Impact of CSI on Distributed Space-Time Coding in Wireless Relay Networks

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Abstract

We consider a two-hop wireless network where a transmitter communicates with a receiver via M relays with an amplify-and-forward (AF) protocol. Recent works have shown that sophisticated linear processing such as beamforming and distributed space-time coding (DSTC) at relays enables to improve the AF performance. However, the relative utility of these strategies depend on the available channel state information at transmitter (CSIT), which in turn depends on system parameters such as the speed of the underlying fading channel and that of training and feedback procedures. Moreover, it is of practical interest to have a single transmit scheme that handles different CSIT scenarios. This motivates us to consider a unified approach based on DSTC that potentially provides diversity gain with statistical CSIT and exploits some additional side information if available. Under individual power constraints at the relays, we optimize the amplifier power allocation such that pairwise error probability conditioned on the available CSIT is minimized. Under perfect CSIT we propose an on-off gradient algorithm that efficiently finds a set of relays to switch on. Under partial and statistical CSIT, we propose a simple waterfilling algorithm that yields a non-trivial solution between maximum power allocation and a generalized STC that equalizes the averaged amplified noise for all relays. Moreover, we derive closed-form solutions for M = 2 and in certain asymptotic regimes that enable an easy interpretation of the proposed algorithms. It is found that an appropriate amplifier power allocation is mandatory for DSTC to offer sufficient diversity and power gain in a general network topology.

I. MOTIVATION

We consider a wireless relay network illustrated in Fig.1 where a transmitter communicates with a receiver via M relays and each terminal has a single antenna. We let **h**, **g** denote the channel vector between the transmitter and the relays, the channel vector between the relays and the receiver. For its simplicity, we focus on an amplify-and-forward (AF) protocol [1], [2] in a two-hop communication: in the first T channel uses the transmitter broadcasts a codeword, then in the second T channel uses relays amplify and forward the observed codeword by applying some linear precoder (to be specified later). We assume that a transmitter and M relays have individual power constraints rather than a total power constraint, since in a practical wireless network terminals are physically distributed and hence are subject to their own power supplies.

In a classical AF protocol, the amplifier coefficients have been determined so as to satisfy required power constraints [1]. In order to improve the performance of the AF protocol, a large number of recent works have considered some additional linear processing at relays [3], [4], [5], [6], [7], [8], [9], [10]. These works can be roughly classified into two classes according to their assumption of channel state information at transmitter (CSIT) and their objective. The first class assumes perfect CSIT and aims to maximize either the achievable rate or the instantaneous receive SNR [3], [4], [5], [6]. The resulting transmit scheme yields beamforming with appropriate power allocation. The second class assumes only statistical CSIT and aims to minimize the error probability by designing some type of linear precoder [7], [8], [9], [10]. In particular, significant attention has been paid to distributed space-time coding (DSTC) in which each relay sends a different column of a STC matrix [11], [8], [9], [10]. With a single antenna at each terminal, we hasten to say that the goal of DSTC is to approach full diversity gain of M offered by the relay-receiver channels g since the multiplexing gain of the MISO channel under the half-duplex constraint is very limited, i.e. 1/2. Notice that an AF based DSTC that achieves the optimal diversity-multiplexing tradeoff [12] has been well studied [13], [14], [15], [16]. It clearly appears that a practical utility of

these two approaches depends on the available CSIT, which in turn depends on the speed of fading and that of feedback and training procedures. Notice that in a two-hop communication model obtaining perfect CSIT is rather challenging because the transmitter needs at least a two-step training and feedback process, i.e. first learns h and then g, which typically induces additional delay and estimation error. Consequently, the perfect CSIT assumption holds only if the underlying fading is quasi-static and a sufficiently fast feedback and training is available. For this particular case, the first approach might be useful. On the contrary, if a rate of feedback and training is much slower than the coherence time of the channel, the second approach based on the statistical channel knowledge is more appropriate.

The above observation motivates us to find a unified approach that can handle different CSIT scenarios, rather than changing a transmit strategy as a function of the quality of side information. To this end, we fix our transmission strategy to DSTC that potentially provides diversity gain with statistical CSIT and further power gain if additional side information is available. Our goal is not to find the optimal strategy for each CSIT case but to propose a unified DSTC scheme that simply adapts the amplifier power allocation to available CSIT. Among a large family of DSTC, we consider linear dispersion (LD) codes [17] because they offer desirable performance in terms of diversity gain and coding gain [10], [18] and moreover keep the amplified noise white. The latter considerably simplifies the power allocation strategy. We assume perfect synchronization between relay terminals and perfect channel state information available at the receiver, which is necessary for coherent detection. Under this setting, we will address the following question: how does the quality of CSIT impact the amplifier power allocation and the resulting performance of the DSTC? To answer the question, we optimize the amplifier power allocation in such a manner that the pairwise error probability (PEP) conditioned on CSIT is minimized. Note that the conditional PEP is a performance criteria widely used in the literature of STC [19], [20], [21] and DSTC [22], [10], [18]. In particular, [21] has provided elegant precoder designs that minimize the PEP conditioned on CSIT for orthogonal STC. Unfortunately the extension of this work to a two-hop relay network appears very difficult due to the nonconvexity of the underlying problem. We examine the following CSIT cases : 1) perfect knowledge of the absolute value of the entries of h and g (*perfect CSIT*), 2) perfect knowledge of absolute value of the entries of h and statistical knowledge of g (partial CSIT), and 3) statistical knowledge of h and g (statistical CSIT).

Under perfect CSIT, the PEP minimization reduces to the maximization of an approximate receive SNR. The optimal power allocation strategy turns out to be an on-off strategy, whereby some relays are switched off and others transmit at maximum available power. We propose an on-off gradient algorithm that efficiently finds the optimal set of relays to switch on. Under partial and statistical CSIT, the conditional PEP minimization appears very difficult due to the self-interference caused by amplified noise and calls for a good heuristic approximation. First, we apply a Laplace-based saddle point approximation of an inherent integral in order to make the problem amenable. Since the approximated problem is still non-convex due to *averaged* amplified noises, we transform it into a convex problem via a log transformation [23] assuming a high transmitter power (which is the regime of interest). For a new objective function, we propose a very simple waterfilling algorithm that yields a non-trivial solution between maximum power allocation and a generalized STC that equalizes the averaged amplified noise, i.e. $p_i \gamma_{g,i}$ with $\gamma_{g,i}$ being the variance of the channel between relay *i* and the receiver (in a classical STC we consider $\gamma_{g,i} = 1, \forall i$). We derive closed-form solutions for M = 2 and in certain asymptotic regimes that enable an easy interpretation of the proposed algorithms. It is found that that an appropriate power allocation is mandatory for DSTC in order to provide diversity and power gains in a general network topology.

In order to situate this work in the context of relevant literature, we note that the LD based DSTC for a two-hop AF network has been addressed for a single-antenna case [10] and for a multiple antenna case [18]. In both works, Jing and Hassibi provided the diversity analysis by optimizing the power partition between the transmitter and the relays under the assumptions that the transmitter and M relays are subject to a total power constraint and that both channels have unit variance. Clearly, the optimal power partition under this setting, letting the transmitter use half of the total power and each relay share the other half, does not hold for a general network topology with unequal variances. We make a progress with respect to this since our proposed waterfilling solution can be applied for any set of variances under statistical CSIT and moreover handles the partial CSIT case. With perfect CSIT, Jing proposed a cooperative beamforming scheme for the same two-hop AF network [5]. Although this beamforming scheme provides a non-negligible power gain compared to our on-off power allocation as shown in Section VI, we remark that our on-off algorithm is much simpler and can be implemented at the receiver without requiring any knowledge at the transmitter. To this end, it suffices that the receiver sends to each relay a feedback of one

bit indicating whether to activate or not. Hence, our on-off algorithm might be appealing due to its robustness and simplicity despite its suboptimal performance.

The rest of the paper is organized as follows. After briefly introducing the two-hop network model in Section II, we derive the the conditional PEP upper bounds for different CSIT cases in Section III. In Section IV we propose efficient algorithms that solve the conditional PEP minimization, namely on-off gradient algorithm for perfect CSIT and waterfilling algorithm for partial and statistical CSIT. We provide some asymptotic properties of these algorithms in Section V and numerical examples in Section VI. Finally we conclude the paper in Section VII.

