

On the Performance of Golden Space-Time Trellis Coded Modulation over MIMO Block Fading Channels

Emanuele Viterbo and Yi Hong

Abstract

The Golden space-time trellis coded modulation (GST-TCM) scheme was proposed in [1] for a high rate 2×2 multiple-input multiple-output (MIMO) system over slow fading channels. In this letter, we present the performance analysis of GST-TCM over block fading channels, where the channel matrix is constant over a fraction of the codeword length and varies from one fraction to another, independently. In practice, it is not useful to design such codes for specific block fading channel parameters and a robust solution is preferable. We then show both analytically and by simulation that the GST-TCM designed for slow fading channels are indeed robust to all block fading channel conditions.

Index Terms

Golden code, Golden space-time trellis coded modulation, union bound, block fading.

I. INTRODUCTION

The Golden code was proposed in [2] as a full rate and full diversity code for 2×2 multiple-input multiple-output (MIMO) systems with *non-vanishing minimum determinant* (NVD). It was shown in [3] how this property guarantees to achieve the diversity-multiplexing gain trade-off. In order to enhance the coding gain, a first attempt to concatenate the Golden code with an outer trellis code was made in [4]. However, the resulting *ad hoc* scheme suffered from a high trellis complexity.

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In [1], a Golden space-time trellis coded modulation (GST-TCM) scheme was designed for slow fading channels. The NVD property of the inner Golden code is essential for a TCM scheme. This property guarantees that the code will not suffer from a reduction of the minimum determinant, when a constellation expansion is required [2]. The systematic design proposed in [1], is based on set partitioning of the Golden code in order to increase the minimum determinant. An outer trellis code is then used to increase the Hamming distance between the codewords. The Viterbi algorithm is applied for trellis decoding, where the branch metrics are computed with a lattice *sphere decoder* [7, 8] for the inner Golden code.

In this letter, we analyze performance of the GST-TCM scheme in *block fading* channels [5]. The block fading channel is a simple and powerful model to describe a variety of wireless fading channels ranging from fast to slow. For example, in OFDM based systems over frequency selective fading channels it can model various channel delay profiles. In particular, low delay spread channels correspond to small frequency selectivity, i.e., many adjacent subcarriers experience similar fading coefficients. On the contrary, channels with long delays profiles correspond to large frequency selectivity, i.e., the fading coefficients vary significantly among adjacent subcarriers.

In practice, it is not useful to design a GST-TCM for specific block fading channel parameters and a robust solution is preferable. We therefore analyze the performance of known GST-TCM, designed for slow fading, over arbitrary block fading channels. The impact of the block fading channel on the code performance is estimated analytically using a two-term truncated union bound (UB). We finally show both analytically and by simulation that the GST-TCM designed for slow fading channels are indeed robust to various block fading channel conditions.

The rest of the letter is organized as follows. Section II introduces the system model for block fading channels. Section III presents an analytic performance estimation of linear STBCs over block fading channels. In Section IV we specialize the result for GST-TCM designed for slow fading. Section V shows simulation results. Conclusions are drawn in Section VI.

Notations: Let T denote transpose and \dagger denote Hermitian transpose. Let \mathbb{Z} , \mathbb{C} and $\mathbb{Z}[i]$ denote the ring of rational integers, the field of complex numbers, and the ring of Gaussian integers, respectively, where $i^2 = -1$. Let $\lceil x \rceil$ denote the smallest integer greater or equal to x . The operator $(\bar{\cdot})$ denotes the algebraic conjugation in a quadratic algebraic number field [2].

II. SYSTEM MODEL

Let us first consider a 2×2 MIMO system ($n_T = 2$ transmit and $n_R = 2$ receive antennas) over a slow fading channel using the Golden code \mathcal{G} . A 2×2 Golden codeword $X \in \mathcal{G}$ is transmitted over two channel uses, where the channel matrix H is constant and

$$Y = HX + Z \quad (1)$$

is received, where Z is a complex white Gaussian noise 2×2 matrix. The Golden codeword $X \in \mathcal{G}$ is defined as [2]

$$X \triangleq \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha(a + b\theta) & \alpha(c + d\theta) \\ i\bar{\alpha}(c + d\bar{\theta}) & \bar{\alpha}(a + b\bar{\theta}) \end{bmatrix} \quad (2)$$

where $a, b, c, d \in \mathbb{Z}[i]$ are the information symbols, $\theta \triangleq 1 - \bar{\theta} = \frac{1+\sqrt{5}}{2}$, $\alpha \triangleq 1 + i\bar{\theta}$, $\bar{\alpha} \triangleq 1 + i\theta$, and the factor $1/\sqrt{5}$ is used to normalize energy [2]. As information symbols, Q -QAM constellations are used, where $Q = 2^\eta$. The QAM constellation is assumed to be scaled to match $\mathbb{Z}[i] + (1+i)/2$.

