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BER Analysis of QAM on Fading Channels with Transmit Diversity

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***Abstract*— In this letter, we derive analytical expressions for the bit error rate (BER) of space-time block codes (STBC) from complex orthogonal designs (COD) using quadrature amplitude modulation (QAM) on Rayleigh fading channels. We take a bit log-likelihood ratio (LLR) based approach to derive the BER expressions. The approach presented here can be used in the BER analysis of any STBC from COD with linear processing for any value of M in an M-QAM system. Here, we present the BER analysis and results for a 16-QAM system with i) (2-Tx, L-Rx) antennas using Alamouti code (rate-1 STBC), ii) (3-Tx, L-Rx) antennas using a rate-1/2 STBC, and iii) (5-Tx, L-Rx) antennas using a rate-7/11 STBC. In addition to being used in the BER analysis, the LLRs derived can also be used as soft inputs to decoders for various coded QAM schemes, including turbo coded QAM with space-time coding as in high speed downlink packet access (HSDPA) in 3G.**

***Index Terms*— Space-time block codes, transmit diversity, QAM, bit log-likelihood ratio.**

I. INTRODUCTION

THE potential capacity gains achieved by using multiple antenna systems have led to considerable attention in the area of space-time coding [1]. Space-time block codes (STBC) from complex orthogonal designs (COD) are of interest as they can be used for complex constellations such as quadrature amplitude modulation (QAM) to achieve higher data rates in wireless communication systems [2],[3]. Recent works have reported analytical expressions for the symbol error rate (SER) and the bit error rate (BER) of orthogonal STBCs. In [4], Shin and Lee derived expressions for the SER of orthogonal STBCs on Rayleigh fading channels. They derived the SER by converting the multiple input multiple output (MIMO) system model to an equivalent single input single output (SISO) model. Recently, Simon in [5], and Taricco and Biglieri in [6], have reported exact expressions for the pairwise error probability (PEP) as well as approximate expressions for the BER for space-time codes.

In this letter, we derive analytical expressions for the BER for linear STBCs from COD using QAM on Rayleigh fading channels. We adopt a bit log-likelihood ratio (LLR) based approach, where we first derive expressions for the LLRs of the individual bits forming the QAM symbol, and then use these LLRs to obtain the BER expressions. We point out that this approach can be used in the BER analysis of any STBC from COD with linear processing for any value of M in an M-QAM system. Here, we present the BER analysis and results for a 16-QAM system with i) (2-Tx, L-Rx) antennas using the rate-1 Alamouti code, ii) (3-Tx, L-Rx) antennas using a rate-1/2 code, and iii) (5-Tx, L-Rx) antennas using a rate-7/11 code. Another major usefulness of this contribution is that the derived LLRs provide a soft metric for each bit in the mapping, which can be used as soft inputs to decoders for various coded QAM schemes with space-time coding. Examples of such schemes include turbo coded QAM with transmit diversity in high speed downlink packet access (HSDPA) in 3G, and convolutionally coded QAM with orthogonal frequency division multiplexing (OFDM) in digital video broadcasting (DVB) and IEEE 802.11.

II. SYSTEM MODEL

We consider a wireless communication system with L_t transmit and L_r receive antennas. We consider space-time block codes, where each codeword is a matrix with P rows and L_t columns, with complex valued symbols as its entries. Here, P is the number of time slots required to transmit one codeword. For some K information symbols, $s_1, s_2, \cdots, s_K$, which are selected from the 16-QAM constellation (see Fig. 1)[1], the entries of the codeword $\mathbf{X} = \{x_t^i, t = 1, 2, \cdots, P;\ i = 1, 2, \cdots, L_t\}$ are a linear combination of the information symbols s_k, $k = 1, 2, \cdots, K$, and their complex conjugates. At time slot t, $t = 1, 2, \cdots, P$, the t^{th} row of the codeword $\mathbf{X}$ (i.e., $x_t^1, x_t^2, \cdots, x_t^{L_t}$) is transmitted simultaneously from L_t antennas. The symbol transmission rate, R, is defined as the number of information symbols transmitted per time slot, i.e., $R = K/P$. The channel fade coefficients are assumed to

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[1]Four bits, (r_1, r_2, r_3, r_4) are mapped on to a complex symbol $s_k = s_{kI} + js_{kQ}$. The horizontal/vertical line pieces in Fig. 1 denote that all bits under these lines take the value 1, and the rest take the value 0.

