Impact of CFO Estimation on the Performance of ZF Receiver in Massive MU-MIMO Systems

Sudarshan Mukherjee, Saif Khan Mohammed and Indra Bhushan Abstract

In this paper, we study the impact of carrier frequency offset (CFO) estimation/compensation on the information rate performance of the zero-forcing (ZF) receiver in the uplink of a multi-user massive multiple-input multiple-output (MIMO) system. Analysis of the derived closed-form expression of the per-user information rate reveals that with increasing number of BS antennas M, an $\mathcal{O}(\sqrt{M})$ array gain is achievable, which is same as that achieved in the ideal zero CFO scenario. Also it is observed that compared to the ideal zero CFO case, the performance degradation in the presence of residual CFO (after CFO compensation) is the same for both ZF and MRC.

Index Terms

Massive MIMO, carrier frequency offset (CFO), multi-user, array gain, zero-forcing (ZF).

I. INTRODUCTION

In the past few years, massive multiple-input multiple-output (MIMO) systems have emerged as one of the key technologies in the evolution of the next generation 5G wireless systems due to their ability to support high data rate and improved energy efficiency [1], [2]. In a massive multi-user (MU) MIMO system, the base station (BS) is provided with hundreds of antennas to simultaneously serve only a few tens of single-antenna user terminals (UTs) in the same timefrequency resource [3]. Increasing the number of BS antennas open up more available degrees of freedom, which helps accommodate more number of users, thus improving the achievable spectral efficiency [4], [5]. At the same time, the required radiated power to achieve a fixed desired information rate can be reduced with increasing number of BS antennas, M (array gain). It has been shown that even with imperfect channel state information (CSI), the achievable array gain for any sub-optimal linear receiver (e.g. zero-forcing (ZF), maximum ratio combining (MRC) etc.) is $O(\sqrt{M})$ [6].

Above results assume perfect frequency synchronization at the BS receiver, without which the performance of the system would deteriorate rapidly. In practice acquiring perfect knowledge of the carrier frequency offsets (CFOs) between the received user signals at the BS and the frequency of the BS oscillator is however a challenging task. There exists various techniques

The authors are with the Department of Electrical Engineering, Indian Institute of Technology (I.I.T.) Delhi, India. Saif Khan Mohammed is also associated with Bharti School of Telecommunication Technology and Management (BSTTM), I.I.T. Delhi. Email: saifkmohammed@gmail.com. This work is supported by EMR funding from the Science and Engineering Research Board (SERB), Department of Science and Technology (DST), Government of India.

for CFO estimation and compensation for conventional small MIMO systems in the literature [7]-[10]. However these algorithms incur tremendous increase in computational complexity with increasing number of BS antennas, M and increasing number of UTs, K (i.e. massive MIMO scenario). Recently in [11] an approximation to the joint ML (Maximum Likelihood) CFO estimation has been proposed for massive MIMO system. However this technique requires a multi-dimensional grid search and therefore has high complexity with large number of UTs.

In [12] the authors propose a simple low complexity algorithm for CFO estimation and a corresponding communication strategy for massive MU-MIMO uplink. It has been shown that with sufficiently large M, the algorithm has only $\mathcal{O}(M)$ complexity (independent of the number of UTs). However the impact of the residual CFO (due to CFO compensation) on the performance of massive MIMO is yet to be studied. The most common linear suboptimal receivers used in massive MIMO uplink are MRC (maximum ratio combining) and ZF (zero-forcing) receivers. With the MRC receiver, system performance is limited by the multi-user interference (MUI) in the high SNR regime. For the ideal zero CFO scenario, the ZF receiver is known to remove this limitation by eliminating the MUI [6]. In this work we therefore study the impact of the residual CFO error (due to the CFO estimation strategy proposed in [12]) on the achievable information rate of the ZF receiver and compare it to that of the MRC receiver. To the best of our knowledge, this paper is the first to report such a study.

The contributions of our paper are as follows: (i) we have derived a closed-form expression for an achievable information rate for the ZF receiver with MMSE (minimum mean square error) channel estimation and CFO compensation. A closed-form expression for the same is also derived for MRC; (ii) analysis of the ZF information rate expression reveals that an $\mathcal{O}(\sqrt{M})$ array gain is achievable. This is very interesting since even for the ideal zero CFO scenario, the best possible array gain is known to be $\mathcal{O}(\sqrt{M})$ only [6]; (iii) for the same desired per-user information rate, the SNR gap (i.e. the extra SNR required by MRC when compared to ZF) does not degrade with CFO estimation/compensation, when compared to the ideal zero CFO case. This suggests that compared to the ideal zero CFO case, the performance degradation in the presence of residual CFO (due to compensation) is the same for both ZF and MRC. [Notations: \mathbb{C} denotes the set of complex numbers. \mathbb{E} denotes the expectation operator. (.)^H denotes the complex conjugate transpose operation, while (.)* denotes the complex conjugate operator. Also, I_N denotes the $N \times N$ identity matrix and A_{mk} (or $(A)_{mk}$) denotes the (m, k)-th element of matrix A.] II. SYSTEM MODEL

We consider a frequency-flat massive MU-MIMO uplink (UL) channel, where the massive MIMO BS is equipped with M BS antennas and is coherently communicating with K single

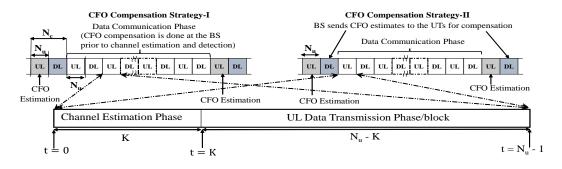


Fig. 1 The communication strategy: CFO Estimation and Compensation Strategies and Data Communication. antenna UTs simultaneously in the same time-frequency resource. Therefore for a massive MU-MIMO BS, acquisition and compensation of CFOs from different UTs is important. Since in massive MIMO, the BS is expected to operate in time division duplexed (TDD) mode, the coherence interval (of N_c channel uses) consists of a UL slot (N_u channel uses), followed by a downlink (DL) slot ($N_c - N_u$ channel uses). As for the communication strategy (see in Fig. 1), we perform CFO estimation in a special UL slot prior to the data communication. For CFO estimation, we adopt the CFO estimation strategy presented in [12]. CFO compensation can be performed in two different ways - (i) at the BS (prior to channel estimation and multiuser detection); or (ii) at the respective UTs prior to data transmission (this however requires transmission of CFO estimates from the BS to the UTs over a control channel in the DL slot, following the special UL slot for CFO estimation). Data communication starts from the first UL slot, following the special CFO estimation UL slot. In these UL slots, prior to UL data transmission, the UTs transmit pilots for channel estimation.¹ The special UL/DL slot for CFO estimation might be repeated every few coherence intervals, depending on how fast the CFOs change.

A. CFO Estimation Strategy in [12]

For the CFO estimation phase, special pilots are transmitted by the UTs in the uplink. A pilot sequence of length $N \leq N_u$ is divided into B = N/K pilot-blocks, where each pilot-block is K channel uses long.² Each UT transmits only a single impulse of amplitude $\sqrt{Kp_u}$ in each pilot-block. Therefore in the b^{th} pilot-block, the k^{th} UT transmits impulse at $t = \tau(b, k) =$ (b-1)K+k-1, where k = 1, 2, ..., K and b = 1, 2, ..., B. The pilot signal received at the m^{th} BS antenna at time $\tau(b, k)$ is therefore given by $r_m[\tau(b, k)] = \sqrt{Kp_u} g_{mk} e^{j\omega_k \tau(b,k)} + w_m[\tau(b, k)]$, where $\omega_k \stackrel{\Delta}{=} 2\pi \Delta f_k T_s$ is the CFO for the k^{th} user $(T_s = 1/B_c)$, where B_c is the communication bandwidth and Δf_k is the frequency offset of the k^{th} UT). Also $g_{mk} \stackrel{\Delta}{=} h_{mk} \sqrt{\beta_k}, m = 1, 2, ..., M$ and k = 1, 2, ..., K, is independent complex baseband frequency-flat channel gain coefficient

¹The same CFO estimates can be used for CFO compensation prior to precoding in the DL slot of each coherence interval.