In this letter we will consider linear codes of length L over an alphabet \mathcal{G} in a block fading channel, i.e., the transmitted codewords are given by $\mathbf{X} = (X_1, \dots, X_t, \dots, X_L) \in \mathbb{C}^{2 \times 2L}$:

- if the elements $X_t \in \mathcal{G}$ are selected independently, we have the *uncoded Golden code*;
- if a trellis outer code is used to constrain the X_t 's, we have a GST-TCM [1].

Let $\mathbf{Z} = (Z_1, \dots, Z_t, \dots, Z_L) \in \mathbb{C}^{2 \times 2L}$ denote a complex white Gaussian noise matrix with i.i.d. samples distributed as $\mathcal{N}_{\mathbb{C}}(0, N_0)$, where Z_t are the complex white Gaussian noise 2×2 matrices. At the receiver, we have the following received signal matrix

$$\mathbf{Y} = (Y_1, \dots, Y_t, \dots, Y_L) \in \mathbb{C}^{2 \times 2L}$$

where Y_t is given by

$$Y_t = H_t X_t + Z_t \quad t = 1, \dots, L \quad (3)$$

where H_t are assumed to be i.i.d. circularly symmetric Gaussian random variables $\sim \mathcal{N}_{\mathbb{C}}(0, 1)$.

In a *block fading channel*, the matrices $H_t \in \mathbb{C}^{2 \times 2}$ are assumed to be constant in a block of N consecutive alphabet symbols in \mathcal{G} (i.e., $2N$ channel uses) and vary independently from one block to another, i.e.,

$$H_{kN+1} = \dots = H_{(k+1)N} \quad \text{for } k = 0, \dots, L/N - 1$$

where we assume for convenience that N divides L . This implies that the number of blocks within a codeword experiencing independent fading is $B = L/N$. For $N = L$ ($B = 1$) we have a *slow* fading channel and for $N = 1$ ($B = L$) a *fast* fading channel. In this letter, we assume that the channel is known at the receiver but not at the transmitter.

III. PERFORMANCE OF LINEAR STBC OVER BLOCK FADING CHANNELS

In this section we analyze performance of linear STBC over block fading channels. In the following we will make the analysis specific to the GST-TCM.

Assuming that a codeword \mathbf{X} is transmitted over a *slow fading* channel ($N = L$), the maximum-likelihood receiver might decide erroneously in favor of another codeword $\hat{\mathbf{X}}$, resulting in a *pairwise error event*. Let r denote the rank of the *codeword difference matrix* $\mathbf{X} - \hat{\mathbf{X}}$. Let $\lambda_j, j = 1, \dots, r$, be the non-zero eigenvalues of the *codeword distance matrix* $\mathbf{A} = (\mathbf{X} - \hat{\mathbf{X}})(\mathbf{X} - \hat{\mathbf{X}})^\dagger$. The *pairwise error probability* (PEP) depends on the determinant $\det(\mathbf{A})$ for full rank codes ($r = 2$) [6].

The UB gives an upper bound to the performance of the STBC, while a truncated UB gives an asymptotic approximation [9]. The dominant term in the UB is the PEP that depends on the *minimum determinant* of the codeword distance matrix

$$\Delta_{\min}^{(s)} = \min_{\mathbf{X} \neq \hat{\mathbf{X}}} \det(\mathbf{A})$$

where the superscript s denotes the slow fading case. The traditional code design criterion for space-time codes in [6] is based on the minimization of the dominant term in the UB, which in turn depends on the *diversity gain* $n_T n_R$ and the *coding gain* $\left(\Delta_{\min}^{(s)}\right)^{\frac{1}{n_T}}$.

In this letter, we will consider the truncated UB with two terms

$$P(e) \approx N_{s_1} P_1 + N_{s_2} P_2 \quad (4)$$

where the $P_i, i = 1, 2$, are the two largest PEPs of the two dominating events depending on and N_{s_i} the corresponding multiplicities. We assume that P_1 depends on $\Delta_1 = \Delta_{\min}^{(s)}$ and P_2 depends on Δ_2 the second smallest value of $\det(\mathbf{A})$.

Since we focus on *full rank* (i.e., $r = n_T = 2$ for all \mathbf{A}) and *linear* (i.e., the sum of any two codewords is a codeword) codes, we can simply consider the PEP from the all-zero transmitted codeword matrix.

