# Wireless Network Reliability Analysis for Arbitrary Network Topologies

Semiha Tedik Basaran<sup>†</sup>, *Student Member, IEEE*, Gunes Karabulut Kurt<sup>†</sup>, *Senior Member, IEEE*, and Frank R. Kschischang<sup>‡</sup>, *Fellow, IEEE* 

#### Abstract

The outage performance of wireless networks with unstructured network topologies is investigated. The network reliability perspective of graph theory is used to obtain the network outage polynomial of generalized wireless networks by enumerating paths and cut-sets of its graph representation for both uncorrelated and correlated wireless channels. A relation is established between the max-flow min-cut theorem and key communication performance indicators. The diversity order is equal to the size of the minimum cut-set between source and destination, and the coding gain is the number of cut-sets with size equal to the minimum cut. An ergodic capacity analysis of networks with arbitrary topologies based on the network outage polynomial is also presented. Numerical results are used to illustrate the technical definitions and verify the derivations.

#### **Index Terms**

Diversity gain, ergodic capacity, network reliability, outage probability, terminal reliability polynomial.

#### I. INTRODUCTION

Network topologies in wireless environments are generally dynamic in nature, as the connectivity between nodes is determined according to their time-varying link signal-to-noise ratio

<sup>&</sup>lt;sup>†</sup> S. Tedik Basaran and G. Karabulut Kurt are with the Department of Communications and Electronics Engineering, Istanbul Technical University, Turkey, e-mail: {tedik, gkurt}@itu.edu.tr.

<sup>&</sup>lt;sup>‡</sup> Frank R. Kschischang is with the Edward S. Rogers Sr. Department of Electrical and Computer Engineering, University of Toronto, Canada, e-mail: frank@ece.utoronto.ca.

(SNR) value. Channel impairments such as fading and path loss make it essential to monitor the quality of each link.

Based on the corresponding SNR value, the outage status can be determined and used as a performance indicator for each link. In case of an outage (where the SNR value falls below a certain threshold), two nodes are deemed to be disconnected; otherwise, they remain connected. Outage probability is a convenient measure of communication system performance [1]. Here we investigate *network outage*, i.e., the outage probability of communication between a source node and a terminal node over a network of relay nodes. Assuming links are in outage independently, network outage can be measured by using individual link outage probabilities. The behavior of the outage probability in the high SNR regime also gives an intuitive understanding of performance limits of the network [2]. In [3], high SNR error performance of any communication network (coded or uncoded) is represented by diversity and coding gains. *Diversity gain* is a measure of the number of independent copies of the transmitted signal captured by the receiver [4], while *coding gain* represents the difference in the outage probability curve relative to a benchmark performance in the high SNR region [3].

In this paper, we show that the diversity gain and the coding gain between a source node and a terminal node can be determined through the *network outage polynomial*. This approach dates back at least to Shannon and Moore [5], who provide a reliability analysis of relayaided systems by considering the unreliability probabilities of relay nodes. It is proven that the end-to-end reliability of a given network can be increased through these unreliable relay nodes. When a sufficient number of relay nodes is used, the probability of network unreliability approaches zero [5]. However, transforming a complex network into equivalent series-parallel projection may not always be possible. When the series-parallel representation of a given network is not available, the reliability analysis of generalized networks becomes more difficult. There are various methods proposed to calculate network reliability, such as state enumeration, factorizing, path enumeration, and cut-set enumeration [6]–[9].

Although network reliability is a well-studied subject, its extension to wireless networks it is still relatively unexplored [10]–[14]. As the popularity of wireless communication systems increases when compared to their wired counterparts in many different areas, the reliability analysis of wireless communications becomes more important, yet challenging, as wireless links are more prone to errors and erasures. Firstly, an unrealistic deterministic channel model is used when investigating the interference effect of the wireless channels [10]. The reliability analysis of

wireless multi-hop networks is conducted regarding shadowing effect of the wireless channel in [11], [12]. Both [11], [12] do not consider the correlation effect of shadowing and this gap is filled by [13]. The reliability analysis of wireless multi-hop networks, which proposes a mathematical model to represent the network reliability of correlated shadowing wireless channel, is given in [14].

In [8], path-enumeration and cut-set enumeration methods are used to calculate network reliability of generalized schemes. An algorithm based on a path-enumeration method is presented in [15] to determine the reliability of telecommunication networks from the capacity of the networks by considering different link capacities. In [16], a path-based algorithm with a reduced computational time is modeled to obtain network reliability of wired communications. Instead of considering all cut-sets of the network, some cut-sets that have as many elements as the size of the size of the minimum cut are used to obtain an approximate network reliability expression with reduced complexity [17]. Hence, a lower bound for network reliability is attained by providing a more practical algorithm.

The network outage polynomial gives the probability that the network has zero instantaneous capacity. The investigation of network capacity is an attractive problem since the maximum capacity of any network is restricted by the size of the minimum cut of the graph. Hence, the ergodic capacity of any network can be calculated by using zero-to-m capacity polynomials, where the *i*th polynomial gives the probability that the network has instantaneous capacity *i*. In the literature, there are some works about the calculation of the capacity polynomials that determine the value of the maximum flow of arbitrary networks with random capacity edges by utilizing subset decomposition method [18], [19]. In [18], a subspace decomposition principle is used to determine the value of the maximum flow of arbitrary networks with random capacity edges. The value of maximum flow analysis of arbitrary networks with random edge capacities is conducted in [19], based upon Bernoulli statistics.