²Although the CFO estimation method assumes N/K to be integer, we can accommodate non-integer values of N/K, by defining the number of blocks $B \stackrel{\Delta}{=} \lceil N/K \rceil$. Hence for non-integer N/K, the B^{th} block is less than K channel uses long. Therefore the effective N/K is $\lceil N/K \rceil$ for UTs allowed to transmit in the B^{th} block, and it is $(\lceil N/K \rceil - 1)$ for all other UTs.

between the m^{th} BS antenna and the k^{th} UT and $h_{mk} \sim C\mathcal{N}(0,1)$.³ $\sqrt{\beta_k} > 0$ models the geometric attenuation factor for the k^{th} user and $w_m[\tau(b,k)] \sim C\mathcal{N}(0,\sigma^2)$ is the complex circular symmetric AGWN noise with variance σ^2 . The estimate of the CFO of the k^{th} UT, $\hat{\omega}_k$, is obtained as the principal argument of the block-wise correlation term of the pilot sequence received from the

$$k^{\text{th}} \text{ user, i.e., } \widehat{\omega}_k = \frac{1}{K} \arg(\rho_k),^4 \text{ where}^5 \rho_k \triangleq \frac{\sum\limits_{b=1}^{N} \sum\limits_{m=1}^{m} r_m^*[\tau(b,k)]r_m[\tau(b+1,k)]}{MK(B-1)p_{\mathrm{u}}\beta_k} = G_k e^{j\omega_k K} + \nu_k, \text{ and}$$
$$G_k \triangleq \frac{1}{M} \sum\limits_{m=1}^{M} |h_{mk}|^2.$$

Remark 1. Note that the above CFO estimate is well-defined iff $|\omega_k K| < \pi$. For most practical massive MIMO systems, this condition will hold true [12]. Also, from the strong law of large numbers it can be shown that for i.i.d. h_{mk} , $G_k \to 1$ as $M \to \infty$.

Result 1. (Approximation of the CFO Estimate in [12]): If $|\omega_k K| \ll \pi$ and $\gamma \stackrel{\Delta}{=} \frac{p_u}{\sigma^2} \gg \gamma^0$, then the above CFO estimate can be approximated by $\widehat{\omega}_k = \frac{1}{K} \arg(\rho_k) \approx \omega_k + \frac{\nu_k^Q}{G_k K}$, where $\gamma^0 \stackrel{\Delta}{=} \frac{\frac{B-1}{2B-3}}{KG_k \left[\sqrt{1+2M\frac{(B-1)^3}{(2B-3)^2}} - 1\right]}$ and $\nu_k^Q \stackrel{\Delta}{=} \Im(\nu_k)$. Note that $(\widehat{\omega}_k - \omega_k) \sim \mathcal{N}(0, \sigma_{\omega_k}^2)$, where $\sigma_{\omega_k}^2$ is the mean square error (MSE) given by

b) given by

$$\sigma_{\omega_k}^2 \stackrel{\Delta}{=} \mathbb{E}[(\widehat{\omega}_k - \omega_k)^2] \approx \frac{\frac{1}{\gamma\beta_k} \left(\frac{G_k}{B-1} + \frac{1}{2K\gamma\beta_k}\right)}{M(N-K)K^2 G_k^2}.$$
(1)

Remark 2. Clearly with $M \to \infty$, we have $\gamma^0 \propto \frac{1}{\sqrt{M}}$. Therefore we choose some constant $c_0 > 0$ such that $\gamma = \frac{c_0}{\sqrt{M}} \gg \gamma^0$ as $M \to \infty$, thereby satisfying the required condition $\gamma \gg \gamma^0$ in Result 1. From (1) we note that with $\gamma = \frac{c_0}{\sqrt{M}}$ and $M \to \infty$ (fixed N, K), we have $\mathbb{E}[(\hat{\omega}_k - \omega_k)^2] \approx \frac{1/c_0^2}{2K^3(N-K)\beta_k^2}$, since $\lim_{M\to\infty} G_k = 1$. This shows that the MSE for CFO estimation approaches a constant value as $M \to \infty$ with $\gamma \propto 1/\sqrt{M}$ for fixed K and fixed N. Note that with $M \to \infty$, the desired MSE $\propto 1/c_0^2$, i.e., a smaller desired MSE can be attained using a higher value of c_0 . Therefore for a target/desired MSE, a sufficiently large M must be chosen so that the required power ($\propto 1/\sqrt{M}$) for CFO estimation is within the desired limits.

B. Uplink Data Communication

After CFO estimation, CFO compensation can be performed in one of the following two ways: (a) the BS can feed the individual CFO estimates back to the corresponding UTs over a control channel in the DL slot, following the special UL slot (see Fig. 1). In this way, the k^{th} UT would correct its CFO by rotating the transmit signal at the t^{th} channel use by $e^{-j\widehat{\omega}_k t}$. However in this method there is possibility of corruption of the estimates due to error in the control channel; (b) another way of correcting frequency offsets is to perform CFO compensation at the BS,

³Independent and identically distributed Rayleigh fading is a commonly used model for the distribution of channel gains in a massive MIMO system [3]–[5].

⁴Here $\arg(c)$ denotes the 'principal argument' of the complex number c.

⁵For expression of ν_k see [12].

prior to channel estimation and multi-user detection. In this paper, we would use this second technique for CFO compensation and study the impact of the residual CFOs (i.e. $\hat{\omega}_k - \omega_k$) on the performance of massive MIMO uplink.⁶ The uplink data communication starts at t = 0(see Fig. 1). We assume that in the first K consecutive channel uses, the UTs transmit pilots for channel estimation sequentially in time⁷, i.e., the kth UT transmits an impulse of amplitude $\sqrt{K p_u}$ only in the $(k-1)^{\text{th}}$ channel use. The received pilot at the mth BS antenna at t = k - 1is therefore given by $r_m[k-1] = \sqrt{K p_u} g_{mk} e^{j\omega_k(k-1)} + w_m[k-1]$, where $k = 1, 2, \ldots, K$ and $m = 1, 2, \ldots, M$. $w_m[k-1] \sim C\mathcal{N}(0, \sigma^2)$ is the circular symmetric AWGN.

