations. In such situations, a few beacons can be employed that estimate or even precisely measure their distances from the BS. What can be said about the distance distributions of the other nodes given this information?

Suppose we know that the k^{th} nearest neighbor is at distance s from the center. Then, clearly, the first n-1 nodes are uniformly randomly distributed in $\mathbb{B}_d(o, s)$ while the more distant nodes are uniformly randomly distributed in $\mathbb{B}_d(o, s)$. Following (3), the distance distributions

of the first k-1 nearest neighbors from the origin can be written as

$$f_{R_n}(r|R_k = s) = \frac{d}{s} \frac{B(n - 1/d + 1, k - n)}{B(k - n, n)} \beta\left(\left(\frac{r}{s}\right)^d; n - \frac{1}{d} + 1, k - n\right), \quad n < k$$

for $0 \le r \le s$, which again follows a generalized beta distribution.

For the remaining nodes i.e., for n > k, we have in $r \in [s, R]$,

$$f_{R_n}(r|R_k = s) = -\frac{d}{dr} I_{1-q}(N-n+1, n-k)$$

= $\frac{dr^{d-1}}{R^d - s^d} \frac{(1-q)^{N-n} q^{n-k-1}}{B(N-n+1, n-k)}$

where $q = (r^{d} - s^{d})/(R^{d} - s^{d})$.

The moments of R_n are also straightforward to obtain. Following (10), we see that for n < kand $n + \alpha/d > 0$,

$$\mathbb{E}[R_n^{\alpha}|R_k=s] = \frac{s^{\alpha}n^{[\alpha/d]}}{(k+1)^{[\alpha/d]}}.$$
(13)

For n > k, we have

$$\mathbb{E}[R_n^{\alpha}|R_k = s] = \int_s^R \frac{dr^{\alpha+d-1}}{R^d - s^d} \frac{(1-q)^{N-n} q^{n-k-1}}{B(N-n+1, n-k)} dr$$

= $\frac{1}{B(N-n+1, n-k)} \int_0^1 q^{n-k-1} (1-q)^{N-n} \left(q \left(R^d - s^d\right) + s^d\right)^{\alpha/d} dt$
= $\frac{s^{\alpha}}{(n-k)B(N-n+1, n-k)} F_1\left(n-k; n-N, -\frac{\alpha}{d}; n-k+1; 1, 1-\frac{R^d}{s^d}\right),$

where $F_1[a; b_1, b_2; c; x, y]$ is the Appell hypergeometric function of two variables⁴.

Often, it is easiest to measure the nearest neighbor distance. Give this distance as s, we have for n > 1,

$$f_{R_n}(r|R_1 = s) = \frac{dr^{d-1}}{R^d - s^d} \frac{\left(1 - \left(\frac{r^d - s^d}{R^d - s^d}\right)\right)^{N-n} \left(\frac{r^d - s^d}{R^d - s^d}\right)^{n-2}}{B(N - n + 1, n - 1)}$$

⁴Mathematica: AppellF1[a, b1, b2, c, x, y].

for $r \in [s, R]$. Also, the mean conditional distances to the remaining neighbors are

$$\mathbb{E}[R_n|R_1=s] = \frac{s}{(n-1)B(N-n+1,n-1)}F_1\left(n-1;n-N,-\frac{1}{d};n-1;1,1-\frac{R^d}{s^d}\right).$$

C. Interference

In order to accurately determine network parameters such as outage, throughput or transmission capacity, the interference in the system I needs to be known. The mean interference as seen at the center of the network is given by

$$\mu_{I} = \mathbb{E}\left[\sum_{n=1}^{N} (T_{n}R_{n}^{-\alpha})\right] = \sum_{n=1}^{N} \mathbb{E}[T_{n}]\mathbb{E}\left[R_{n}^{-\alpha}\right],$$
$$= p\sum_{n=1}^{N} \mathbb{E}\left[R_{n}^{-\alpha}\right],$$

where $T_n \in \{0, 1\}$ are i.i.d Bernoulli variables (with parameter p) representing whether the n^{th} nearest node transmits or not in a particular time slot.

Setting $\gamma = -\alpha$ in (10), we can conclude that the mean interference is infinite for $d \leq \alpha$. When the number of dimensions is greater than the PLE, we have

$$\mu_I = \frac{pR^{-\alpha}\Gamma(N+1)}{\Gamma(N+1-\alpha/d)} \sum_{n=1}^N \frac{\Gamma(n-\alpha/d)}{\Gamma(n)}.$$

One can inductively verify that

$$\sum_{n=1}^{k} \frac{\Gamma(n-\alpha/d)}{\Gamma(n)} = \frac{\Gamma(k-\alpha/d)}{\Gamma(k)} \frac{k-\alpha/d}{1-\alpha/d} \quad \forall k \in \mathbb{Z},$$
(14)

and we obtain after some simplifications,

$$\mu_I = \frac{NpdR^{-\alpha}}{d-\alpha}, \quad d > \alpha.$$
(15)

The unboundedness of the mean interference at practical values of d and α (i.e., $d < \alpha$) actually occurs due to the fact that the path loss model we employ breaks down for very small distances.