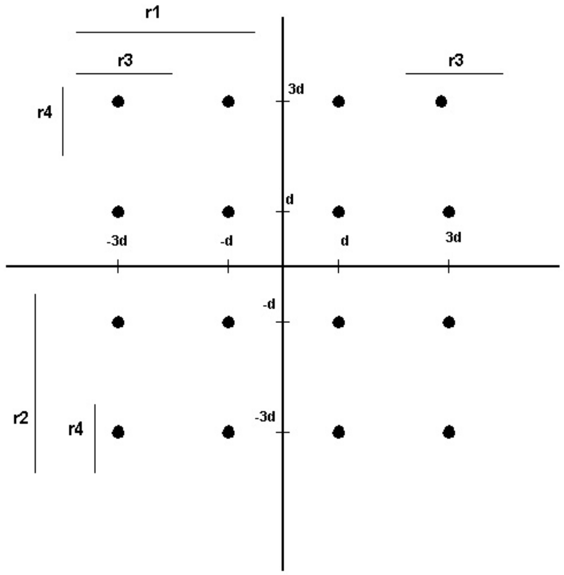

Fig. 1. 16-QAM Constellation.

remain constant over P time slots. The received codeword, $\mathbf{Y}$, can be written as

$$\mathbf{Y} = \mathbf{X}\mathbf{H} + \mathbf{N}, \tag{1}$$

where $\mathbf{Y} = \{y_t^j : t = 1, 2, \cdots, P;\ j = 1, 2, \cdots, L_r\}$ is a matrix of size $P \times L_r$, whose entry y_t^j is the signal received at antenna j at time slot t; $\mathbf{H} = \{h_{i,j}\}$ is the channel matrix of size $L_t \times L_r$, whose entry $h_{i,j}$ is the complex channel coefficient from the transmit antenna i to the receive antenna j. The random variables $|h_{i,j}|$'s are assumed to be i.i.d Rayleigh distributed with $E(|h_{i,j}|^2) = \Omega$. $\mathbf{N} = \{n_t^j\}$ is the noise matrix of size $P \times L_r$, whose entries are i.i.d complex Gaussian noise with zero mean and variance σ^2.

Let $\mathcal{C}(.)$ be a mapping from a K-tuple complex message vector $\mathbf{s} = (s_1, s_2, \cdots, s_K)$ to the columnwise orthogonal $P \times L_t$ codeword $\mathbf{X} = \mathcal{C}(\mathbf{s})$. Due to the columnwise orthogonality of the linear orthogonal space-time block codes considered, the $L_t \times L_t$ matrix $\mathcal{C}(s)^H \mathcal{C}(s)$ is given by

$$\mathcal{C}(s)^H \mathcal{C}(s) = diag\left\{\sum_{k=1}^{K}(g_{k,1}|s_k|^2), \cdots, \sum_{k=1}^{K}(g_{k,L_t}|s_k|^2)\right\}, \tag{2}$$

where $(.)^H$ denotes the Hermitian operator, and $\mathbf{G} = \{g_{m,n}\}$ is a matrix of size $K \times L_t$ whose entries can take non-negative integer values (for example, for the Alamouti code [7] $g_{m,n} = 1, \forall m, n$). Assuming perfect knowledge of the channel coefficients at the receiver, the combined signal output for the symbol s_k is given by

$$\widehat{s}_k = \Delta_k\, s_k + \zeta_k, \tag{3}$$

where

$$\Delta_k = \sum_{j=1}^{L_r}\left[g_{k,1}|h_{1,j}|^2 + g_{k,2}|h_{2,j}|^2 + \cdots + g_{k,L_t}|h_{L_t,j}|^2\right], \tag{4}$$

and ζ_k is a complex Gaussian random variable with zero mean and variance $\Delta_k\, \sigma^2$.

III. Bit Log-Likelihood Ratios

We define the LLR for the bit $r_i,\ i = 1, 2, 3, 4$, of symbol $s_k,\ k = 1, 2, \cdots, K$, as

$$\begin{aligned} LLR_{s_k}(r_i) &= \log\left(\frac{\Pr\left(r_i = 1|\mathbf{Y}, \mathbf{H}\right)}{\Pr\left(r_i = 0|\mathbf{Y}, \mathbf{H}\right)}\right) \\ &= \log\left(\frac{\Pr\left(r_i = 1|\widehat{s}_k, \mathbf{H}\right)}{\Pr\left(r_i = 0|\widehat{s}_k, \mathbf{H}\right)}\right). \end{aligned} \tag{5}$$

Assuming that all the symbols are equally likely and that the fading is independent of the transmitted symbols, using Bayes' rule, we have

$$LLR_{s_k}(r_i) = \log\left(\frac{\sum_{\alpha \in S_i^{(1)}} f_{\hat{s}_k|\mathbf{H},s_k}\left(\widehat{s}_k|\mathbf{H}, s_k = \alpha\right)}{\sum_{\beta \in S_i^{(0)}} f_{\hat{s}_k|\mathbf{H},s_k}\left(\widehat{s}_k|\mathbf{H}, s_k = \beta\right)}\right). \tag{6}$$

Since $f_{\hat{s}_k|\mathbf{H},s_k}\left(\widehat{s}_k|\mathbf{H}, s_k = \alpha\right) = \frac{1}{\pi\hat{\sigma}_k^2}\exp\left(\frac{-1}{\hat{\sigma}_k^2}\left\|\widehat{s}_k - \Delta_k\,\alpha\right\|^2\right)$ where $\hat{\sigma}_k^2 = \Delta_k\,\sigma^2$, (6) can be written as

$$LLR_{s_k}(r_i) = \log\left(\frac{\sum_{\alpha \in S_i^{(1)}} \exp\left(\frac{-1}{\hat{\sigma}_k^2}\|\widehat{s}_k - \Delta_k\alpha\|^2\right)}{\sum_{\beta \in S_i^{(0)}} \exp\left(\frac{-1}{\hat{\sigma}_k^2}\|\widehat{s}_k - \Delta_k\beta\|^2\right)}\right). \tag{7}$$