II. SYSTEM MODEL

We consider frequency-flat fading channels and let $\mathbf{g} = [g_1, \ldots, g_M]^T$, $\mathbf{h} = [h_1, \ldots, h_M]^T$ denote the channel vector between transmitter and relays, the channel vector between relays and receiver, respectively. We assume the entries of \mathbf{h} and \mathbf{g} are i.i.d. zero-mean circularly symmetric complex Gaussian with variance $\gamma_h = [\gamma_{h1}, \ldots, \gamma_{hM}], \gamma_g = [\gamma_{g1}, \ldots, \gamma_{gM}]$ respectively. The variance of each channel is assumed to capture path-loss and shadowing. We assume a block fading model, namely \mathbf{h} and \mathbf{g} remain constant over a block of 2T channel uses. In this paper we do not consider a transmitter-receiver direct link for simplicity. It is well known however that the direct link should be taken into account if one aims at optimizing the diversity-multiplexing tradeoff [13], [14], [16], [15]. The communication between the transmitter and the receiver is performed in two steps. The transmitter first broadcasts a symbol vector $\mathbf{s} = [s_1, \ldots, s_T]^T \in \mathbb{C}^{T \times 1}$ with $\mathbb{E}[\mathbf{ss}^H] = \mathbf{I}_T$ and relay *i* receives

$$\mathbf{y}_i = \sqrt{p_s} h_i \mathbf{s} + \mathbf{n}_i$$

where p_s is the power of the transmitter and $\mathbf{n}_i \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, N_0 \mathbf{I}_T)$ is AWGN. In the second T channel uses, M relays amplify and forward the observed codeword by applying a linear precoder. Namely, the transmit vector \mathbf{x}_i of relay i is given by

$$\mathbf{x}_i = q_i \mathbf{A}_i \mathbf{y}_i \tag{1}$$

where q_i denote a complex amplifier coefficient of relay i, $\mathbf{A}_i \in \mathbb{C}^{T \times T}$ is a unitary matrix satisfying $\mathbf{A}_i^H \mathbf{A}_i = \mathbf{I}_T$, and \mathbf{x}_i should satisfy a power constraint, i.e.

$$\mathbb{E}[||\mathbf{x}_i||^2] = |q_i|^2 \mathbb{E}[||\mathbf{y}_i||^2] \le T p_r$$

where the expectation is with respect to \mathbf{n}_i for a short-term constraint and with respect to both \mathbf{n}_i and \mathbf{h}_i for a long-term constraint. This in turns imposes a constraint on $\{q_i\}$ such that $|q_i|^2 \leq P_i$ where P_i denotes the maximum amplifier power of relay *i* given by

$$P_i = \begin{cases} \frac{p_r}{p_s |h_i|^2 + N_0}, & \text{short-term} \\ \frac{p_r}{p_s \gamma_{hi} + N_0}, & \text{long-term} \end{cases}$$

The received signal at the final destination is given by

$$\mathbf{r} = \sum_{i=1}^{M} g_i \mathbf{x}_i + \mathbf{w} = \sum_{i=1}^{M} g_i h_i q_i \mathbf{A}_i \mathbf{s} + \sum_{i=1}^{M} g_i q_i \mathbf{A}_i \mathbf{n}_i + \mathbf{w}$$

where $\mathbf{w} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, N_0 \mathbf{I}_T)$ is AWGN at the receiver uncorrelated with $\{\mathbf{n}_i\}$. The received vector can be further simplified to

$$\mathbf{r} = \sqrt{p_s} \mathbf{S} \mathbf{Q} \mathbf{f} + \mathbf{v} \tag{2}$$

where we let $\mathbf{S} = [\mathbf{A}_1 \mathbf{s}, \dots, \mathbf{A}_M \mathbf{s}] \in \mathbb{C}^{T \times M}$ denote a LD codeword, $\mathbf{Q} = \text{diag}(q_1, \dots, q_M)$ is a diagonal matrix with M amplifier coefficients, $\mathbf{f} = [h_1 g_1, \dots, h_M g_M]^T$ is a composite channel vector, and we let

$$\mathbf{v} = \sum_{i=1}^{M} q_i g_i \mathbf{A}_i \mathbf{n}_i + \mathbf{w}$$
(3)

denote the overall noise whose covariance is given by

$$\mathbb{E}[\mathbf{v}\mathbf{v}^H] = N_0 \left(\sum_{i=1}^M |q_i|^2 |g_i|^2 + 1\right) \mathbf{I}_T = \sigma_v^2 \mathbf{I}_T$$

It follows that the overall noise seen by the receiver is white and this considerably simplifies the amplifier power allocation in the following sections.

III. CONDITIONAL PEP

With perfect knowledge of both h and g, the receiver can perform Maximum Likelihood decoding ¹ by estimating a codeword according to

$$\hat{\mathbf{S}} = \arg\min_{\mathbf{S}\in\mathcal{S}} ||\mathbf{r} - \sqrt{p_s} \mathbf{S} \mathbf{Q} \mathbf{f}||^2$$
(4)

When the transmitter has only partial knowledge of the channels, it is reasonable to consider the pairwise error probability (PEP) conditioned on the available CSIT. In the following we derive the expressions of the conditional PEP for three different CSIT cases: 1) perfect CSIT, where the transmitter knows the absolute values of the entries of h, g, 2) partial CSIT where the transmitter knows the absolute values of th and γ_g , 3) statistical CSIT where the transmitter knows γ_h and γ_g . Perfect CSIT corresponds to the case of quasi-static fading, while statistical CSIT corresponds to the case of fast fading so that the transmitter can track only the second order statistics of the channel. Finally, partial CSIT is an intermediate case relevant to a time-division duplexing system where the transmitter learns perfectly h by reciprocity but only statistically g due to a low-rate feedback.

A. Perfect CSIT

The PEP conditioned on \mathbf{h}, \mathbf{g} for any $k \neq l$ is defined by

$$P(\mathbf{S}_k \to \mathbf{S}_l | \mathbf{h}, \mathbf{g}) \stackrel{\Delta}{=} \Pr\left(||\mathbf{r} - \sqrt{p_s} \mathbf{S}_l \mathbf{Q} \mathbf{H} \mathbf{g}||^2 \le ||\mathbf{r} - \sqrt{p_s} \mathbf{S}_k \mathbf{Q} \mathbf{H} \mathbf{g}||^2 |\mathbf{S}_k, \mathbf{h}, \mathbf{g} \right) = \Pr\left(d^2(\mathbf{S}_k, \mathbf{S}_l) \le \kappa \right)$$

where the composite channel **f** is decoupled into **Hg** with $\mathbf{H} = \text{diag}(\mathbf{h})$, where we define squared Euclidean distance between \mathbf{S}_l and \mathbf{S}_k as

$$d^{2}(\mathbf{S}_{k}, \mathbf{S}_{l}) = p_{s}\mathbf{g}^{H}\mathbf{H}^{H}\mathbf{Q}^{H}(\mathbf{S}_{k} - \mathbf{S}_{l})^{H}(\mathbf{S}_{k} - \mathbf{S}_{l})\mathbf{Q}\mathbf{H}\mathbf{g}$$

and where $\kappa = 2\sqrt{p_s} \operatorname{Re}\{\mathbf{v}^H(\mathbf{S}_k - \mathbf{S}_l)\mathbf{Q}\mathbf{f}\}\$ is a real Gaussian random variable distributed as $\mathcal{N}_{\mathbb{R}}(0, 2\sigma_v^2 d^2)$. In order to obtain a upper bound of the PEP, we assume that the term $(\mathbf{S}_k - \mathbf{S}_l)^H(\mathbf{S}_k - \mathbf{S}_l)$ has a full rank M, i.e. the LD code achieves a full diversity (for a special case of orthogonal STC this always holds). By letting λ_{min} denote the smallest singular value of $(\mathbf{S}_k - \mathbf{S}_l)^H(\mathbf{S}_k - \mathbf{S}_l)$ over all possible codewords, we obtain the inequality

$$d^{2}(\mathbf{S}_{k}, \mathbf{S}_{l}) \geq p_{s} \lambda_{min} \mathbf{g}^{H} \mathbf{H}^{H} \mathbf{Q}^{H} \mathbf{Q} \mathbf{H} \mathbf{g}$$

which yields a Chernoff bound

$$P(\mathbf{S}_k \to \mathbf{S}_l | \mathbf{h}, \mathbf{g}) \le \exp\left(-\frac{p_s \lambda_{min} \mathbf{g}^H \mathbf{H}^H \mathbf{Q}^H \mathbf{Q} \mathbf{H} \mathbf{g}}{4\sigma_v^2}\right).$$
(5)