Let us now consider a *block fading* channel, where H_t is constant for $2N$ channel uses and changes independently in the $B = L/N$ blocks. For a given codeword \mathbf{X} , we define the matrices

$$F_\ell \triangleq \sum_{t=(\ell-1)N+1}^{\ell N} X_t X_t^\dagger \quad \ell = 1, \dots, B \quad (5)$$

Following [6], it can be easily shown that the dominating term in the UB will be driven by the quantity

$$\Delta_{\min}^{(b)} \triangleq \min_{\det(F_\ell) \neq 0} \prod_{\ell=1}^B \det(F_\ell) \quad (6)$$

where the superscript b denotes the block fading case. The above performance metric $\Delta_{\min}^{(b)}$ could be hard to exploit, due to the non-additive nature of the determinant metric in (6). Since $X_t X_t^\dagger$ are positive definite matrices, we resort to the following determinant inequality [10]

$$\det(F_\ell) \geq \sum_{t=(\ell-1)N+1}^{\ell N} \det(X_t X_t^\dagger) \triangleq a_\ell \quad (7)$$

and use the simpler lower bound:

$$\Delta_{\min}^{(b)} \geq \min_{a_\ell \neq 0} \prod_{\ell=1}^B a_\ell \triangleq \Delta_{\min}^{(b)'} \quad (8)$$

We can see that the $\Delta_{\min}^{(b)'}$ is not only determined by the code structure, but also by the block fading channel parameters B and N . Note that $\Delta_{\min}^{(b)'}$ coincides with the Δ'_{\min} defined in [1], when $B = 1$ (slow fading).

Finally, we note that for a specific value of B and N the design of a good linear STBC is clearly impractical and a robust solution is preferable.

IV. PERFORMANCE ANALYSIS OF GST-TCM ON BLOCK FADING CHANNELS

In this section we show the specific analysis concerning GST-TCM [1]. The design of GST-TCM for slow fading ($B = 1$) was based on:

- the design of a trellis code that maximizes the number of non-zero $\det(X_t X_t^\dagger)$ in (7)
- the design of partitions of the Golden code with increasing values of $\det(X_t X_t^\dagger)$

In particular, the trellis design focused on the *shortest simple error event*, i.e., a path diverging from the zero state and remerging into the zero state in the trellis diagram. We will show here

how the length S of such event influences the performance of the code over a block fading channel.

Lemma 1: A GST-TCM of length $L \geq S \geq 2$ can have $N_s = L - S + 1$ shortest simple error events. ■

Proof – The shortest simple error events with length S can only start in a position $\{1, 2, \dots, L - S + 1\}$, thereby we obtain $N_s = L - S + 1$. ■

Since the codeword spans $B = L/N$ independent fading blocks of length N , the simple error events will affect different blocks depending on their starting position and length. We obtain the following lemma.

Lemma 2: A shortest simple error event of length S is either affecting

- 1) $n_1 = \lceil S/N \rceil$ consecutive blocks, or
- 2) $n_2 = n_1 + 1 = \lceil S/N \rceil + 1$ consecutive blocks. ■

Proof – Depending on the starting position of the shortest simple error event we have

- if $S \leq N$ then either $n_1 = 1$, if it is fully within one block, or $n_2 = 2$.
- if $S > N$ then it will either cross $n_1 = \lceil S/N \rceil$ or $n_2 = n_1 + 1$ consecutive blocks.

For example, if $S = 2$ over a block fading channel where $B = 4$ and $N = 4$, as shown in Fig. 1, we have some simple error events (solid arrows), in $n_1 = 1$ consecutive blocks and others (dashed lines) in $n_2 = 2$ consecutive block. ■

Lemma 3: The corresponding numbers of simple error events in Case 1 and Case 2 of the previous lemma are respectively

$$N_{s_1} = B' \times \ell \quad N_{s_2} = N_s - N_{s_1} \quad (9)$$

where

$$B' = B - \left\lceil \frac{S}{N} \right\rceil + 1$$

$$\ell = \left\lceil \frac{S}{N} \right\rceil \times N - S + 1$$

Proof – We first recall from Lemma 2 for Case 1, that a simple error event occupies $\lceil \frac{S}{N} \rceil$ consecutive blocks of length N . Now, let us define a *group* as $\lceil \frac{S}{N} \rceil$ consecutive blocks. Hence, a group has length $\lceil \frac{S}{N} \rceil \times N$ and contains $\ell = \lceil \frac{S}{N} \rceil \times N - S + 1$ distinct shortest simple error

events. Since there are $B' = B - \lceil \frac{S}{N} \rceil + 1$ distinct groups, we have $N_{s_1} = B' \times \ell$ shortest simple error events of Case 1. The other case directly derives from the identity $N_s = N_{s_1} + N_{s_2}$. ■

Using the same example illustrated in Fig. 1 with $S = 2$, $B = 4$ and $N = 4$, it is shown that we have $N_{s_1} = 12$ simple error events crossing $n_1 = 1$ consecutive block (Case 1) and $N_{s_2} = 3$ simple error events crossing $n_2 = 2$ consecutive blocks (Case 2).

