The Deterministic Time-Linearity of Service Provided by Fading Channels

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Abstract

In the paper, we study the service process $S(t)$ of an independent and identically distributed (*i.i.d.*) Nakagami-m fading channel, which is defined as the amount of service provided, i.e., the integral of the instantaneous channel capacity over time t. By using the Characteristic Function (CF) approach and the infinitely divisible law, it is proved that, other than certain generally recognized curve form or a stochastic process, the channel service process $S(t)$ is a deterministic linear function of time t, namely, $S(t) = c_m^* \cdot t$ where c_m^* is a constant determined by the fading parameter m. Furthermore, we extend it to general *i.i.d.* fading channels and present an explicit form of the constant service rate c_p^*. The obtained work provides such a new insight on the system design of joint source/channel coding that there exists a coding scheme such that a receiver can decode with zero error probability and zero high layer queuing delay, if the transmitter maintains a constant data rate no more than c_p^*. Finally, we verify our analysis through Monte Carlo simulations.

Index Terms

i.i.d. fading channels; Nakagami-m fading; channel service process; time linearity.

I. Introduction

In a wireless communication system, signal from a transmitter usually travels over multiple reflective, diffracted and scattered paths to a receiver. As a result, the received signal may fluctuate violently and the received *signal to noise ratio* (SNR) varies randomly over time. This phenomenon is referred to as the multipath propagation. In particular, when these multipath signals naturally arrived at the receiver, the fading usually occurs, which is characterized by Rayleigh, Rician, or the Nakagami-m fading models. In general, the Nakagami-m fading [1] includes a wide range of multipath channels via adjusting parameter m. For instance, the Nakagami-m distribution includes the one-sided Gaussian distribution ($m = 1/2$), which corresponds to the highest amount of multipath fading scenario. The Rayleigh distribution is also included by setting $m = 1$, which is most applicable when there is no dominant propagation along the line of sight (LOS) between a transmitter and the receiver. Moreover, if one takes $\frac{(K+1)^2}{2K+1}$ as the value of m, i.e., $m = \frac{(K+1)^2}{2K+1}$, then the Nakagami-m distribution reduces approximately to the Rician distribution with parameter K, which models the situation

when there exists a fixed LOS component in the received signal[1]. In fact, for the modeling of fading channels in practical communication systems, the Nakagami-m distribution often gives the best fit to both urban and indoor multipath propagation.

For a wireless link over fading channel, the study of its capacity has always been of interest. In literature, the *Shannon (ergodic) capacity* with the channel side information (CSI) at a receiver for an average power constrain $\overline{P}$ is given in [2]. In this case, the transmission data rate over the channel is constant regardless of instantaneous SNR at the receiver. The capacity-achieving code has to be sufficiently long so that a received codeword is affected by all possible fading states. Besides the ergodic capacity, the *outage capacity* defines the maximal (fixed) rate achievable in all non-outage states with asymptotically small error probability [3], [4]. This capacity usually applies to slowly-varying channels where instantaneous SNR is constant over a number of transmission slots and changes to a new value following certain fading distribution. Moreover, in [5], the authors considered the capacity of fading channels with side information at both encoder and decoder with optimal water filling power allocation. In [6], it provided an exhaustive review on the information-theoretic and communication features of fading channels and derived the capacities of fading channels with and without the channel side information.

Furthermore, from a viewpoint of cross layer design method, the authors in [7] investigated fading channels in terms of the high layer Qos parameters. According to the large deviation theory and Legendre transformation, they proposed a link-layer channel model termed *effective capacity* (EC) which specifies the maximum transmission data rate supported by a fading channel under certain Qos metrics, such as the maximum delay and the buffer overflow probability. However, their result involves a lot of inevitable approximation, which limits its application. Consequently, to facilitate the application of EC theory, the authors [8], [9] derived the close form of EC function for both correlated Rayleigh and Nakagami-m fading channels with the consideration of Doppler effect and presented the error analysis of the measurement-based estimation algorithm of the EC function.

In this paper, we first investigate the service process $S(t)$ of the *i.i.d.* fading channels with the CSI available at the receiver. Here, the service process $S(t)$ is defined as the integral of the instantaneous channel capacity over time t. As the Nakagami-m fading model is widely used and includes several classic fading models by different m parameters, this paper is focused on the characteristics of Nakagami-m fading channels. For such channels, the channel magnitude gain varies randomly following Nakagami-m distribution, the service process $S(t)$ should also be a stochastic process. However, by using the infinitely divisible law [10] and the

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[1]K is the ratio between the power in the LOS component and the average non-LOS multipath components [13]. For $K = 0$ it is Rayleigh fading, and for $K = +\infty$ it has no fading.
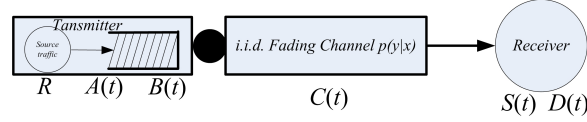
Fig. 1. The transmission system model

CF (Characteristic Function) approach, we prove that the service process $S(t)$ is itself a deterministic linear function of time t, i.e., $S(t) = c_m^* \cdot t$ where c_m^* is a constant determined by the fading parameter m. Then the result is applied to three special cases of Nakagami-m fading channel, namely, the Rayleigh fading channel, the Rician fading channel and the AWGN channel. Also, the corresponding service rates are derived. Finally, we prove that the time linearity nature of $S(t)$ maintains for all kinds of *i.i.d.* fading channels more than the Rayleigh, Rician and Nakagami fading. In terms of our analysis, it indicates in theory that there exists a channel coding scheme such that a constant data rate can be supported by an *i.i.d.* fading channel with no queuing delay in the viewpoint of application layer. Therefore, the *i.i.d.* fading channels have constant and stable transmission ability, just like the AWGN channels.