C. MMSE Channel Estimation

To estimate the channel gains, firstly, CFO compensation is performed on the received pilots at the BS. Since the pilots from different UTs are separated in time, pilot from the k^{th} UT after compensation is given by $y_m[k-1] = r_m[k-1]e^{-j\widehat{\omega}_k(k-1)} = \sqrt{K p_u} \widetilde{g}_{mk} + n_{mk}[k-1]$, where⁸ $\widetilde{g}_{mk} \stackrel{\Delta}{=} g_{mk} e^{-j\Delta\omega_k(k-1)} \sim C\mathcal{N}(0,1)$, $n_{mk}[k-1] \stackrel{\Delta}{=} w_m[k-1]e^{-j\widehat{\omega}_k(k-1)} \sim C\mathcal{N}(0,\sigma^2)$ and $\Delta\omega_k \stackrel{\Delta}{=} \widehat{\omega}_k - \omega_k$ is the residual CFO error. Next we compute the minimum mean square estimate (MMSE) of the effective channel gain coefficient \widetilde{g}_{mk} as given below:

$$\widehat{g}_{mk} = \frac{\sqrt{Kp_{u}}\beta_{k}}{Kp_{u}\beta_{k} + \sigma^{2}}y_{m}[k-1] = \frac{\sqrt{Kp_{u}}\beta_{k}}{Kp_{u}\beta_{k} + \sigma^{2}}\Big(\sqrt{Kp_{u}}g_{mk}e^{-j\Delta\omega_{k}(k-1)} + n_{mk}[k-1]\Big).$$
(2)

where m = 1, 2, ..., M and k = 1, 2, ..., K. Using (2), the estimate of the effective channel gain matrix is given by $\widehat{C} = (\sqrt{Kn}C\Phi_0 + N)\widetilde{D}$ (3)

where
$$\widehat{\boldsymbol{G}} \triangleq [\widehat{g}_{mk}]_{M \times K}$$
, $\boldsymbol{G} \triangleq [g_{mk}]_{M \times K}$, and $\Phi_0 \triangleq diag(1, e^{-j\Delta\omega_2}, \cdots, e^{-j\Delta\omega_K(K-1)})$. Here
 $\boldsymbol{N} = [n_{mk}[k-1]]_{M \times K}$ and $\widetilde{\boldsymbol{D}} \triangleq (\sqrt{Kp_u}\boldsymbol{I}_K + \frac{\sigma^2}{\sqrt{Kp_u}}\boldsymbol{D}^{-1})^{-1}$, where $\boldsymbol{D} \triangleq diag(\beta_1, \beta_2, \cdots, \beta_K)$.

III. UPLINK RECEIVER PROCESSING

In this section we formulate a generalized approach towards multi-user receiver processing at the massive MIMO BS. From Fig. 1 it is clear that uplink data transmission begins at the $t = K^{\text{th}}$ channel use and continues till the $(N_u - 1)^{\text{th}}$ channel use, where N_u is the duration of the UL slot. Let $\sqrt{p_u} x_k[t]$ be the information symbol transmitted by the k^{th} UT at the t^{th} channel use. The signal received at the m^{th} BS antenna in the t^{th} channel use is given by $r_m[t] = \sqrt{p_u} \sum_{q=1}^{K} g_{mq} e^{j\omega_q t} x_q[t] + w_m[t]$, where m = 1, 2, ..., M. To detect information symbols

⁶Note that the information theoretic performance is identical for both the CFO compensation techniques.

⁷Though impulse type pilots are not amenable to practical implementation (due to high peak-to-average-power ratio (PAR)), we use them because our main objective is to study the first order effects of system parameters M, K, N, p_u etc. on the information rate performance of ZF and MRC detectors in the presence of residual CFOs.

⁸Both g_{mk} and $w_m[k-1]$ have uniform phase distribution (i.e. circular symmetric) and are independent of each other. Clearly, rotating these random variables by fixed angles (for a given realization of CFOs and its estimates) would not change the distribution of their phases and they will remain independent. Therefore the distribution of \tilde{g}_{mk} and $n_{mk}[k-1]$ would be same as that of g_{mk} and $w_m[k-1]$ respectively.

of the k^{th} UT, first, CFO compensation is performed, followed by detection using the detector for the k^{th} UT. In this paper, we only consider linear detectors. We also assume $x_k[t] \sim C\mathcal{N}(0,1)$, for $t = K, \dots, N_u - 1$ and are i.i.d. Let $\mathbf{r}[t] \triangleq (r_1[t], r_2[t], \dots, r_M[t])^T$. The detected signal from the k^{th} UT at the BS after CFO compensation is given by

$$\widehat{x}_{k}[t] = \underbrace{\mathbf{a}_{k}^{H} \mathbf{r}[t]}_{\text{Detection}} \underbrace{\underbrace{\mathbf{e}_{cro}}_{\text{compensation}}}_{\text{cmpensation}} = \sum_{m=1}^{M} a_{mk}^{*} r_{m}[t] \, e^{-j\widehat{\omega}_{k}t} = \sqrt{p_{u}} \sum_{q=1}^{K} \left(\underbrace{\sum_{m=1}^{M} a_{mk}^{*} g_{mq}}_{\stackrel{\text{det}}{=} \mathbf{a}_{k}^{H} \mathbf{g}_{q}} e^{j(\omega_{q} - \widehat{\omega}_{k})t} x_{q}[t] \right) + \underbrace{\sum_{m=1}^{M} a_{mk}^{*} n_{mk}[t]}_{\stackrel{\text{det}}{=} \mathbf{a}_{k}^{H} \mathbf{g}_{q}}$$

$$= \sqrt{p_{u}} \boldsymbol{a}_{k}^{H} \boldsymbol{g}_{k} e^{-j\Delta\omega_{k}t} x_{k}[t] + \sqrt{p_{u}} \sum_{q=1,q\neq k}^{K} \boldsymbol{a}_{k}^{H} \boldsymbol{g}_{q} e^{-j\Delta\omega_{q}t} x_{q}[t] e^{j(\widehat{\omega}_{q}-\widehat{\omega}_{k})t} + \boldsymbol{a}_{k}^{H} \boldsymbol{n}_{k}[t]$$

$$= \sqrt{p_{u}} \boldsymbol{a}_{k}^{H} \widetilde{\boldsymbol{g}}_{k} e^{-j\Delta\omega_{k}(t-(k-1))} x_{k}[t] + \sqrt{p_{u}} \sum_{q=1,q\neq k}^{K} \boldsymbol{a}_{k}^{H} \widetilde{\boldsymbol{g}}_{q} e^{-j\Delta\omega_{q}(t-(q-1))} x_{q}[t] e^{j(\widehat{\omega}_{q}-\widehat{\omega}_{k})t} + \boldsymbol{a}_{k}^{H} \boldsymbol{n}_{k}[t], \qquad (4)$$

where $\boldsymbol{a}_{k} \stackrel{\Delta}{=} (a_{1k}, a_{2k}, \cdots, a_{Mk})^{T} \in \mathbb{C}^{M \times 1}$ is the linear detector for the k^{th} user, $\Delta \omega_{k} = \widehat{\omega}_{k} - \omega_{k}$ and $n_{mk}[t] \stackrel{\Delta}{=} w_{m}[t]e^{-j\widehat{\omega}_{k}t}$. Also, $\boldsymbol{n}_{k}[t] \stackrel{\Delta}{=} (n_{1k}[t], n_{2k}[t], \cdots, n_{Mk}[t])^{T}$, $\boldsymbol{g}_{q} \stackrel{\Delta}{=} (g_{1q}, g_{2q}, \cdots, g_{Mq})^{T}$ (the q^{th} column of \boldsymbol{G}) and $\widetilde{\boldsymbol{g}}_{q} \stackrel{\Delta}{=} (\widetilde{g}_{1q}, \widetilde{g}_{2q}, \cdots, \widetilde{g}_{Mq})^{T} = \boldsymbol{g}_{q} e^{-j\Delta\omega_{q}(q-1)}$.