In order to evaluate the two dominant terms in (4) we look at the contribution of the simple error events in the trellis together with their multiplicity. We get N_{s_1} terms with the corresponding minimum determinant

$$\Delta_1^{(b)'} = \min_{\ell} \prod_{n=0}^{n_1-1} a_{\ell+n} \quad (10)$$

and N_{s_2} terms with the corresponding minimum determinant

$$\Delta_2^{(b)'} = \min_{\ell} \prod_{n=0}^{n_2-1} a_{\ell+n} \quad (11)$$

Depending on the length and structure of the simple error events, the $\Delta_1^{(b)'}$ and $\Delta_2^{(b)'}$, together with their multiplicity N_{s_1}, N_{s_2} , will dominate the performance of the coding scheme.

Even if we have $\Delta_2^{(b)'}$ smaller than $\Delta_1^{(b)'}$ its contribution to the overall performance can be mitigated by the fact that $N_{s_1} \gg N_{s_2}$. We will see in the following section how the a_{ℓ} s are affected by the trellis code structure.

V. SIMULATION RESULTS

In this section we show the performance of different GST-TCM schemes over block fading channels. Signal-to-noise ratio per bit is defined as $\text{SNR}_b = n_T E_b / N_0$, where $E_b = E_s / q$ is the energy per bit and q denotes the number of information bits per QAM symbol of energy E_s .

We consider two types of GST-TCM based on the two and three level partitions \mathbb{Z}^8 / E_8 and \mathbb{Z}^8 / L_8 in [1]. For each case we consider trellises with 4 or 16 states and 16 or 64 states, respectively. The length of the simple error events is $S = 2, 3, 4$ for 4, 16 and 64 state trellises, respectively. We assume the codeword length is $L = 120$ and the block fading channels are characterized by $N = 1, 3, 5, 20, 40, 120$. The GST-TCM were optimized in [1] for the slow fading channel, i.e., for $N = 120$ (or $B = 1$).

In Figures 2-5 we can see that the best performance is obtained in the slow fading case ($N = 120$), for which the codes were explicitly optimized. The worst performance appears in

the fast fading case ($N = 1$), although the difference is about 1.5-2dB at FER of 10^{-2} and only about 1dB at FER of 10^{-3} . Note that the slow and fast fading curves will eventually cross, since the fast fading exhibits a higher diversity order. The intermediate cases of block fading exhibit a performance between the fast and slow, which degrades as N decreases.

Let us analyze these simulation results using the truncated UB (4). The sequences of values of $\det(X_t \hat{X}_t)$ in the shortest simple error events of the GST-TCMs in Figs. 2 to 5 are given in Table I, where $\delta = 1/5$ is the minimum determinant of the Golden code.

Tables II-III show all the code parameters. When $N = 1$ or $N = 120$, the term $\Delta_1^{(b)'}$ and its multiplicity N_{s_1} dominate the performance. We see that $\Delta_1^{(b)'}$ for $N = 120$ is always greater than that for $N = 1$, provided $\delta = 1/5$ and a fixed N_{s_1} . This results in a better performance when $N = 120$. The same observation can be found for 64-state GST-TCM when $N = 3$.

For the remaining cases, we note that $\Delta_2^{(b)'}$ is always smaller than $\Delta_1^{(b)'}$ since $\delta = 1/5$. As N increases the multiplicity N_{s_2} of the $\Delta_2^{(b)'}$ term decreases, while N_{s_1} of the $\Delta_1^{(b)'}$ term increases, which results in a better performance. This analysis qualitatively agrees with the actual performance of the codes.

VI. CONCLUSIONS

In this letter, we analyzed the impact of a block fading channel on the performance of GST-TCM by using a truncated UB technique. The analysis shows that the performance of the GST-TCM designed for slow fading channel varies slightly if the channel condition varies from slow to fast. It is further demonstrated by simulation that the performance degrades at most 1 dB at the FER of 10^{-3} , when block fading varies from slow to fast. This robust coding scheme can be particularly beneficial for high rate transmission in WLANs using OFDM to combat widely variable multipath fading.