The aforementioned works have focused on obtaining only network reliability expressions. On the other hand, these works do not introduce any fundamental performance analysis. In this work, the essential goal is to obtain performance limits of an arbitrary network topology comprised of links that are prone to errors and erasures. The main contributions of this work can be listed as follows:

• We establish a framework to calculate the *network outage polynomial*, as a tool to obtain network outage performance of communication networks.

- We determine the network outage polynomial of some simple directed networks, in both correlated channels and uncorrelated channels. Three methods, namely the path-enumeration method, the cut-set enumeration method, and the terminal reliability based method are proposed.
- We extract the diversity order and the coding gain of a wireless network for arbitrary topology based on its graph properties.
- We establish a relationship between the max-flow min-cut theorem of graph theory and the diversity gain definition and show that the diversity order corresponds to the size of the minimum cut of the wireless network graph. We also prove that the coding gain is equal to the number of cut-sets which have the size of the minimum cut, and also be easily determined from the network graph.
- We provide the ergodic capacity analysis of networks in terms of individual link outage probability. Hence, an upper bound for the achievable transmission rate is determined.

Using this analysis, optimization of resource utilization can be realized thanks to the information about the diversity order and the ergodic capacity of any topology in wireless networks. For example, efficient multiple access schemes can be obtained by considering user demands and network limitations (the diversity order and the ergodic capacity).

The rest of the paper is organized as follows. Firstly, in Section II, methods for the calculation of outage polynomials of wireless networks are given. In Section III, diversity order analysis and ergodic capacity derivations are presented. In Section IV, to demonstrate the validity of theoretical results, numerical results are presented. Finally, the paper concludes with a summary of the findings and suggestions for future work in Section V.

#### **II. OUTAGE POLYNOMIALS OF WIRELESS NETWORKS**

Graph representations of communication systems are frequently used to analyze system performance; hence, key graph theory concepts can often be matched with the elements of communication systems. In literature pertaining to wired networks, the link outage probability is generally ignored since links are generally highly reliable. Thus, for wired networks, the connections between nodes can be represented by deterministic edges. The links in wireless channels, on the other hand, are subject to random SNR values, and so the connections between nodes must be modeled probabilistically. We model a communications network  $\mathcal{N} = (\mathcal{V}, \mathcal{E}, s, t)$  as a directed acyclic network compromising of a finite vertex set  $\mathcal{V}$  of communication nodes, a multi-set of n directed edges  $\mathcal{E} = \{e_1, e_2, \dots, e_n\} \subseteq \mathcal{V} \times \mathcal{V}$  representing communication links between nodes, a designated source vertex s and a designated terminal vertex t where  $s, t \in V, s \neq t$ . An edge e from vertex v to vertex w is denoted as  $v \to w$ .

A directed path in  $\mathcal{N}$  from s to t is a sequence of edges  $(v_0 \to v_1), (v_1 \to v_2), \dots, (v_{\ell-1} \to v_\ell)$ with  $v_0 = s$  and  $v_\ell = t$ . We suppose that there are g distinct paths  $\mathcal{P}_1, \dots, \mathcal{P}_g$  in  $\mathcal{N}$  from s to t. Nodes s and t are said to be *connected* if  $g \ge 1$ .

A subset  $C \subseteq \mathcal{E}$  of edges whose removal from the network disconnects s and t is called an s-t-separating cut, or simply a cut-set. We suppose that there are k distinct cut-sets  $C_1, \ldots, C_k$ ; the collection of all cut-sets is denoted as  $\mathcal{K}$ .

A cut-set  $C \in K$  is called *minimal* if no proper subset of C is itself a cut-set. The collection of all minimal cut-sets is denoted as  $\mathcal{L}$ . A cut-set  $C \in K$  is called a *minimum cut-set* if it is a cut-set of minimum possible size, i.e., having the least number of edges among all cut-sets. The collection of all minimum cut-sets is denoted as  $\mathcal{M}$ , and the size of any minimum cut-set is denoted as m. Although each minimum cut-set is certainly a minimal cut-set, the converse is not true in general, thus  $\mathcal{M} \subseteq \mathcal{L} \subseteq \mathcal{K}$ .

Network outage is a convenient measure of a communication system's performance, as the overall system performance can be obtained using individual outage probabilities of the links in the system. To enable communication between a source node s and a terminal node t, there must be at least one path from s to t. Hence, we can obtain an overall performance result by considering individual link outages. The network outage polynomial concept, which has been proposed for switching networks [5], [20], is also suitable as performance observation tool for wireless communication. Network outage is random due to individual link outages. In order to obtain the network outage polynomial for an arbitrary topology, we use three different methods: path enumeration, cut-set enumeration, and reliability polynomial calculation. The required method can be selected to realize the target aim, as detailed below.

In the following, we consider the network at a given time instant, and denote by  $p_j$  the probability that link  $e_j$  is in outage at that instant. For example, if the wireless channel gain  $|h_j|$  has a Rayleigh distribution (a frequent assumption in the wireless communication literature),

then the outage probability of  $e_j$  is equal to

$$p_j = 1 - \exp\left(-\gamma_j^{-1}\right),$$

where  $\gamma_j$  represents the average SNR of the link  $e_j$  [4].