Minimizing the RHS of (5) corresponds to maximizing approximated receive SNR, given by

$$\frac{p_s \lambda_{\min} \mathbf{g}^H \mathbf{H}^H \mathbf{Q}^H \mathbf{Q} \mathbf{H} \mathbf{g}}{4\sigma_v^2} = \eta \frac{\sum_{i=1}^M p_i |g_i|^2 |h_i|^2}{\sum_{i=1}^M p_i |g_i|^2 + 1}$$
(6)

where we let $\eta = \lambda_{min} p_s / 4N_0$ and let $p_i = |q_i|^2$ denote the amplifier power of relay *i*. Notice that the above function depends on the absolute values of channels and of amplifier coefficients.

¹In practice, efficient decoding techniques such as sphere decoding can be implemented to achieve near ML results [17].

B. Partial CSIT

The PEP upper bound conditioned on \mathbf{h}, γ_g is obtained by averaging (5) over the distribution of \mathbf{g}

$$P(\mathbf{S}_{k} \to \mathbf{S}_{l} | \mathbf{h}, \boldsymbol{\gamma}_{g}) \stackrel{(a)}{\approx} \int \frac{1}{\det(\pi \operatorname{diag}(\boldsymbol{\gamma}_{g}))} \exp\left\{-\mathbf{g}^{H}\left(\frac{\eta \mathbf{H}^{H} \mathbf{P} \mathbf{H}}{1 + \sum_{i=1}^{M} \gamma_{gi} p_{i}} + \operatorname{diag}(\boldsymbol{\gamma}_{g})^{-1}\right) \mathbf{g}\right\} d\mathbf{g}$$
$$= \det\left(\frac{\eta}{1 + \sum_{i=1}^{M} \gamma_{gi} p_{i}} \operatorname{diag}(\boldsymbol{\gamma}_{g}) \mathbf{H}^{H} \mathbf{P} \mathbf{H} + \mathbf{I}_{M}\right)^{-1} = \prod_{i=1}^{M} \left(1 + \eta \frac{|h_{i}|^{2} \gamma_{gi} p_{i}}{1 + \sum_{i=1}^{M} \gamma_{gi} p_{i}}\right)^{-1} (7)$$

where in (a) we let $\mathbf{P} = \text{diag}(p_1, \dots, p_M)$ and apply a Laplace-based saddle point approximation that becomes accurate as the number of relays increases without bound (see further Appendix I). This saddle point approximation is inspired by the approximation method suggested in [24] to evaluate the expectation of quotients of quadratic forms in Gaussian random variables. We remark that, in order to maximize the corresponding cost function in (7), only the absolute values of the entries of \mathbf{h} are needed.

C. Statistical CSIT

The PEP upper bound conditioned on γ_h, γ_g is obtained by averaging (5) over the distribution of h and g

$$P(\mathbf{S}_{k} \to \mathbf{S}_{l} | \boldsymbol{\gamma}_{h}, \boldsymbol{\gamma}_{g}) \leq \mathbb{E}_{\mathbf{g}} \left[\det^{-1} \left(\mathbf{I}_{M} + \frac{\eta}{\operatorname{tr}(\mathbf{G}^{H} \mathbf{P} \mathbf{G})} \operatorname{diag}(\boldsymbol{\gamma}_{h}) \mathbf{G}^{H} \mathbf{P} \mathbf{G} \right) \right]$$

$$\stackrel{(a)}{\approx} \mathbb{E}_{\mathbf{g}} \left[\prod_{j=1}^{M} \left(1 + \frac{\eta \gamma_{hj} |g_{j}|^{2} p_{j}}{1 + \sum_{i=1}^{M} \gamma_{gi} p_{i}} \right)^{-1} \right] \stackrel{(b)}{=} \prod_{j=1}^{M} \frac{1}{\rho_{j}} e^{1/\rho_{j}} E_{1}(1/\rho_{j})$$

$$\stackrel{(c)}{=} \prod_{j=1}^{M} \frac{1}{\rho_{j}} \left[-\gamma + \ln(\rho_{j}) + O\left(\frac{1}{\rho_{j}}\right) \right] \stackrel{(d)}{\approx} \prod_{j=1}^{M} \frac{1 + \sum_{i=1}^{M} \gamma_{gi} p_{i}}{\eta \gamma_{gj} \gamma_{hj} p_{j}} \ln\left(\frac{\eta \gamma_{gj} \gamma_{hj} p_{j}}{1 + \sum_{i=1}^{M} \gamma_{gi} p_{i}}\right) \quad (8)$$

where in (a) we apply the Laplace-based saddle-point approximation of the integral mentioned above, (b) follows by noticing that $|g_i|^2/\gamma_{gi}$ is an exponential random variable with unit mean and using [25, 3.352] where we write $\rho_j = \frac{\eta \gamma_{hj} \gamma_{gj} p_j}{1+\sum_{i=1}^M \gamma_{gi} p_i}$ and let $E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt$ denote the exponential integral. In (c) we assume that ρ_j is large $(\eta \to \infty)$ so that

$$e^{1/\rho_j} E_1\left(\frac{1}{\rho_j}\right) = -\gamma + \ln(\rho_j) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (\rho_j)^{-k}}{kk!} = -\gamma + \ln(\rho_j) + O\left(\frac{1}{\rho_j}\right)$$

where γ is the Euler constant, and finally in (d) we assume $\ln(\rho_i) \gg \gamma$.

IV. POWER ALLOCATION ALGORITHMS

This section proposes efficient power allocation algorithms to optimize (6), (7), and (8).

A. Perfect CSIT

Under perfect CSIT, the optimal p^* is obtained by maximizing

$$f_0(\mathbf{p}) \stackrel{\Delta}{=} \frac{\sum_{i=1}^M \alpha_i p_i}{1 + \sum_{i=1}^M \beta_i p_i} \tag{9}$$

where p_i is subject to the maximum amplifier power of relay *i*, i.e. $p_i \leq P_i = \frac{p_r}{p_s|h_i|^2 + N_0}$ and we let $\alpha_i = |h_i|^2 |g_i|^2$ and $\beta_i = |g_i|^2$. We remark that the linear constraints form a feasible region \mathcal{V} composed by *M* half-spaces with $2^M - 1$ vertices. For M = 2 the feasible region \mathcal{V} is a rectangular region with 3 vertices $(P_1, 0), (0, P_2), (P_1, P_2)$ plus the origin. Since this problem is quasi-linear, it is possible to transform it into a linear program. By exploiting the structure of the problem, we propose a more efficient algorithm to find the solution. First, we start with the following proposition.

Proposition 1 The solution to (9) is always found in the one of $2^M - 1$ vertices of the feasible region \mathcal{V} . Moreover, at the solution \mathbf{p}^{\star} , the entries of the gradient satisfy the following inequality for $i = 1, \ldots, M$.

$$\frac{\partial f_0}{\partial p_i}(\mathbf{p}^{\star}) \begin{cases} > 0, & \text{if } p_i^{\star} = P_i \\ \le 0, & \text{if } p_i^{\star} = 0 \end{cases}$$
(10)

Proof see Appendix II.

For M = 2 we have a closed form solution of the optimal power allocation as an obvious result of Proposition 1.