In summary, traditionally the ergodic is only considered as one statistic characteristic of fading channels. However, it is proved in this paper that the ergodic capacity is more than that. In fact, *i.i.d.* fading channels have the deterministic ability to support a constant traffic rate c_m^*, which is equal to the ergodic capacity. Particularly, there are two points that should be noted. Firstly, the deterministic time linearity of the service process is related to the fluid traffic model on the large time scales. That is to say, the traffic data is infinitely divisible and we are interested in the channel performance over long periods rather than its performance on the order of symbols. Compared with the time scale interested, the duration when channel gain is constant is very short. Therefore, the channel can be treated as very fast fading channel without loss of generality. However, the duration when channel gain is constant still contains many channel uses so that the c_p^* achievable coding scheme can be realized by adapting coding rate according to the channel condition. Secondly, what we are saying is about the deterministic time linearity of $S(t)$ rather than $E[S(t)]$ or $\frac{S(t)}{t}$. Specifically, given an arbitrary time t_0, one has $S(t_0)$ is a constant equal to $c_p^* \cdot t_0$ instead of a random variable. And as t_0 varies, $S(t_0)$ is a deterministic function of t_0 instead of a stochastic process. However, the statements that $E[S(t)]$ is a linear function of t and $\frac{S(t)}{t}$ reduces to a constant are straightforward.

The rest of the paper is organized as follows. In Section II, we describe the system model, including the fading channel model. The theoretical analysis is developed in Section III and IV. More specifically, we prove the time-linearity of the channel service process $S(t)$ for *i.i.d.* Nakagimi-m fading channels by using the CF approach and the infinitely divisible law in Section III. Furthermore, we derive the corresponding linear
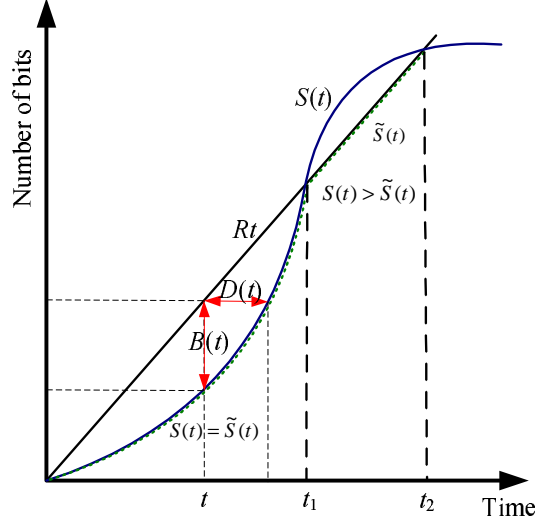
Fig. 2. Characterization of the source traffic and channel service process

coefficients i.e., the constant service rate c_m^*. Then, in Section IV, we show that the deterministic time-linearity

nature exists for all kinds of *i.i.d.* fading channels. Numerical results via Monte Carlo simulations provided

in Section V to confirm the time-linearity nature of $S(t)$. This section also provides some related discussions.

Finally, we conclude the paper in Section VI.

II. SYSTEM MODEL

We consider an adaptive transmission model shown in Fig. 1. The source maintains a constant data rate R and

the transmitter adapts the transmission rate $C(t)$ according to the channel condition. The source traffic stream

and channel service ability are matched by a First In First Out (FIFO) buffer. Let $B(t)$ denote the queuing

length in the buffer in nats at the moment t and $D(t)$ denote the latency that the data arriving in the system at

the moment t will suffer from, namely, the delay between its arrival time and the moment it has been served.

Next, we define $S(t)$ as the amount of service provided by the fading channel until the moment t, namely,

the solid curve shown in Fig. 2, and $\tilde{S}(t)$ as the amount of actual utilized service of the channel until the

moment t for the source traffic, namely, the dotted curve. Intuitively, the relationship among $B(t)$, $D(t)$, $\tilde{S}(t)$

and $S(t)$ is depicted in Fig. 2. Note that we have $\tilde{S}(t) \leq S(t)$ always satisfied. The reason is that, until any

moment t, the amount of actually provided service, i.e., $\tilde{S}(t)$, should always be jointly upper-bounded by the

amount of source traffic (Rt), and the potential amount of service which the channel can provide, namely, $S(t)$.

By its physical meaning, $S(t)$ is given by

$$S(t) = \int_0^t C(\tau)d\tau, \tag{1}$$

where $C(\tau)$ is the instantaneous capacity of the time varying channel at time τ.

Note that since the instantaneous capacity $C(t)$ is a stochastic process for $t \geq 0$. Therefore, $S(t)$ is also a stochastic process which is equivalent to the stochastic integral in the sense of zero mean-square error, based on the stochastic calculus theory. That is, for any time $t \geq 0$, the following relation holds,

$$E[(S(t) - \int_0^t C(\tau)d\tau)^2] = 0. \tag{2}$$

However, one amazing thing is that it will be proved that $S(t)$ has time-linearity nature which means its profile is a line, other than certain generally recognized curve shown in Fig. 2.

In particular, for the channel service process $S(t)$ associated with the instantaneous channel capacity $C(t)$ mentioned above, we consider a continuous time Nakagami-m fading channel with the stationary and ergodic time varying channel gain $g(t)$ and additive white Gaussian noise (AWGN) $n(t)$. For any moment t_0, the channel gain $g(t_0)$ is Nakagami-m distributed and the corresponding power gain is denoted by $\gamma(t_0) = g^2(t_0)$. For $m \geq \frac{1}{2}$, their *probability density functions (p.d.f)* are given by

$$p_g^m(g) = 2(\frac{m}{P_r})^m \frac{g^{2m-1}}{\Gamma(m)} e^{\frac{-mg^2}{P_r}}, \tag{3a}$$

$$p_\gamma^m(\gamma) = (\frac{m}{P_r})^m \frac{\gamma^{m-1}}{\Gamma(m)} e^{\frac{-m\gamma}{P_r}}, \tag{3b}$$

where $g(t)$ is independent of both the channel input process and any other $g(t')$ for $t \neq t'$, and so is $\gamma(t)$. Here, P_r is the average received signal power of a receiver unit distance away and $\Gamma(\cdot)$ is the Gamma function. Let P_t denote the average transmit power of the signal, and N_0 denote the noise power spectral density. W is the limited bandwidth of the received signal. Note that, without loss of generality, let d and α denote the distance between the transmitter and receiver and the path loss exponent, respectively. Then, the instantaneous received SNR is given by

$$\text{SNR} = \frac{\gamma(t)P_t d^{-\alpha}}{WN_0}, \tag{4}$$

and the corresponding instantaneous capacity of the fading channel in $nats$ is given by

$$C(t) = W\ln(1 + \frac{\gamma(t)P_t d^{-\alpha}}{WN_0}). \tag{5}$$

In particular, for $m = 1$, the Nakagami-m distribution is reduced to the Rayleigh distribution, which is given by