Link outages induce a random subgraph of  $\mathcal{N}$ , called the *residual network*, with edges that are in outage removed. In the residual network, it may happen that s and t are not connected. The *network outage polynomial*, which gives the probability that no path exists between s and t in the residual network, is then formally a polynomial function of  $p_1, \ldots, p_n$ , denoted as  $O(p_1, \ldots, p_n)$ .

Throughout this paper, for any positive integer  $\ell$ , we will denote the set  $\{1, 2, \dots, \ell\}$  as  $[\ell]$ .

#### A. Network Outage Polynomial Calculation Based on Path Enumeration

Firstly, we investigate the path enumeration method to obtain the network outage polynomial. We suppose that the edges comprising a path  $\mathcal{P}_r$  in  $\mathcal{N}$  from s to t are indexed by the set  $\mathscr{P}_r \subseteq [n]$ , i.e.,  $\mathcal{P}_r = \{e_j : j \in \mathscr{P}_r\}, r \in [g]$ .

Let  $Q_r$  denote the event that path  $\mathcal{P}_r$  is available, i.e., that none of its links are in outage. The outage probability of the network is then given by

$$O(p_1,\ldots,p_n) = 1 - \Pr[Q_1 \cup Q_2 \cup \cdots \cup Q_g].$$

By the principle of inclusion-exclusion [21], we have

$$\Pr[Q_{1} \cup Q_{2} \cup \dots \cup Q_{g}] = \sum_{i_{1} \in [g]} \Pr[Q_{i_{1}}] - \sum_{\substack{i_{1}, i_{2} \in [g] \\ i_{1} \neq i_{2}}} \Pr[Q_{i_{1}} \cap Q_{i_{2}}] + \dots + (-1)^{\beta - 1} \sum_{\substack{i_{1}, i_{2}, \dots, i_{\beta} \in [g] \\ i_{1}, i_{2}, \dots, i_{\beta} \text{ distinct}}} \Pr[Q_{i_{1}} \cap Q_{i_{2}} \cap \dots \cap Q_{i_{\beta}}] + \dots + (-1)^{g - 1} \Pr[Q_{1} \cap Q_{2} \cap \dots \cap Q_{g}].$$
(1)

Assuming that individual links are in outage (or not) independently, we have

$$\Pr[Q_{i_1} \cap Q_{i_2} \cap \dots \cap Q_{i_\beta}] = \prod_{j \in \mathscr{P}_{i_1} \cup \mathscr{P}_{i_2} \cup \dots \cup \mathscr{P}_{i_\beta}} (1 - p_j).$$
(2)

### B. Network Outage Polynomial Calculation Based on Cut-Set Enumeration

The network outage polynomial of an arbitrary network can also be calculated by enumerating cut-sets of the network, which is dual to the process of path enumeration. If the edges of any cut-set are all in outage, the network is in outage.

We suppose that the edges comprising a cut-set  $C_r$  are indexed by the set  $\mathscr{C}_r \subseteq [n]$ , i.e.,  $C_r = \{e_j : j \in \mathscr{C}_r\}, r \in [k].$ 

Let  $D_r$  denote the event that cut-set  $C_r$  is active, i.e., that all of its links are in outage. The outage probability of the network is then given by

$$O(p_1,\ldots,p_n) = \Pr[D_1 \cup D_2 \cup \cdots \cup D_k].$$

Again by the principle of inclusion-exclusion we have

$$\Pr[D_{1} \cup D_{2} \cup \dots \cup D_{k}] = \sum_{i_{1} \in [k]} \Pr[D_{i_{1}}] - \sum_{\substack{i_{1}, i_{2} \in [k] \\ i_{1} \neq i_{2}}} \Pr[D_{i_{1}} \cap D_{i_{2}}] + \dots$$
$$+ (-1)^{\beta - 1} \sum_{\substack{i_{1}, i_{2}, \dots, i_{\beta} \in [k] \\ i_{1}, i_{2}, \dots, i_{\beta} \text{ distinct}}} \Pr[D_{i_{1}} \cap D_{i_{2}} \cap \dots \cap D_{i_{\beta}}] + \dots$$
$$+ (-1)^{k - 1} \Pr[D_{1} \cap D_{2} \cap \dots \cap D_{k}],$$
(3)

where

$$\Pr[D_{i_1} \cap D_{i_2} \cap \dots \cap D_{i_\beta}] = \prod_{j \in \mathscr{C}_{i_1} \cup \mathscr{C}_{i_1} \cup \dots \cup \mathscr{C}_{i_\beta}} p_j.$$
(4)

#### C. Network Outage Polynomial Calculation Based on Two-Terminal Polynomial

Finally, we derive the network outage polynomial expressions of a network based on the reliability polynomial concept [20], which is a useful function to reflect the performance of a network.