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Figures

- 1) Comparison of 4-state trellis codes using 16-QAM constellation at the rate 7 bpcu form a three level partition \mathbb{Z}^8/E_8 ($S = 2$).
- 2) Comparison of 16-state trellis codes using 16-QAM constellation at the rate 7 bpcu form a three level partition \mathbb{Z}^8/E_8 ($S = 3$).
- 3) Comparison of 16-state trellis codes using 16-QAM constellation at the rate 6 bpcu form a three level partition \mathbb{Z}^8/L_8 ($S = 3$).
- 4) Comparison of 64-state trellis codes using 16-QAM constellation at the rate 6 bpcu form a three level partition \mathbb{Z}^8/L_8 ($S = 4$).
- 5) Enumeration of simple error events of a GST-TCM with $S = 2$ over a block fading channel with $B = 4$ and $N = 4$.

Tables

- 1) Sequences of $\det(X_t \hat{X}_t)$ for the simple error events of the GST-TCMs in Figs. 2-5 ($\delta = 1/5$).
- 2) Simple error events for 4, 16 states \mathbb{Z}^8/E_8 GST-TCM, $S = 2, 3$ and different block fading channels ($N = 1, 3, 5, 20, 40, 120$).
- 3) Simple error events for 16, 64 states \mathbb{Z}^8/L_8 GST-TCM, $S = 3, 4$ and different block fading channels ($N = 1, 3, 5, 20, 40, 120$).

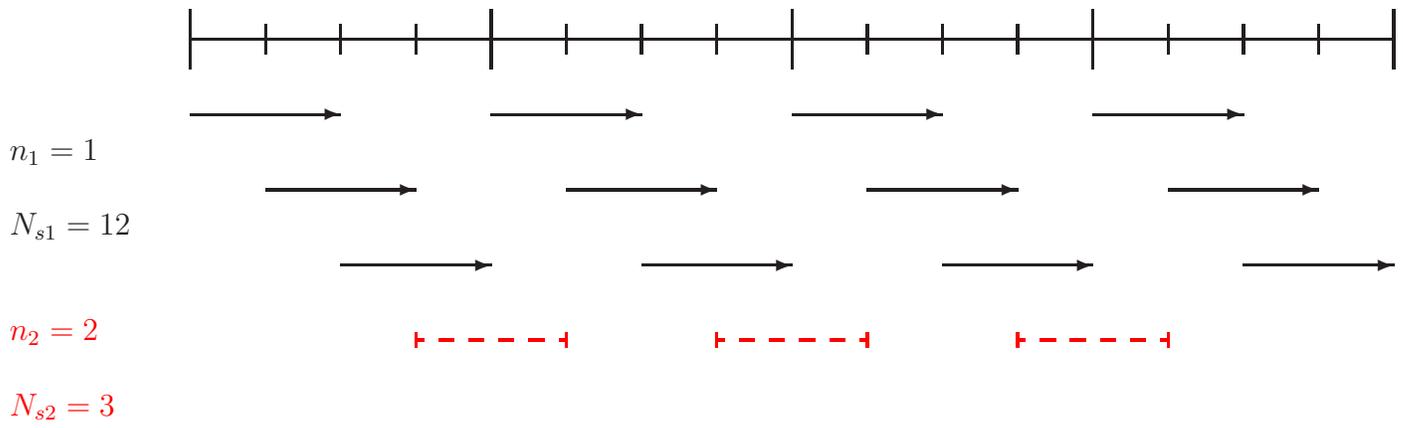


Fig. 1. Enumeration of simple error events of a GST-TCM with $S = 2$ over a block fading channel with $B = 4$ and $N = 4$.