Corollary 1 For M = 2, we find a closed-form solution given by

$$(p_1, p_2) = \begin{cases} (P_1, 0), & \text{if } |h_1|^2 < \frac{p_r |g_2|^2 |h_2|^2}{p_s |h_2|^2 + p_r |g_2|^2 + N_0} \\ (0, P_2), & \text{if } |h_2|^2 < \frac{p_r |g_1|^2 |h_1|^2}{p_s |h_1|^2 + p_r |g_1|^2 + N_0} \\ (P_1, P_2), & \text{if } |h_1|^2 > \frac{p_r |g_2|^2 |h_2|^2}{p_s |h_2|^2 + p_r |g_2|^2 + N_0} \text{ and } |h_2|^2 > \frac{p_r |g_1|^2 |h_1|^2}{p_s |h_1|^2 + p_r |g_1|^2 + N_0} \end{cases}$$
(11)

Proof see Appendix III.

In order to visualize the conditions of activating relay 1 and/or relay 2, we provide a graphical representation of the on-off region in Fig. 5. Interestingly, it can be observed there is a minimum value of $|h_i|^2$ in order for relay i to be activated. Namely relay 1 is activated independently of relay 2 if

$$|h_1|^2 > \frac{p_r}{p_s} |g_2|^2$$

which is readily obtained when letting $|h_2|^2 \to \infty$ in the first inequality of the third condition, i.e. $|h_1|^2 >$ $\frac{p_r|g_2|^2|h_2|^2}{p_s|h_2|^2+p_r|g_2|^2+N_0}$, and vice versa. For M > 2, the solution does not lead itself to a simple closed form expression. Nevertheless, as a straightforward

result of Proposition 1 we propose the following algorithm to solve (9).

On-off gradient algorithm

- 1) Initialize $\mathbf{p}^{(0)}$ to an arbitrary vertex $\in \mathcal{V}$ 2) At iteration *n*, compute the gradients $\frac{\partial f}{\partial \mathbf{p}}(\mathbf{p}^{(n)})$ and update

$$p_i^{(n+1)} = \left[p_i^{(n)} + \nabla_i(\mathbf{p}^{(n)}) \right]_0^{P_i}, \quad i = 1, \dots, M$$
(12)

where we let $\nabla_i(\mathbf{p}^{(n)}) = -P_i$ if $\frac{\partial f_0}{\partial p_i}(\mathbf{p}^{(n)}) < 0$ and $\nabla_i(\mathbf{p}^{(n)}) = P_i$ if $\frac{\partial f}{\partial p_i}(\mathbf{p}^{(n)}) > 0$, and $[x]_a^b$ denotes the value of x truncated to the interval [a, b].

3) Stop if $\mathbf{p}^{(n)}$ satisfies (10).

Proposition 2 The on-off gradient algorithm converges to the global maximum.

Proof See Appendix IV.

Fig. 2 plots the convergence behavior of the proposed on-off algorithm when we let $p_s = p_r = 10$ and consider equal variances $\gamma_{h,i} = \gamma_{g,i} = 1$ for all *i*. The objective values are normalized with respect to the optimal values and averaged over a large number of random channel realizations. Fig. 2 as well as other examples show that the proposed on-off gradient algorithm converges only after a few iterations irrespectively of M and of the initialization.

It is worth noticing that this on-off algorithm can be implemented at the receiver without any knowledge at the transmitter. To this end, it suffices that the receiver sends to each relay a feedback of one bit indicating whether to activate or not.

B. Partial CSIT

When the transmitter knows h and γ_a , the problem reduces to minimizing (7) or maximizing

$$f_1(\mathbf{p}) = \sum_{i=1}^M \ln\left(1 + \eta |h_i|^2 \frac{\gamma_{gi} p_i}{1 + \sum_{j=1}^M \gamma_{gj} p_j}\right)$$
(13)

The term $\frac{\eta |h_i|^2 \gamma_{gi} p_i}{1+\sum_{j=1}^M \gamma_{gj} p_j}$, similar to $\rho_i = \frac{\eta \gamma_{hi} \gamma_{gi} p_i}{1+\sum_{j=1}^M \gamma_{gj} p_j}$ defined in (b) of (7) for the statistical CSIT case, can be interpreted as the contribution of relay *i* to the receive SNR. With some abuse of notation, we let ρ_i denote

 $\frac{\eta |h_i|^2 \gamma_{g_i} p_i}{1 + \sum_{j=1}^M \gamma_{g_j} p_j}$ under partial CSIT. Unfortunately, the function f_1 is neither concave or convex in **p**. Nevertheless, assuming $\eta \to \infty$ (which is the regime of our interest), let us consider a new objective function, given by

$$J(\mathbf{p}) = \sum_{i=1}^{M} \ln\left(\frac{a_i p_i}{1 + \sum_j \gamma_{gj} p_j}\right)$$
(14)

where we let $a_i = \eta \gamma_{gi} |h_i|^2$ for notation simplicity. It is well known that the function $J(\mathbf{p})$ can be transformed into a *concave* function through a log transformation [23]. In the following, we use the notation \tilde{x} to express $\ln x$ for any variable x (equivalently $x = e^{\tilde{x}}$). The objective function can be expressed in terms of $\tilde{\mathbf{p}}$ as

$$J(\widetilde{\mathbf{p}}) = \sum_{i=1}^{M} \ln(a_i e^{\widetilde{p}_i}) - M \ln\left(1 + \sum_j \exp(\widetilde{p}_j)\gamma_{gj}\right)$$
(15)

Proposition 3 The optimal $\tilde{\mathbf{p}}$ that maximizes (15) is given

$$\tilde{p}_i = \left[\tilde{\mu}^\star - \tilde{\gamma}_{gi}\right]_{-\infty}^{\tilde{P}_i}, \quad p_i = \left[\frac{\mu^\star}{\gamma_{gi}}\right]_0^{P_i} \tag{16}$$

where $\tilde{\mu}^{\star}$ is the *water level* that is determined as follows. Let π denote a permutation such that

$$P_{\pi(1)}\gamma_{g,\pi(1)} \leq \ldots \leq P_{\pi(M)}\gamma_{g,\pi(M)}.$$
(17)

and define $\tilde{\mu_j} = \ln \mu_j$, where μ_j is given by

$$\mu_j = \frac{1 + \sum_{i=1}^j P_{\pi(i)} \gamma_{g,\pi(i)}}{j} \quad j = 1, \dots, M.$$
(18)

The optimal water level $\tilde{\mu}^*$ is obtained as that the value out of these M possible ones that maximizes the objective function, namely

$$\tilde{\mu}^{\star} = \arg \max_{\tilde{\mu}_1, \dots, \tilde{\mu}_M} J(\tilde{\mu}_j) \tag{19}$$

where $J(\tilde{\mu})$ is the objective function (15) parameterized by the water level defined in (31).

Proof see Appendix V.

Fig. 3 illustrates an example of our waterfilling solution for the case M = 3. The power curve of relay *i* increases linearly with slope $1/\gamma_{gi}$ and then is bounded at its maximum amplifier power P_i . In this example, relays 1 and 2 with $P_i\gamma_{gi} < \mu^*$ are allocated their maximum amplifier powers while relay 3 is allocated $\frac{\mu^*}{\gamma_{g,3}}$. Depending on the water level, this waterfilling yields a non-trivial solution between maximum power allocation ($p_i = P_i, \forall i$) and a generalized STC that equalizes the averaged amplified noise $p_i\gamma_{gi} = \mu^*$ (notice a classical STC considers $\gamma_{gi} = 1$ for all *i*). Note that the proposed waterfilling approach only requires a search over *M* values in order to determine the water level, and consequently it is extremely simple.

Corollary 2 For M = 2, we find a closed-form solution given by

$$(p_{1}, p_{2}) = \begin{cases} \left(\frac{1+P_{2}\gamma_{g_{2}}}{\gamma_{g_{1}}}, P_{2}\right), & \text{if } P_{2}\gamma_{g_{2}} \ge P_{1}\gamma_{g_{1}} + 1\\ \left(P_{1}, \frac{1+P_{1}\gamma_{g_{1}}}{\gamma_{g_{2}}}\right), & \text{if } P_{2}\gamma_{g_{2}} \le P_{1}\gamma_{g_{1}} - 1\\ (P_{1}, P_{2}), & \text{otherwise} \end{cases}$$
(20)

Proof see Appendix VI.