$$p_g^{ra}(g) = \frac{2g}{P_r} e^{\frac{-g^2}{P_r}}, \tag{6a}$$

$$p_\gamma^{ra}(\gamma) = \frac{1}{P_r} e^{\frac{-\gamma}{P_r}}. \tag{6b}$$

The Rician distribution can also be approximated by the Nakagami-m distribution by setting $m = \frac{(K+1)^2}{2K+1}$. The Rician distribution in terms of parameter K is given by

$$
\begin{aligned}
p_g^{ri} &= \frac{2g(K+1)}{P_r} \exp[-K - \frac{(K+1)g^2}{P_r}] \\
&\quad I_0(2g\sqrt{\frac{K(K+1)}{P_r}}), \quad g \geq 0,
\end{aligned}
\tag{7}
$$

where $K = \frac{s^2}{\sigma^2}$ is the ratio between the power in the LOS component and the average non-LOS multipath components.

Upon all of the preparation above, we will begin to analyze the service process $S(t)$ and study its property next.

III. TIME-LINEARITY OF $S(t)$ FOR NAKAGAMI FADING CHANNELS

In this paper, we are interested in the statistic characteristics of $S(t)$ and study some properties of such a process by exploiting its Characteristic Function (CF). Specifically, for an *i.i.d.* Nakagami-m fading channel, we give the following theorem which is one of the main contributions of the paper.

Theorem **1**: *For an i.i.d. Nakagami-m fading channel, the amount of service provided by the channel during time t, namely, $S(t)$, is a deterministic linear function of time t, i.e., $S(t) = c_m^* \cdot t$, where c_m^* is a constant and given by*

$$
c_m^* = \frac{W}{\Gamma(m)} \int_0^\infty \ln(\frac{P_r P_t}{mW N_0 d^\alpha} y + 1) y^{m-1} e^{-y} dy,
\tag{8}
$$

where P_r is the average received signal power of the signal. N_0 is the noise power spectral density, W is the received signal bandwidth, P_t is the average transmit power and d is the distance between the transmitter and receiver. The function $\Gamma(m) = \int_0^\infty t^{m-1} e^{-t} dt$ is the Gamma function.

First, note that, c_m^* is not the instantaneous capacity of the channel, namely, $C(t)$, which means that we have $c_m^* \neq dS(t)/dt$. However, there does not exist any contradiction. Actually, $S(t)$ is continuous and non-differentiable in $[0, t]$, just like the Brown motion process, which is an integration of a Gaussian process. It is easy to understand since the signal obeying Nakagami-m distribution can be viewed as the square root of the summation of $2m$ squared independent Gaussian random variables. Hence, it has the similar property as Brown motion naturally.

To prove *Theorem* 1, we first show that $S(t)$ is a Levy Process. Then, its CF is derived based on the infinitely divisible law for the Levy process. With the CF, the theorem can be proved then.

A. Review of the Levy Process and Infinitely Divisibility

We first review the definition of *Levy process and infinitely divisibility* as below.

***Definition* 1** [10]: $X = \{X(t)\}_{t \geq 0}$ is said to be a *Levy process* if

1) X has independent increments.

2) $X(0)=0$ is satisfied almost surely.

3) X is stochastically continuous, i.e., for $s \geq 0$, $X(t+s) - X(s) \xrightarrow{P} 0$ as $t \to 0$.

4) X is time homogeneous, i.e., for $t \geq 0$, $\mathcal{L}(X(t+s) - X(s))$ does not depend on $s \geq 0$.

5) X is Right Continuous with Left Limits (RCLL) almost surely.

***Definition* 2** [12]: a probability distribution F of a random variable on the real line is *infinitely divisible*, if, for every positive integer n, there exists n *i.i.d.* random variables $X_1, ..., X_n$ whose sum is equivalent to X in distribution. Note that these n random variables do not have to obey the same probability distribution as X.

According to the definition of *Levy process and infinitely divisibility*, the condition of the infinitely divisibility for a Levy process is given by the following proposition.

***Proposition* 1**: *For a random vector Y in $\mathbb{R}^d$, the following three statements are equivalent* (Theorem 1.3 [10]).

1) Y is infinitely divisible.

2) $Y_{n,1} + \cdots + Y_{n,r_n} \xrightarrow{d} Y$ for some *i.i.d.* array $(Y_{n,j})_{n \geq 1, r_n \geq j \geq 1}$, where $r_n \to \infty$.

3) $Y \stackrel{d}{=} X_1$ for some Levy process X in $\mathbb{R}^d$.

Next, we shall show that the service process $\{S(t)\}_{t \geq 0}$ satisfies the five conditions given in *Definition* 1 and it is a Levy process. First, for the channel magnitude gain $g(t)$ which is *i.i.d.* in our discussion, the service process $S(t)$ has independent increments. Second, the condition $S(0) = 0$ is easily satisfied. Third, it is clear that $S(t)$ is stochastically continuous due to the condition $X(t+s) - X(s) = \int_s^{s+t} C(\tau)d\tau \xrightarrow{P} 0$ as $t \to 0$. Besides, $S(t)$ is also time homogeneous since its components $C(\tau)$ is *i.i.d.* over τ. Finally, it is easy to see that $S(t)$ is Right Continuous with Left Limits (RCLL), since there is no leap in $S(t)$ and for any $t \geq 0$, $S(t)$ exists.

Thus, according to *Proposition* 1, the distribution of $S(t)$ is infinite divisible and, for any $n > 0$, $S(t)$ can be decomposed into the sum of n *i.i.d.* random variables.

Now, for sufficiently large N, we define the time resolution $\Delta\tau = \frac{t}{N}$ and obtain the samples $\Delta s_n = C(t_n)\Delta t$ where $t_n = n\Delta\tau, n = 0, 1, \cdots, N$. Then, we get

$$S(t) = \lim_{N \to \infty} \sum_{n=0}^{N} C(t_n)\Delta t = \lim_{N \to \infty} \sum_{n=0}^{N} \Delta s_n, \tag{9}$$

where Δs_n are *i.i.d.* random variables by the same probability distribution, which will be given in Section III-B.