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In fact, it exhibits a singularity at x = 0. In order to overcome this, some authors employ a modified path loss law given by $\min\{1, ||x||^{-\alpha}\}$. Employing the modified path loss law, the mean interference is given by [12] (assuming R > 1)

$$\mu_{I} = \begin{cases} \frac{Npd}{R^{d}} \left[\frac{1}{d} + \frac{R^{d-\alpha} - 1}{d-\alpha} \right] & d \neq \alpha \\ \frac{Npd}{R^{d}} \left[1/d + \ln(R) \right] & d = \alpha \end{cases}$$
(16)

D. Connectivity

We now study the connectivity properties of the binomial network. Define a node to be connected to the origin if the SINR at the BS is greater than a threshold Θ . Let the nodes transmit at unit power and assume noise to be AWGN with variance N_0 . In the absence of interference, the probability that the BS is connected to its n^{th} nearest neighbor is

$$= \Pr(R_{n}^{-\alpha} > N_{0}\Theta)$$

$$= 1 - \Pr(R_{n} > (N_{0}\Theta)^{-1/\alpha})$$

$$= \begin{cases} 1 - I_{1-p'}(N - n + 1, n) & \Theta > R^{-\alpha}/N_{0} \\ 1 & \Theta \le R^{-\alpha}/N_{0}, \end{cases}$$
(17)

where $p' = \left((N_0 \Theta)^{-1/\alpha} / R \right)^d$. Fig. 3 plots the connectivity probability in a two-dimensional binomial network with 25 nodes.

E. Outage Probability

An outage \mathcal{O} is defined to occur if the SINR at the BS is lower than a certain threshold Θ . Let the desired transmitter be located at unit distance from the origin, transmit at unit power and also not be a part of the original point process. Assuming that the system is interference-limited,

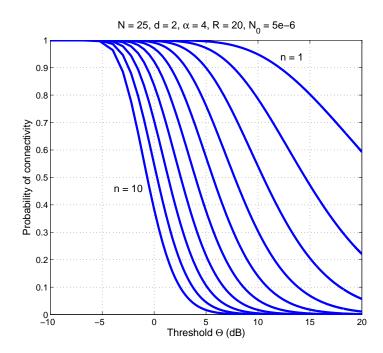


Fig. 3. The probability of the n^{th} nearest neighbor being connected to the BS for a binomial network with 25 nodes.

we have

$$Pr(\mathcal{O}) = Pr[1/I < \Theta]$$

= $Pr[I > 1/\Theta].$ (18)

Considering only the interference contribution from the nearest neighbor to the origin, a simple lower bound is established on the outage probability as

$$\Pr(\mathcal{O}) \geq \Pr\left(T_1 R_1^{-\alpha} > 1/\Theta\right)$$

$$= p \Pr\left(R_1 < \Theta^{1/\alpha}\right)$$

$$= \begin{cases} p\left(1 - \left(1 - \frac{\Theta^{d/\alpha}}{R^d}\right)^N\right) & \Theta \le R^{\alpha} \\ p & \Theta > R^{\alpha}. \end{cases}$$
(19)

The empirical values of success probabilities and their upper bounds (19) are plotted for different values of N in Fig. 4. As the plot depicts, the bounds are tight for lower values of N and Θ , and therefore we conclude that the nearest neighbor contributes to a major portion of the network interference. However, note that as α decreases, the bound gets looser since the contributions from the farther neighbors are also increased.

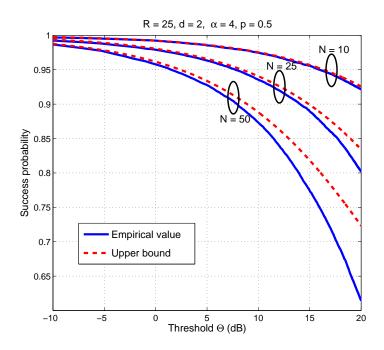


Fig. 4. Comparison of exact success probabilities versus their upper bounds for different values of the system parameters.

F. Other Applications

We now list a few other areas where knowledge of the distance distributions is useful.

• *Routing*: The question of whether to route over smaller or longer hops is an important, yet a nontrivial issue [13], [14]. The knowledge of internodal distances is necessary for evaluating the optimum hop distance and maximizing the progress of a packet towards its destination.

• *Path loss exponent estimation*: The issue of PLE estimation is a very important and relevant problem [15]. Several PLE estimation algorithms are based on received signal strength techniques and these require the knowledge of distances between nodes.

V. CONCLUDING REMARKS

We point out that the Poisson model for nodal distributions in wireless networks is not accurate in many practical situations, and consider the more realistic binomial network model. We derive exact analytical expressions for the pdfs of the distances to the n^{th} nearest neighbor from the origin in a general *d*-dimensional isotropic random network of radius *R*. We also analytically express the moments of these generalized beta-distributed variables in closed-form. Our findings have applications in several problems related to wireless networks such as energy consumption, connectivity, localization, interference characterization and outage evaluation.

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