Compute-and-Forward Network Coding Design over Multi-Source Multi-Relay Channels

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Abstract

Network coding is a new and promising paradigm for modern communication networks by allowing intermediate nodes to mix messages received from multiple sources. Compute-and-forward strategy is one category of network coding in which a relay will decode and forward a linear combination of source messages according to the observed channel coefficients, based on the algebraic structure of lattice codes. The destination will recover all transmitted messages if enough linear equations are received. In this work, we design in a system level, the compute-and-forward network coding coefficients by Fincke-Pohst based candidate set searching algorithm and network coding system matrix constructing algorithm, such that by those proposed algorithms, the transmission rate of the multi-source multi-relay system is maximized. Numerical results demonstrate the effectiveness of our proposed algorithms.

Index Terms

Compute-and-forward, network coding, linear network coding, lattice codes, cooperative, relay channel.

I. Introduction

Since the pioneering research work of Ahlswede *et al.* in 2000 [1], network coding (NC) has rapidly emerged as a major research area in electrical engineering and computer science. NC is a generalized routing approach that breaks the traditional assumption of simply forwarding data, and allows intermediate nodes to send out functions of their received packets, by which the multicast capacity can be achieved. Subsequent works of [2]-[4] made the important observation that, for multicasting, intermediate nodes can simply send out a linear combination of their received packets. Linear network coding with random coefficients is considered in [5]. Physical layer network coding is presented in [6]. Complex field network coding is proposed in [7]. Several other network coding realizations in wireless networks are discussed in [8]-[12].

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There is also a large body of works on lattice codes [13]-[14] and their applications in communications. For many AWGN networks of interest, lattice codes with linear structure can approach the performance of standard random coding arguments. It has been shown that nested lattice codes (combined with lattice decoding) can achieve the capacity of the point-to-point AWGN channel [15]-[16]. Also, another appealing aspect of linear lattice codes lies in their lower decoding complexity by a class of efficient decoders [17]-[20]. In the two-way relay networks, a nested lattice based strategy has been developed that the achievable rate is near the optimal upper bound [21]-[23].

Recently, a new strategy of compute-and-forward (CPF) [24]-[25], beneficial from both network coding and lattice codes, attracts great attention. The main idea is that a relay will decode a linear function of transmitted messages according to the observed channel coefficients rather than ignoring the interference as noise. Upon utilizing the algebraic structure of lattice codes, i.e. the integer combination of lattice codewords is still a codeword as well, the intermediate relay node decodes and forwards an integer combination of original messages. With enough linear independent equations, the destination can recover the original messages respectively. Subsequent works for design and analysis of the CPF strategy have been given in [26]-[29]. The idea of MIMO compute-and-forward is presented in [30].

Those previous works in CPF only consider the integer network coding coefficients optimization of each relay locally/separately. However, for a multi-source multi-relay system with L sources, the previous separate optimizations cannot guarantee the network coding system matrix, which is constructed by all the integer network coding coefficient vectors, is of rank L such that the destination can decode all messages. In this work, the compute-and-forward network coding strategy is considered in a system level. First, by our proposed Fincke-Pohst [17] based candidate set searching algorithm, instead of one optimal network coding coefficient vector, for each relay we will provide a network coding vector candidate set with corresponding computation rate in descending order. Then, by our proposed network coding system matrix constructing algorithm, we will try to choose network coding vectors from those candidate sets to construct network coding system matrix with rank L, while in the meantime the transmission rate of the multi-source multi-relay system is maximized. The underlying codes are based on lattice codes whose algebraic structure ensures that integer combinations of messages can be decoded reliably.

The notations used in this work are as follows. $\{\cdot\}^T$ denotes the transpose operation, $|\cdot|$ represents the cardinality of a set, $\mathbb{Z}^n$ denotes the n dimensional integer ring, $\mathbb{R}^n$ denotes the n dimensional real field, $\mathbb{F}_p$ denotes a finite field of size p. $\mathbf{I}_n$ denotes the identity matrix of size $n \times n$, and $\mathbf{0}$ denotes the vectors with all zeros elements. Assume that the $\log$ operation is with respect to base 2. We use boldface lowercase letters to denote column vectors and boldface uppercase letters to denote matrices.

II. Multi-Source Multi-Relay Channel

A. System Model

We consider the multi-source multi-relay (MSMR) system model as shown in Fig. 1, where L sources $\mathcal{S}_1$, $\mathcal{S}_2$, $\cdots$, $\mathcal{S}_L$ are communicating to one destination $\mathcal{D}$ through L relays $\mathcal{R}_1$, $\mathcal{R}_2$, $\cdots$, $\mathcal{R}_L$. Each node is equipped with a single antenna and works in half-duplex mode. There are no direct links from sources to the destination.

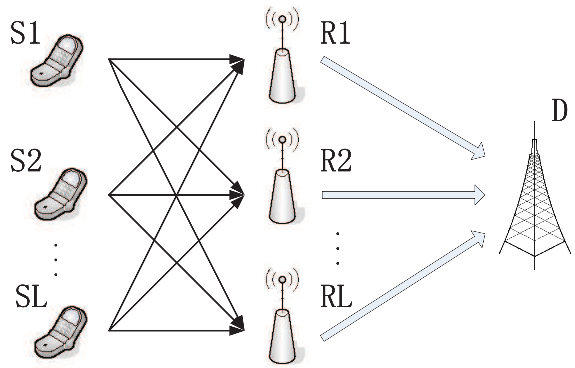

Fig. 1. System model of a MSMR network

The information transmission, which we call one transmission realization, is performed in two phases. The first phase is for the transmissions from all sources $\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_L$ to the relays $\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_L$. Each relay will receive signals from all sources due to the wireless medium. In the second phase, assume each relay has a point-to-point AWGN channel or orthogonal access to the destination, for example, in different time slots as shown in Fig. 2. Every relay will obtain a linear combination of original messages and forward towards the destination by orthogonal channels. With enough linear combinations, the destination is able to recover the desired original messages from all sources.

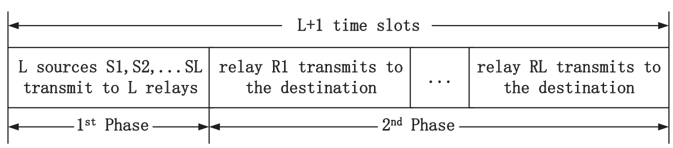

Fig. 2. Time division allocation for one transmission realization

Without loss of generality, in one transmission realization, each source has a length-k message vector that is drawn independently and uniformly over a prime size finite field,

$$\mathbf{w}_l \in \mathbb{F}_p^k, \quad l = 1, 2, \cdots, L, \tag{1}$$

where $\mathbb{F}_p$ denotes the finite field with a set of p elements. Each source is equipped with an encoder $\Psi_l : \mathbb{F}_p^k \to \mathbb{R}^n$ that maps the length-k message $\mathbf{w}_l$ into a length-n real valued lattice codeword $\mathbf{x}_l = \Psi_l(\mathbf{w}_l)$. The lattice codeword $\mathbf{x}_l$ must satisfy the power constraint, $\frac{1}{n}||\mathbf{x}_l||^2 \leq P$ for $P \geq 0$ and $l = 1, 2, \cdots, L$. The message rate, defined as the length of the message measured in bits normalized by the number of channel uses $R = \frac{k}{n} \log p$ [24], is the same for all sources.