By expressing the power constraint $P_i = \frac{p_r}{p_s|h_i|^2 + N_0}$, the above allocation policy can be graphically represented as a function of $|h_1|^2$ and $|h_2|^2$ in Fig. 6. Similarly to Fig. 5 for the case of perfect CSIT, there exists a minimum value of $|h_i|^2$ so that relay *i* is allocated its maximum amplifier power. Namely, relay *i* is allocated its maximum amplifier power independently of relay $j \neq i$ if $|h_i|^2 > \frac{p_r \gamma_{g,i} - N_0}{p_s}$. Interestingly, the threshold associated to relay *i* depends only on the *i*-th channel, as opposed of what happened in the perfect CSIT case. This means that the power allocation is more selfish under partial/statistical CSIT in order to increase the reliability of the wireless link.

C. Statistical CSIT

When the transmitter only knows the variances γ_g , γ_h of the channels, we minimize the expression (8) which is equivalent to maximizing

$$f_2(\mathbf{p}) = \sum_{i=1}^M \ln\left(\frac{\eta \gamma_{hi} \gamma_{gi} p_i}{1 + \sum_{j=1}^M p_j \gamma_{gj}}\right)$$
(21)

where we ignored doubly logarithmic terms and the amplifier power p_i of relay *i* is subject to a long-term individual power constraint $P_i = \frac{p_r}{p_s \gamma_{hi} + N_0}$ for all *i*. Again, by performing a log-transformation we obtain precisely the same objective function $J(\tilde{\mathbf{p}})$ in (15) where $a_i = \eta \gamma_{gi} |h_i|^2$ defined in the previous partial CSIT case is replaced with $\eta \gamma_{gi} \gamma_{hi}$. Hence, the waterfilling solution proposed for the partial CSIT case can be directly applied to the statistical CSIT case and needs to be implemented once for a given set of variances γ_h, γ_g . The power allocation region for M = 2 is given in Fig. 6 where the axes are replaced by γ_{h1} and γ_{h2} .

V. ASYMPTOTIC BEHAVIOR

This section studies the asymptotic behavior of the proposed power allocation algorithms and gives an informal discussion on the resulting error rate performance.

A. Relays close to transmitter $\gamma_h \rightarrow \infty$

We consider the regime where $\gamma_{h,i} \to \infty$ or equivalently $|h_i|^2 \to \infty$ for all *i* at the same rate while treating other parameters finite. First we examine a two relay case. Under prefect CSIT, Fig. 5 implies that both relays tend to be switched on in this regime. Under partial and statistical CSIT cases, it follows immediately from Fig. 6 that two relays shall transmit at their maximum powers.

For M > 2 the same conclusion can be drawn as a straightforward result of Proposition 1 for the perfect CSIT case. The condition for which all M relays are switched on under perfect CSIT is given by

$$|h_i|^2 > \frac{\sum_{j=1}^M P_j |h_j g_j|^2}{1 + \sum_{j=1}^M P_j |g_j|^2}, \ i = 1, \dots, M$$

where the RHS corresponds to the objective value $f_0(P_1, \ldots, P_M)$ when letting all relays transmit with maximum power. The RHS is upper bounded by

$$f_0(P_1, \dots, P_M) \stackrel{\text{(a)}}{\leq} \frac{p_r}{p_s} \frac{\sum_{j=1}^M |g_j|^2}{1 + \sum_{j=1}^M P_j |g_j|^2} \stackrel{\text{(b)}}{\leq} \frac{p_r}{p_s} \sum_{j=1}^M |g_j|^2$$
(22)

where (a) follows from $P_j|h_j|^2 = \frac{p_r|h_j|^2}{p_s|h_j|^2+N_0} \leq \frac{p_r}{p_s}$ and (b) follows from $\frac{1}{1+\sum_{j=1}^M |g_j|^2P_j} \leq 1$. In the limit of $|h_j|^2 \to \infty, \forall j$, both (a) and (b) hold with equality and we have

$$|h_i|^2 > \frac{p_r}{p_s} \sum_{j=1}^M |g_j|^2, \ i = 1, \dots, M$$

so that all relays are allocated their maximum powers. From the upper bound (22) of the objective function, it can be expected that the performance of distributed LD code improves for a larger M. Under partial and statistical CSIT, the proposed waterfilling tends to allocate the maximum power to each relay. This can be seen immediately from the waterfilling solution depicted in Fig.3. As $P_j \rightarrow 0$, the values $\{P_i \gamma_{g,i}\}$ above which the power curves are bounded become much smaller than the lowest water level $\mu_{\min} = 1/M$. This means that all relays are allocated the maximum powers. The following remarks are in order;

- 1) Since the waterfilling algorithm under partial CSIT and on-off gradient algorithm coincide, both algorithms yield the same error performance. This implies that the knowledge of g has a negligible effect on the performance in the regime.
- 2) As a final remark, the same behavior can be observed in the following cases.
 - the transmitter power increases $p_s \to \infty$.
 - the variance γ_g of the relay-receiver channel decreases, i.e. $\gamma_g \to 0$ or equivalently $|g_i|^2 \to 0$ for all i at the same rate.

B. Relays get close to receiver $\gamma_q \rightarrow \infty$

We consider the regime where $\gamma_{g,i} \to \infty$ or equivalently $|g_i|^2 \to \infty$ for all *i* at the same rate. First we examine a two-relay case M = 2. Under perfect CSIT, it can be observed that the threshold values $\frac{p_r|g_i|^2}{p_s}$ above which each relay becomes activated (represented by straight lines in Fig. 5) get large and the on-off algorithm converges to relay selection. Similarly, the waterfilling algorithm also tends to allocate only one relay with maximum power under partial and statistical CSIT cases as expected from Fig. 6. The only exception is a symmetric variance case $\gamma_{h,1} = \gamma_{h,2}$.

For M > 2 under perfect CSIT, we next show that the on-off strategy converges to single relay selection as $|g_i|^2 \to \infty, \forall i$. To see this, first observe that if the one-off algorithm chooses only one relay to switch on, it selects the relay

$$i^{\star} = \arg\max_{i} \frac{|h_i|^2}{1 + 1/|g_i|^2 P_i}$$

The objective value is given by (with some abuse of notation in the argument of the cost function)

$$f_0(P_{i^\star}, \mathbf{0}_{M-1}) = \frac{|h_{i^\star}|^2}{1 + 1/|g_{i^\star}|^2 P_{i^\star}} \le |h_{i^\star}|^2 \tag{23}$$

where the inequality holds with equality for $|g_{i^*}|^2 \to \infty$. If the on-off algorithm activates any arbitrary set of m > 1 relays, the corresponding objective is upper bounded by

$$f_0(P_1, \dots, P_m, \mathbf{0}_{M-m}) = \frac{\sum_{i=1}^m |h_i|^2 |g_i|^2 P_i}{1 + \sum_{j=1}^m |g_j|^2 P_j} < \frac{\sum_{i=1}^m |h_i|^2 |g_i|^2 P_i}{\sum_{j=1}^m |g_j|^2 P_j} \stackrel{(a)}{=} \sum_{i=1}^m \theta_i |h_i|^2$$
(24)

where in (a) we define $\theta_i = \frac{|g_i|^2 P_i}{\sum_{j=1}^m |g_j|^2 P_j}$ with $0 < \theta_i < 1$ and $\sum_i \theta_i = 1$. We see that the last expression (24) is strictly smaller than (23) for any m and regardless of a set of relays. This implies that as the relays get close to the receiver, the on-off algorithm converges to single relay selection.