Consider, for any cut-set  $C_r$ ,  $r \in [k]$ , the event  $E_r$  that all the edges of  $C_r$  are in outage while all *other* edges of the network are *not* in outage. Since  $E_r$  is disjoint from  $E_s$  when  $r \neq s$ , we have

$$O(p_1,\ldots,p_n) = \Pr\left[\bigcup_{r\in[k]} E_r\right] = \sum_{r\in[k]} \Pr[E_r].$$

Again assuming that individual links are in outage (or not) independently, we have

$$\Pr[E_r] = \prod_{j \in \mathscr{C}_r} p_j \cdot \prod_{i \in [k] \setminus \mathscr{C}_r} (1 - p_i).$$

In the special case where  $p_j = p$  for all  $j \in [n]$ , we have

$$\Pr[E_r] = p^{|\mathcal{C}_r|} (1-p)^{n-|\mathcal{C}_r|}.$$

Writing O(p) for the outage polynomial in this case, we get

$$O(p) = \sum_{r \in [k]} p^{|\mathcal{C}_r|} (1-p)^{n-|\mathcal{C}_r|} = \sum_{i=m}^n A_i p^i (1-p)^{n-i}$$
$$= (1-p)^n A\left(\frac{p}{1-p}\right),$$
(5)

where

$$A(x) = \sum_{\mathcal{C} \in \mathcal{K}} x^{|\mathcal{C}|} = A_m x^m + A_{m+1} x^{m+1} + \dots + A_n x^n,$$
(6)

and where the coefficient  $A_i$  of  $x^i$  enumerates the number of cut-sets of size *i*.

It can be deduced from the minimum cut-set definition that  $A_m$  is equal to the number of distinct minimum cut-sets and  $A_m \neq 0$ . In addition,  $A_n$  is equal to 1. The outage polynomial can be also expressed in terms of the reliability polynomial associated with the  $\text{Conn}_2(\mathcal{N})$  s-t connectedness problem,  $O(p) = 1 - \text{Rel}(\mathcal{N}, 1-p)$  [20, Sec. 1.2].

The computational complexity of the outage polynomial depends on the determination of  $\mathcal{K}$ . The complexity per cut is given as  $\mathcal{O}(n)$  in [22]. Hence, the enumeration of cut-sets can be found as  $\mathcal{O}(kn)$  where the number of all cut-sets ( $k = |\mathcal{K}|$ ) depends on the size of  $\mathcal{N}$  [22], [23].

#### D. Bounds on the Outage Polynomial

We may write some simple bounds on the outage polynomial as follows.

Firstly, if we use the inequality of  $(1-p) \leq 1$  in (5), we get

$$O(p) \le \sum_{i=m}^{n} A_i p^i = A(p) \tag{7}$$

To derive another upper bound expression, we can use the fact that every cut-set must contain a

minimal cut-set. Since the probability that edges of a cut-set C are in outage is  $p^{|C|}$ , we get that

$$O(p) \le \sum_{C \in \mathcal{L}} p^{|C|}.$$
(8)

We also have the lower bound

$$O(p) \ge A_m p^m (1-p)^{n-m} \tag{9}$$

which is obtained by retaining just the first term in the expansion  $O(p) = \sum_{i=m}^{n} A_i p^i (1-p)^{n-i}$ .

# E. Presence of Correlated Channels

In the previous subsections, we have assume that the state of each link is independent of the others. This assumption may be unrealistic in many situations (e.g., multi-antenna systems) because of spatial correlation. The correlated channel case needs to be considered to determine the limitations of the wireless networks.

We adopt a simple correlation model, as follows. Firstly, the set  $\mathcal{E}$  of links is partitioned into disjoint nonempty subsets,  $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_f$ , so that

$$\bigcup_{i=1}^{f} \mathcal{B}_i = \mathcal{E} \quad \text{and} \quad i \neq j \text{ implies } \mathcal{B}_i \cap \mathcal{B}_j = \emptyset.$$

To subset  $\mathcal{B}_i$  is associated a Bernoulli ( $\{0, 1\}$ -valued) random variable  $S_i$ , with  $\Pr[S_i = 1] = \rho$ . If  $S_i = 1$ , the link states (in outage or not) for all links in  $\mathcal{B}_i$  are chosen to be equal, while if  $S_i = 0$ , the link states for the links in  $\mathcal{B}_i$  are chosen independently at random. Suppose that  $\mathcal{B}_i$  has size  $|\mathcal{B}_i| = x$ , and let  $\mathcal{S}_i$  be any subset of  $\mathcal{B}_i$  of size  $|\mathcal{S}_i| = y$ , where  $0 \le y \le x$ . Then the probability  $p_o(x, y)$  that the links of  $\mathcal{S}_i$  are in outage while the links of  $\mathcal{B}_i \setminus \mathcal{S}_i$  are not in outage is given as

$$p_{o}(x,y) = \begin{cases} \rho(1-p) + (1-\rho)(1-p)^{x} & \text{if } y = 0\\ \rho p + (1-\rho)p^{x} & \text{if } y = x\\ (1-\rho)p^{y}(1-p)^{x-y} & \text{otherwise.} \end{cases}$$
(10)

We assume that the random variables  $S_1, \ldots, S_f$  are mutually independent. Note that the previously considered case (of independent link-states) is obtained by considering  $\rho = 0$ , or, equivalently, by partitioning  $\mathcal{E}$  into singleton sets where  $|\mathcal{B}_i| = 1$  for all *i*. Now, given any subset  $C \subset \mathcal{E}$  of edges (e.g., a cut-set), the probability that all edges of C are in outage while all edges in  $\mathcal{E} \setminus C$  are *not* in outage is given by

$$\prod_{i=1}^{f} p_o(|\mathcal{B}_i|, |\mathcal{C} \cap \mathcal{B}_i|).$$

Thus the network outage polynomial is obtained as

$$O(p) = \sum_{\mathcal{C} \in \mathcal{K}} \prod_{i=1}^{f} p_o(|\mathcal{B}_i|, |\mathcal{C} \cap \mathcal{B}_i|).$$
(11)

# III. DIVERSITY ORDER AND ERGODIC CAPACITY ANALYSES FOR ARBITRARY NETWORK TOPOLOGIES

In this section, performance limitations of an arbitrary network are determined via the outage polynomial. Firstly, expressions for diversity gain and coding gain are derived. Secondly, the ergodic capacity is considered.