Using the approximation $\log\left(\sum_j \exp(-X_j)\right) \approx -\min_j(X_j)$, $LLR_{s_k}(r_i)$ can be approximated as[2]

$$\begin{aligned} LLR_{s_k}(r_i) = \frac{1}{\hat{\sigma}_k^2}\Bigg(&\min_{\beta \in S_i^{(0)}}\left\|\widehat{s}_k - \Delta_k\,\beta\right\|^2 \\ &- \min_{\alpha \in S_i^{(1)}}\left\|\widehat{s}_k - \Delta_k\,\alpha\right\|^2\Bigg). \end{aligned} \tag{8}$$

Define k complex variables, $\widehat{z}_k,\ k = 1, 2, \cdots, K$, as

$$\widehat{z}_k \triangleq \frac{\widehat{s}_k}{\Delta_k}. \tag{9}$$

Using (9) in (8) and normalizing by $4/\hat{\sigma}_k^2$, $LLR_{s_k}(r_i)$ is written as

$$LLR_{s_k}(r_i) = \frac{\Delta_k}{4}\left(\min_{\beta \in S_i^{(0)}}\|\widehat{z}_k - \beta\|^2 - \min_{\alpha \in S_i^{(1)}}\|\widehat{z}_k - \alpha\|^2\right). \tag{10}$$

Note that the set partitions $S_i^{(1)}$ and $S_i^{(0)}$ are delimited by horizontal or vertical boundaries. As a consequence, two symbols in different sets closest to the received symbol always lie either on the same row (if the delimiting boundaries are vertical) or on the same column (if the delimiting boundaries are horizontal). Using the above fact, the LLRs for each of the bits forming the symbol, s_k, are obtained as

$$LLR_{s_k}(r_1) = \begin{cases} -d\widehat{z}_{kI}\Delta_k, & |\widehat{z}_{kI}| \le 2d \\ 2d(d - \widehat{z}_{kI})\Delta_k, & \widehat{z}_{kI} > 2d \\ -2d(d + \widehat{z}_{kI})\Delta_k, & \widehat{z}_{kI} < -2d \end{cases} \tag{11}$$

$$LLR_{s_k}(r_2) = \begin{cases} -d\widehat{z}_{kQ}\Delta_k, & |\widehat{z}_{kQ}| \le 2d \\ 2d(d - \widehat{z}_{kQ})\Delta_k, & \widehat{z}_{kQ} > 2d \\ -2d(d + \widehat{z}_{kQ})\Delta_k, & \widehat{z}_{kQ} < -2d \end{cases} \tag{12}$$

[2]As we will see in Sec. V, the analytical BER evaluated using this approximate LLR is almost the same as the BER evaluated through simulations without this approximation.

$$LLR_{s_k}(r_3) = d(|\widehat{z}_{kI}| - 2d)\Delta_k \quad (13)$$

$$LLR_{s_k}(r_4) = d(|\widehat{z}_{kQ}| - 2d)\Delta_k. \quad (14)$$

In the above equations, $\widehat{z}_{kI}$ and $\widehat{z}_{kQ}$ are the real and imaginary parts of $\widehat{z}_k$, respectively, and $2d$ is the minimum distance between pairs of signal points. We note that, likewise, the LLR expressions for other values of M in M-QAM can be derived. For example, we have derived the bit LLR expressions for the 32-QAM constellation in Fig. 4 of [10] as well as the 64-QAM constellation in Fig. 4 of [11] and presented them in Table I. These LLR expressions can be used to derive the BER expressions for M-QAM as illustrated in the following section.

IV. Derivation of BER

In this section, we derive the probability of error for the bit r_i, $i = 1, 2, 3, 4$, forming a 16-QAM symbol. The probability of error for bit r_1 in symbol s_k, P^k_{b1}, can be written as

$$\begin{aligned} P^k_{b1} &= P^k_{b1|s_{kI}=-d}\Pr(s_{kI}=-d) + P^k_{b1|s_{kI}=-3d}\Pr(s_{kI}=-3d) \\ &+ P^k_{b1|s_{kI}=d}\Pr(s_{kI}=d) + P^k_{b1|s_{kI}=3d}\Pr(s_{kI}=3d), \end{aligned} \quad (15)$$

where s_{kI} represents the real part of s_k. Let us first consider $P^k_{b1|s_{kI}=-d}$, which is given by

$$P^k_{b1|s_{kI}=-d} = \overline{P^k_{b1|s_{kI}=-d,\mathbf{H}}} \quad (16)$$

where the overline indicates averaging over the complex random variables $\{h_{i,j}\}$. $P^k_{b1|s_{kI}=-d,\mathbf{H}}$ can be written as