The waterfilling algorithm under partial and statistical CSIT lets only one relay transmit with the maximum power as we see in the following. Let us first consider the permutation π given in (17) sorting relays according to $P_{\pi(1)}\gamma_{g,\pi(1)} < P_{\pi(2)}\gamma_{g,\pi(2)} < \cdots < P_{\pi(M)}\gamma_{g,\pi(M)}$ with strict inequalities. As $\gamma_{g,i} \to \infty$ for all *i*, the possible water level in (18) is roughly given by $\mu_j \approx \frac{\sum_{i=1}^{j} P_{\pi(i)}\gamma_{g,\pi(i)}}{j}$ and the levels tend to be sorted as

$$\mu_{\min} \ll \mu_1 < \mu_2 \le \dots < \mu_M = \mu_{\max} \tag{25}$$

Notice that the inequality $\mu_j < \mu_{j+1}$ holds if $\sum_{i=1}^{j} (P_{\pi(j+1)}\gamma_{g,\pi(j+1)} - P_{\pi(i)}\gamma_{g,\pi(i)}) > 1$. We show that the function J is monotonically decreasing for $\mu_1 \le \mu \le \mu_M$ and the optimal water level is always given by μ_1 . We recall that the derivative of J with respect to $\tilde{\mu}$ in (32) can be expressed as a function of μ as

$$\nabla J = 1 - \frac{M\mu}{1 + |\mathcal{C}(\mu)|\mu + \sum_{i \in \overline{\mathcal{C}}(\mu)} \gamma_{gi} P_i}$$

where we assumed $|\mathcal{C}(\mu)| > 0$. Under the specific order of the water levels given in (25), we can further express the derivative ∇J_j for each interval $\mu_j < \mu \leq \mu_{j+1}$ for $j = 1, \dots, M-1$ such that

$$\nabla J_j = \frac{j(\mu_j - \mu)}{1 + (M - j)\mu + \sum_{i=1}^j \gamma_{g,\pi(i)} P_{\pi(i)}}$$

Since we have $\nabla J_j < 0$ for any interval j, it clearly appears that the function is monotonically decreasing thus the waterfilling algorithm allocates maximum power only to the relay $\pi(1)$ by letting $\mu^* = \mu_1$. In order to have an insight on the error rate performance achieved by the waterfilling letting $p_i = \frac{\mu_1}{\gamma_{g_i}}$ for all i, we evaluate the approximated receive SNR value f_0 .

$$f_0\left(\left\{\frac{\mu_1}{\gamma_{gi}}\right\}\right) = \frac{\mu_1 \sum_{i=1}^M |h_i|^2 \frac{|g_i|^2}{\gamma_{gi}}}{1 + \mu_1 \sum_{j=1}^M \frac{|g_j|^2}{\gamma_{gj}}} \stackrel{(a)}{\leq} \frac{\sum_{i=1}^M |h_i|^2 |g_i|^2 / \gamma_{gi}}{\sum_{j=1}^M |g_j|^2 / \gamma_{gj}} \stackrel{(b)}{=} \sum_{i=1}^M \theta_i |h_i|^2$$
(26)

where (a) holds with equality as $\mu_1 \to \infty$ ($P_i \gamma_{gi} \to \infty$ for all *i*), and in (c) we define $\theta_i = \frac{|g_i|^2 / \gamma_{gi}}{\sum_{j=1}^{M} |g_j|^2 / \gamma_{gj}}$ with $0 < \theta_i < 1$ and $\sum_i \theta_i = 1$. We see that the final expression is dominated by (23) achieved by single relay selection under perfect CSIT. The following remarks are in order;

- 1) The optimal transmit scheme in this regime is single relay selection that chooses roughly the relay with the largest $|h_i|^2$. Activating more than one relay becomes highly suboptimal due to large amplified noise.
- 2) The power allocation under partial and statistical CSIT is the same and equalizes $p_i \gamma_{gi}$.
- 3) As a final remark, the same behavior can be observed in the following cases.
 - the relay power increases $p_r \to \infty$.
 - the variance of the transmitter-relay channel decreases $m{\gamma}_h o m{0}.$

which yield $P_i|g_i|^2 \to \infty$ under perfect CSIT and $P_i\gamma_{q,i} \to \infty$ under partial and statistical CSIT.

VI. NUMERICAL RESULTS

In this section, we provide some numerical results to illustrate the behavior of the proposed power allocation algorithms. Assuming a homogeneous network, we let $p_s = p_r$. We consider BPSK modulation and generate randomly a LD code with M = T drawn from an isotropic distribution.

First, we compare the proposed on-off algorithm with other schemes in a system with M = 2 relays and equal variances $\gamma_{hi} = \gamma_{gi} = 1$ for i = 1, 2. Fig. 7 shows the block error probability versus per-relay SNR p_r/N_0 with the on-off algorithm, network beamforming of [5], and maximum power allocation that lets both relays transmit with their peak powers. For a reference we also plot the performance of our waterfilling algorithm under statistical CSIT. We observe that network beamforming outperforms the on-off gradient algorithm by roughly 3 dB by exploiting full channel knowledge and that both schemes achieve the same diversity gain. On the contrary, maximum power allocation has a substantial performance loss and fails to achieve full diversity gain. This clearly shows that an appropriate power allocation is essential for distributed LD code to provide diversity gain.

Next, we examine how the network topology impacts the proposed power allocation algorithms and the resulting BER performance. To model a simple network topology, we consider a unit transmitter-receiver distance and let the transmitter-relay distance varies in the range 0 < r < 1. The resulting variances are $\gamma_{hi} = 1/r^2$ and $\gamma_{qi} = 1/(1-r)^2$ for all i. For the sake of fair comparison between systems with different M, we assume that the whole network power \mathcal{P} is equally shared between the transmitter and M relays so that $p_r = \mathcal{P}/(M+1)$. Fig. 8 shows the BER performance of the proposed power allocation algorithms with M = 2, 4, 6 and $\mathcal{P}/N_0 = 15$ dB along with the performance of the direct transmission with a fixed power P. Fig. 9 shows the averaged allocated power ratio, i.e. $\sum_{i=1}^{M} \mathbb{E}[p_i(t)/P_i(t)]$, or equivalently the effective number of relays. The following remarks are in order: 1) As the relays get closer to the transmitter $r \to 0$, the transmitter activates all relays with their maximum power. The waterfilling solution under partial CSIT converges to the on-off gradient algorithm in the limit of $r \rightarrow 0$, which implies that the knowledge of g has a negligible impact on the performance. The result agrees well with the analysis provided in subsection V-A. 2) As the relays get closer to the receiver $r \to 1$, the optimal strategy activates only one relay to limit the amplified noise. As seen in Fig. 9 the on-off gradient algorithm indeed reduces to relay selection. On the contrary, the waterfilling solution equalizes $p_1\gamma_{g1} = \cdots = p_M\gamma_{gM}$ both under partial and statistical CSIT, and moreover it converges to the same error performance independently the number of relays. Under the given setting where $P_1\gamma_{g1} = \cdots = P_M\gamma_{gM}$, the waterfilling solution under statistical CSIT lets all relays transmit with maximum power. The result is in a good agreement with the analysis of subsection V-B.

Fig. 10 shows the BER performance versus \mathcal{P}/N_0 for M = 2, 4, 8. Here, we randomly choose the relay-receiver distances and let $\gamma_g = [0.85, 3.17, 1.50, 1.89, 2.06, 2.36, 3.19, 3.99]$. The transmitter-relay distance r = 0.5 is fixed $(\gamma_{hi} = 4 \text{ for any } i)$. Compared to the direct transmission, DSTC with our proposed power allocation algorithms yields significant diversity gain at moderate to high power regime. Moreover, additional CSIT yields a considerable power gain.

VII. CONCLUSIONS

We considered a two-hop wireless network where M relays aid one transmitter-receiver pair to communicate via DSTC together with the AF protocol. In order to study the impact of CSIT on the design and the performance of DSTC, we optimized the amplifier power allocation under individual power constraints so that the PEP conditioned to the available CSIT is minimized. Under perfect CSIT we proposed the on-off gradient algorithm that efficiently

finds a subset of relays to switch on. It turns out that this algorithm can be implemented at the receiver if the receiver can send a one-bit feedback to each relay indicating whether to switch on or not. Under partial and statistical CSIT we derived a simple waterfilling algorithm that yields a non-trivial solution between maximum power allocation and a generalized STC that equalizes the averaged amplified powers for all relays. Closed-form solutions were derived for M = 2 and in certain asymptotic regimes. Namely, when the relays are physically close to the transmitter, the on-off algorithm and the waterfilling algorithm coincide and both let all relays transmit with maximum amplifier powers. When the relay are close to the receiver, the on-off algorithm converges to relay selection in order to minimize the amplified noise seen by the receiver while the waterfilling equalizes the averaged amplified noise and becomes highly suboptimal. The proposed amplifier power allocation algorithms were derived for a particular type of linear dispersion STC but can be extended to more general LD code as well as other classes of STC as long as the amplified noise remains white.