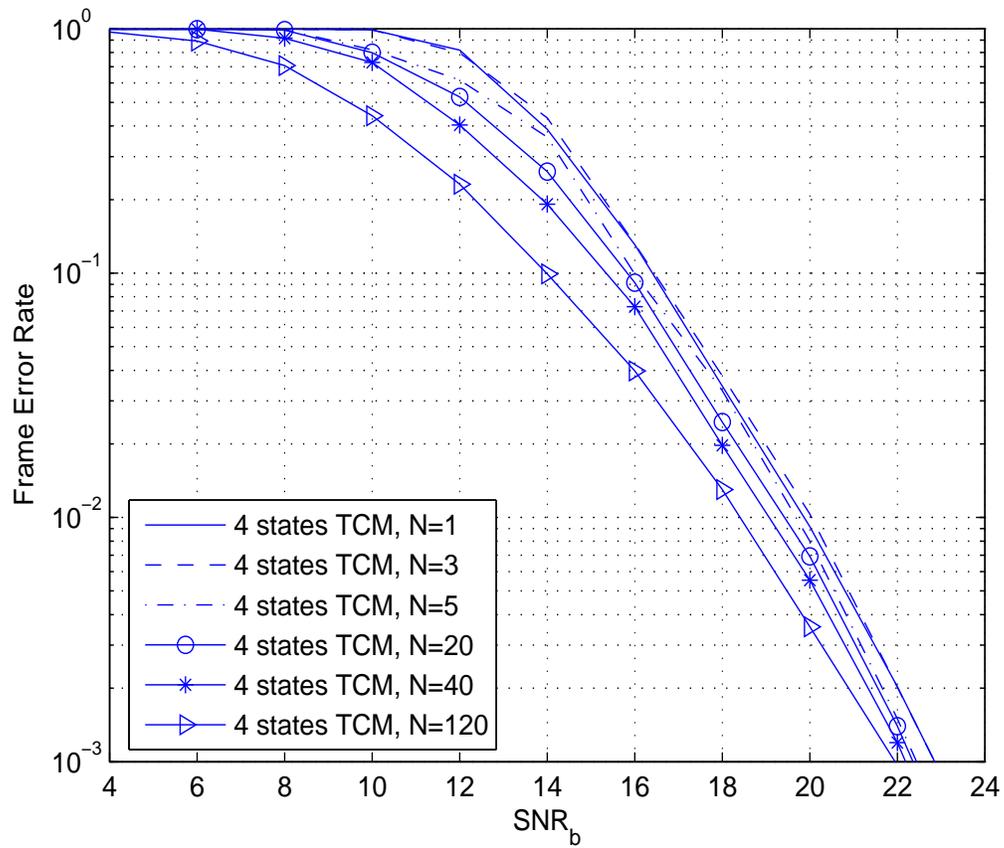


Fig. 2. Comparison of 4-state trellis codes using 16-QAM constellation at the rate 7 bpcu from a three level partition \mathbb{Z}^8/E_8 ($S = 2$).

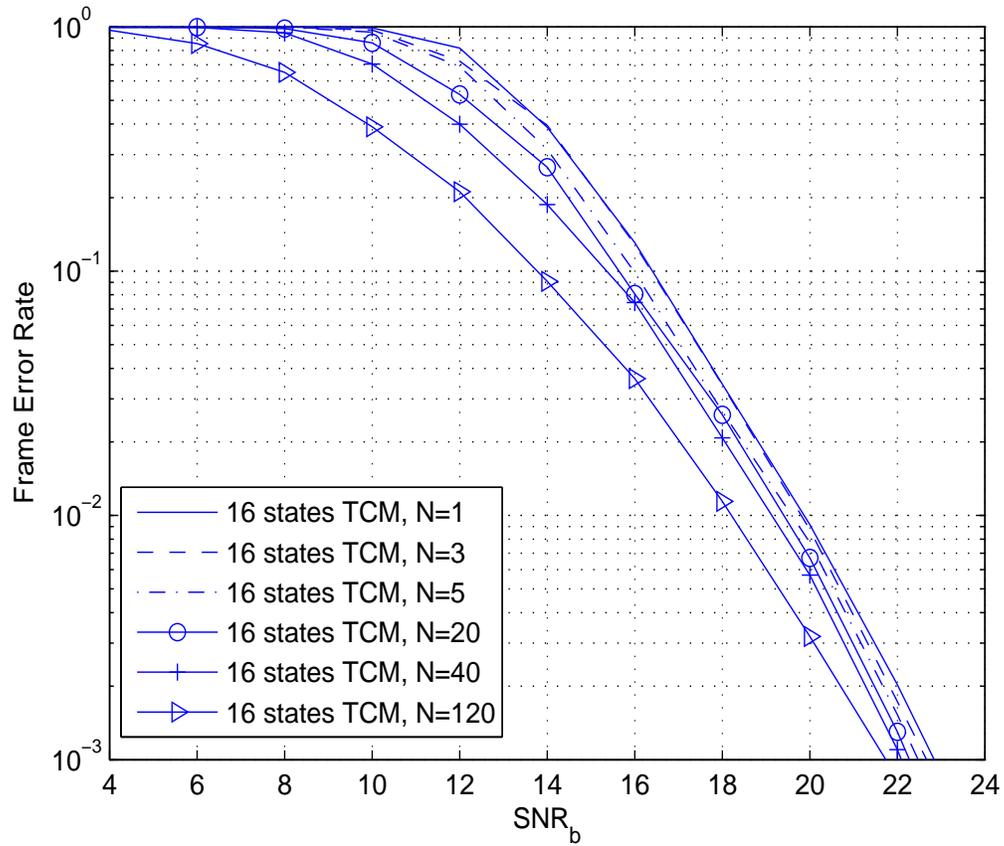


Fig. 3. Comparison of 16-state trellis codes using 16-QAM constellation at the rate 7 bpcu form a three level partition \mathbb{Z}^8/E_8 ($S = 3$).