After mapping its message $\mathbf{w}_l \in \mathbb{F}_p^k$ into a lattice codeword $\mathbf{x}_l \in \mathbb{R}^n$, the source $\mathcal{S}_l$ will send the codeword $\mathbf{x}_l$ across the channel. Due to the broadcast nature of wireless medium, the m-th relay will observe a noisy combination of the transmitted signals at the end of the first phase,

$$\mathbf{y}_m = \sum_{l=1}^{L} h_{ml}\mathbf{x}_l + \mathbf{z}_m, \qquad m = 1, 2, \cdots, L, \tag{2}$$

where $h_{ml} \in \mathbb{R}$ denotes real valued fading channel coefficient from $\mathcal{S}_l$ to relay $\mathcal{R}_m$, generated i.i.d. according to a normal distribution $\mathcal{N}(0,1)$; $\mathbf{z}_m \in \mathbb{R}^n$ denotes additive Gaussian noise vector, $\mathbf{z}_m \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n)$. Let

$$\mathbf{h}_m = [h_{m1}, \cdots, h_{mL}]^T \tag{3}$$

denote the vector of channel coefficients from all sources to the m-th relay. We assume this channel state information $\mathbf{h}_m$ is available at relay m.

B. *Compute-and-Forward Scheme*

In a recent work, Nazer and Gastpar propose the *compute-and-forward* approach [24] which exploits the property that any integer combination of lattice points is again a lattice point. After receiving the noisy vector $\mathbf{y}_m$ of (2), the m-th relay will first select a scalar $\beta_m \in \mathbb{R}$ and an integer network coding coefficient vector $\mathbf{a}_m = [a_{m1}, a_{m2}, \cdots, a_{mL}]^T \in \mathbb{Z}^L$, then attempt to decode the lattice point $\sum_{l=1}^{L} a_{ml}\mathbf{x}_l$ from

$$\beta_m \mathbf{y}_m = \sum_{l=1}^{L} \beta_m h_{ml}\mathbf{x}_l + \beta_m \mathbf{z}_m \tag{4}$$

$$= \sum_{l=1}^{L} a_{ml}\mathbf{x}_l + \underbrace{\sum_{l=1}^{L} (\beta_m h_{ml} - a_{ml})\,\mathbf{x}_l + \beta_m \mathbf{z}_m}_{\textit{Effective Noise}}. \tag{5}$$

Note that we do not need to conduct joint maximum likelihood (ML) decoding to get $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \cdots, \hat{\mathbf{x}}_L)$ for network coding. Instead we decode $\sum_{l=1}^{L} a_{ml}\mathbf{x}_l$ as one regular codeword due to the lattice algebraic structure. In other words, the network coded codeword is still in the same field as original source codeword.

In the finite field, it is equivalent that each relay is desired to reliably recover a linear combination of the messages,

$$\mathbf{u}_m = \bigoplus_{l=1}^{L} q_{ml}\mathbf{w}_l = \left[\sum_{l=1}^{L} a_{ml}\mathbf{w}_l\right] mod\ p, \tag{6}$$

where $\bigoplus$ denotes summation over the finite field, q_{ml} is a coefficient taking values in $\mathbb{F}_p$ and $q_{ml} = a_{ml}\ mod\ p$.

Each relay is equipped with a decoder, $\Pi_m : \mathbb{R}^n \to \mathbb{F}_p^k$, that maps the observed channel output $\mathbf{y}_m \in \mathbb{R}^n$ to an estimate $\hat{\mathbf{u}}_m = \Pi_m(\mathbf{y}_m) \in \mathbb{F}_p^k$ of the message combination $\mathbf{u}_m$. The diagram of compute-and-forward scheme is given in Fig. 3.

We are interested in the rate of $\sum_{l=1}^{L} a_{ml}\mathbf{x}_l$ as a whole and will capture the performance of the computation scheme by what we refer to as the *computation rate*, namely, the number of bits of the linear function successfully recovered per channel use. The work of [24] shows that a relay can often recover an equation of messages at a higher rate than any individual message (or subset of message). The rate is highest when the equation coefficients closely approximate the effective channel coefficients. The formal statements are given in the following theorems

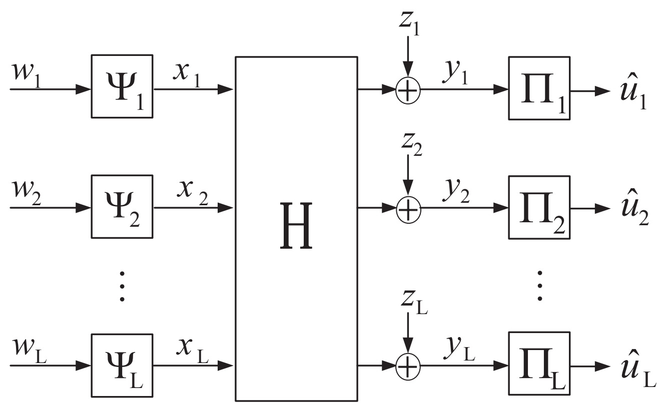

Fig. 3. Compute-and-Forward Diagram

[24]-[26]. Let $\log^+(x) \triangleq \max(\log(x), 0)$.

Theorem 2.1: For real-valued AWGN networks with channel coefficient vector $\mathbf{h}_m \in \mathbb{R}^L$ and desired network coding coefficient vector $\mathbf{a}_m \in \mathbb{Z}^L$, the following computation rate is achievable

$$\mathscr{R}_m(\mathbf{a}_m) = \max_{\beta_m \in \mathbb{R}} \frac{1}{2} \log^+ \left(\frac{P}{\beta_m^2 + P||\beta_m \mathbf{h}_m - \mathbf{a}_m||^2} \right). \tag{7}$$

Theorem 2.2: The computation rate given in Theorem 2.1 is uniquely maximized by choosing β_m to be the MMSE coefficient

$$\beta_{MMSE} = \frac{P\,\mathbf{h}_m^T \mathbf{a}_m}{1 + P||\mathbf{h}_m||^2}, \tag{8}$$

which results in a computation rate of

$$\mathscr{R}_m(\mathbf{a}_m) = \frac{1}{2} \log^+ \left(||\mathbf{a}_m||^2 - \frac{P(\mathbf{h}_m^T \mathbf{a}_m)^2}{1 + P||\mathbf{h}_m||^2} \right)^{-1}. \tag{9}$$

Theorem 2.3: For a given channel coefficient vector $\mathbf{h}_m = [h_{m1}, h_{m2}, \cdots, h_{mL}]^T \in \mathbb{R}^L$, $\mathscr{R}_m(\mathbf{a}_m)$ is maximized by choosing the integer network coding coefficient vector $\mathbf{a}_m \in \mathbb{Z}^L$ as

$$\mathbf{a}_m = arg \min_{\mathbf{a}_m \in \mathbb{Z}^L, \mathbf{a}_m \neq \mathbf{0}} \left(\mathbf{a}_m^T \mathbf{G}_m \mathbf{a}_m \right), \tag{10}$$

where

$$\mathbf{G}_m \triangleq \mathbf{I} - \frac{P}{1 + P\,||\mathbf{h}_m||^2} \mathbf{H}_m, \tag{11}$$

and $\mathbf{H}_m = [H_{ij}^{(m)}]$, $H_{ij}^{(m)} = h_{mi} h_{mj}$, $1 \leq i, j \leq L$.