Appendix I

SADDLE POINT APPROXIMATION

The objective of this appendix is to justify the approximations (a) in (7) and (8) and to show that the approximation is valid as the number of relays increases without bound. Let us denote

$$a = \mathbf{g}^H \mathbf{H}^H \mathbf{P} \mathbf{H} \mathbf{g} \quad b = 1 + \mathbf{g}^H \mathbf{P} \mathbf{g}$$

We need to evaluate

$$\mathbb{E}\left[\exp\left(-\eta\frac{a}{b}\right)\right] = \mathbb{E}\left[\exp\left(-\eta\frac{\mathbf{g}^{H}\mathbf{H}^{H}\mathbf{P}\mathbf{H}\mathbf{g}}{1+\mathbf{g}^{H}\mathbf{P}\mathbf{g}}\right)\right]$$

where the expectation is with respect to the statistics of g. First of all, observe that we can write

$$\exp\left(-\eta \frac{\mathbf{g}^H \mathbf{H}^H \mathbf{P} \mathbf{H} \mathbf{g}}{1 + \mathbf{g}^H \mathbf{P} \mathbf{g}}\right) = \lim_{n \to \infty} X_n$$

where

$$X_n = \sum_{k\geq 0}^n \frac{(-\eta)^k}{k!} \left(\frac{a}{b}\right)^k.$$

On the other hand,

$$|X_n| < \sum_{k\geq 0}^n \frac{\eta^k}{k!} \left(\frac{a}{b}\right)^k < \exp\left[\eta \frac{\mathbf{g}^H \mathbf{H}^H \mathbf{P} \mathbf{H} \mathbf{g}}{1 + \mathbf{g}^H \mathbf{P} \mathbf{g}}\right] < \exp\left[\eta \frac{\mathbf{g}^H \mathbf{H}^H \mathbf{P} \mathbf{H} \mathbf{g}}{\mathbf{g}^H \mathbf{P} \mathbf{g}}\right] \leq \exp\left(\eta \max_{1\leq j\leq M} |h_j|^2\right) < \infty$$

Hence, the bounded convergence theorem ensures that we can write

$$\mathbb{E}\left[\exp\left(-\eta\frac{a}{b}\right)\right] = \mathbb{E}\left[\lim_{n \to \infty} X_n\right] = \lim_{n \to \infty} \mathbb{E}\left[X_n\right] = \sum_{k \ge 0} \frac{(-\eta)^k}{k!} \mathbb{E}\left[\left(\frac{a}{b}\right)^k\right]$$
(27)

and we can therefore concentrate on the study of the moments

$$r_k = \mathbb{E}\left[\left(\frac{a}{b}\right)^k\right].$$

In particular, we can follow the procedure introduced in [24], which is based on a Laplace approximation of the integral. More specifically, in [24] it was shown that a Laplace approximation of r_k about the origin leads to the identity

$$r_k = \mathbb{E}\left[\left(\frac{a}{\mathbb{E}\left[b\right]}\right)^k\right] + R_k \tag{28}$$

where both expectations are with respect to g and we have $R_k \to 0$ as $M \to \infty$. Although the procedure is somewhat tedious, one can extend this to show that $\sup_k R_k \to 0$. As a direct consequence of this theorem, we have that as $M \to \infty$, it holds that²

$$\mathbb{E}\left[\exp\left(-\eta\frac{a}{b}\right)\right] = \sum_{k\geq 0} \frac{(-\eta)^k}{k!} \mathbb{E}\left[\left(\frac{a}{b}\right)^k\right]$$
$$= \sum_{k\geq 0} \frac{(-\eta)^k}{k!} \left(\mathbb{E}\left[\left(\frac{a}{\mathbb{E}[b]}\right)^k\right] + R_k\right) = \mathbb{E}\left[\exp\left(-\eta\frac{a}{\mathbb{E}[b]}\right)\right] + \sum_{k\geq 0} \frac{(-\eta)^k}{k!} R_k$$

where now

$$\left|\sum_{k\geq 0} \frac{(-\eta)^k}{k!} R_k\right| \leq \sup_k R_k \exp\left(-\eta\right) \to 0$$

and this justifies the approximation used in the paper.

APPENDIX II

PROOF OF PROPOSITION 1

In order to prove Proposition 1, we consider the *i*-th entry of the gradient of the objective function, given by

$$\frac{\partial f_0}{\partial p_i} = \frac{\alpha_i + \alpha_i \sum_{j \neq i} \beta_j p_j - \beta_i \sum_{j \neq i} \alpha_j p_j}{(1 + \beta_i p_i + \sum_{j \neq i} \beta_j p_j)^2}.$$

When we treat the variables $\{p_j\}_{j \neq i}$ fixed, the *i*-th gradient can be expressed as a function of p_i in the form $\frac{\xi_i}{(\beta_i p_i + \zeta_i)^2}$, where $\zeta_i > 0$ and ξ_i are constants. Depending on the sign of ξ_i , the gradient is always negative or positive, i.e. the function is monotonically decreasing or increasing in each p_i^3 . Since the objective function cannot be maximized at $0 < p_i < P_i$, the solution of (9) is achieved only at one of the vertices. The second part follows directly from the monotonicity of the function in each component p_i . Namely, the solution is achieved by the vertex at which the objective function cannot further increase beyond the thresholds.

APPENDIX III

Proof of Corollary 1 : A closed-form solution of M = 2 under perfect CSIT

From Proposition 1, we see immediately that the power allocation of two relays depend on the sign of

$$\xi_1(p_2) = \alpha_1 + \Delta p_2, \ \xi_2(p_1) = \alpha_2 - \Delta p_1$$

where we let $\Delta = \alpha_1 \beta_2 - \beta_1 \alpha_2 = |g_1 g_2|^2 (|h_1|^2 - |h_2|^2)$. Moreover, it is sufficient to check the sign of ξ_1 and ξ_2 at each vertex to determine the optimum power allocation. Table I summarizes the optimal solution and the conditions ; the optimal solution is given by $(P_1, 0)$ if and only if $\xi_2(P_1) = \alpha_2 - \Delta P_1 < 0$ holds while it is given by a vertex $(0, P_2)$ if and only if we have $\xi_1(P_2) = \alpha_1 + \Delta P_2 < 0$. Finally, both relays are activated if $\frac{-\alpha_1}{P_2} < \Delta < \frac{\alpha_2}{P_1}$. These inequalities yield (11).

APPENDIX IV

PROOF OF PROPOSITION 2 : CONVERGENCE OF ON-OFF GRADIENT ALGORITHM

We have to first prove that the objective is non-decreasing, i.e. $f(\mathbf{p}^{(n+1)}) \ge f(\mathbf{p}^{(n)})$ for any iteration *n*. Identifying that the update in (12) is nothing than a discrete steepest ascent algorithm with a fixed step size, the objective always increases. It remains to prove that the converged point is the global maximum. In other words, the stopping criterion above is sufficient to guarantee a global convergence. It is not difficult to see that there is a *unique* \mathbf{p}^* satisfying the condition (10) such that the signs of the gradients and the powers match. Otherwise, we can always increase the objective by switching on (off) the power with a positive (negative) gradient.

 2 In the last equality of the following section, one should justify the fact that expectation and sum can be interchanged. This is not difficult to see, but the proof is omitted due to space constraints.

³For $\xi_i = 0$, the function is constant in p_i , then we let $p_i = 0$.