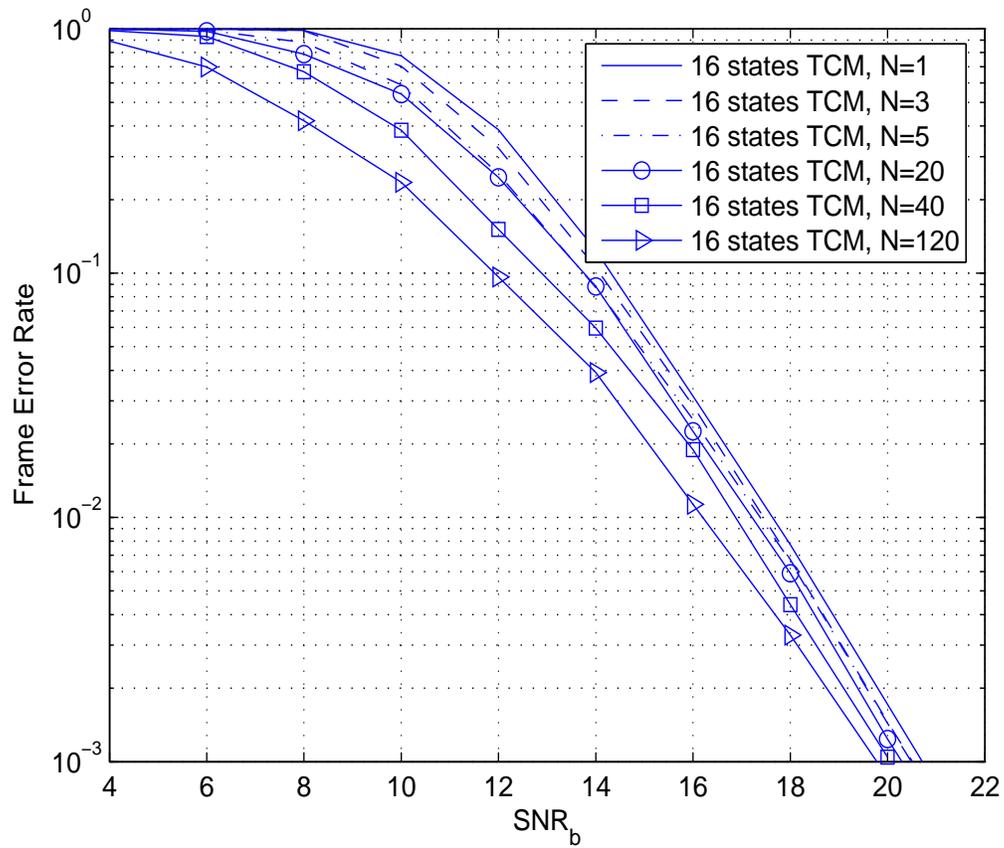


Fig. 4. Comparison of 16-state trellis codes using 16-QAM constellation at the rate 6 bpcu from a three level partition \mathbb{Z}^8/L_8 ($S = 3$).