C. Problem Statement

Theorems 2.1-2.3 only give the optimal network coding integer coefficient vector $\mathbf{a}_m$ and achievable computation rate $\mathscr{R}_m$ for each relay locally/separately and do not take consideration of the overall system constraints. For the multi-source multi-relay system, at the destination, enough linear combinations of the original messages need to be collected. Let $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots$, $\mathbf{a}_L$ be the integer network coding coefficients vector for each relay, then the network coding system matrix $\mathbf{A}$ at the destination can be denoted as

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_L]^T . \tag{12}$$

Hence, the destination can solve for the original packets if the network coding system matrix $\mathbf{A}$ has full rank L, i.e. $|\mathbf{A}| \neq 0$. In which case, as the same rate of source-relay channels in phase I is available for relay-destination channels in phase II, the transmission rate at the destination is dominated/bottlenecked by

$$\mathscr{R}_D = \min\{\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_L\} . \tag{13}$$

We can easily understand that after calculating the integer network coding coefficient vector $\mathbf{a}_m$ for each relay by theorems 2.1-2.3 to maximize its own computation rate, the network coding system matrix $\mathbf{A}$ constructed by those integer vectors may not have full rank L, in which case the destination cannot decode the original messages by those linear equations. In other words, we cannot fix the optimal integer network coding vector $\mathbf{a}_m$ for each relay separately, since it cannot guarantee that the system constraint of full rank $\mathbf{A}$.

Therefore, we need to optimize the integer network coding vectors for L relays in a overall system level. Instead of distributed calculations, to construct the full rank network coding system matrix that maximize the overall message rate at destination, $\mathbf{A}$ will be designed according to the following criteria

$$\begin{aligned} \mathbf{A} &= arg \max_{|\mathbf{A}|\neq 0} \mathscr{R}_D \\ &= arg \max_{|\mathbf{A}|\neq 0} \left(\min\{\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_L\}\right) \\ &= arg \max_{|\mathbf{A}|\neq 0} \min_{m=1,\cdots L} \left(\frac{1}{2}\log^+\left(||\mathbf{a}_m||^2 - \frac{P(\mathbf{h}_m^T\mathbf{a}_m)^2}{1+P||\mathbf{h}_m||^2}\right)^{-1}\right). \end{aligned} \tag{14}$$

In other words, we need to find the integer network coding vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots$, $\mathbf{a}_L$, under the system level constraint of full rank $\mathbf{A}$, to maximize the computation rate of each relay $\mathscr{R}_1$, $\mathscr{R}_2$, $\cdots$, $\mathscr{R}_L$ jointly, such that the minimum value of $\mathscr{R}_1$, $\mathscr{R}_2$, $\cdots$, $\mathscr{R}_L$ is maximized.

Equivalently, the optimum network coding system matrix $\mathbf{A}$ should be

$$\mathbf{A} = arg \min_{|\mathbf{A}|\neq 0} \max_{m=1,\cdots L} \mathbf{a}_m^T \mathbf{G}_m \mathbf{a}_m, \tag{15}$$

where $\mathbf{G}_m$ is defined in (11).

III. Proposed Strategy

In this work, to approach the overall system optimization of (14)-(15), we propose the following novel strategy which includes two steps. In the first step, for relay m, instead of finding one optimal network coding coefficient vector $\mathbf{a}_m$ to maximize its own computation rate, we are trying to find a candidate set

$$\Omega_m^{T_{max}} = \{\mathbf{a}_m^{(1)}, \mathbf{a}_m^{(2)}, \cdots, \mathbf{a}_m^{(T_{max})}\}, \tag{16}$$

with $|\Omega_m^{T_{max}}| = T_{max}$. The network coding coefficient vectors with the top T_{max} maximum computation rates for relay m are within the candidate set $\Omega_m^{T_{max}}$. Note that T_{max} is a parameter to control the candidate set length for each relay and currently set by experience/simulation. We will propose an algorithm based on Fincke-Pohst Method [17] to find the network coding coefficient vector candidate set for each relay.

After we get all the candidate vector sets $\Omega_1^{T_{max}}$, $\Omega_2^{T_{max}}$, $\cdots$, $\Omega_L^{T_{max}}$, in the second step, we will try to pick up $\mathbf{a}_1 \in \Omega_1^{T_{max}}$, $\mathbf{a}_2 \in \Omega_2^{T_{max}}$, $\cdots$, $\mathbf{a}_L \in \Omega_L^{T_{max}}$, to construct the full rank network coding coefficient matrix $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_L]^T$, while in the meantime, the minimum value of corresponding $\mathscr{R}_1(\mathbf{a}_1)$, $\mathscr{R}_2(\mathbf{a}_2)$, $\cdots$, $\mathscr{R}_L(\mathbf{a}_L)$ is maximized.

A. Searching Candidate Set $\Omega_m^{T_{max}}$ for Each Relay

For relay m, we are trying to find the candidate set $\Omega_m^{T_{max}} = \{\mathbf{a}_m^{(1)}, \mathbf{a}_m^{(2)}, \cdots, \mathbf{a}_m^{(T_{max})}\}$ with $|\Omega_m^{T_{max}}| = T_{max}$, such that the network coding coefficient vectors with the top T_{max} maximum computation rate for relay m are within. According to Theorem 2.3, it is equivalent to find the set $\Omega_m^{T_{max}}$ with T_{max} vectors, such that those vectors give the bottom T_{max} minimum $\mathbf{a}_m^T \mathbf{G}_m \mathbf{a}_m$ values, where $\mathbf{G}_m$ is defined in (11).

The searching of candidate set Ω_m^{max} with fixed length T_{max} can be decomposed into following steps.

(1) Enumerate all vectors $\mathbf{t} \in \mathbb{Z}^L$ ($\mathbf{t} \neq \mathbf{0}$) in Ω_m, such that $\mathbf{t}^T \mathbf{G}_m \mathbf{t} \leq C$ for a given positive constant C, i.e.,

$$\Omega_m = \left\{ \mathbf{t} : \ \mathbf{t}^T \mathbf{G}_m \mathbf{t} \leq C, \ \mathbf{t} \neq \mathbf{0}, \ \mathbf{t} \in \mathbb{Z}^L \right\}. \tag{17}$$

(2) Adjust the constant C to guarantee that $|\Omega_m| \geq T_{max}$.