$$\begin{aligned} P^k_{b1|s_{kI}=-d,\mathbf{H}} &= \Pr\Big(LLR_{s_k}(r_1) < 0 \,|\, s_{kI} = -d, \mathbf{H}\Big) \\ &= \Pr\left(\frac{\zeta_{kI}}{\Delta_k} \geq d\right) = Q\left(\frac{d\sqrt{\Delta_k}}{\sigma_I}\right), \end{aligned} \quad (17)$$

where $\sigma_I^2 = \sigma^2/2$. Let us define

$$\xi = \frac{1}{P}\sum_{i=1}^{L_t}\sum_{k=1}^{K} g_{k,i}. \quad (18)$$

We then have $\frac{d}{\sigma_I} = \sqrt{\frac{4E_b R}{5N_o L_r \xi}}$, where E_b is the energy per bit per transmit antenna and R is the rate of the STBC used. From the above, we can write

$$P^k_{b1|s_{kI}=-d,\mathbf{H}} = Q\left(\sqrt{\frac{4E_b R\,\Delta_k}{5N_o L_r\,\xi}}\right). \quad (19)$$

To obtain $P^k_{b1|s_{kI}=-d}$, we need to uncondition $P^k_{b1|s_{kI}=-d,\mathbf{H}}$ w.r.t Δ_k, which is given by

$$\begin{aligned} \Delta_k &= \sum_{j=1}^{L_r}\Big(g_{k,1}|h_{1,j}|^2 + g_{k,2}|h_{2,j}|^2 + \cdots + g_{k,L_t}|h_{L_t,j}|^2\Big) \\ &= g_{k,1}\left(\sum_{j=1}^{L_r}|h_{1,j}|^2\right) + \cdots + g_{k,L_t}\left(\sum_{j=1}^{L_r}|h_{L_t,j}|^2\right) \end{aligned} \quad (20)$$

Let us define $\theta_n = \sum_{j=1}^{L_r}|h_{n,j}|^2, n = 1, 2, \cdots, L_t$. Since $|h_{i,j}|^2$ are i.i.d exponential with mean Ω, the random variables θ_n are i.i.d Gamma random variables with density function

$$f_{\theta_n}(x) = \frac{1}{\Gamma(L_r)\Omega^{L_r}}\exp\Big(-\frac{x}{\Omega}\Big)x^{L_r-1}, \quad (21)$$

and the moment generating function (MGF) is given by

$$\mathcal{M}_{\theta_n}(s) = \left(\frac{1}{1+s\Omega}\right)^{L_r}. \quad (22)$$

Since $\Delta_k = \sum_{n=1}^{L_t} g_{k,n}\theta_n$, its MGF, $\mathcal{M}_{\Delta_k}$, is given by

$$\mathcal{M}_{\Delta_k} = \prod_{n=1}^{L_t}\left(\frac{1}{1+s\Omega g_{k,n}}\right)^{L_r}. \quad (23)$$

Using the above and Craig's formula [8], we can show that

$$\begin{aligned} P^k_{b1|s_{kI}=-d} &= \overline{Q\left(\sqrt{\frac{4E_b R\,\Delta_k}{5N_o L_r\,\xi}}\right)} \\ &= \frac{1}{\pi}\int_{\phi=0}^{\frac{\pi}{2}}\prod_{n=1}^{L_t}\left(\frac{\sin^2\phi}{\sin^2\phi + \mu_1 g_{k,n}}\right)^{L_r} d\phi, \end{aligned} \quad (24)$$

where $\mu_1 = \frac{2\gamma_b R}{5L_r\xi}$ and $\gamma_b = \frac{\Omega E_b}{N_o}$. Similarly, the conditional error probability $P^k_{b1|s_{kI}=-3d,\mathbf{H}}$ is given by

$$\begin{aligned} P^k_{b1|s_{kI}=-3d,\mathbf{H}} &= \Pr\Big(LLR_{s_k}(r_1) < 0 \,|\, s_{kI} = -3d, \mathbf{H}\Big) \\ &= \Pr\left(\frac{\zeta_{kI}}{\Delta_k} \geq 3d\right) = Q\left(\sqrt{\frac{36E_b R\,\Delta_k}{5N_o L_r\,\xi}}\right) \end{aligned} \quad (25)$$

Unconditioning $P^k_{b1|s_{kI}=-3d,\mathbf{H}}$ w.r.t Δ_k, it can be shown that

$$\begin{aligned} P^k_{b1|s_{kI}=-d} &= \overline{Q\left(\sqrt{\frac{36E_b R\,\Delta_k}{5N_o L_r\,\xi}}\right)} \\ &= \frac{1}{\pi}\int_{\phi=0}^{\frac{\pi}{2}}\prod_{n=1}^{L_t}\left(\frac{\sin^2\phi}{\sin^2\phi + \mu_2 g_{k,n}}\right)^{L_r} d\phi, \end{aligned} \quad (26)$$

where $\mu_2 = \frac{18\gamma_b R}{5L_r\xi}$. It can further be shown that $P^k_{b1|s_{kI}=-d}$ = $P^k_{b1|s_{kI}=d}$ and $P^k_{b1|s_{kI}=-3d}$ = $P^k_{b1|s_{kI}=3d}$. Moreover, for the 16-QAM constellation considered, it can be shown that $P^k_{b1} = P^k_{b2}$ and $P^k_{b3} = P^k_{b4}$. With the above, the BER expressions for the bits r_1, r_2, r_3, r_4 of the symbol s_k can be written as