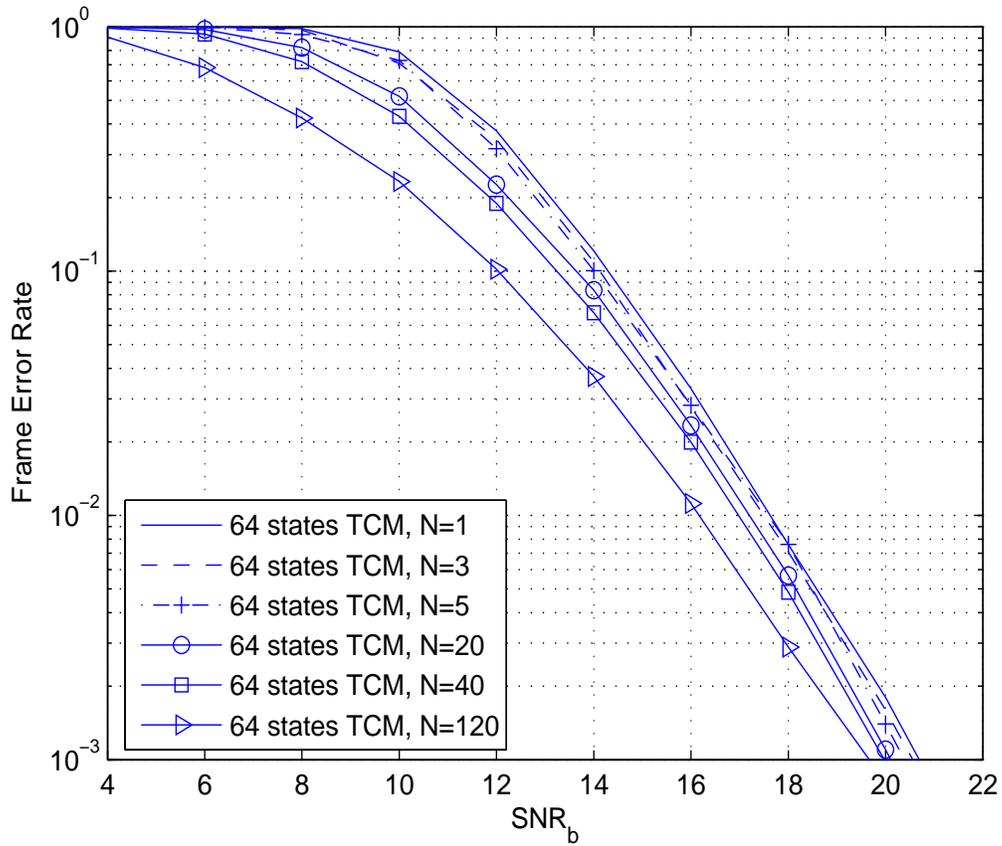


Fig. 5. Comparison of 64-state trellis codes using 16-QAM constellation at the rate 6 bpcu from a three level partition \mathbb{Z}^8/L_8 ($S = 4$).

S	step 1	step 2	step 3	step 4
2	δ	2δ		
3	2δ	δ	2δ	
3	4δ	δ	2δ	
4	4δ	δ	2δ	4δ

TABLE I

SEQUENCES OF $\text{DET}(X_t X_t^\dagger)$ FOR THE SIMPLE ERROR EVENTS OF THE GST-TCMS IN FIGS. 2-5 ($\delta = 1/5$).

St.	N	N_{s_1}	N_{s_2}	n_1	n_2	$\Delta_1^{(b)'}$	$\Delta_2^{(b)'}$
4	1	119	—	2	—	$2\delta^2$	—
4	3	80	39	1	2	3δ	$2\delta^2$
4	5	96	23	1	2	3δ	$2\delta^2$
4	20	114	5	1	2	3δ	$2\delta^2$
4	40	117	2	1	2	3δ	$2\delta^2$
4	120	119	—	1	—	3δ	—
16	1	118	—	3	—	$4\delta^3$	—
16	3	40	78	1	2	5δ	$2\delta^2 + 2\delta$
16	5	72	46	1	2	5δ	$2\delta^2 + 2\delta$
16	20	108	10	1	2	5δ	$2\delta^2 + 2\delta$
16	40	114	4	1	2	5δ	$2\delta^2 + 2\delta$
16	120	118	—	1	—	5δ	—

TABLE II

SIMPLE ERROR EVENTS FOR 4, 16 STATES \mathbb{Z}^8/E_8 GST-TCM, $S = 2, 3$ AND DIFFERENT BLOCK FADING CHANNELS ($N = 1, 3, 5, 20, 40, 120$).

St.	N	N_{s_1}	N_{s_2}	n_1	n_2	$\Delta_1^{(b)'}$	$\Delta_2^{(b)'}$
16	1	118	–	3	–	$8\delta^3$	–
16	3	40	78	1	2	7δ	$4\delta^2 + 2\delta$
16	5	72	46	1	2	7δ	$4\delta^2 + 2\delta$
16	20	108	10	1	2	7δ	$4\delta^2 + 2\delta$
16	40	114	4	1	2	7δ	$4\delta^2 + 2\delta$
16	120	118	–	1	–	7δ	–
64	1	117	–	4	–	$32\delta^4$	–
64	3	117	–	2	–	$28\delta^2, 40\delta^2$	–
64	5	48	69	1	2	11δ	$28\delta^2, 40\delta^2$
64	20	102	15	1	2	11δ	$28\delta^2, 40\delta^2$
64	40	111	6	1	2	11δ	$28\delta^2, 40\delta^2$
64	120	117	–	1	–	11δ	–

TABLE III

SIMPLE ERROR EVENTS FOR 16, 64 STATES \mathbb{Z}^8/L_8 GST-TCM, $S = 3, 4$ AND DIFFERENT BLOCK FADING CHANNELS
($N = 1, 3, 5, 20, 40, 120$).