(3) Sort all the vectors $\mathbf{t}_1$, $\mathbf{t}_2$, $\cdots$, $\mathbf{t}_{|\Omega_m|}$ in Ω_m in descending order corresponding to the computation rate value $\mathscr{R}_m$ in (9), such that

$$\mathscr{R}_m(\mathbf{t}_1) \geq \mathscr{R}_m(\mathbf{t}_2) \geq \cdots \geq \mathscr{R}_m(\mathbf{t}_{|\Omega_m|}). \tag{18}$$

(4) Pick the first T_{max} vectors of Ω_m to form the set $\Omega_m^{T_{max}}$.

The procedure of enumerating all vectors $\mathbf{t} \in \mathbb{Z}^L$ ($\mathbf{t} \neq \mathbf{0}$) in Ω_m, such that $\mathbf{t}^T \mathbf{G}_m \mathbf{t} \leq C$ for a given positive constant C is based on the Fincke-Pohst Method and derived as follows.

We operate Cholesky's factorization of matrix $\mathbf{G}_m$, $\mathbf{G}_m = \mathbf{U}^T \mathbf{U}$, where $\mathbf{U}$ is an upper triangular matrix. Denote $||\cdot||_F$ for the Frobenius norm. Let u_{ij}, $i, j = 1, 2, \cdots, L$, be the entries of the upper triangular matrix $\mathbf{U}$ and $\mathbf{t} = [t_1, t_2, \cdots, t_L]^T$. Then, the searching vector $\mathbf{t}$ that makes $\mathbf{t}^T \mathbf{G}_m \mathbf{t} \leq C$ can be expressed as

$$\begin{aligned} \mathbf{t}^T \mathbf{G}_m \mathbf{t} &= ||\mathbf{U}\ \mathbf{t}||_F^2 = \sum_{i=1}^{L} \left(u_{ii} t_i + \sum_{j=i+1}^{L} u_{ij} t_j \right)^2 \\ &= \sum_{i=1}^{L} g_{ii} \left(t_i + \sum_{j=i+1}^{L} g_{ij} t_j \right)^2 \\ &= \sum_{i=k}^{L} g_{ii} \left(t_i + \sum_{j=i+1}^{L} g_{ij} t_j \right)^2 + \sum_{i=1}^{k-1} g_{ii} \left(t_i + \sum_{j=i+1}^{L} g_{ij} t_j \right)^2 \\ &\leq C \end{aligned} \tag{19}$$

where $g_{ii} = u_{ii}^2$ and $g_{ij} = u_{ij}/u_{ii}$ for $i = 1, 2, \cdots, L$, $j = i+1, \cdots, L$. Obviously the second term of (19) is non-negative, hence, to satisfy (19), it is equivalent to consider for every $k = L, L-1, \cdots, 1$,

$$\sum_{i=k}^{L} g_{ii} \left(t_i + \sum_{j=i+1}^{L} g_{ij} t_j \right)^2 \leq C. \tag{20}$$

Then, we can start work backwards to find the bounds for vector entries $t_L, t_{L-1}, \cdots, t_1$ one by one.

We begin to evaluate the last element t_L of the searching vector $\mathbf{t}$. Referring to (20) and let $k = L$, we have

$$g_{LL} t_L^2 \leq C. \tag{21}$$

Set $\Delta_L = 0$, $C_L = C$, and we will get

$$LB_L \leq t_L \leq UB_L, \tag{22}$$

with

$$UB_L = \left\lfloor \sqrt{\frac{C_L}{g_{LL}}} - \Delta_L \right\rfloor, LB_L = \left\lceil -\sqrt{\frac{C_L}{g_{LL}}} - \Delta_L \right\rceil, \tag{23}$$

where $\lceil x \rceil$ is the smallest integer no less than x and $\lfloor x \rfloor$ is the greatest integer no bigger than x.

Next, we evaluate the element t_{L-1} of the searching vector $\mathbf{t}$. Referring to (20) and let $k = L-1$, we have

$$g_{LL} t_L^2 + g_{L-1,L-1} \left(t_{L-1} + g_{L-1,L} t_L \right)^2 \leq C, \tag{24}$$

which leads to

$$\left\lceil -\sqrt{\frac{C - g_{LL} t_L^2}{g_{L-1,L-1}}} - g_{L-1,L} t_L \right\rceil \leq t_{L-1} \leq \left\lfloor \sqrt{\frac{C - g_{LL} t_L^2}{g_{L-1,L-1}}} - g_{L-1,L} t_L \right\rfloor. \tag{25}$$

If we denote $\Delta_{L-1} = g_{L-1,L} t_L$, $C_{L-1} = C - g_{LL} t_L^2$, the bounds for s_{L-1} can be expressed as

$$LB_{L-1} \leq t_{L-1} \leq UB_{L-1}, \tag{26}$$

where

$$UB_{L-1} = \left\lfloor \sqrt{\frac{C_{L-1}}{g_{L-1,L-1}}} - \Delta_{L-1} \right\rfloor, LB_{L-1} = \left\lceil -\sqrt{\frac{C_{L-1}}{g_{L-1,L-1}}} - \Delta_{L-1} \right\rceil. \tag{27}$$

We can see that given radius $\sqrt{C}$ and matrix $\mathbf{U}$, the bounds for t_{L-1} only depends on the previous evaluated t_L, and not correlated with $t_{L-2}, t_{L-3}, \cdots, t_1$.

In a similar fashion, we can proceed for t_{L-2} evaluation, and so on.

To evaluate the element t_k of the searching vector $\mathbf{t}$, referring to (20) we will have

$$\sum_{i=k}^{L} g_{ii} \left(t_i + \sum_{j=i+1}^{L} g_{ij} t_j \right)^2 \leq C, \tag{28}$$

which leads to

$$\begin{aligned} &\left\lceil -\sqrt{\tfrac{1}{g_{kk}} \left(C - \textstyle\sum_{i=k+1}^{L} g_{ii} \left(t_i + \sum_{j=i+1}^{L} g_{ij} t_j \right)^2 \right)} - \textstyle\sum_{j=k+1}^{L} g_{kj} t_j \right\rceil \\ &\leq t_k \leq \left\lfloor \sqrt{\tfrac{1}{g_{kk}} \left(C - \textstyle\sum_{i=k+1}^{L} g_{ii} \left(t_i + \sum_{j=i+1}^{L} g_{ij} t_j \right)^2 \right)} - \textstyle\sum_{j=k+1}^{L} g_{kj} t_j \right\rfloor. \end{aligned}$$

If we denote

$$\begin{aligned}\Delta_k &= \sum_{j=k+1}^{L} g_{kj} t_j, \\ C_k &= C - \sum_{i=k+1}^{L} g_{ii} \left(t_i + \sum_{j=i+1}^{L} g_{ij} t_j \right)^2, \end{aligned} \tag{29}$$

the bounds for s_k can be expressed as

$$LB_k \leq t_k \leq UB_k, \tag{30}$$

where

$$UB_k = \left\lfloor \sqrt{\frac{C_k}{g_{kk}}} - \Delta_k \right\rfloor, LB_k = \left\lceil -\sqrt{\frac{C_k}{g_{kk}}} - \Delta_k \right\rceil. \tag{31}$$

Note that for given radius $\sqrt{C}$ and matrix $\mathbf{U}$, the bounds for t_k only depends on the previous evaluated $t_{k+1}, t_{k+2}, \cdots, t_L$.