B. Derivation of the CDF of $C(t)$ and Δs_n

To develop the CF of $C(t)$ and Δs_n, we need to derive their Cumulative Distribution Function (CDF) first. More specifically, for the CDF of $C(t)$, it is given by

$$
\begin{aligned}
F_C(c) &= P\{C(t) < c\} \\
&= P\{\gamma(t) < \frac{WN_0 d^\alpha}{P_t}(e^{\frac{c}{W}} - 1)\} \\
&= \frac{1}{\Gamma(m)}(\frac{m}{P_r})^m \int_0^{\frac{WN_0 d^\alpha}{P_t}(e^{\frac{c}{W}} - 1)} \gamma^{m-1} e^{-\frac{m}{P_r}\gamma} d\gamma,
\end{aligned}
\tag{10}
$$

and, for the CDF of Δs_n, it is given by

$$
\begin{aligned}
F_{\Delta s_n}(\Delta s) &= P\{\Delta s_n < \Delta s\} \\
&= F_C(\frac{\Delta s}{\Delta \tau}) \\
&= \frac{1}{\Gamma(m)}(\frac{m}{P_r})^m \int_0^{\frac{WN_0 d^\alpha}{P_t}(e^{\frac{\Delta s}{W\Delta \tau}} - 1)} \gamma^{m-1} e^{-\frac{m}{P_r}\gamma} d\gamma.
\end{aligned}
\tag{11}
$$

According to the CDF expression of Δs_n given in Eqn.(11), we introduce the following proposition.

Proposition 2: *For an arbitrary $\varepsilon > 0$, the equation below*

$$
\lim_{N \to \infty} P\{\Delta s_n > \varepsilon\} = 0
\tag{12}
$$

is always satisfied.

Proof: According to the CDF of Δs_n in Eqn.(11), we get

$$
\begin{aligned}
&\lim_{N \to \infty} P\{\Delta s_n > \varepsilon\} \\
&= \lim_{N \to \infty} \{1 - F_{\Delta s_n}(\varepsilon)\} \\
&= \lim_{\Delta \tau \to 0} \{1 - \frac{1}{\Gamma(m)}(\frac{m}{P_r})^m \int_0^{\frac{WN_0 d^\alpha}{P_t}(e^{\frac{\varepsilon}{W\Delta \tau}} - 1)} \gamma^{m-1} e^{-\frac{m}{P_r}\gamma} d\gamma\} \\
&= 0.
\end{aligned}
\tag{13}
$$

Thus, no leap exists at any time t for $S(t)$, which indicates that $S(t)$ is continuous in mean-square right continuous with left limits. $\square$

Until now, with the CDF of Δs_n, we can derive the CF of $S(t)$ in Section III-C.

C. Derivation of the CF of $S(t)$

First, let us calculate the CF of Δs_n as follows,

$$
\begin{aligned}
\varphi_{\Delta s_n}(\lambda) &= E[e^{i\lambda \Delta s}] \\
&= \int_0^\infty e^{i\lambda \Delta s} dF_{\Delta s_n}(\Delta s) \\
&\stackrel{(a)}{=} \frac{1}{\Gamma(m)}(\frac{mWN_0 d^\alpha}{P_r P_t})^m \int_1^\infty x^{i\lambda W\Delta \tau}(x-1)^{m-1} e^{-\frac{mWN_0 d^\alpha}{P_r P_t}(x-1)} dx \\
&\stackrel{(b)}{=} \frac{1}{\Gamma(m)}(\frac{mWN_0 d^\alpha}{P_r P_t})^m \int_0^\infty (y+1)^{i\lambda W\Delta \tau} y^{m-1} e^{-\frac{mWN_0 d^\alpha}{P_r P_t}y} dy,
\end{aligned}
\tag{14}
$$

where some variable substitutions are used, e.g. $x = e^{\frac{\Delta s}{W\Delta\tau}}$ in (a) and $y = x - 1$ in (b), respectively. Note that $i = \sqrt{-1}$ is the imaginary unit and $\Gamma(m) = \int_0^\infty t^{m-1}e^{-t}dt$ is the Gamma Function.

Next, as $N \to \infty$, we have the following result.

$$
\begin{aligned}
&\lim_{N\to\infty} \frac{1}{\Gamma(m)}(\frac{mWN_0d^\alpha}{P_rP_t})^m \int_0^\infty (y+1)^{i\lambda W\Delta\tau}y^{m-1}e^{-\frac{mWN_0d^\alpha}{P_rP_t}y}dy \\
&= \lim_{\Delta\tau\to 0} \frac{1}{\Gamma(m)}(\frac{mWN_0d^\alpha}{P_rP_t})^m \int_0^\infty (y+1)^{i\lambda W\Delta\tau}y^{m-1}e^{-\frac{mWN_0d^\alpha}{P_rP_t}y}dy \\
&= \frac{1}{\Gamma(m)}(\frac{mWN_0d^\alpha}{P_rP_t})^m \int_0^\infty y^{m-1}e^{-\frac{mWN_0d^\alpha}{P_rP_t}y}dy \\
&\overset{(a)}{=} \frac{1}{\Gamma(m)}(\frac{mWN_0d^\alpha}{P_rP_t})^m (\frac{P_rP_t}{mWN_0d^\alpha})^m \Gamma(m) \\
&= 1,
\end{aligned}
\tag{15}
$$

where the variable substitution $z = \frac{mWN_0d^\alpha}{P_rP_t}y$ is used in (a).

With the CF of Δs_n, to derive the CF of $S(t)$, we first give the following lemma.

Lemma 1: *The following item*

$$
\frac{1}{\Gamma(m)}(\frac{mWN_0d^\alpha}{P_rP_t})^m \int_0^\infty (y+1)^{i\lambda w\Delta\tau}y^{m-1}e^{-\frac{mWN_0d^\alpha}{P_rP_t}y}dy - 1
$$

is an infinitesimal of the same order with $\Delta\tau$, where $\Delta\tau = \frac{t}{N}$.