A. Coding Strategy

We define the effective channel estimation error as $\epsilon_{mk} \triangleq \widehat{g}_{mk} - \widetilde{g}_{mk}$ (see Section II-C). Let $\epsilon_k \triangleq (\epsilon_{1k}, \epsilon_{2k}, \cdots, \epsilon_{Mk})^T = \widehat{g}_k - \widetilde{g}_k$, where $\widehat{g}_k = (\widehat{g}_{1k}, \widehat{g}_{2k}, \cdots, \widehat{g}_{Mk})^T$ (kth column of \widehat{G}). The mean vector and the covariance matrix of ϵ_k are respectively given by $\mathbb{E}[\epsilon_k] = 0$ and

$$\mathbb{E}[\boldsymbol{\epsilon}_{k}\boldsymbol{\epsilon}_{k}^{H}] = \mathbb{E}\left[\left(\underbrace{\left(\frac{Kp_{u}\beta_{k}}{Kp_{u}\beta_{k}+\sigma^{2}}-1\right)\widetilde{\boldsymbol{g}}_{k}+\frac{\sqrt{Kp_{u}}\beta_{k}}{Kp_{u}\beta_{k}+\sigma^{2}}\boldsymbol{n}_{k}[k-1]}_{\widehat{\boldsymbol{g}}_{k}-\widetilde{\boldsymbol{g}}_{k}}\right)^{H}\right] \stackrel{(a)}{=} \frac{\beta_{k}\sigma^{2}}{Kp_{u}\beta_{k}+\sigma^{2}}\boldsymbol{I}_{M}, \quad (5)$$

where (a) follows from (2) and $n_k[k-1] = (n_{1k}[k-1], \cdots, n_{Mk}[k-1])^T$. Using $\tilde{g}_k = \hat{g}_k - \epsilon_k$ in (4), we get

$$\widehat{x}_{k}[t] = \underbrace{\sqrt{p_{u}} a_{k}^{H} \widehat{g}_{k} e^{-j\Delta\omega_{k}(t-(k-1))}}_{\triangleq S_{k}[t]} x_{k}[t] + \underbrace{a_{k}^{H} n_{k}[t]}_{\triangleq EN_{k}[t]} \\
+ \sqrt{p_{u}} \left(\sum_{\substack{q=1, \\ q \neq k}}^{K} \left(a_{k}^{H} \left(\widehat{g}_{q} - \epsilon_{q} \right) e^{-j\Delta\omega_{q}(t-(q-1))} x_{q}[t] \right) e^{j(\widehat{\omega}_{q} - \widehat{\omega}_{k})t} - a_{k}^{H} \epsilon_{k} e^{-j\Delta\omega_{k}(t-(k-1))} x_{k}[t] \right) \\
\stackrel{\triangleq \text{MUI}_{k}[t]}{= \underbrace{\mathbb{E}} \left[S_{k}[t] \right] x_{k}[t] + \underbrace{\left(S_{k}[t] - \mathbb{E} \left[S_{k}[t] \right] \right) x_{k}[t]}_{\triangleq SIF_{k}[t]} + \underbrace{\text{MUI}_{k}[t] + EN_{k}[t], \quad (6)$$

where $\text{SIF}_k[t]$ is the time-varying self-interfering component of the desired signal and $\text{SIF}_k[t] + \text{MUI}_k[t] + \text{EN}_k[t] \stackrel{\Delta}{=} W_k[t]$ is the overall effective noise term. Further $\mathbb{E}\left[S_k[t]\right]$ is the average value of $S_k[t]$, across several uplink data transmission blocks, i.e., several channel realizations, and is a function of t. The same is also true for the variance of $W_k[t]$. Furthermore for a given t, across multiple uplink data transmission blocks, the realizations of $W_k[t]$ are i.i.d. Hence for each channel use, $t = K, K + 1, \ldots, N_u - 1$, we have a additive noise SISO (single-input single-output) channel in (6). Thus for each user there are $N_u - K$ different SISO channels

with distinct channel statistics. Therefore we consider $N_u - K$ channel codes for each user, one for each SISO channel. The data received for each user in the t^{th} channel use across multiple coherence intervals is jointly decoded at the receiver [13]. This coding strategy albeit not practical is useful in computing an achievable information rate.⁹

IV. ACHIEVABLE INFORMATION RATE

In essence, from the above coding strategy, we have $N_u - K$ parallel channel decoders for each user. For the t^{th} SISO channel of the k^{th} user, we note that the correlation between the desired signal term $\text{ES}_k[t]$ and the overall effective noise $W_k[t]$ is zero, i.e., from (6) we have

$$\mathbb{E}\left[\mathrm{ES}_{k}^{*}[t]W_{k}[t]\right] \stackrel{(a)}{=} \mathbb{E}\left[S_{k}^{*}[t]\right] \mathbb{E}\left[\underbrace{|x_{k}[t]|^{2}\left\{S_{k}[t] - \mathbb{E}\left[S_{k}[t]\right]\right\}}_{= 0, \text{ since } x_{k}[t] \text{ and } S_{k}[t] \text{ are independent}} + x_{k}^{*}[t]\mathrm{MUI}_{k}[t] + \underbrace{x_{k}^{*}[t]\mathrm{EN}_{k}[t]}_{= 0, \text{ since } n_{k}[t] \text{ is zero mean and independent of } x_{k}[t]}_{= 0, \text{ since } n_{k}[t] \text{ is zero mean and } x_{k}[t]}$$

$$\stackrel{(b)}{=} \mathbb{E}\left[S_{k}^{*}[t]\right] \mathbb{E}\left[\sqrt{p_{u}}\left(\underbrace{x_{k}^{*}[t]\sum_{q=1,q\neq k}^{K}\left(a_{k}^{H}(\hat{g}_{q}-\epsilon_{q})e^{-j\Delta\omega_{q}(t-(q-1))}x_{q}[t]\right)}_{= 0, \text{ since } x_{i}[t] \text{ are all i.i.d.}}} - \underbrace{a_{k}^{H}\epsilon_{k}e^{-j\Delta\omega_{k}(t-(k-1))}|x_{k}[t]|^{2}}_{= 0, \text{ since } \hat{g}_{k} \text{ and } \epsilon_{k} \text{ are orthogonal due to } MMSE \text{ estimation and } a_{k} \text{ is function of } \hat{g}_{k}}\right] = 0. \quad (7)$$

where (a) and (b) follow from the definitions of $SIF_k[t]$ and $MUI_k[t]$ in (6). With Gaussian information symbols $x_k[t]$, a lower bound on the information rate of the effective channel in (6) is obtained by considering the *worst case uncorrelated additive noise* (in terms of mutual information), having the same variance as $W_k[t]$. With Gaussian information symbols, this worst case uncorrelated noise is also Gaussian [14]. The variance of $W_k[t]$ is given by $\mathbb{E}[|W_k[t]|^2] =$ $\mathbb{E}[|SIF_k[t] + MUI_k[t] + EN_k[t]|^2]$. Since all $x_k[t]$ and $n_k[t]$ are independent and zero mean, it can be shown that $\mathbb{E}[SIF_k^*[t]EN_k[t]] = \mathbb{E}[MUI_k^*[t]EN_k[t]] = 0$. Also due to MMSE channel estimate it can be shown that $\mathbb{E}[SIF_k^*[t]MUI_k[t]] = 0$. Therefore $\mathbb{E}[|W_k[t]|^2] = \mathbb{E}[|SIF_k[t]|^2] + \mathbb{E}[|MUI_k[t]|^2] +$ $\mathbb{E}[|EN_k[t]|^2]$. Also, $\mathbb{E}\left[ES_k[t]\right] = \mathbb{E}\left[SIF_k[t]\right] = \mathbb{E}\left[MUI_k[t]\right] = \mathbb{E}\left[MUI_k[t]\right] = 0$. An achievable rate is therefore given by the following lower bound on $I(\hat{x}_k[t]; x_k[t])$

$$I(\widehat{x}_k[t]; x_k[t]) \ge \log_2(1 + \text{SINR}_k[t]), \text{ where } \text{SINR}_k[t] \stackrel{\Delta}{=} \mathbb{E}\left[|\text{ES}_k[t]|^2\right] / \mathbb{E}\left[|W_k[t]|^2\right], \quad (8)$$

and the overall information rate for the k_{-}^{th} user is thus given by¹⁰

$$I_{k} = \frac{1}{N_{u}} \sum_{t=K}^{N_{u}-1} \log_{2}(1 + \text{SINR}_{k}[t]).$$
(9)