Finally, we evaluate the element t_1 of the searching vector $\mathbf{t}$. Referring to (20) and let $k = 1$, we will have

$$\sum_{i=1}^{L} g_{ii} \left(t_i + \sum_{j=i+1}^{L} g_{ij} t_j \right)^2 \leq C, \tag{32}$$

which leads to

$$\begin{aligned} &\left\lceil -\sqrt{\tfrac{1}{g_{11}} \left(C - \textstyle\sum_{i=2}^{L} g_{ii} \left(t_i + \sum_{j=i+1}^{L} g_{ij} t_j \right)^2 \right)} - \textstyle\sum_{j=2}^{L} g_{1j} t_j \right\rceil \\ &\leq t_1 \leq \left\lfloor \sqrt{\tfrac{1}{g_{11}} \left(C - \textstyle\sum_{i=2}^{L} g_{ii} \left(t_i + \sum_{j=i+1}^{L} g_{ij} t_j \right)^2 \right)} - \textstyle\sum_{j=2}^{L} g_{1j} t_j \right\rfloor \end{aligned} \tag{33}$$

If we denote

$$\begin{aligned}\Delta_1 &= \sum_{j=2}^{L} g_{1j} t_j, \\ C_1 &= C - \sum_{i=2}^{L} g_{ii} \left(t_i + \sum_{j=i+1}^{L} g_{ij} t_j \right)^2, \end{aligned} \tag{34}$$

the bounds for t_1 can be expressed as

$$LB_1 \leq t_1 \leq UB_1, \tag{35}$$

where

$$UB_1 = \left\lfloor \sqrt{\frac{C_1}{g_{11}}} - \Delta_1 \right\rfloor, \quad LB_1 = \left\lceil -\sqrt{\frac{C_1}{g_{11}}} - \Delta_1 \right\rceil. \tag{36}$$

In practice, C_L, C_{L-1}, $\cdots$, C_1 can be updated recursively by the following equations

$$\Delta_k = \sum_{j=k+1}^{L} g_{kj} t_j, \tag{37}$$

$$\begin{aligned} C_k &= C - \sum_{i=k+1}^{L} g_{ii} \left(t_i + \sum_{j=i+1}^{L} g_{ij} t_j \right)^2 \\ &= C_{k+1} - g_{k+1,k+1} \left(\Delta_{k+1} + t_{k+1} \right)^2, \end{aligned} \tag{38}$$

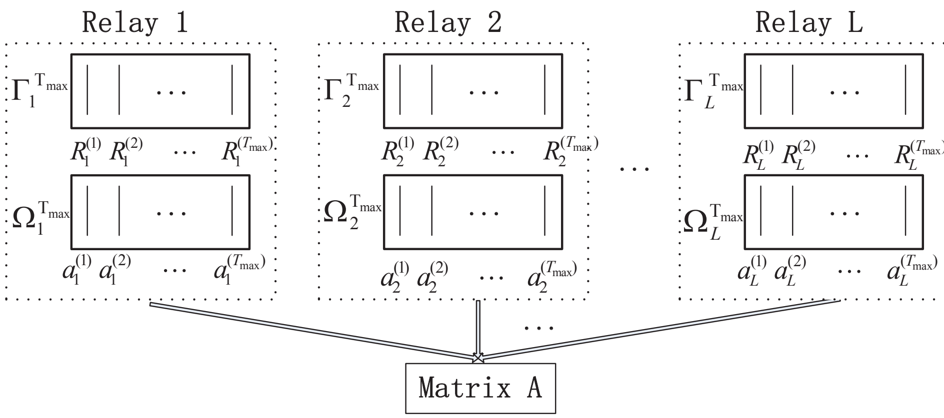

Fig. 4. Candidate sets and rate tables for all relays

for $k = L-1, L-2, \cdots, 1$ and $\Delta_L = 0$, $C_L = C$.

The entries $t_L, t_{L-1}, \cdots, t_1$ are chosen as follows: for a chosen candidate of t_L satisfying the bounds (22)-(23), we can choose a candidate for t_{L-1} satisfying the bounds (26)-(27). If a candidate value for t_{L-1} does not exist, we go back to (22)-(23) and choose other candidate value t_L. Then search for t_{L-1} that meets the bounds (26)-(27) for the given t_L. If t_L and t_{L-1} are chosen as candidates, we follow the same procedure to choose t_{L-2}, and so on. When a set of $t_L, t_{L-1}, \cdots, t_1$ is chosen and satisfies all corresponding bounds requirements, one candidate vector $\mathbf{t} = [t_1, t_2, \cdots, t_L]^T$ is obtained. We record all the candidate vectors satisfying their bounds requirements, such that all vectors meet $\mathbf{t}^T \mathbf{G}_m \mathbf{t} \leq C$ will be in Ω_m.

Regarding the setting of positive constant C, we will set it based on the binary vector obtained by applying the direct sign operator of the real minimum-eigenvalue eigenvector of $\mathbf{G}_m$, denoted as $\mathbf{t}_{quant}$, such that

$$C = \mathbf{t}_{quant}^T \mathbf{G}_m \ \mathbf{t}_{quant}. \tag{39}$$

By setting the searching sphere radius this way, it is big enough to have at least one searching vector $\mathbf{t}_{quant}$ falls inside, while in the meantime small enough to have not too many searching vectors within.

Note that this searching procedure will return *all* candidates that satisfy $\mathbf{t}^T \mathbf{G}_m \mathbf{t} \leq C$. There is at least one candidate vector $\mathbf{t}_{quant}$ such that its entries satisfy all the bounds requirements. On the other hand, the maximum likelihood (ML) exhaustive search among all $\mathbf{t} \in \mathbb{Z}^L$, with optimal result $\mathbf{t}_{ML}$ that returns the minimum metric $\mathbf{t}^T \mathbf{G}_m \mathbf{t}$, or equivalently maximum the computation rate for one relay, will also fall inside the search bounds, since

$$\mathbf{t}_{ML}^T \mathbf{G}_m \mathbf{t}_{ML} \leq \mathbf{t}_{quant}^T \mathbf{G}_m \ \mathbf{t}_{quant} = C. \tag{40}$$

Hence, we are guaranteed to include the local optimal network coding coefficient vector, which maximizes the computation rate for one relay m, in $\Omega_m^{T_{max}}$.

We summarize our proposed algorithm for the searching candidate set $\Omega_m^{T_{max}}$ for relay m based on Fincke-Pohst method as follows.

Algorithm 1 FP Based Candidate Set Searching Algorithm

Input: Matrix $\mathbf{G}_m$, $T_{max} = |\Omega_m^{T_{max}}|$.

Output: The candidate vector set $\Omega_m^{T_{max}}$ and corresponding computation rate set $\Gamma_m^{T_{max}}$.