APPENDIX V

PROOF OF PROPOSITION 3 : WATERFILLING SOLUTION UNDER PARTIAL CSIT

Since $J(\tilde{\mathbf{p}})$ is a strictly concave function of $\tilde{\mathbf{p}}$, we solve the KKT conditions which are necessary and sufficient for optimality. By letting $\lambda_i \ge 0$ a Lagrangian variable associated to the individual power constraint $p_i \le P_i$, we obtain the KKT conditions given by

$$1 - \frac{M\gamma_{gi}\exp(\tilde{p}_i)}{1 + \sum_j \gamma_{gj}\exp(\tilde{p}_j)} = \lambda_i, \quad i = 1, \dots, M$$
⁽²⁹⁾

Summing the above equation over all i and defining $I = \sum_{j} \gamma_{gj} \exp(\tilde{p}_j)$ and $\mu = \frac{1}{\sum_i \lambda_i}$ we obtain

$$I = M\mu - 1 \tag{30}$$

It follows that μ is lower bounded by $\mu^{\min} = \frac{1}{M}$ and upper bounded by $\mu^{\max} = \frac{1+\sum_{j} \gamma_{gj} P_j}{M}$. Plugging (30) into (29) and using the inequality $\lambda_i \ge 0$, we readily obtain the optimal power given in (16).

It remains to determine the optimal *water level* $\tilde{\mu}^*$ that maximizes the objective function J (note that the individual power constraint is always satisfied for any μ). To this end, we define $\mathcal{C}(\tilde{\mu})$ and $\overline{\mathcal{C}}(\tilde{\mu})$ as

$$\mathfrak{C}(\tilde{\mu}) = \{i | \tilde{p}_i = \tilde{\mu} - \tilde{\gamma}_{gi}\}, \quad \overline{\mathfrak{C}}(\tilde{\mu}) = \{i | \widetilde{p}_i = \widetilde{P}_i\}$$

Plugging (16) into (15), the function can be expressed in terms of $\tilde{\mu}$

$$J(\tilde{\mu}) = |\mathcal{C}(\tilde{\mu})|\tilde{\mu} + \sum_{i \in \overline{\mathcal{C}}(\tilde{\mu})} \tilde{P}_i - M \ln \left(1 + |\mathcal{C}(\tilde{\mu})| \exp(\tilde{\mu}) + \sum_{i \in \overline{\mathcal{C}}(\tilde{\mu})} \gamma_{gi} \exp(\tilde{P}_i) \right) + \sum_{i=1}^M \ln a_i$$
(31)

where $|\mathcal{C}|, |\overline{\mathcal{C}}|$ denotes the cardinality of the set $\mathcal{C}, \overline{\mathcal{C}}$ respectively. Since the function $J(\tilde{\mu})$ is strictly concave in $\tilde{\mu}$, the optimal $\tilde{\mu}$ must satisfy $\frac{\partial J}{\partial \tilde{\mu}} = 0$ where

$$\frac{\partial J}{\partial \tilde{\mu}} = |\mathcal{C}(\tilde{\mu})| \left(1 - \frac{M \exp(\tilde{\mu})}{1 + |\mathcal{C}(\tilde{\mu})| \exp(\tilde{\mu}) + \sum_{i \in \overline{\mathcal{C}}(\tilde{\mu})} \gamma_{gi} \exp(\tilde{P}_i)} \right)$$
(32)

which yields

$$\exp(\tilde{\mu}) = \frac{1 + \sum_{i \in \overline{\mathbb{C}}(\tilde{\mu})} \exp(P_i + \tilde{\gamma}_{gi})}{|\overline{\mathbb{C}}(\tilde{\mu})|}$$
(33)

for $|\overline{\mathbb{C}}(\tilde{\mu})| > 0$. Notice that for $|\overline{\mathbb{C}}(\tilde{\mu})| = 0$, it can be shown that the objective function is a monotonically increasing concave function and maximized at $\tilde{\mu}^{\text{max}}$. By sorting $\{P_i\gamma_{gi}\}$ in an increasing order according to the permutation (17), we remark that the RHS of (33) has at most M possible values ⁴ in (18) and we choose the optimal $\tilde{\mu}^*$ according to (19).

APPENDIX VI

Proof of Corollary 2 : A closed-form solution of M = 2 under partial CSIT

Without loss of generality, we assume $P_1\gamma_{g,1} < P_2\gamma_{g,2}$. We recall that the possible values of the water level (18) for M = 2 are

$$\mu_1 = 1 + P_1 \gamma_{g,1} \ \mu_2 = \mu^{\max} = \frac{1 + P_1 \gamma_{g,1} + P_2 \gamma_{g,2}}{2}$$

First we consider the case $\mu_1 < \mu_2$. This inequality reduces to $P_2\gamma_{g,2} > P_1\gamma_{g,1} + 1$, and further yields

$$P_1 \gamma_{g,1} < \mu_1 < \mu_2 < P_2 \gamma_{g,2}.$$

Obviously we obtain $p_1 = P_1$ no matter which of the two values is the optimum water level. It is not difficult to see that the optimal water level is given by μ_1 by comparing the two objective values $J(\tilde{\mu_1})$ and $J(\tilde{\mu_2})$. Hence the amplifier power of relay 2 is $p_2 = \frac{\mu_1}{\gamma_{g^2}} = \frac{1+P_1\gamma_{g_1}}{\gamma_{g^2}}$. Next we consider the case $\mu_2 < \mu_1$. This inequality is equivalent to $P_2\gamma_{g,2} < P_1\gamma_{g,1} + 1$, and yields $P_2\gamma_{g,2} < \mu_2$. In this case, the water level (either μ_1 or μ_2) is larger than

⁴Some of the *M* values might be unfeasible if they are not in the domain $\tilde{\mu} \in [\tilde{\mu}_{\min}, \tilde{\mu}_{\max}]$.

 $P_1\gamma_{q,1}$ and $P_2\gamma_{q,2}$. Hence, the algorithm lets both relays transmit at the maximum power, $p_1 = P_1$ and $p_2 = P_2$. In summary, we have the following power allocation possibilities:

- $p_1 = P_1$ and $p_2 = \frac{1+P_1\gamma_{g_1}}{\gamma_{g_2}} < P_2$ if $P_2\gamma_{g_2} > P_1\gamma_{g,1} + 1$ $p_1 = P_1$ and $p_2 = P_2$ if $P_2\gamma_{g_2} < P_1\gamma_{g,1} + 1$

By symmetry, when $P_1\gamma_{q1} > P_2\gamma_{q2}$ we obtain

- $p_1 = P_1$ and $p_2 = P_2$ if $P_2 \gamma_{g2} < P_1 \gamma_{g,1} < P_2 \gamma_{g2} + 1$ $p_1 = \frac{1 + P_2 \gamma_{g2}}{\gamma_{g1}} < P_1$ and $p_2 = P_2$ if $P_1 \gamma_{g1} > P_2 \gamma_{g2} + 1$

These conditions and the corresponding power allocation are summarized in (20) and depicted in Fig. 4. Since the parameter $P_i \gamma_{qi}$ does not provide an easy interpretation (note that P_i is a function of h), we express the power allocation region in terms of γ_g , h explicitly. The subregion where only relay 1, 2 is allocated its maximum power is given respectively by

$$|h_2|^2 < \frac{p_r \gamma_{g2} \left(|h_1|^2 + \frac{N_0}{p_s} \right)}{p_s |h_1|^2 + p_r \gamma_{g1} + N_0} - \frac{N_0}{p_s}, \quad |h_1|^2 < \frac{p_r \gamma_{g1} \left(|h_2|^2 + \frac{N_0}{p_s} \right)}{p_s |h_2|^2 + p_r \gamma_{g2} + N_0} - \frac{N_0}{p_s}$$

These conditions yield the power allocation region in terms of h in Fig. 6.

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Fig. 1. A wireless relay network



Fig. 2. Convergence of on-off algorithm for different M



Fig. 3. Proposed waterfilling solution with M = 3

vertex	ξ_1	ξ_2	Δ
$(P_1, 0)$	+	-	$(\alpha_2/P_1,\infty]$
(P_1, P_2)	+	+	$(-\alpha_1/P_2, \alpha_2/P_1)$
$(0, P_2)$	-	+	$[-\infty, -\alpha_1/P_2)$

 TABLE I

 Optimal solutions and corresponding conditions



Fig. 4. Power allocation region as a function of $P_i \gamma_{gi}$



Fig. 5. Two-relay ON/OFF region under perfect CSIT



Fig. 6. Two-relay power region under partial CSIT



Fig. 7. Block error rate vs SNR



Fig. 8. BER vs. transmitter-relay distance



Fig. 9. Normalized allocated power vs. transmitter-relay distance



Fig. 10. BER performance vs \mathcal{P}/N_0