$$P^k_{b1} = P^k_{b2} = \frac{1}{2}\Big(P^k_1 + P^k_2\Big) \quad (27)$$

$$P^k_{b3} = P^k_{b4} = \frac{1}{2}\Big(2P^k_1 + P^k_2 - P^k_3\Big), \quad (28)$$

where P^k_j, $j = 1, 2, 3$, are given by

$$P^k_j = \frac{1}{\pi}\int_{\phi=0}^{\frac{\pi}{2}}\prod_{n=1}^{L_t}\left(\frac{\sin^2\phi}{\sin^2\phi + \mu_j g_{k,n}}\right)^{L_r} d\phi, \quad (29)$$

and

$$\mu_1 = \frac{2\gamma_b R}{5L_r\xi}, \quad \mu_2 = \frac{18\gamma_b R}{5L_r\xi}, \quad \mu_3 = \frac{10\gamma_b R}{L_r\xi}. \quad (30)$$

Note that for STBCs where $g_{k,n} = g, \forall k, n$, the integral in (29) has a closed-form expression given by [9]

$$P^k_j = \left(\frac{1-\lambda_j}{2}\right)^{L_r L_t} \cdot \sum_{k=0}^{L_r L_t - 1}\binom{L_r L_t - 1 + k}{k}\left(\frac{1+\lambda_j}{2}\right)^k, \quad (31)$$

TABLE I

BIT LLR EXPRESSIONS FOR THE 32-QAM CONSTELLATION IN FIG. 4 OF [10] AND THE 64-QAM CONSTELLATION IN FIG. 4 OF [11] IN RAYLEIGH FADING. FOR 32-QAM, THE MAPPING OF BITS r_i'S TO BITS i_j'S AND q_j'S IN FIG. 4 OF [10] IS AS FOLLOWS: $r_1 = i_1, r_2 = q_1, r_3 = i_2, r_4 = q_2, r_5 = i_3$.

Bit	LLR	Expressions for 32-QAM		Expressions for 64-QAM	
r_1	$LLR_{s_k}(r_1) =$	$-d\widehat{z}_{jI}\Delta_k,$ $2d(d-\widehat{z}_{jI})\Delta_k,$ $3d(2d-\widehat{z}_{jI})\Delta_k,$ $4d(3d-\widehat{z}_{jI})\Delta_k,$ $-2d(d+\widehat{z}_{jI})\Delta_k,$ $-3d(2d+\widehat{z}_{jI})\Delta_k,$ $-4d(3d+\widehat{z}_{jI})\Delta_k,$	$\lvert\widehat{z}_{jI}\rvert \le 2d$ $2d < \widehat{z}_{jI} \le 4d$ $4d < \widehat{z}_{jI} \le 6d$ $\widehat{z}_{jI} > 6d$ $-4d \le \widehat{z}_{jI} < -2d$ $-6d \le \widehat{z}_{jI} < -4d$ $\widehat{z}_{jI} < -6d$	$-d\widehat{z}_{jI}\Delta_k,$ $2d(d-\widehat{z}_{jI})\Delta_k,$ $3d(2d-\hat{z}_{jI})\Delta_k,$ $4d(3d-\widehat{z}_{jI}\Delta_k,$ $-2d(d+\widehat{z}_{jI})\Delta_k,$ $-3d(2d+\widehat{z}_{jI})\Delta_k,$ $-4d(3d+\widehat{z}_{jI})\Delta_k,$	$\lvert\widehat{z}_{jI}\rvert \le 2d$ $2d < \widehat{z}_{jI} \le 4d$ $4d < \widehat{z}_{jI} \le 6d$ $\widehat{z}_{jI} > 6d$ $-4d \le \widehat{z}_{jI} < -2d$ $-6d \le \widehat{z}_{jI} < -4d$ $\widehat{z}_{jI} < -6d$
r_2	$LLR_{s_k}(r_2) =$	$-d\widehat{z}_{jQ}\Delta_k,$ $2d(d-\widehat{z}_{jQ})\Delta_k,$ $-2d(d+\widehat{z}_{jQ})\Delta_k,$	$\lvert\widehat{z}_{jQ}\rvert \le 2d$ $\widehat{z}_{jQ} > 2d$ $\widehat{z}_{jQ} < -2d$	$-d\widehat{z}_{jQ}\Delta_k,$ $2d(d-\widehat{z}_{jQ})\Delta_k,$ $3d(2d-\widehat{z}_{jQ})\Delta_k,$ $4d(3d-\widehat{z}_{jQ})\Delta_k,$ $-2d(d+\widehat{z}_{jQ})\Delta_k,$ $-3d(2d+\widehat{z}_{jQ})\Delta_k,$ $-4d(3d+\widehat{z}_{jQ})\Delta_k,$	$\lvert\widehat{z}_{jQ}\rvert \le 2d$ $2d < \widehat{z}_{jQ} \le 4d$ $4d < \widehat{z}_{jQ} \le 6d$ $\widehat{z}_{jQ} > 6d$ $-4d \le \widehat{z}_{jQ} < -2d$ $-6d \le \widehat{z}_{jQ} < -4d$ $\widehat{z}_{jQ} < -6d$
r_3	$LLR_{s_k}(r_3) =$	$2d(-3d+\lvert\widehat{z}_{jI}\rvert)\Delta_k,$ $d(-4d+\lvert\widehat{z}_{jI}\rvert)\Delta_k,$ $2d(-5d+\lvert\widehat{z}_{jI}\rvert)\Delta_k,$	$\lvert\widehat{z}_{jI}\rvert \le 2d$ $2d < \lvert\widehat{z}_{jI}\rvert \le 6d$ $2d < \lvert\widehat{z}_{jI}\rvert > 6d$	$2d(-3d+\lvert\widehat{z}_{jI}\rvert)\Delta_k,$ $d(-4d+\lvert\widehat{z}_{jI}\rvert)\Delta_k,$ $2d(-5d+\lvert\widehat{z}_{jI}\rvert)\Delta_k,$	$\lvert\widehat{z}_{jI}\rvert \le 2d$ $2d < \lvert\widehat{z}_{jI}\rvert \le 6d$ $2d < \lvert\widehat{z}_{jI}\rvert > 6d$
r_4	$LLR_{s_k}(r_4) =$	$d\left(\lvert\widehat{z}_{jQ}\rvert - 2d\right)\Delta_k$		$2d(-3d+\lvert\widehat{z}_{jQ}\rvert)\Delta_k,$ $d(-4d+\lvert\widehat{z}_{jQ}\rvert)\Delta_k,$ $2d(-5d+\lvert\widehat{z}_{jQ}\rvert)\Delta_k,$	$\lvert\widehat{z}_{jQ}\rvert \le 2d$ $2d < \lvert\widehat{z}_{jQ}\rvert \le 6d$ $2d < \lvert\widehat{z}_{jQ}\rvert > 6d$
r_5	$LLR_{s_k}(r_5) =$	$d(2d-\lvert\widehat{z}_{jI}\rvert)\Delta_k,$ $d(-6d+\lvert\widehat{z}_{jI}\rvert)\Delta_k,$	$\lvert\widehat{z}_{jI}\rvert \le 4d$ $\lvert\widehat{z}_{jI}\rvert > 4d$	$d(2d-\lvert\widehat{z}_{jI}\rvert)\Delta_k,$ $d(-6d+\lvert\widehat{z}_{jI}\rvert)\Delta_k,$	$\lvert\widehat{z}_{jI}\rvert \le 4d$ $\lvert\widehat{z}_{jI}\rvert > 4d$
r_6	$LLR_{s_k}(r_6) =$			$d(2d-\lvert\widehat{z}_{jQ}\rvert)\Delta_k,$ $d(-6d+\lvert\widehat{z}_{jQ}\rvert)\Delta_k,$	$\lvert\widehat{z}_{jQ}\rvert \le 4d$ $\lvert\widehat{z}_{jQ}\rvert > 4d$