A. Mutual Information Analysis for the ZF Receiver

For a ZF receiver, the detector matrix is defined as $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_K) = \widehat{\mathbf{G}}(\widehat{\mathbf{G}}^H \widehat{\mathbf{G}})^{-1}$. Clearly, for ZF receiver, $\mathbf{A}^H \widehat{\mathbf{G}} = \mathbf{I}_K$, i.e., $\mathbf{a}_k^H \widehat{\mathbf{g}}_q = \delta_{k,q} = 1$ if k = q and 0 if $k \neq q$,

⁹In practice, coding could be performed across a group of consecutive channel uses within each transmission block, since the statistics of $W_k[t]$ and $S_k[t]$ would not change significantly within a small group of consecutive channel uses.

¹⁰In a wireless channel of bandwidth 200 KHz and a coherence interval of duration 1 millisecond, even with K = 10 UTs, the channel estimation overhead is only 5%. Further, CFO estimation is performed at a 5 to 10 times slower rate than channel estimation and therefore its overhead is expected to be less than 1% [15]. We have therefore neglected the CFO estimation overhead in (9), since it is a mere scaling factor, which does not impact the main conclusions of our work.

where q = 1, 2, ..., K and k = 1, 2, ..., K. Substituting this result in (6), we get $\text{ES}_{k}[t] = \sqrt{p_{u}}e^{-\sigma_{\omega_{k}}^{2}(t-(k-1))^{2}/2} x_{k}[t]$, where we have used the fact that $\mathbb{E}[e^{-j\Delta\omega_{k}(t-(k-1))}] = e^{-\sigma_{\omega_{k}}^{2}(t-(k-1))^{2}/2}$. Clearly, $\mathbb{E}[|\text{ES}_{k}[t]|^{2}] = p_{u}e^{-\sigma_{\omega_{k}}^{2}(t-(k-1))^{2}}$. Similarly $\mathbb{E}[|\text{SIF}_{k}[t]|^{2}] = p_{u}\left(1 - e^{-\sigma_{\omega_{k}}^{2}(t-(k-1))^{2}}\right)$, $\mathbb{E}[|\text{MUI}_{k}[t]|^{2}] = p_{u}\mathbb{E}\left[\left\{(\widehat{G}^{H}\widehat{G})^{-1}\right\}_{kk}\right]\sum_{i=1}^{K}\frac{\beta_{i}\sigma^{2}}{Kp_{u}\beta_{i}+\sigma^{2}}$, and $\mathbb{E}[|\text{EN}_{k}[t]|^{2}] = \sigma^{2}\mathbb{E}\left[\left\{(\widehat{G}^{H}\widehat{G})^{-1}\right\}_{kk}\right]$.

Lemma 1. With MMSE channel estimates, it can be shown that $\mathbb{E}\left[\{(\widehat{G}^H \widehat{G})^{-1}\}_{kk}\right] = \left(\frac{1}{\beta_k} + \frac{\sigma^2}{Kp_u\beta_k^2}\right)/(M-K)$, where \widehat{G} is the MMSE estimate of effective channel gain matrix (see (3)).

Proof: See Appendix A.

Proposition 1. For the ZF receiver, the lower bound in (8) is given by the achievable information rate $R_k^{\text{zf}}[t] = \log_2(1 + \text{SINR}_k^{\text{zf}}[t])$, where

$$\operatorname{SINR}_{k}^{\mathrm{zf}}[t] = \frac{e^{-\sigma_{\omega_{k}}^{2}(t-(k-1))^{2}}}{\left[1 - e^{-\sigma_{\omega_{k}}^{2}(t-(k-1))^{2}}\right] + \frac{1}{M-K} \left(\frac{1}{\beta_{k}} + \frac{1}{K\beta_{k}^{2}\gamma}\right) \left[\sum_{i=1}^{K} \frac{\beta_{i}}{K\gamma\beta_{i}+1} + \frac{1}{\gamma}\right]}.$$
(10)

where $\gamma = \frac{p_u}{\sigma^2}$ and $t = K, K + 1, \dots, N_u - 1$.

Proof: Using the expression for $\mathbb{E}\left[\{(\widehat{G}^H \widehat{G})^{-1}\}_{kk}\right]$ from Lemma 1, we get the expressions for $\mathbb{E}\left[|\mathsf{MUI}_k[t]|^2\right]$ and $\mathbb{E}\left[|\mathsf{EN}_k[t]|^2\right]$ (see paragraph before Lemma 1). Using expressions of $\mathbb{E}\left[|\mathsf{ES}_k[t]|^2\right]$, $\mathbb{E}\left[|\mathsf{SIF}_k[t]|^2\right]$, $\mathbb{E}\left[|\mathsf{MUI}_k[t]|^2\right]$ and $\mathbb{E}\left[|\mathsf{EN}_k[t]|^2\right]$ (see paragraph before Lemma 1) in the expression of $\mathsf{SINR}_k[t]$ in (8) we obtain (10).

B. Mutual Information Analysis for the MRC Receiver

For MRC receiver, $\boldsymbol{A} = \widehat{\boldsymbol{G}}$, or, $\boldsymbol{a}_{k} = \widehat{\boldsymbol{g}}_{k}$, $\forall k = 1, 2, ..., K$. Substituting this result in (6), we get $S_{k}[t] = \sqrt{p_{u}} ||\widehat{\boldsymbol{g}}_{k}||^{2} e^{-j\Delta\omega_{k}(t-(k-1))}$. The desired signal $\mathrm{ES}_{k}[t]$ is therefore given by $\mathrm{ES}_{k}[t] = \mathbb{E}[S_{k}[t]]x_{k}[t] = \sqrt{p_{u}}\mathbb{E}[||\widehat{\boldsymbol{g}}_{k}||^{2}]e^{-\sigma_{\omega_{k}}^{2}(t-(k-1))^{2}/2}x_{k}[t]$, or, $\mathrm{ES}_{k}[t] = \sqrt{p_{u}}\mathbb{E}[(\widehat{G}^{H}\widehat{G})_{kk}]e^{-\sigma_{\omega_{k}}^{2}(t-(k-1))^{2}/2}x_{k}[t]$. Therefore $\mathbb{E}[|\mathrm{ES}_{k}[t]|^{2}] = p_{u}\left(\mathbb{E}\left[(\widehat{G}^{H}\widehat{G})_{kk}\right]\right)^{2}e^{-\sigma_{\omega_{k}}^{2}(t-(k-1))^{2}}$. Similarly, $\mathbb{E}[|\mathrm{SIF}_{k}[t]|^{2}] = p_{u}\left[\mathbb{E}\left[\left|(\widehat{G}^{H}\widehat{G})_{kk}\right|^{2}\right] - \left(\mathbb{E}\left[(\widehat{G}^{H}\widehat{G})_{kk}\right]\right)^{2}e^{-\sigma_{\omega_{k}}^{2}(t-(k-1))^{2}}\right]$, $\mathbb{E}[|\mathrm{MUI}_{k}[t]|^{2}] = p_{u}\left[\sum_{i=1,i\neq k}^{K}\mathbb{E}\left[\left|(\widehat{G}^{H}\widehat{G})_{ki}\right|^{2}\right] + \sum_{i=1}^{K}\frac{\beta_{i}\sigma^{2}}{Kp_{u}\beta_{i}+\sigma^{2}}\mathbb{E}\left[(\widehat{G}^{H}\widehat{G})_{kk}\right]\right]$ and $\mathbb{E}[|\mathrm{EN}_{k}[t]|^{2}] = \sigma^{2}\mathbb{E}\left[(\widehat{G}^{H}\widehat{G})_{kk}\right]$.