Step 1: Calculate the binary quantized vector obtained by applying the direct sign operator of the real minimum-eigenvalue eigenvector of $\mathbf{G}_m$, denoted as $\mathbf{t}_{quant}$, and set C as

$$C = \mathbf{t}_{quant}^T \mathbf{G}_m \, \mathbf{t}_{quant}. \tag{41}$$

Step 2: Operate Cholesky's factorization of matrix $\mathbf{G}_m$, $\mathbf{G}_m = \mathbf{U}^T\mathbf{U}$, where $\mathbf{U}$ is an upper triangular matrix. Let u_{ij}, $i, j = 1, 2, \cdots, L$ denote the entries of matrix $\mathbf{U}$. Set

$$g_{ii} = u_{ii}^2, \qquad g_{ij} = u_{ij}/u_{ii},$$

for $i = 1, 2, \cdots, L$, $j = i+1, \cdots, L$.

Step 3: Search set $\Omega_m = \left\{\mathbf{t} : \ \mathbf{t}^T\mathbf{G}_m\mathbf{t} \le C, \ \mathbf{t} \ne \mathbf{0}, \ \mathbf{t} \in \mathbb{Z}^L\right\}$ according to the following Fincke-Pohst procedure.

(i) Start from $\Delta_L = 0$, $C_L = C$, $k = L$ and $\Omega_m = \emptyset$.

(ii) Set the upper bound UB_k and the lower bound LB_k as follows

$$UB_k = \left\lfloor \sqrt{\frac{C_k}{g_{kk}}} - \Delta_k \right\rfloor, LB_k = \left\lceil -\sqrt{\frac{C_k}{g_{kk}}} - \Delta_k \right\rceil,$$

and $t_k = LB_k - 1$.

(iii) Set $t_k = t_k + 1$. For $t_k \le UB_k$, go to (v); else go to (iv).

(iv) If $k = L$, terminate and output Ω_m; else set $k = k+1$ and go to (iii).

(v) For $k = 1$, go to (vi); else set $k = k-1$, and

$$\begin{aligned} \Delta_k &= \sum_{j=k+1}^{L} g_{kj} t_j, \\ C_k &= C_{k+1} - g_{k+1,k+1}\left(\Delta_{k+1} + t_{k+1}\right)^2, \end{aligned}$$

then go to (ii).

(vi) If $\mathbf{t} = \mathbf{0}$ terminate, else we get a candidate vector $\mathbf{t} \ne \mathbf{0}$ that satisfies all the bounds requirements and put it inside Ω_m, i.e. $\Omega_m = \{\Omega_m, \mathbf{t}\}$. Go to (iii).

Step 4: If $|\Omega_m| < T_{max}$, set $C = 2C$ and repeat Step 3.

Step 5: Sort all the vectors $\mathbf{t}_1$, $\mathbf{t}_2$, $\cdots$, $\mathbf{t}_{|\Omega_m|}$ in Ω_m in descending order corresponding to the computation rate value $\mathscr{R}_m$ in (9), such that

$$\mathscr{R}_m(\mathbf{t}_1) \ge \mathscr{R}_m(\mathbf{t}_2) \ge \cdots \ge \mathscr{R}_m(\mathbf{t}_{|\Omega_m|}). \tag{42}$$

Pick the first T_{max} vectors of Ω_m to form the set $\Omega_m^{T_{max}}$ and construct the corresponding computation rate $\Gamma_m^{T_{max}}$ as

$$\begin{cases} \Omega_m^{T_{max}} & = & \{\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_{T_{max}}\}, \\ \Gamma_m^{T_{max}} & = & \{\mathscr{R}_m(\mathbf{t}_1), \mathscr{R}_m(\mathbf{t}_2), \cdots, \mathscr{R}_m(\mathbf{t}_{T_{max}})\}. \end{cases} \tag{43}$$

B. Constructing Network Coding Matrix $\mathbf{A}$

According to our proposed FP Based Candidate Set $\Omega_m^{T_{max}}$ Searching Algorithm 1, for relay m, we get the candidate set $\Omega_m^{T_{max}}$ for integer network coding coefficient vector $\mathbf{a}_m$. The set $\Omega_m^{T_{max}}$ consists T_{max} candidates vectors $\Omega_m^{T_{max}} = \{\mathbf{a}_m^{(1)}, \mathbf{a}_m^{(2)}, \cdots, \mathbf{a}_m^{(T_{max})}\}$, in which $\mathbf{a}_m^{(1)}$, $\mathbf{a}_m^{(2)}$, $\cdots$, $\mathbf{a}_m^{(T_{max})}$ have been sorted such that $\mathscr{R}_m(\mathbf{a}_m^{(1)}) \geq \mathscr{R}_m(\mathbf{a}_m^{(2)}) \geq \cdots \geq \mathscr{R}_m(\mathbf{a}_m^{(T_{max})})$. Denote $\mathscr{R}_m^{(i)} = \mathscr{R}_m(\mathbf{a}_m^{(i)}), i = 1, 2, \cdots, T_{max}$. Then for each relay we can have two length-T_{max} tables as shown in Fig. 4,

$$\textit{Table 1:} \qquad \Gamma_m^{T_{max}} = \{\mathscr{R}_m^{(1)}, \mathscr{R}_m^{(2)}, \cdots, \mathscr{R}_m^{(T_{max})}\}, \tag{44}$$

$$\textit{Table 2:} \qquad \Omega_m^{T_{max}} = \{\mathbf{a}_m^{(1)}, \mathbf{a}_m^{(2)}, \cdots, \mathbf{a}_m^{(T_{max})}\}. \tag{45}$$

The second table consists the sorted candidate vector set $\Omega_m^{T_{max}}$, while the first one consists the corresponding computation rate set $\Gamma_m^{T_{max}}$ with elements $\mathscr{R}_m^{(1)} \geq \mathscr{R}_m^{(2)} \geq \cdots \geq \mathscr{R}_m^{(T_{max})}$.

After we get all the candidate vector sets $\Omega_1^{T_{max}}$, $\Omega_2^{T_{max}}$, $\cdots$, $\Omega_L^{T_{max}}$ and computation rate sets $\Gamma_1^{T_{max}}$, $\Gamma_2^{T_{max}}$, $\cdots$, $\Gamma_L^{T_{max}}$, we will try to pick up $\mathbf{a}_1 \in \Omega_1^{T_{max}}$, $\mathbf{a}_2 \in \Omega_2^{T_{max}}$, $\cdots$, $\mathbf{a}_L \in \Omega_L^{T_{max}}$, to construct the network coding system matrix $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_L]^T$ with full rank, while at the same time, the minimum corresponding rate $\mathscr{R}_1(\mathbf{a}_1)$, $\mathscr{R}_2(\mathbf{a}_2)$, $\cdots$, $\mathscr{R}_L(\mathbf{a}_L)$ is maximized.