where $\lambda_j = \sqrt{\frac{g\mu_j}{1+g\mu_j}}$. It is noted that, for STBCs including rate-1 Alamouti code ($\mathcal{C}_1$ given in the next section) and rate-1/2 STBC ($\mathcal{C}_2$ given in the next section), $g_{k,n}$ are constants ($g = 1$ for $\mathcal{C}_1$ and $g = 2$ for $\mathcal{C}_2$), and hence the closed-form expression in (31) can be used to compute the BER for these STBCs. For STBCs where $g_{k,n}$ is not a constant (e.g., rate-7/11 STBC $\mathcal{C}_3$ given in the next section), (29) can be evaluated numerically and accurately using the Gauss-Chebyshev Quadrature rule. The average BER for symbol s_k, $k = 1, 2, \cdots, K$, P_b^k, is then given by

$$P_b^k = \frac{1}{4}\left(P_{b1}^k + P_{b2}^k + P_{b3}^k + P_{b4}^k\right). \qquad (32)$$

Finally, the average BER of the system, P_b, is given by

$$P_b = \frac{1}{K}\sum_{k=1}^{K} P_b^k. \qquad (33)$$

The BER expressions for other values of M in M-QAM can be derived likewise.

V. RESULTS AND DISCUSSIONS

We computed the BER performance of 16-QAM on Rayleigh fading channels as a function of average SNR for the following space time block codes:

$$\mathcal{C}_1 = \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix}, \quad \mathcal{C}_2 = \begin{pmatrix} s_1 & s_2 & s_3 \\ -s_2 & s_1 & -s_4 \\ -s_3 & s_4 & s_1 \\ -s_4 & -s_3 & s_2 \\ s_1^* & s_2^* & s_3^* \\ -s_2^* & s_1^* & -s_4^* \\ -s_3^* & s_4^* & s_1^* \\ -s_4^* & -s_3^* & s_2^* \end{pmatrix},$$

and

$$\mathcal{C}_3 = \begin{pmatrix} s_1 & s_2 & s_3 & 0 & s_4 \\ -s_2^* & s_1^* & 0 & s_3 & s_5 \\ s_3^* & 0 & -s_1^* & s_2 & s_6 \\ 0 & s_3^* & -s_2^* & -s_1 & s_7 \\ s_4^* & 0 & 0 & -s_7^* & -s_1^* \\ 0 & s_4^* & 0 & s_6^* & -s_2^* \\ 0 & 0 & s_4^* & s_5^* & -s_3^* \\ 0 & -s_5^* & -s_6^* & 0 & s_1 \\ s_5^* & 0 & s_7^* & 0 & s_2 \\ -s_6^* & -s_7^* & 0 & 0 & s_3 \\ s_7 & -s_6 & -s_5 & s_4 & 0 \end{pmatrix}.$$