Lemma 2. With MMSE channel estimate \widehat{G} of the channel gain matrix G (see (3)), it can be shown that $\mathbb{E}\left[(\widehat{G}^{H}\widehat{G})_{kk}\right] = M \frac{Kp_{u}\beta_{k}^{2}}{Kp_{u}\beta_{k}+\sigma^{2}}, \mathbb{E}\left[\left|(\widehat{G}^{H}\widehat{G})_{kk}\right|^{2}\right] = M(M+1)\left(\frac{Kp_{u}\beta_{k}^{2}}{Kp_{u}\beta_{k}+\sigma^{2}}\right)^{2}$ and, $\mathbb{E}\left[\left|(\widehat{G}^{H}\widehat{G})_{ki}\right|^{2}\right] = M\left(\frac{Kp_{u}\beta_{k}^{2}}{Kp_{u}\beta_{k}+\sigma^{2}}\right)\left(\frac{Kp_{u}\beta_{i}^{2}}{Kp_{u}\beta_{i}+\sigma^{2}}\right).$

Proof: See Appendix B.

Proposition 2. For the MRC receiver, the lower bound in (8) is given by the achievable information rate $R_k^{\text{mrc}}[t] = \log_2(1 + \text{SINR}_k^{\text{mrc}}[t])$, where $t = K, K + 1, \dots, N_u - 1$, and

$$\operatorname{SINR}_{k}^{\operatorname{mrc}}[t] = \frac{e^{-\sigma_{\omega_{k}}^{2}(t-(k-1))^{2}}}{\left[1 - e^{-\sigma_{\omega_{k}}^{2}(t-(k-1))^{2}}\right] + \frac{1}{M}\left(\frac{1}{\beta_{k}} + \frac{1}{K\beta_{k}^{2}\gamma}\right)\left[\sum_{i=1}^{K}\beta_{i} + \frac{1}{\gamma}\right]}.$$
(11)

Proof: Firstly we substitute the expressions of $\mathbb{E}\left[(\hat{G}^H\hat{G})_{kk}\right]$, $\mathbb{E}\left[\left|(\hat{G}^H\hat{G})_{ki}\right|^2\right]$ and $\mathbb{E}\left[\left|(\hat{G}^H\hat{G})_{kk}\right|^2\right]$ from Lemma 2 in the expressions for $\mathbb{E}\left[|\mathbf{ES}_k[t]|^2\right]$, $\mathbb{E}\left[|\mathbf{SIF}_k[t]|^2\right]$, $\mathbb{E}\left[|\mathbf{MUI}_k[t]|^2\right]$ and $\mathbb{E}\left[|\mathbf{EN}_k[t]|^2\right]$ (see paragraph before Lemma 2). Using these in (8), we get (11). **Theorem 1.** (Achievable Array Gain) Consider $|\omega_k K| \ll \pi$, a fixed K, N (length of pilot sequence) and a fixed desired information rate for the t^{th} channel code of the k^{th} user ($R_k^{\text{zf}}[t]$ and $R_k^{\text{mrc}}[t]$ defined in Proposition 1 and Proposition 2 respectively). For both the ZF and MRC receivers, as $M \to \infty$, the minimum required SNR γ to achieve the fixed desired information rate for the t^{th} channel code, i.e., $R_k^{\text{zf}}[t]$ or $R_k^{\text{mrc}}[t]$, approaches a constant value.

Proof: We have observed from Remark 2 that as $M \to \infty$ with $\gamma = \frac{c_0}{\sqrt{M}}$ (constant $c_0 > 0$), the MSE for CFO estimation converges to a constant limiting value, i.e., $\lim_{M\to\infty,\gamma=\frac{c_0}{\sqrt{M}}} \sigma_{\omega_k}^2 = \zeta_0 > 0$ (constant). Substituting this result in the expression for SINR_k^{zf}[t] in (10) and also in the expression for SINR_k^{mrc}[t] in (11) with $\gamma = \frac{c_0}{\sqrt{M}}$ we have

$$\lim_{M \to \infty} \text{SINR}_k^{\text{mrc}}[t] = \lim_{M \to \infty} \text{SINR}_k^{\text{zf}}[t] = \frac{e^{-\zeta_0(t - (k-1))^2}}{1 - e^{-\zeta_0(t - (k-1))^2} + \frac{1}{K\beta_k^2 c_0^2}} > 0 \text{ (constant)}.$$
(12)

From (12) it is clear that $R_k^{\text{zf}}[t] = \log_2(1 + \text{SINR}_k^{\text{zf}}[t])$ and $R_k^{\text{mrc}}[t] = \log_2(1 + \text{SINR}_k^{\text{mrc}}[t])$ would also approach constant limiting values as $M \to \infty$ with $\gamma \propto \frac{1}{\sqrt{M}}$.

Remark 3. From Theorem 1, it is clear that with every doubling in the number of BS antennas, the minimum required SNR to achieve a fixed per-user information rate decreases by approximately 1.5 dB as long as the number of BS antennas M is sufficiently large. This shows that with the CFO estimation technique proposed in [12], the ZF receiver (also the MRC receiver) yields an $\mathcal{O}(\sqrt{M})$ array gain in the massive MIMO uplink. This is interesting since even for the ideal zero CFO scenario with ZF/ MRC receiver, the maximum achievable array gain is known to be only $\mathcal{O}(\sqrt{M})$ [6].

Remark 4. From (12) we have $\lim_{M\to\infty} \text{SINR}_k^{\text{mrc}}[t] = \lim_{M\to\infty} \text{SINR}_k^{\text{zf}}[t]$. Clearly, as $M \to \infty$ with $\gamma \propto 1/\sqrt{M}$, the achievable information rate for both the ZF and MRC receiver approach the same lower bound. This shows us the new result that even with CFO estimation/compensation, MRC and ZF receivers have the same performance when M is sufficiently large.

V. NUMERICAL RESULTS AND DISCUSSIONS

We present a comparative discussion on the performance of the ZF and MRC receivers, with CFO estimation/compensation in frequency-flat massive MIMO uplink. For monte-carlo simulations, we assume an operating carrier frequency $f_c = 2$ GHz and a maximum CFO of 1 PPM of f_c . The communication bandwidth is $B_c = 200$ KHz. The coherence interval and the maximum delay spread are 1 ms and 5 μ s respectively. Thus $|\omega_k| \leq \frac{\pi}{50}$ and $N_c = 1 \text{ms}/B_c = 200$

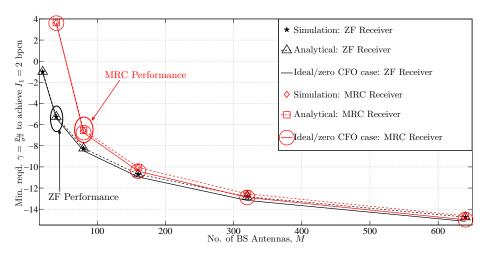


Fig. 2 Plot of $\gamma = \frac{p_u}{\sigma^2}$ required to achieve $I_k = 2$ bpcu (for the first user (k = 1)) vs. M, fixed K = 10, N = 100. channel uses. The duration of uplink is $N_u = 100$ channel uses. The length of pilot sequence for CFO estimation is taken as N = 100 and the number of UTs is K = 10. At the start of each CFO estimation phase ω_k assumes a random value uniformly distributed in $\left[-\frac{\pi}{50}, \frac{\pi}{50}\right]$. Also for simplicity, we assume $\beta_k = 1, \forall k = 1, 2, ..., K$. The information rate for each user is also computed analytically using Propositions 1 and 2 in (9) with $\sigma_{\omega_k}^2 = \mathbb{E}[(\widehat{\omega}_k - \omega_k)^2]$ replaced by its approximation in (1) with $G_k = 1$ (see Remark 1).