# A. Diversity Order Analysis

In order to provide further insight into the obtained outage probability expression, an asymptotic expression of outage probability is derived. The network is in outage if there is no defined path between a source and terminal nodes. Coding and diversity gains can represent the network outage probability in the limit as  $p \rightarrow 0$ , referred to as the high SNR regime. The high SNR performance of any system determines the performance limits of a wireless network. In the high SNR regime, the outage probability expression of an arbitrary given network is given as

$$O(p) \approx \alpha \gamma^{-d},$$

where d, the diversity gain, measures the number of independent copies of the transmitted signal that are received at the terminal node, and where  $\alpha$ , the coding gain (usually expressed on a decibel scale), is a measure of the performance difference between the given system and a baseline system having  $O(p) \approx \gamma^{-d}$  [24].

For the purposes of the following theorem, we say that two functions f(p) and g(p) are asymptotically equal, written  $f(p) \sim g(p)$ , if

$$\lim_{p \to 0} \frac{f(p)}{g(p)} = 1.$$

**Theorem 1.** In a network with outage polynomial  $O(p) = \sum_{i=m}^{n} A_i p^i (1-p)^{n-i}$ ,

$$O(p) \sim A_m p^m$$
.

Thus the diversity order of such a network is equal to the size of a minimum cut-set, i.e., d = m, and the coding gain is equal to the number of distinct minimum cut-sets, i.e.,  $\alpha = A_m$ .

Proof. We have

$$\lim_{p \to 0} \frac{O(p)}{A_m p^m} = \lim_{p \to 0} \frac{A_m p^m (1-p)^{n-m} + A_{m+1} p^{m+1} (1-p)^{n-m-1} + \dots + p^n}{A_m p^m}$$
$$= \lim_{p \to 0} (1-p)^{n-m} + \lim_{p \to 0} \frac{A_{m+1}}{A_m} p (1-p)^{n-m-1} + \dots + \lim_{p \to 0} \frac{1}{A_m} p^{n-m}$$
$$= 1.$$
(12)

The value of maximum flow (the size of the minimum cut) can be calculated by enumerating the number of cut-sets in a dual manner for unit capacity graphs. For dense network graphs, the Ford-Fulkerson algorithm can be used to determine the size of the minimum cut value [25].

It is obvious that adding new edges to a network cannot reduce the size of any cut-sets. If newly added edges (e.g., a line-of-sight edge) provide a new edge-disjoint path from s to t, then the cardinality of all cut-sets, and hence the diversity order of the network, increases by one.

#### B. Ergodic Network Capacity

Suppose now that each network link (when not in outage) provides unit transmission capacity. It is well known, e.g., [26], that the instantaneous s-t unicast capacity C is equal to the size of the minimum s-t-separating cut in the network subgraph induced by the links that are *not* in outage; this transmission rate can be achieved by routing information along edge-disjoint paths between s and t (which, by Menger's Theorem, exist in sufficient number). As the link-state is random, the instantaneous capacity C is a random variable. Indeed, the outage polynomial O(p) gives the probability that C = 0. It is also clear that C is bounded by m, the minimum cut-set size. As C takes integer values in a bounded set, it has a well-defined expected value, called the *ergodic network capacity*.

For  $i \in \{0, 1, ..., m\}$ , the event C = i arises when all minimal cut-sets  $C \in \mathcal{L}$  contain at least *i* links not in outage, and at least one of these cut-sets contains exactly *i* links not in outage.

In other words, C = i arises when the minimum number of non-outage links among minimal cut-sets is equal to *i*. More precisely, let  $\mathcal{E}'$  denote the set of edges *not* in outage at a given time instant. For any minimal cut  $\mathcal{C} \in \mathcal{L}$ , let

$$\delta_{i}(\mathcal{C}) = \begin{cases} 0 & |\mathcal{C} \cap \mathcal{E}'| < i \\ 1 & |\mathcal{C} \cap \mathcal{E}'| \ge i \end{cases}$$
(13)

be the function that indicates whether C contains at least i edges not in outage. The event C = i then arises if

$$\forall \mathcal{C} \in \mathcal{L}(\delta(\mathcal{C}) = 1) \tag{14}$$

$$\min_{\mathcal{C}\in\mathcal{L}}|\mathcal{C}\cap\mathcal{E}'|=i.$$
(15)

For every *i*, the probability that C = i is given by some polynomial  $C_i(p)$ . The ergodic capacity can then be obtained, in terms of *p*, as

$$E[C](p) = \sum_{i=0}^{m} iC_i(p).$$
 (16)