Regarding this problem, first, we will sort the overall computation rate set for all relays $\{\Gamma_1^{T_{max}}, \Gamma_2^{T_{max}}, \cdots, \Gamma_L^{T_{max}}\}$ in a descending order into $\{\gamma_1, \gamma_2, \cdots, \gamma_{L\times T_{max}}\}$, such that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{L\times T_{max}}$. Then, starting from the largest possible achievable rate γ_{index} with *index* $= L$ (the first $L-1$ rates are obviously not achievable), we will check one by one whether the rate γ_{index} is achievable, which means we can find L vectors $\mathbf{a}_1 \in \Omega_1^{T_{max}}$, $\mathbf{a}_2 \in \Omega_2^{T_{max}}$, $\cdots$, $\mathbf{a}_L \in \Omega_L^{T_{max}}$, such that the following two constraints are satisfied:

(i) The system network coding coefficient matrix $\mathbf{A}$ is of full rank;

(ii) $\mathscr{R}_1(\mathbf{a}_1)$, $\mathscr{R}_2(\mathbf{a}_2)$, $\cdots$, $\mathscr{R}_L(\mathbf{a}_L)$ all greater or equal to γ_{index}.

If not, we move to the next largest possible achievable rate $\gamma_{index+1}$ and check in the same way, until the first achievable rate is found.

When we are checking one possible achievable rate γ_{index}, we will reduce/cut the network coding candidate set $\Omega_m^{T_{max}}$ into Ω_m^{cut} such that any $\mathbf{a}_m \in \Omega_m^{cut}$ will satisfy that $\mathscr{R}_m(\mathbf{a}_m)$ greater or equal to γ_{index}. In other words, the sets of Ω_1^{cut}, Ω_2^{cut}, $\cdots$, Ω_L^{cut} are constructed such that the constraint *(ii)* will definitely be satisfied if $\mathbf{a}_1 \in \Omega_1^{cut}$, $\mathbf{a}_2 \in \Omega_2^{cut}$, $\cdots$, $\mathbf{a}_L \in \Omega_L^{cut}$.

Suppose $\gamma_{index} = \mathscr{R}_m^{(n)} \in \Gamma_m^{T_{max}}$, i.e. γ_{index} is taken from Table 1 of relay m with table index n, then the network coding vector $\mathbf{a}_m^{(n)}$ is taken from Table 2 with the same index n, i.e. $\mathbf{a}_m^{(n)} \in \Omega_m^{max}$ is fixed for that relay and

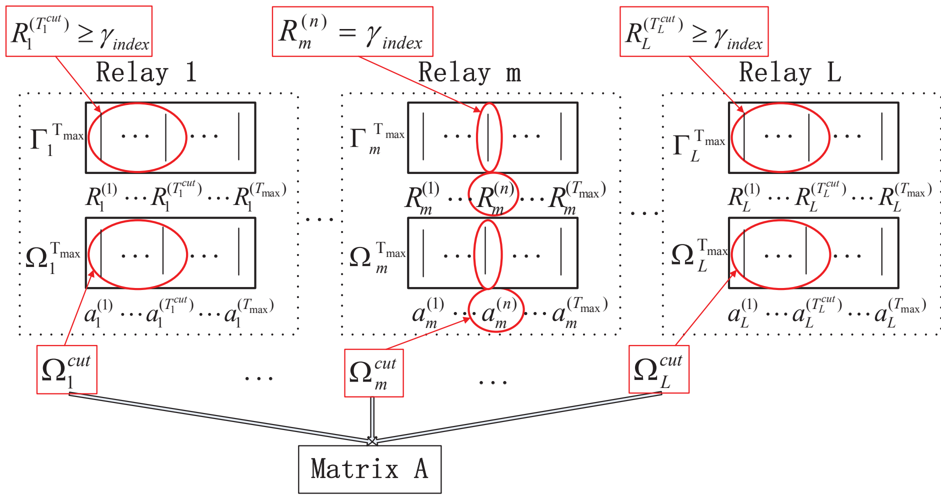

Fig. 5. Constructing network coding system matrix $\mathbf{A}$

$\Omega_m^{cut} = \{\mathbf{a}_m^{(n)}\}$. For other relays $i \neq m$, the candidate set will reduce/cut to length T_i^{cut} such that $\mathscr{R}_i^{(1)}$, $\mathscr{R}_i^{(2)}$, $\cdots$, $\mathscr{R}_i^{(T_i^{cut})}$ all greater or equal to γ_{index}.

Denote $\Omega_i^{cut} = \{\mathbf{a}_i^{(1)}, \mathbf{a}_i^{(2)}, \cdots, \mathbf{a}_i^{(T_i^{cut})}\}$. We can start to check the constraint *(i)* of the system network coding matrix $\mathbf{A}$ constructed by any $\mathbf{a}_1 \in \Omega_1^{cut}$, $\mathbf{a}_2 \in \Omega_2^{cut}$, $\cdots$, $\mathbf{a}_L \in \Omega_L^{cut}$. If there exists one constructed $\mathbf{A}$ with full rank, then this rate γ_{index} is achievable. The procedure is shown in Fig. 5.

We summarize this procedure to constructing the full rank network coding system matrix $\mathbf{A}$ with candidate sets $\Omega_1^{T_{max}}$, $\Omega_2^{T_{max}}$, $\cdots$, $\Omega_L^{T_{max}}$ and the corresponding computation rate sets $\Gamma_1^{T_{max}}$, $\Gamma_2^{T_{max}}$, $\cdots$, $\Gamma_L^{T_{max}}$ as follows.

Algorithm 2

Network Coding System Matrix Constructing Algorithm

Input: Candidate vector sets $\Omega_1^{T_{max}}$, $\Omega_2^{T_{max}}$, $\cdots$, $\Omega_L^{T_{max}}$;

Computation rate sets $\Gamma_1^{T_{max}}$, $\Gamma_2^{T_{max}}$, $\cdots$, $\Gamma_L^{T_{max}}$.

Output: The network coding system matrix $\mathbf{A}$ constructed from $\mathbf{a}_1 \in \Omega_1^{T_{max}}$, $\mathbf{a}_2 \in \Omega_2^{T_{max}}$, $\cdots$, $\mathbf{a}_L \in \Omega_L^{T_{max}}$ with full rank that gives the maximum transmission rate $\mathscr{R}_D^{max}$.

Step 1: Sort the overall computation rate set for all relays $\{\Gamma_1^{T_{max}}, \Gamma_2^{T_{max}}, \cdots, \Gamma_L^{T_{max}}\}$ in a descending order into $\{\gamma_1, \gamma_2, \cdots, \gamma_{L \times T_{max}}\}$, such that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{L \times T_{max}}$. Initialize *index* $= L$.

Step 2: Check whether the rate of γ_{index} is achievable by the following procedure. Suppose $\gamma_{index} = \mathscr{R}_m^{(n)} \in \Gamma_m^{T_{max}}$. Then, for relay i, the reduced candidate set $\Omega_i^{cut}, i = 1, 2, \cdots, L$ will be constructed as follows.

(i) For relay m, set $\Omega_m^{cut} = \{\mathbf{a}_m^{(n)}\}$.