Proof: To prove one is an infinitesimal of the same order with the other, we need to compute

$$
\begin{aligned}
&\lim_{\Delta\tau\to 0} \frac{\frac{1}{\Gamma(m)}(\frac{mWN_0d^\alpha}{P_rP_t})^m \int_0^\infty (y+1)^{i\lambda W\Delta\tau}y^{m-1}e^{-\frac{mWN_0d^\alpha}{P_rP_t}y}dy - 1}{\Delta\tau} \\
&= \lim_{\Delta\tau\to 0} \frac{1}{\Gamma(m)}(\frac{mWN_0d^\alpha}{P_rP_t})^m \times \\
&\quad \int_0^\infty i\lambda W(y+1)^{i\lambda W\Delta\tau}\ln(y+1)y^{m-1}e^{-\frac{mWN_0d^\alpha}{P_rP_t}y}dy \\
&\overset{(a)}{=} \frac{1}{\Gamma(m)}(\frac{mWN_0d^\alpha}{P_rP_t})^m \cdot i\lambda W \int_0^\infty \ln(y+1)y^{m-1}e^{-\frac{mWN_0d^\alpha}{P_rP_t}y}dy \\
&\overset{(b)}{=} \frac{1}{\Gamma(m)} \cdot i\lambda W \int_0^\infty \ln(\frac{P_rP_t}{mWN_0d^\alpha}z+1)z^{m-1}e^{-z}dz \\
&\overset{(c)}{=} i\lambda W c_0,
\end{aligned}
\tag{16}
$$

where, in (a) we apply the L'Hôpital's rule and in (b), we use variable substitutions $z = \frac{mWN_0d^\alpha}{P_rP_t}y$ and in (c),

$$
c_0 = \frac{1}{\Gamma(m)} \int_0^\infty \ln(\frac{P_rP_t}{mWN_0d^\alpha}z+1)z^{m-1}e^{-z}dz.
\tag{17}
$$

If c_0 is finite, then the limit in (b) should also be finite and the proof is completed. This can be assured by *Lemma* 2. $\quad\square$

Lemma 2: *c_0 is finite and its lower and upper bound are given by*

$$
a \le c_0 \le \frac{P_rP_t}{WN_0d^\alpha},
\tag{18}
$$

where $\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}dt$ is the incomplete Gamma Function and $a = \frac{\Gamma(m,1)}{\Gamma(m)}\ln(\frac{P_rP_t}{mWN_0d^\alpha}+1)$.

Proof: It is easy to see that c_0 is a finite number if it is finitely lower and upper bounded. Firstly, for its lower bound we have

$$c_0 = \frac{1}{\Gamma(m)} \int_0^\infty \ln(\frac{P_r P_t}{mW N_0 d^\alpha} z + 1) z^{m-1} e^{-z} dz$$

$$> \frac{1}{\Gamma(m)} \int_1^\infty \ln(\frac{P_r P_t}{mW N_0 d^\alpha} z + 1) z^{m-1} e^{-z} dz$$

$$\overset{(a)}{\geq} \frac{1}{\Gamma(m)} \int_0^\infty \ln(\frac{P_r P_t}{mW N_0 d^\alpha} \cdot 1 + 1) z^{m-1} e^{-z} dz \qquad (19a)$$

$$= \frac{1}{\Gamma(m)} \Gamma(m, 1) \ln(\frac{P_r P_t}{mW N_0 d^\alpha} + 1)$$

$$\overset{d}{=} a,$$

where (a) comes from the fact that $z \geq 1$.

Similarly, for its upper bound we have

$$c_0 = \frac{1}{\Gamma(m)} \int_0^\infty \ln(\frac{P_r P_t}{mW N_0 d^\alpha} z + 1) z^{m-1} e^{-z} dz$$

$$\overset{(a)}{\leq} \frac{1}{\Gamma(m)} \int_0^\infty \frac{P_r P_t}{mW N_0 d^\alpha} z \cdot z^{m-1} e^{-z} dz \qquad (19b)$$

$$= \frac{P_r P_t}{mW N_0 d^\alpha} \frac{\Gamma(m+1)}{\Gamma(m)}$$

$$\overset{(b)}{=} \frac{P_r P_t}{W N_0 d^\alpha},$$

where we have (a) by inequality $\ln(\frac{P_r P_t}{mW N_0 d^\alpha} z + 1) \leq \frac{P_r P_t}{mW N_0 d^\alpha} z$ and (b) follows $\Gamma(m+1) = m\Gamma(m)$. Based on *Lemma* 1 and *Lemma* 2, it is assured that $(\frac{1}{\Gamma(m)}(\frac{mW N_0 d^\alpha}{P_r P_t})^m \int_0^\infty (y+1)^{i\lambda w \Delta\tau} y^{m-1} e^{-\frac{mW N_0 d^\alpha}{P_r P_t} y} dy - 1)$ and $\Delta\tau$ are of the same order. $\square$

Finally, by using the properties given by (15) and (III-C) in *Lemma* 1, the CF of $S(t)$ can be derived as follows.

$$\varphi_{s(t)}(\lambda) = E[e^{i\lambda s(t)}]$$

$$= \lim_{N \to \infty} [\varphi_{\Delta s_n}(\lambda)]^N$$

$$= \lim_{\Delta\tau \to 0} [\varphi_{\Delta s_n}(\lambda)]^{\frac{t}{\Delta\tau}}$$

$$= \lim_{\Delta\tau \to 0} [\frac{1}{\Gamma(m)}(\frac{mW N_0 d^\alpha}{P_r P_t})^m \cdot \qquad (20)$$

$$\int_0^\infty (y+1)^{i\lambda W \Delta\tau} y^{m-1} e^{-\frac{mW N_0 d^\alpha}{P_r P_t} y} dy]^{\frac{t}{\Delta\tau}}$$

$$\overset{(a)}{=} \lim_{\Delta\tau \to 0} [1 + i\lambda W c_0 \Delta\tau + o(\Delta\tau)]^{\frac{t}{\Delta\tau}}$$

$$\overset{(b)}{=} e^{i\lambda W c_0 t}$$

where in (a), we use the property given by (III-C) and (b) follows the known result $\lim_{x \to 0}(1 + kx)^{\frac{1}{x}} = e^k$.