When the minimal cut sets  $C \in \mathcal{L}$  are disjoint, the *i*th capacity polynomial can be calculated as follows. For any minimal cut  $C \in \mathcal{L}$  of size |C|, let q(i, |C|, p) denote the probability that Ccontains *at least i* links not in outage; thus

$$q(i, |\mathcal{C}|, p) = \sum_{j \ge i} {|\mathcal{C}| \choose j} p^{|\mathcal{C}| - j} (1 - p)^j.$$

The probability that *every* minimal cut contains i or more links not in outage is then given as  $\prod_{C \in \mathcal{L}} q(i, |C|, p)$ . The probability that C = i is then given as the probability that *every* minimal cut contains i or more links in non-outage but not every minimal cut contains i + 1 more links in non-outage, namely

$$C_i(p) = \prod_{\mathcal{C} \in \mathcal{L}} q(i, |\mathcal{C}|, p) - \prod_{\mathcal{C} \in \mathcal{L}} q(i+1, |\mathcal{C}|, p).$$

The computational complexity of the ergodic network capacity is made up of the enumeration of all minimal cut-sets, (13), (14), and (15). The enumeration of all minimal cut-sets has a complexity of  $\mathcal{O}(|\mathcal{L}||\mathcal{V}|^3)$  as given in [27]. The total complexity of the functions defined in (13), (14), and (15) is equal to  $\mathcal{O}(m(|\mathcal{L}| + \zeta))$ , where  $\zeta = \sum_{\mathcal{C} \in \mathcal{L}} |\mathcal{C}|$ . Hence the total computational



**Fig. 1:** There are three example networks denoted by  $N_1$ ,  $N_2$ ,  $N_3$ ,  $N_4$ ,  $N_5$ , and  $N_6$  which are presented in (a), (b), (c), (d), and (e), respectively.  $N_1$ ,  $N_2$ , and  $N_3$  respectively have 3 edges, 4 edges, 2 edges, 6 edges, 4 edges and 5 edges.

complexity of the ergodic network capacity expression is equal to  $\mathcal{O}(|\mathcal{L}||\mathcal{V}|^3 + m(|\mathcal{L}| + \zeta))$ . Note that k and  $|\mathcal{L}|$  increase exponentially with the size of  $\mathcal{N}$  [22].

### **IV. NUMERICAL RESULTS**

In this section, we present numerical results to clarify the theoretical expressions on the network performance. We provide two instructive examples to clarify theoretical expressions derived in the previous sections.

Firstly, consider the example network  $N_1$  presented in Fig. 1(a) with edges as labeled. In this network, there are n = 3 edges with the size of the minimum cut m = 1. The cut-sets, minimal



Fig. 2: The comparative results of upper and lower bounds of the outage polynomial of the  $\mathcal{N}_1$ .

cut-sets, and minimum cut-sets are

$$\mathcal{K} = \{\{e_1\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}\},\$$
$$\mathcal{L} = \{\{e_1\}, \{e_2, e_3\}\}, \text{ and}$$
$$\mathcal{M} = \{\{e_1\}\}, \text{ respectively.}$$

We have  $A(x) = x + 3x^2 + x^3$ , thus, the outage polynomial for  $\mathcal{N}_1$  is calculated as

$$O(p) = p(1-p)^{2} + 3p^{2}(1-p) + p^{3} = p + p^{2} - p^{3}.$$

The bound expressions are also given by

$$O(p) \leq A(p) = p + 3p^2 + p^3$$
$$O(p) \leq p + p^2$$
$$O(p) \geq p(1-p)^2.$$

Fig. 2 shows that the given upper bounds become tight when p > 0.5. As  $p \rightarrow 0$ , all bounds



Fig. 3: (a) The capacity polynomials results of  $\mathcal{N}_1$  versus varying p value are presented. (b) The ergodic capacity results of  $\mathcal{N}_1$ .

have the same outage performance with the exact O(p) expression. In addition, the first order approximation of O(p) given as  $O(p) \sim p$  has a close performance with O(p) along with p.

Based on the given capacity assurance sets, capacity polynomials of the network can be calculated as:

$$C_0(p) = O(p) = p + p^2 - p^3$$
  
$$C_1(p) = 2p(1-p)^2 + (1-p)^3 = 1 - p - p^2 + p^3$$

By using (16), the ergodic capacity of  $\mathcal{N}_1$  can be found as:

$$E[C](p) = 1 - p - p^{2} + p^{3}.$$

The obtained capacity polynomials of  $\mathcal{N}_1$  are presented in Fig. 3 (a). While p < 0.5,  $C_0(p)$  is highly probable when compared to  $C_m(p)$  for m = 1. On the other hand,  $C_1(p) \rightarrow 1$  in the case of  $p \rightarrow 0$ . It can be deduced from Fig. 3 (b), the average capacity of the network increases while p is decreasing. In addition, the maximum value of the average capacity of the network is equal to m = 1 for p = 0.

We give another example to illustrate the correlated case results. The depicted extended graph of Fig. 1 (a) with 4 edges labeled as  $N_2$  is given in Fig. 1 (b).



Fig. 4: The outage polynomial results of  $N_2$  are presented in various correlation coefficient,  $\rho$ .