$\mathcal{C}_1$ is the well known Alamouti code with parameters $P = K = L_t = 2$, $R = 1$, and $\mathcal{C}_1^H\mathcal{C}_1$ is a 2×2 diagonal matrix with the $(i,i)^{th}$ diagonal element, $D(i,i)$, of the form $\sum_{k=1}^{2}\|s_k\|^2$. $\mathcal{C}_2$ is a rate-1/2 STBC with parameters $P = 8$, $K = 4$, $L_t = 3$, $R = 1/2$, and $\mathcal{C}_2^H\mathcal{C}_2$ is a 3×3 diagonal matrix with the $(i,i)^{th}$

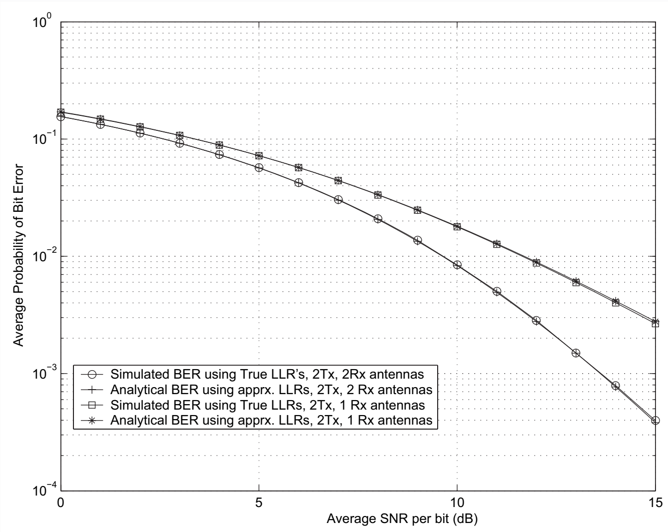

Fig. 2. Comparison of the analytical BER evaluated using approximate LLRs vs the simulated BER using the LLRs without approximation. 16-QAM with rate-1 STBC (Alamouti code) in Rayleigh fading. 2-Tx/2-Rx and 2-Tx/1-Rx antennas.

diagonal element, $D(i,i)$, of the form $\sum_{k=1}^{4}(2\cdot\|s_k\|^2)$. $\mathcal{C}_3$ is a rate-7/11 STBC with parameters $P = 11, K = 7, L_t = 5$, $R = 7/11$, and $\mathcal{C}_3^H\mathcal{C}_3$ is a 5×5 diagonal matrix with the $(i,i)^{th}$ diagonal element, $D(i,i)$, of the form

$$D(1,1) = D(2,2) = D(3,3) = D(4,4) = \sum_{k=1}^{7}\|s_k\|^2, \quad (34)$$

$$D(5,5) \;=\; \sum_{k=1}^{3}(2.\|s_k\|^2) + \sum_{k=3}^{7}\|s_k\|^2. \quad (35)$$

In Fig. 2, we compare the analytical BER evaluated using the approximate LLRs derived versus the simulated BER using the LLRs without approximation for rate-1 STBC (Alamouti code) using 16-QAM for 2-Tx/2-Rx and 2-Tx/1-Rx antennas. It is observed that the analytically computed BER is almost the same as the simulated BER, indicating that the approximation to the LLRs results in insignificant difference between the analytically computed BER and the true BER. We would like to point out that the BER obtained using the approximate LLR expression is the same as that of the 'traditional BER results for M-QAM' (as published, for example, in the paper by Cho and Yoon [10]). The reason for this observation is that the decision statistic for each bit forming the QAM symbol with Gray coding and approximate LLR is the same as that of the conventional symbol-to-bit demapping approach. In other words, without the approximation, the average BER performance for M-QAM will be slightly better than the conventional symbol-to-bit demapping approach. In [12], it is shown that for all practical values of the bit SNR this improvement can be negligible. We further point out that [13] presents BER results for Gray-coded M-QAM by dividing the SER by the number of bits per symbol. However, this result is only approximate, as the exact BER analysis requires evaluating the number of bit errors occurring for each possible transmitted symbol. We also note that the approximate BER results in [13] match the exact BER only at large-enough SNR values.