In Fig. 2 we plot the variation of the minimum required SNR $\gamma = p_u/\sigma^2$ (both analytical and simulated) to achieve a fixed information rate of 2 bpcu (bits per channel use) for the 1st user versus the number of BS antennas, M (fixed K = 10 and N = 100). Observe that the analytical approximation to the required γ for both ZF and MRC is quite tight. Also for Msufficiently large, with every doubling in M, the required γ decreases roughly by 1.5 dB (note the decrease in required SNR γ from M = 320 to M = 640). This supports Theorem 1 and shows that with the discussed CFO estimation/compensation technique, an $\mathcal{O}(\sqrt{M})$ array gain is achievable. Also note that the required γ for ZF and MRC is the same for sufficiently large M > 320 (see Remark 4). However for finite M, ZF is more power efficient compared to MRC. For example at M = 80, ZF requires approximately 1.7 dB less power than MRC. Next we consider this extra SNR required by MRC when compared to ZF (denoted as SNR gap) for the same desired information rate $I_1 = 1, 2, 2.5$ bpcu (for the 1st UT) for fixed M = 80 and K = 10(see Table I). From Table I, we make an interesting observation that the SNR gap between the ZF and MRC receivers is almost the same irrespective of whether we have the ideal zero CFO

TABLE I SNR gap between ZF and MRC Receiver for fixed M = 80, K = 10.

Desired Per-User Information Rate	SNR gap for Ideal/zero CFO case	SNR gap with CFO compensation
1 bpcu	0.1 dB	0.12 dB
2 bpcu	1.7 dB	1.71 dB
2.5 bpcu	4.57 dB	4.59 dB

scenario or the residual CFO (after CFO compensation) scenario. Therefore the new result in this paper is that with CFO compensation, there is no significant degradation in the SNR gap when compared to the SNR gap in the ideal/zero CFO scenario.

VI. CONCLUSION

In this paper, we study the impact of low-complexity CFO estimation and compensation on the performance of ZF receiver in massive MIMO uplink in a flat fading environment and compare it to that of the MRC receiver. The tight closed-form analytical expressions for information rates of ZF and MRC reveal that an $\mathcal{O}(\sqrt{M})$ array gain is indeed achievable with CFO estimation. This is interesting since the best possible array gain for ideal zero CFO scenario is also known to be $\mathcal{O}(\sqrt{M})$. Finally the study of the SNR gap between ZF and MRC receivers for the same per-user information rate suggests that compared to the ideal zero CFO case, the performance degradation due to residual CFO is same for both the ZF and MRC receivers.

APPENDIX A

PROOF OF LEMMA 1

From the relation $g_{mk} = h_{mk}\sqrt{\beta_k}$ we have $G = HD^{1/2}$, where $H \stackrel{\Delta}{=} [h_{mk}]_{M \times K}$. Clearly from (3) we have $\hat{G} = (\sqrt{Kp_u}G\Phi_0 + N)\tilde{D} \stackrel{\text{(a)}}{=} (\sqrt{Kp_u}H\underbrace{D^{1/2}\Phi_0}_{\triangleq X} + \underbrace{N\Phi_0^H D^{-1/2}}_{\triangleq V}D^{1/2}\Phi_0)\tilde{D} = (\underbrace{\sqrt{Kp_u}H + V}_{\triangleq X})X\tilde{D} = ZX\tilde{D}$, where (a) follows from the fact that $\Phi_0^H\Phi_0 = \Phi_0\Phi_0^H = I_K$ and $\tilde{D} = (\sqrt{Kp_u}I_K + \frac{\sigma^2}{\sqrt{Kp_u}}D^{-1})^{-1}$. Let n_k and v_k be the k^{th} columns of N and Vrespectively. Since $n_k \sim C\mathcal{N}(0, \sigma^2 I_M), k = 1, 2, \cdots, K$ are all i.i.d. random vectors, $v_k = (\Phi_0^H D^{-1/2})_{kk}n_k, k = 1, 2, \cdots, K$, are also independently distributed as $C\mathcal{N}(0, \frac{\sigma^2}{\beta_k}I_M)$.

We also note that the columns of \boldsymbol{H} and \boldsymbol{V} are independent of each other. Clearly, the same is also true for the columns of \boldsymbol{Z} . Therefore we can write $\boldsymbol{Z} = \boldsymbol{U}\boldsymbol{Q}$, where $\boldsymbol{Q} \stackrel{\Delta}{=} (Kp_{u}\boldsymbol{I}_{K} + \sigma^{2}\boldsymbol{D}^{-1})^{1/2}$ and $\boldsymbol{U} \stackrel{\Delta}{=} (\boldsymbol{u}_{1}, \boldsymbol{u}_{2}, \cdots, \boldsymbol{u}_{K})$, where $\boldsymbol{u}_{k} \sim \mathcal{CN}(0, \boldsymbol{I}_{M}) \forall k = 1, 2, \dots, K$ are i.i.d. random vectors. Now, we have $\left\{ (\widehat{\boldsymbol{G}}^{H}\widehat{\boldsymbol{G}})^{-1} \right\}_{kk} = \left\{ (\widetilde{\boldsymbol{D}}\Phi_{0}^{H}\boldsymbol{D}^{1/2}\boldsymbol{Q}\boldsymbol{U}^{H}\boldsymbol{U}\boldsymbol{Q}\boldsymbol{D}^{1/2}\Phi_{0}\widetilde{\boldsymbol{D}})^{-1} \right\}_{kk}$, which follows from the fact that $\widehat{\boldsymbol{G}} = \boldsymbol{Z}\boldsymbol{X}\widetilde{\boldsymbol{D}}, \, \boldsymbol{X} = \boldsymbol{D}^{1/2}\Phi_{0}$ and $\boldsymbol{Z} = \boldsymbol{U}\boldsymbol{Q}$. Since $\boldsymbol{D}, \, \Phi_{0}, \, \boldsymbol{Q}$ and $\widetilde{\boldsymbol{D}}$ are all diagonal, $\left\{ (\widehat{\boldsymbol{G}}^{H}\widehat{\boldsymbol{G}})^{-1} \right\}_{kk} = \left[|\boldsymbol{T}_{kk}|^{2} \right]^{-1} (\boldsymbol{W}^{-1})_{kk}$, where $\boldsymbol{T} \stackrel{\Delta}{=} \boldsymbol{Q}\boldsymbol{D}^{1/2}\Phi_{0}\widetilde{\boldsymbol{D}}$ and $\boldsymbol{W} \stackrel{\Delta}{=} \boldsymbol{U}^{H}\boldsymbol{U} \sim \mathcal{W}_{M}(M, \boldsymbol{I}_{M})$ is a $K \times K$ central Wishart matrix with M degrees of freedom. Clearly, $\mathbb{E}\left[\left\{ (\widehat{\boldsymbol{G}}^{H}\widehat{\boldsymbol{G}})^{-1} \right\}_{kk} \right] = \left[|\boldsymbol{T}_{kk}|^{2} \right]^{-1} \mathbb{E}[(\boldsymbol{W}^{-1})_{kk}] = \left(\frac{Kp_{u}\beta_{k} + \sigma^{2}}{Kp_{u}\beta_{k}^{2}} \right) \frac{1}{K} \mathbb{E}\left[tr(\boldsymbol{W}^{-1}) \right] \stackrel{\text{(b)}}{=} \frac{Kp_{u}\beta_{k} + \sigma^{2}}{(M-K)Kp_{u}\beta_{k}^{2}},$ where (b) follows from $\mathbb{E}[tr(\boldsymbol{W}^{-1})] = \frac{K}{M-K}$ [16].