The network has the following sets:

$$\mathcal{K} = \{\{e_1, e_2\}, \{e_3, e_4\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_2, e_3, e_4\}, \{e_1, e_3, e_4\}, \{e_1, e_2, e_3, e_4\}\}$$
$$\mathcal{L} = \{\{e_1, e_2\}, \{e_3, e_4\}\}$$
$$\mathcal{M} = \{\{e_1, e_2\}, \{e_3, e_4\}\}$$

In the uncorrelated case, the outage polynomial can be calculated as:

$$O(p) = 2p^{2}(1-p)^{2} + 4p^{3}(1-p) + p^{4} = 2p^{2} - p^{4}$$

where m = 2 and  $A_m = 2$ . If the correlated edge assumption given in (10) is used, the disjoint edge sets are given as

$$\mathcal{B}_1 = \{e_1, e_2\}, \ \mathcal{B}_2 = \{e_3, e_4\}.$$

where  $\mathcal{B}_1 \cup \mathcal{B}_2 = \mathcal{E}$  and  $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ . By using (11), the outage polynomial of the correlated case is derived as:

$$O(p) = (\rho p + p^2 - \rho p^2) \left[2 - \rho p - p^2 + \rho p^2\right]$$
(17)



Fig. 5: (a) The capacity polynomials results of the  $N_2$  versus varying p value are presented. (b) The ergodic capacity results of the network  $N_2$ .

The numerical results of (17) are presented in Fig. 4. The uncorrelated case ( $\rho = 0$ ) has the best outage performance as expected. When  $\rho = 0.1$ , the outage performance is worse than the uncorrelated case. As  $\rho$  increases, the outage performance gets worse. When  $\rho = 0.5$ , 4-edges network has the close performance of 3-edges system handled in Example 1. On the other hand, if  $\rho$  is equal to 0.9 it means that highly correlated links are available, the outage performance of 4-edges networks approaches to 2-edges system model labeled as  $\mathcal{N}_3$  is given in Fig. 1 (c).

The capacity polynomials of  $\mathcal{N}_2$  is given in Fig. 4 can be calculated as:

$$C_0(p) = p^4 + 4p^3(1-p) + 2p^2(1-p)^2 = 2p^2 - p^4$$
  

$$C_1(p) = 4p^2(1-p)^2 + 4p(1-p)^3 = 4p - 8p^2 + 4p^3$$
  

$$C_2(p) = (1-p)^4.$$

Hence, the ergodic capacity of  $\mathcal{N}_2$  is given by

$$E[C](p) = 2 - 4p + 4p^2 - 4p^3 + 2p^4.$$

The numerical results of the given polynomials are shown in Fig. 5 (a). While p < 0.5,  $C_m(p) < m$  is high than  $C_m(p)$ . On the other hand,  $C_m(p) \to 1$  in the case of  $p \to 0$ . Hence, the maximum value of the ergodic capacity of the network which is shown in Fig. 5 (b) is equal to m = 2 for  $p \to 0$ .

In order to obtain further insight about the derivations, the outage polynomial and the ergodic



Fig. 6: (a) The outage polynomial results of all the networks given in Fig. 1. (b) The ergodic capacity results of all the networks given in Fig. 1.

capacity results of  $\mathcal{N}_4$ ,  $\mathcal{N}_5$ , and  $\mathcal{N}_6$  depicted in Fig.s 1 (d), (e) and (f), respectively, are investigated. By using (5), the outage polynomial expressions of  $\mathcal{N}_4$ ,  $\mathcal{N}_5$ , and  $\mathcal{N}_6$  can be respectively calculated as:

$$O(p) = 4p^{2} - 2p^{3} - 4p^{4} + 4p^{5} - p^{6},$$
  

$$O(p) = 4p^{2} - 4p^{3} + p^{4}$$
  

$$O(p) = 4p^{3} - 4p^{4} + p^{5},$$

Here, the three graphs have the same coding gain with  $A_m = 4$ . On the other hand,  $\mathcal{N}_4$  and  $\mathcal{N}_5$  have the same diversity order equal to 2 and the diversity order of  $\mathcal{N}_6$  is equal to 3. The ergodic capacity results of the three networks can be respectively given as:

$$E[C](p) = 2 - 5p + 6p^{2} - 8p^{3} + 9p^{4} - 5p^{5} + p^{6},$$
  

$$E[C](p) = 2 - 4p + 2p^{2}$$
  

$$E[C](p) = 3 - 5p + 2p^{2}$$

The outage polynomial and the ergodic capacity results of all the networks shown in Fig. 1 are presented in Fig. 6. It can be deduced from Fig. 1(a) that  $\mathcal{N}_6$  has the best outage performance with the highest diversity order m = 3. The two worst outage performance with m = 1 belongs to  $\mathcal{N}_3$  and  $\mathcal{N}_1$ , as expected.  $\mathcal{N}_2$ ,  $\mathcal{N}_4$ , and  $\mathcal{N}_5$  have close outage performance results with m = 2.

The ergodic capacity results are in accordance with the outage polynomial results. Hence, the best performance belongs to  $N_6$ .

## V. CONCLUSION

In this paper, we have obtained the performance limits of generalized wireless communication networks by using the concepts of graph theory. We have evaluated the network outage polynomial by utilizing individual link outages, through the use of path enumeration, cut-set enumeration and terminal-reliability approaches. For high-SNR region, diversity order and coding gain have been extracted from the graph model of wireless networks. We have proven that the diversity order of any wireless communication network is minimum cut-set size of the network graph and the coding gain is the number of distinct minimum cut-sets. We have also presented the ergodic capacity analysis of arbitrary networks to obtain the ergodic capacity polynomials. The theoretical expressions have been illustrated by numerical examples. Hence, we have provided a comprehensive tool can be used to determine asymptotic performance of unstructured wireless networks and to specify their performance limitations under various implementation schemes.