Besides, the relationship between the moments of a random variable X and its CF $\varphi_X(\lambda)$ is given by

$$E(X) = \frac{1}{i} \varphi_X'(\lambda)|_{\lambda=0}, \qquad (21a)$$

$$E(X^2) = \frac{1}{i^2} \varphi''_X(\lambda)|_{\lambda=0}. \qquad (21b)$$

Then we get the following numerical characteristics of $S(t)$ directly,

$$E(S(t)) = \frac{1}{i}\varphi'_{S(t)}(\lambda)|_{\lambda=0} = Wc_0 t, \tag{22a}$$

$$E(S^2(t)) = \frac{1}{i^2}\varphi''_{S(t)}(\lambda)|_{\lambda=0} = [Wc_0 t]^2. \tag{22b}$$

By using the results given by (22a) and (22b), the variance of $S(t)$ is

$$D(S(t)) = E(S^2(t)) - E^2(S(t)) = 0. \tag{23}$$

It is clear that for any given t, $S(t)$ is a random variable with zero variance, namely,

$$S(t) = E[S(t)] = Wc_0 t. \tag{24}$$

This means that it is a deterministic linear function of t in accord with the expression in (24), where c_0 is given by (17) and the linear coefficient c_m^* is given by (8) in *Theorem* 1. Up to now, we complete the proof of *Theorem* 1.

Next, based on *Theorem* 1, we investigate three special cases of Nakagami-m fading channel, namely $m = 1$, $m = \frac{(K+1)^2}{2K+1}$ and $m = \infty$, which corresponds to the Rayleigh fading channel, the Rician fading channel and the channel with no fading, respectively.

Firstly, for the *i.i.d.* Rayleigh fading channel, we have the following corollary.

Corollary **1**: *For an i.i.d. Rayleigh fading channel, the service process $S(t)$ is a deterministic linear function of time t given by $S(t) = c_{ra}^* \cdot t$, where c_{ra}^* is a constant and given by*

$$c_{ra}^* = We^{\frac{WN_0 d^\alpha}{P_r P_t}}\text{Ei}(\frac{WN_0 d^\alpha}{P_r P_t}), \tag{25}$$

where P_r is the average received signal power, N_0 is the noise power spectral density, W is the received signal bandwidth, P_t is the average transmit power and d is the distance between the transmitter and receiver. The function

$$\text{Ei}(x) = \int_x^{+\infty} \frac{e^{-x}}{x}dx = \eta + \ln x - x + \frac{1}{2}\frac{x^2}{2!} - \frac{1}{3}\frac{x^3}{3!} + \cdots$$

is the exponential integration where the Euler's *constant is $\eta = \int_{+\infty}^0 \frac{e^{-x}}{x}dx = 0.577215655$.*

Proof: It is known that Rayleigh distribution is a special case of Nakagami distribution for $m = 1$. Then,

according to (8), we have

$$
\begin{aligned}
c_{ra}^* &= c_m^*|_{m=1} \\
&= \{\frac{W}{\Gamma(m)} \int_0^\infty \ln(\frac{P_r P_t}{mWN_0 d^\alpha} y + 1) y^{m-1} e^{-y} dy\}|_{m=1} \\
&\overset{(a)}{=} W \cdot \int_0^\infty \ln(\frac{P_r P_t}{WN_0 d^\alpha} y + 1) e^{-y} dy \\
&\overset{(b)}{=} W \frac{P_r P_t}{WN_0 d^\alpha} \int_0^\infty e^{-y} \frac{1}{\frac{P_r P_t}{WN_0 d^\alpha} y + 1} dy \\
&\overset{(c)}{=} W e^{\frac{WN_0 d^\alpha}{P_r P_t}} \int_{\frac{WN_0 d^\alpha}{P_r P_t}}^\infty \frac{e^{-z}}{z} dz \\
&= -W e^{\frac{WN_0 d^\alpha}{P_r P_t}} \text{Ei}(\frac{WN_0 d^\alpha}{P_r P_t}),
\end{aligned}
\tag{26}
$$

where (a) follows $\Gamma(1) = 1$. We apply the integration by parts in (b) and variable substitution $z = y + \frac{P_r P_t}{WN_0 d^\alpha}$ in (c). $\qquad\square$

Secondly, for the *i.i.d.* Rician fading case, we have the following corollary.

***Corollary* 2**: *For an i.i.d. Rician fading channel, the service process $S(t)$ is a deterministic linear function of time t given by $S(t) = c_{ri}^* \cdot t$, where c_{ri}^* is a constant and given by*

$$
c_{ri}^* = \frac{W}{\Gamma(\frac{(K+1)^2}{2K+1})} \int_0^\infty \ln(\frac{P_r P_t}{\frac{(K+1)^2}{2K+1} WN_0 d^\alpha} y + 1) y^{\frac{K^2}{2K+1}} e^{-y} dy,
\tag{27}
$$

where Rician parameter $K = \frac{s^2}{\sigma^2}$ is the ratio between the power in the LOS component and the average non-LOS multipath components, N_0 is the noise power spectral density, W is the received signal bandwidth, P_t is the average transmit power and d is the distance between the transmitter and receiver.

For this corollary, we simply substitute $m = \frac{(K+1)^2}{2K+1}$ in (8) and this complets the proof. Note that we did not get an explicit closed form expression here. Some upper and lower bounds may be needed. We shall discuss it in the future.

Furthermore, for the case when there is no fading, we have the following corollary.

***Corollary* 3**: *For an i.i.d. Nakagami-m fading channel with negligible fading, i.e., $m \to \infty$, the service process $S(t)$ is a deterministic linear function of time t given by $S(t) = c_\infty^* \cdot t$, where c_∞^* is a constant and given by*

$$
c_\infty^* = W \ln(1 + \frac{P_r P_t}{WN_0 d^\alpha}),
\tag{28}
$$

where P_r is the average received signal power, N_0 is the noise power spectral density, P_t is the average transmit power and d is the distance between the transmitter and receiver.

Proof: Let $m \to \infty$ in (8), we have

$$
\begin{aligned}
c_\infty^* &= c_m^*|_{m\to\infty} \\
&= \lim_{m\to\infty} \frac{W}{\Gamma(m)} \int_0^\infty \ln(\frac{P_r P_t}{mWN_0 d^\alpha}y + 1)y^{m-1}e^{-y}dy \\
&\overset{(a)}{\leq} \lim_{m\to\infty} W\ln(\frac{P_r P_t}{mWN_0 d^\alpha}\frac{1}{\Gamma(m)}\int_0^\infty y \cdot y^{m-1}e^{-y}dy + 1) \\
&= \lim_{m\to\infty} W\ln(\frac{P_r P_t}{WN_0 d^\alpha}\frac{1}{m\Gamma(m)}\Gamma(m+1) + 1) \\
&\overset{(b)}{=} W\ln(1 + \frac{P_r P_t}{WN_0 d^\alpha}),
\end{aligned}
\tag{29}
$$

where (a) follows the Jensen' inequality and the unction $\ln(\frac{P_r P_t}{mWN_0 d^\alpha}y + 1)$ is an concave function of y and (b) follows $\Gamma(m+1) = m\Gamma(m)$.