APPENDIX B

Proof of Lemma 2

From Appendix A we know that $W = U^H U$ is a central Wishart matrix with M degrees of freedom, i.e., W_{kk} is $\chi^2(2M)$ (chi-squared) distributed. Therefore from definition of \hat{G} we have

$$\mathbb{E}\left[\left(\widehat{G}^{H}\widehat{G}\right)_{kk}\right] \stackrel{\text{(a)}}{=} |\mathbf{T}_{kk}|^{2} \mathbb{E}[\mathbf{W}_{kk}] = M\left(\frac{Kp_{u}\beta_{k}^{2}}{Kp_{u}\beta_{k} + \sigma^{2}}\right), \quad \mathbb{E}\left[\left|\left(\widehat{G}^{H}\widehat{G}\right)_{kk}\right|^{2}\right] = M(M+1)\left(\frac{Kp_{u}\beta_{k}^{2}}{Kp_{u}\beta_{k} + \sigma^{2}}\right)^{2}, \text{ and} \\ \mathbb{E}\left[\left|\left(\widehat{G}^{H}\widehat{G}\right)_{ki}\right|^{2}\right] \stackrel{\text{(b)}}{=} \mathbb{E}\left[\left|\mathbf{T}_{kk}^{*}\mathbf{W}_{ki}\mathbf{T}_{ii}\right|^{2}\right] = |\mathbf{T}_{kk}|^{2}|\mathbf{T}_{ii}|^{2} \mathbb{E}\left[\left|\mathbf{W}_{ki}\right|^{2}\right] \stackrel{\text{(c)}}{=} \left(\frac{Kp_{u}\beta_{k}^{2}}{Kp_{u}\beta_{k} + \sigma^{2}}\right)\left(\frac{Kp_{u}\beta_{i}^{2}}{Kp_{u}\beta_{i} + \sigma^{2}}\right)M, \\ \text{where } (a) \text{ and } (b) \text{ follow from the facts that } \widehat{G} = \mathbf{Z}\mathbf{X}\widetilde{D}, \quad \mathbf{X} = \mathbf{D}^{1/2}\Phi_{0}, \quad \mathbf{Z} = \mathbf{U}\mathbf{Q} \text{ and } \mathbf{T} = \mathbf{Q}\mathbf{D}^{1/2}\Phi_{0}\widetilde{D} \text{ (see Appendix A). Also } (c) \text{ follows from the fact that } \mathbb{E}[|\mathbf{W}_{ki}|^{2}] = \mathbb{E}[|\mathbf{u}_{k}^{H}\mathbf{u}_{i}|^{2}] = M, \\ \text{since } \mathbf{u}_{k}, \quad \forall k = 1, 2, \dots, K \text{ are i.i.d. } \mathcal{CN}(0, \mathbf{I}_{M}). \end{cases}$$

REFERENCES

- J. Andrews, S. Buzzi, W. Choi, S. Hanly, A. Lozano, A. Soong, and J. Zhang, "What Will 5G Be?" *IEEE J. Sel. Areas Commun.*, vol. 32, no. 6, pp. 1065–1082, June 2014.
- [2] F. Boccardi, R. Heath, A. Lozano, T. Marzetta, and P. Popovski, "Five Disruptive Technology Directions for 5G," *IEEE Commun. Mag.*, vol. 52, no. 2, pp. 74–80, February 2014.
- [3] T. Marzetta, "Noncooperative Cellular Wireless with Unlimited Numbers of Base Station Antennas," *IEEE Trans. Wireless Commun.*, vol. 9, no. 11, pp. 3590–3600, November 2010.
- [4] E. Larsson, O. Edfors, F. Tufvesson, and T. Marzetta, "Massive MIMO for Next Generation Wireless Systems," *IEEE Commun. Mag.*, vol. 52, no. 2, pp. 186–195, February 2014.
- [5] F. Rusek, D. Persson, B. K. Lau, E. Larsson, T. Marzetta, O. Edfors, and F. Tufvesson, "Scaling Up MIMO: Opportunities and Challenges with Very Large Arrays," *IEEE Signal Process. Mag.*, vol. 30, no. 1, pp. 40–60, Jan 2013.
- [6] H. Q. Ngo, E. Larsson, and T. Marzetta, "Energy and Spectral Efficiency of Very Large Multiuser MIMO Systems," *IEEE Trans. Commun.*, vol. 61, no. 4, pp. 1436–1449, April 2013.
- [7] J. Chen, Y.-C. Wu, S. Ma, and T.-S. Ng, "Joint CFO and Channel Estimation for Multiuser MIMO-OFDM Systems with Optimal Training Sequences," *IEEE Trans. Signal Process.*, vol. 56, no. 8, pp. 4008–4019, Aug 2008.
- [8] E. P. Simon, L. Ros, H. Hijazi, and M. Ghogho, "Joint Carrier Frequency Offset and Channel Estimation for OFDM Systems via the EM Algorithm in the Presence of Very High Mobility," *IEEE Trans. Signal Process.*, vol. 60, no. 2, pp. 754–765, Feb 2012.
- M. Ghogho and A. Swami, "Training Design for Multipath Channel and Frequency-Offset Estimation in MIMO Systems," *IEEE Trans. Signal Process.*, vol. 54, no. 10, pp. 3957–3965, Oct 2006.
- [10] Y. Yu, A. Petropulu, H. Poor, and V. Koivunen, "Blind estimation of multiple carrier frequency offsets," in *Personal, Indoor and Mobile Radio Communications*, 2007. PIMRC 2007. IEEE 18th International Symposium on, Sept 2007, pp. 1–5.
- [11] H. Cheng and E. Larsson, "Some Fundamental Limits on Frequency Synchronization in massive MIMO," in Signals, Systems and Computers, 2013 Asilomar Conference on, Nov 2013, pp. 1213–1217.
- [12] S. Mukherjee and S. K. Mohammed, "Low-Complexity CFO Estimation for Multi-User Massive MIMO Systems," accepted in *IEEE Globecom*, 2015 [Online]. Available. http://arxiv.org/abs/1504.05657v2.
- [13] A. Pitarokoilis, S. Mohammed, and E. Larsson, "Uplink performance of time-reversal mrc in massive mimo systems subject to phase noise," *IEEE Trans. Wireless Commun.*, vol. 14, no. 2, pp. 711–723, Feb 2015.
- [14] B. Hassibi and B. Hochwald, "How much training is needed in multiple-antenna wireless links?" *IEEE Trans. Inf. Theory*, vol. 49, no. 4, pp. 951–963, April 2003.
- [15] E. Dahlman, S. Parkvall, and J. Sköld, 4G: LTE/LTE-Advanced for Mobile Broadband, 2nd ed. Academic Press, 2014.
- [16] A. M. Tulino and S. Verdú, "Random Matrix Theory and Wireless Communications," Foundations and Trends in Communications and Information Theory, vol. 1, no. 1, pp. 1–182, June 2004.