#### REFERENCES

- R. C. French, "The effect of fading and shadowing on channel reuse in mobile radio," *IEEE Trans. Veh. Techol.*, vol. 28, no. 3, pp. 171–181, 1979.
- [2] J. G. Proakis, Digital Communications. McGraw-Hill, 1995.
- [3] Z. Wang and G. B. Giannakis, "A simple and general parameterization quantifying performance in fading channels," *IEEE Trans. Commun.*, vol. 51, no. 8, pp. 1389–1398, Aug. 2003.
- [4] L. Zheng and D. N. C. Tse, "Diversity and multiplexing: a fundamental tradeoff in multiple-antenna channels," *IEEE Trans. Inf. Theory*, vol. 49, no. 5, pp. 1073–1096, 2003.
- [5] E. Moore and C. Shannon, "Reliable circuits using less reliable relays," *Journal of the Franklin Institute*, vol. 262, no. 3, pp. 191 208, 1956.
- [6] J. A. Bondy and U. S. R. Murty, Graph theory with applications. Citeseer, 1976, vol. 290.
- [7] O. Wing and P. Demetriou, "Analysis of probabilistic networks," *IEEE Trans. Commun. Tech.*, vol. 12, no. 3, pp. 38–40, 1964.
- [8] P. M. Lin, B. J. Leon, and T. C. Huang, "A new algorithm for symbolic system reliability analysis," *IEEE Trans. Rel.*, vol. R-25, no. 1, pp. 2–15, Apr. 1976.
- [9] F. Moskowitz, "The analysis of redundancy networks," *Transactions of the American institute of electrical engineers, part i: communication and electronics*, vol. 77, no. 5, pp. 627–632, 1958.
- [10] P. Gupta and P. R. Kumar, "Critical power for asymptotic connectivity in wireless networks," in *Stochastic analysis, control, optimization and applications.* Springer, 1999, pp. 547–566.
- [11] C. Bettstetter and C. Hartmann, "Connectivity of wireless multihop networks in a shadow fading environment," Wireless Networks, vol. 11, no. 5, pp. 571–579, 2005.

- [12] R. Hekmat and P. Van Mieghem, "Connectivity in wireless ad-hoc networks with a log-normal radio model," *Mobile networks and applications*, vol. 11, no. 3, pp. 351–360, 2006.
- [13] P. Agrawal and N. Patwari, "Correlated link shadow fading in multi-hop wireless networks," *IEEE Trans. Wireless Commun.*, vol. 8, no. 8, 2009.
- [14] S. Lu, J. May, and R. J. Haines, "Reliability modeling and prediction of wireless multi-hop networks with correlated shadowing," in *Proc. IEEE PIMRC*, 2014, pp. 1663–1668.
- [15] S. Rai and S. Soh, "A computer approach for reliability evaluation of telecommunication networks with heterogeneous link-capacities," *IEEE Trans. Rel.*, vol. 40, no. 4, pp. 441–451, 1991.
- [16] M. Hayashi and T. Abe, "Evaluating reliability of telecommunications networks using traffic path information," *IEEE Trans. Rel.*, vol. 57, no. 2, pp. 283–294, June 2008.
- [17] P. A. Jensen and M. Bellmore, "An algorithm to determine the reliability of a complex system," *IEEE Trans. Rel.*, vol. 18, no. 4, pp. 169–174, 1969.
- [18] P. Doulliez and E. Jamoulle, "Transportation networks with random arc capacities," *Revue française d'automatique, informatique, recherche opérationnelle. Recherche opérationnelle*, vol. 6, no. V3, pp. 45–59, 1972.
- [19] G. Grimmett and D. Welsh, "Flow in networks with random capacities," *Stochastics: An International Journal of Probability and Stochastic Processes*, vol. 7, no. 3, pp. 205–229, 1982.
- [20] C. J. Colbourn, The combinatorics of network reliability. Oxford University Press, Inc., 1987.
- [21] G. E. Andrews, Number theory. Courier Corporation, 1994.
- [22] J. S. Provan and D. R. Shier, "A paradigm for listing (s, t)-cuts in graphs," *Algorithmica*, vol. 15, no. 4, pp. 351–372, 1996.
- [23] M. O. Ball, C. J. Colbourn, and J. S. Provan, "Network reliability," Handbooks in operations research and management science, vol. 7, pp. 673–762, 1995.
- [24] Z. Wang and G. B. Giannakis, "A simple and general parameterization quantifying performance in fading channels," *IEEE Trans Commun*, vol. 51, no. 8, pp. 1389–1398, 2003.
- [25] L. R. Ford and D. R. Fulkerson, "Maximal flow through a network," *Canadian journal of Mathematics*, vol. 8, no. 3, pp. 399–404, 1956.
- [26] R. Koetter and M. Médard, "An algebraic approach to network coding," *IEEE/ACM Transactions on Networking (TON)*, vol. 11, no. 5, pp. 782–795, 2003.
- [27] A. Berry, J.-P. Bordat, and O. Cogis, "Generating all the minimal separators of a graph," *International Journal of Foundations of Computer Science*, vol. 11, no. 03, pp. 397–403, 2000.