However, we can see from (3b) that

$$
\lim_{m\to\infty} p_\gamma^{(m)} = \delta(\gamma - P_r).
\tag{30}
$$

It is easy to understand because no fading exists when $m \to \infty$ and the channel power gain is a constant $\sqrt{P_r}$ between the transmitter and the receiver. Hence, in (a) of (29), the equality holds. $\square$

It is worthy to be noted that this result can be predicated intuitively in AWGN channel, which further comfired our developed theory in *Theorem* 1.

IV. TIME-LINEARITY OF GENERAL *i.i.d.* FADING CHANNELS

In this section, we will show that the deterministic time-linearity is true for arbitrary *i.i.d.* fading channels.

Assume an *i.i.d.* fading channel with its power gain $\gamma(t)$ following distribution $p_\gamma(\gamma)$. Let $C(t)$ and $S(t)$ be the instantaneous channel capacity and the service process, we have the following theorem.

***Theorem* 2**: *For an arbitrary i.i.d. fading channel with its p.d.f. $p_\gamma(\gamma)$, the service process $S(t)$ is a deterministic linear function of time t given by $S(t) = c_p^* \cdot t$, where c_p^* is a constant and given by*

$$
c_p^* = W \int_0^\infty \ln(\frac{P_t}{WN_0 d^\alpha}y + 1)p_\gamma(y)dy,
\tag{31}
$$

where N_0 is the noise power spectral density, W is the received signal bandwidth, P_t is the average transmit power and d is the distance between the transmitter and receiver.

The proof of *Theorem* 2 is similar to that of *Theorem* 1 and its sketch is provided below.

Firstly, we get the CDF of $C(t)$ and Δs_n by

$$
F_C(c) = \int_0^{\frac{WN_0 d^\alpha}{P_t}(e^{\frac{c}{W}}-1)} p_\gamma(\gamma)d\gamma,
\tag{32a}
$$

$$
F_{\Delta s_n}(\Delta s) = \int_0^{\frac{WN_0 d^\alpha}{P_t}(e^{\frac{\Delta s}{W\Delta\tau}}-1)} p_\gamma(\gamma)d\gamma.
\tag{32b}
$$

Then we derive the CF of Δs_n.

$$\varphi_{\Delta s_n}(\lambda) = E(e^{i\lambda\Delta s_n})$$

$$\overset{(a)}{=} \int_0^\infty e^{i\lambda\Delta s}dF_{\Delta s_n}(\Delta s) \tag{33}$$

$$\overset{(b)}{=} \frac{WN_0 d^\alpha}{P_t} \int_1^\infty x^{i\lambda W\Delta\tau} p_\gamma(\frac{WN_0 d^\alpha}{P_t}(x-1))dx,$$

where in (b) we apply the variable substitution $x = \frac{\Delta s}{W\Delta\tau}$.

From the step (a) in (33), it is easy to see that $\lim_{\Delta\tau\to 0}\varphi_{\Delta s_n}(\lambda) = 1$. With the similar computational procedure as (III-C), we have

$$\lim_{\Delta\tau\to 0}\frac{\varphi_{\Delta s_n}(\lambda) - 1}{\Delta\tau} = i\lambda W \int_0^\infty \ln(\frac{P_t}{WN_0 d^\alpha}y + 1)p_\gamma(y)dy. \tag{34}$$

Finally, the CF of $S(t)$ can be derived as

$$\varphi_{S(t)}(\lambda) = \lim_{N\to\infty}[\varphi_{\Delta s_n}(\lambda)]^N$$

$$= \lim_{\Delta\tau\to 0}[\varphi_{\Delta s_n}(\lambda)]^{\frac{t}{\Delta\tau}}$$

$$= \lim_{\Delta\tau\to 0}[1 + i\lambda W\Delta\tau \int_0^\infty \ln(\frac{P_t}{WN_0 d^\alpha}y + 1)p_\gamma(y)dy + o(\Delta\tau)]^{\frac{t}{\Delta\tau}} \tag{35}$$

$$= \exp[i\lambda Wt \int_0^\infty \ln(\frac{P_t}{WN_0 d^\alpha}y + 1)p_\gamma(y)dy].$$

According to (21), we get $E(S(t)) = c_p^* \cdot t$ and $D(S(t)) = 0$, where c_p^* is given by (31). This means that $S(t) = c_p^* \cdot t$ and completes the proof of *Theorem* 2.

Up to now, we have proved that the deterministic time-linearity nature exists for all kinds of *i.i.d.* fading channels and also derived the linear coefficients i.e., the constant service rate c_p^*, for all kinds of fading channels.

V. NUMERICAL RESULTS AND DISCUSSIONS

To demonstrate the time-linearity of the channel service process $S(t)$, we consider a point to point communication system over an *i.i.d.* fading channel, as shown in Fig. 1, where the average LOS received power P_r is $3dB$, the system bandwidth 1KHz and transmitting power is $15dBW$. Suppose that the distance between the transmitter and the receiver is $1000m$ and the pathloss exponent is 4. In particular, another very important parameter which will greatly affect the simulation is the sampling interval, i.e., $\Delta\tau$. As shown previously, *Theorem* 1 is assured only if $\Delta\tau = t/N \to 0$. Therefore, the sampling interval should be as small as possible, or say, for certain fixed observation duration t, the number of samples, i.e., N, should be as large as possible. We select the sampling interval as $0.1\mu s$ namely, $N \geq 10^7$ samples in one second, which is in good agreement with the parameter in practical communication systems.