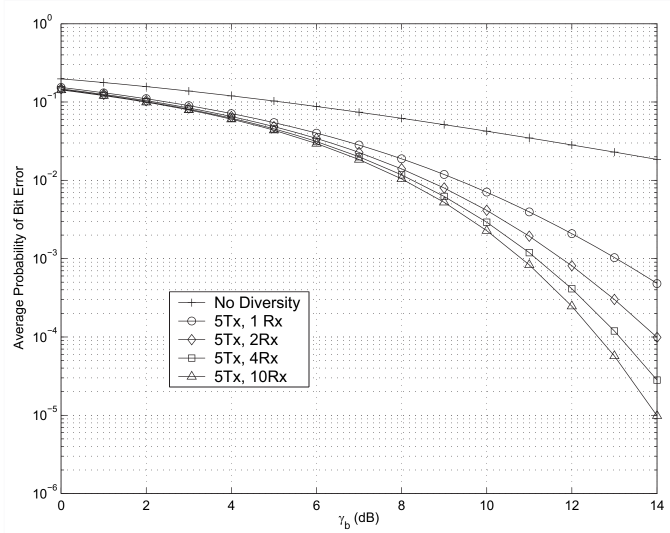

Fig. 3. BER performance of 16-QAM with 5 transmit antennas and $L_r = 1, 2, 4, 10$ receive antennas using rate-7/11 STBC in Rayleigh fading.

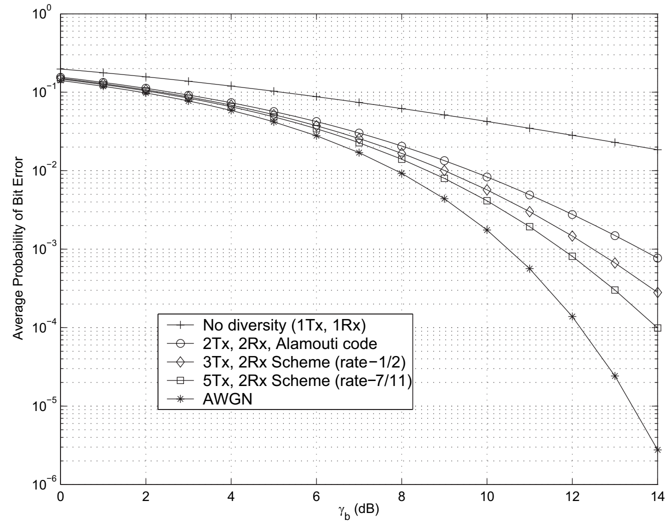

Fig. 4. BER performance of 16-QAM with different STBCs in Rayleigh fading; $i)$ 2 Tx antennas using rate-1 STBC (Alamouti code), $ii)$ 3 Tx antennas using rate-1/2 STBC, $iii)$ 5 Tx antennas using rate-7/11 STBC. Number of receive antennas, $L_r = 2$.

In Fig. 3, we present the analytical results of the average BER performance as a function of the average SNR, γ_b, for the rate-7/11 STBC, $\mathcal{C}_3$. The number of receive antennas considered include $L_r = 1, 2, 4, 10$. Figure 4 presents the comparative BER performance of the different STBCs $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ when the number of receive antennas $L_r = 2$. The performance in AWGN is also shown for comparison. As we pointed out earlier, in addition to being used in the BER analysis, the derived LLRs for the individual bits in the QAM symbols can be used as soft inputs to the decoders in various coded QAM schemes. As an example, we employed the LLRs as soft inputs to the turbo decoder in a rate-1/3 turbo

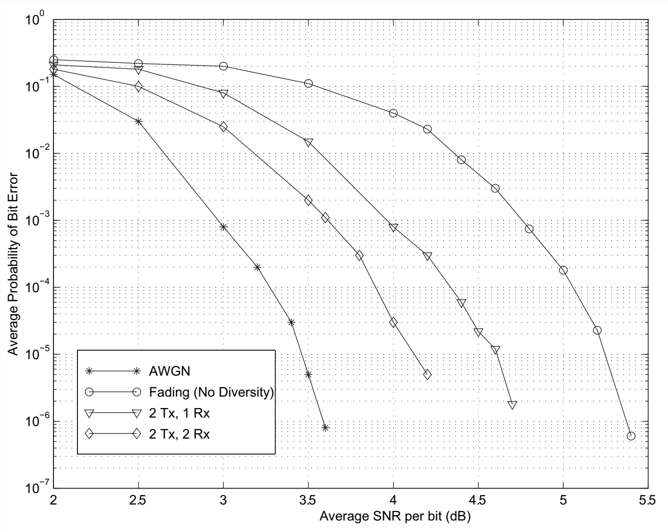

Fig. 5. BER performance of rate-1/3 turbo coded 16-QAM with two transmit antennas and $L_r = 1, 2$ receive antennas using rate-1 STBC (Alamouti code) in Rayleigh fading. LLRs of bits in QAM symbols used as soft inputs to the turbo decoder.

coded 16-QAM scheme on Rayleigh fading without and with transmit diversity using Alamouti code $\mathcal{C}_1$. Figure 5 shows the simulated BER performance of the turbo coded 16-QAM system using the derived LLRs as soft inputs to the decoder. The turbo code used in the simulations is the one specified in the 3GPP standard. Likewise, the LLRs can be used as soft inputs to decoders in DVB and IEEE 802.11a, where convolutionally coded QAM with OFDM is used.

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