We consider the *i.i.d.* Rayleigh fading channel and measure the amount of service $S(t)$ by statistics with an observation duration of $t = 5s$ and run it independently for 500 rounds. It is observed in Fig. 3 that for certain fixed moment such as $t = 5s$, the variance of $S(t)$ is almost zero which means that it is deterministic linear
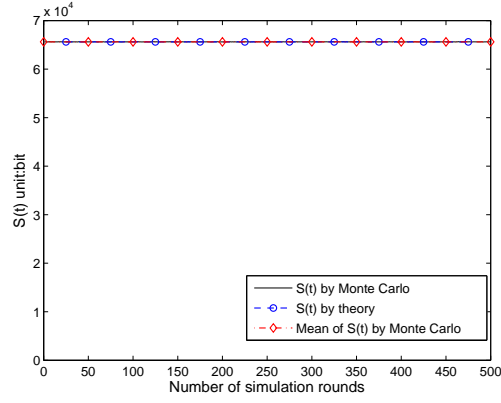
Fig. 3. The Rayleigh channel service process S(t) v.s. number of simulation round. (Observation duration $t = 5$s for each round.)

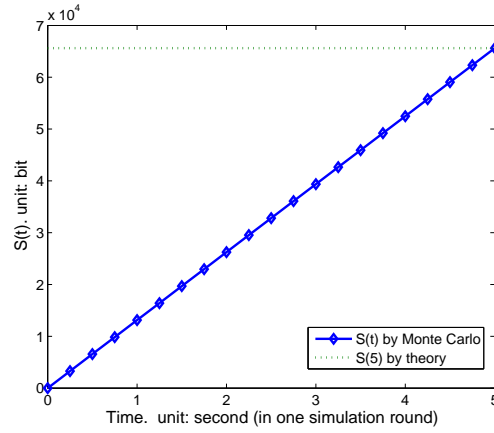Fig. 4. The Rayleigh channel service process S(t) v.s. observation time

function of t. The simulation result also fits the c_{ra}^* by (25) perfectly. It can be further observed in the detail when the vertical coordinate axis is zoomed in that there are still small fluctuations. However, the maximum deviation ratio of $S(t)$ from the average value in all of the simulation rounds is less than 2.6×10^{-5} and will decrease when smaller $\Delta\tau$ is used. Fig.4 illustrates the channel service process $S(t)$ v.s. observation time t. It confirms the deterministic time-linearity of the channel service process $S(t)$ for each time t, which is consistent with our theoretical analysis, namely, the amount of channel service increases linearly with time t.

In Fig.5, we provide the c_m^* v.s. Nakagami parameter m and c_{ri}^* v.s. Rician parameter K. As is known, the Nakagami channel with $m = 1$ and the Rician channel with $K = 0$ both reduce to the Rayleigh channel. It is observed in Fig. 5 that the $c_m^*|_{m=1} = c_{ri}^*|_{K=0}$ and they fit the service provided by a Rayleigh channel in 5 seconds, i.e., $S(t)|_{t=5}$, which is shown in Fig. 3. The AWGN channel capacity c_{awgn} with a average LOS received power Pr is also presented in Fig. 5, and it can be seen that as m (or K) increases, the c_m^* and c_{ri}^* increase accordingly and will convergence to $c_\infty^* = c_{awgn}$, which confirms *Corollary* 2 and *Corollary* 3.
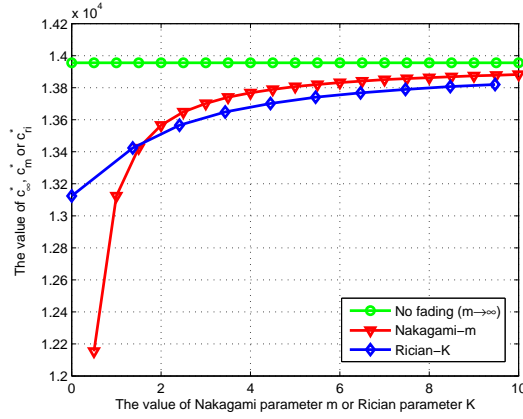
Fig. 5. The c_m^*, c_{ri}^* v.s. m(or K)

By these simulation results, the deterministic time-linearity of the extensively investigated *i.i.d.* fading channels is verified once again. In the paper, $S(t)$ is defined as a stochastic integration and is investigated in the sense of mean square. Therefore, at any time t, $S(t)$ is a constant other than a random variable, so is $\frac{S(t)}{t}$. To make the problem more trackable, we adopt in the paper a sampling method to approximate it, i.e., $S(t) = \lim_{N \to \infty} \sum_1^N C(t_n) \Delta t$. During a sampling time, it is assumed that the channel gain remains unchanged, which is a commonly used processing method for integration. When the sampling interval goes to zero, one can get the integration value. More importantly, such a treating will not change the inherent characteristic of $S(t)$. Besides, we adopt the blocking fading concept for the simplicity of expression. This is similar to the approximation of the Brown motion by random walking, where both the two items are stochastic processes. However, it is really a lucky thing and it can be proved in this paper that the variance of the independent increment process $S(t)$ is zero, i.e., $D[S(t)] = 0$. This is totally different from $D[E[S(t)]] = 0$ or $D[\frac{S(t)}{t}] = 0$ and is a new result. As mentioned previously, there still are some points to be noted. The channel service considered is a concept of large time scales other than the fading property in small scales. In fact, it holds when the ratio of the observation time and the sampling time ($\frac{t}{\Delta t}$) is sufficiently large. And even for a block fading channel, the result holds if the observation time t is sufficiently large. However, if one investigates the channel service on smaller time scales, some physical layer technologies should be used to adapt to the instantaneous channel fluctuations, namely the fading characteristic of the channel. To achieve this goal, buffers must be used at the transmitter. In this way, data can be stored in the buffer at the transmitter when the channel is in bad condition. For this topic, we have obtained some results on the channel utilization and buffer overflow probability, which will be presented in our following works.

VI. Conclusions

This work introduced a new picture of *i.i.d.* fading channels from the viewpoint of the cross layer. That is, we proved that the channel service process $S(t)$ of an *i.i.d.* fading channel is a deterministic linear function of time t, by using the CF approach based on the infinitely divisible law. This work provides some significant insights in both theory and applications. First, different from conventional ergodic capacity or outage capacity, it asserts that the *i.i.d.* fading channel has a deterministic transmission ability. In other words, there exists a coding scheme such that the receiver can decode with zero error probability, if the transmitter maintains a constant data rate no more than c_p^* in the point of view from application layer. Second, in opposite to conventional opinions, this work asserts that the high layer queuing delay is assured to be zero as long as the transmission data rate is less than c_p^*. Otherwise, the queuing delay will be upper-bounded, which is determined by the difference between the transmission data rate and c_p^*.

Acknowledgement

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