(ii) For relay $i \neq m$, compare the value of γ_{index} and the sorted descending set $\Gamma_i^{T_{max}} = \{\mathscr{R}_i^{(1)}$, $\mathscr{R}_i^{(2)}$, $\cdots$,

$\mathscr{R}_i^{(T_{max})}\}$. Find all $\{\mathscr{R}_i^{(1)}, \mathscr{R}_i^{(2)}, \cdots, \mathscr{R}_i^{(T_i^{cut})}\}$ greater or equal to γ_{index}. Set $\Omega_i^{cut} = \{\mathbf{a}_i^{(1)}, \mathbf{a}_i^{(2)}, \cdots, \mathbf{a}_i^{(T_i^{cut})}\}$.

Step 3: Check every $\mathbf{a}_1 \in \Omega_1^{cut}$, $\mathbf{a}_2 \in \Omega_2^{cut}$, $\cdots$, $\mathbf{a}_L \in \Omega_L^{cut}$, until we find one network coding system matrix $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_L]^T$ has full rank, i.e. $|\mathbf{A}| \neq 0$. If so, terminate and output the network coding system matrix $\mathbf{A}$ and the maximum transmission rate $\mathscr{R}_D^{max} = \gamma_{index}$.

Step 4: If for any $\mathbf{a}_1 \in \Omega_1^{cut}$, $\mathbf{a}_2 \in \Omega_2^{cut}$, $\cdots$, $\mathbf{a}_L \in \Omega_L^{cut}$, we cannot construct a full rank network coding system matrix $\mathbf{A}$, then set $index = index + 1$, go to Step 2.

One possible implementation of the whole system will let relays calculate the candidate sets and corresponding computation rate sets, construct the optimal network coding system matrix $\mathbf{A}$, then transmit the $L \times L$ integers matrix $\mathbf{A}$ to the destination. Another possible implementation is to allow the destination work as processing center, that does all calculations, including candidate sets, corresponding computation rate sets, and the optimal network coding system matrix $\mathbf{A}$ construction. The destination will then feedback the optimal network coding vector $\mathbf{a}_m \in \mathbb{Z}^L$ to relay m for $m = 1, 2, \cdots, L$. After system initialization, these optimal network coding vectors can be used for the system when the channels are stationary.

IV. Experimental Studies

A. A Transparent Realization

In this subsection, we will give a detailed experimental example to show our proposed algorithms in a transparent way. For a three-source three-relay system with $L = 3$, we set the power constraints $P = 10dB$ and $T_{max} = 5$. The channel coefficient vector $\mathbf{h}_m$ for each relay is generated as

$$\begin{aligned} \mathbf{h}_1 &= [0.9730, 0.4674, 0.5103]^T, \\ \mathbf{h}_2 &= [-1.7291, 0.7166, -0.5856]^T, \\ \mathbf{h}_3 &= [-0.3912, 1.4407, -0.8115]^T. \end{aligned}$$

After calculating $\mathbf{G}_m$, $m = 1, 2, 3$ and running our proposed FP based candidate set searching algorithm for each relay, we will get the network coding candidate vector sets $\Omega_1^{T_{max}}$, $\Omega_2^{T_{max}}$, $\Omega_3^{T_{max}}$ and corresponding computation rate sets $\Gamma_1^{T_{max}}$, $\Gamma_2^{T_{max}}$, $\Gamma_3^{T_{max}}$ as follows

$$\begin{aligned} \Omega_1^{T_{max}} &= \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}, \\ \Gamma_1^{T_{max}} &= [0.4846, \ 0.4620, \ 0.3408, \ 0.2918, \ 0.2231]; \\ \Omega_2^{T_{max}} &= \begin{bmatrix} 1 & 2 & 3 & -1 & -2 \\ 0 & -1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \\ \Gamma_2^{T_{max}} &= [0.7087, \ 0.6785, \ 0.5572, \ 0.3625, \ 0.2694]; \end{aligned}$$

$$\begin{aligned}\Omega_3^{T_{max}} &= \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ -1 & 1 & -2 & -2 & -3 \\ 1 & 0 & 1 & 1 & 2 \end{bmatrix}, \\ \Gamma_3^{T_{max}} &= [0.5987, \ 0.5935, \ 0.4384, \ 0.4165, \ 0.2902].\end{aligned}$$

We can see that the computation rate set $\Gamma_m^{T_{max}}$, $m = 1, 2, 3$ has elements sorted in descending order where the first element is the maximum computation rate for relay m. The n-th column in $\Omega_m^{T_{max}}$ is a candidate network coding vector $\mathbf{a}_m^{(n)}$ for relay m, while the corresponding computation rate is the n-th element in $\Gamma_m^{T_{max}}$. Note that if we optimize the network coding coefficients separately, which means each relay will use network coding vector that maximizes its own computation rate, $\mathbf{a}_m$ is taken from the first column of $\Omega_m^{T_{max}}$, $m = 1, 2, 3$ and the constructed network coding system matrix

$$\mathbf{A}_{local} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}^T$$

is obviously not of full rank. In this case, the destination actually cannot decode all the messages efficiently.

Then we go forward to run our proposed network coding system matrix constructing algorithm. We sort the computation rates for all relays in a descending order,

$$\{\underbrace{0.7087}_{\gamma_1}, \underbrace{0.6785}_{\gamma_2}, \underbrace{0.5987}_{\gamma_3}, \underbrace{0.5935}_{\gamma_4}, \underbrace{0.5572}_{\gamma_5}, \underbrace{0.4846}_{\gamma_6}, \cdots\}.$$

and start to check the rate from the third maximum value, $\gamma_3 = 0.5987$, then $\gamma_4 = 0.5935$, then $\gamma_5 = 0.5572$, $\cdots$, to see whether it is achievable. If so, terminate and output; if not, move to the next rate.

For example, when we are checking $\gamma_4 = 0.5935 = \mathscr{R}_3^{(2)}$, which is taken from the second element of $\Gamma_3^{T_{max}}$, the reduced candidate sets Ω_1^{cut}, Ω_2^{cut}, Ω_3^{cut} with all corresponding rates greater or equal to $\gamma_4 = 0.5935$ can be constructed as

$$\Omega_1^{cut} = \emptyset, \qquad \Omega_2^{cut} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \qquad \Omega_3^{cut} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

We can easily see that no full rank network coding system matrix $\mathbf{A}$ can be constructed with $\mathbf{a}_1 \in \Omega_1^{cut}$, $\mathbf{a}_2 \in \Omega_2^{cut}$, $\mathbf{a}_3 \in \Omega_3^{cut}$. Hence the rate of $\gamma_4 = 0.5935$ is not achievable. We will move to $\gamma_5 = 0.5572$ and check in the same way.

After running our proposed Network Coding System Matrix $\mathbf{A}$ Constructing Algorithm 2, the network coding system matrix $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]^T$ is finally constructed as

$$\mathbf{A}_{proposed} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}^T$$

and the maximum transmission rate $\mathscr{R}_D^{max} = 0.4846$.

B. Simulation Results

We present numerical results to evaluate the performance of our proposed algorithms. First, we show that if network coding integer coefficient vector is optimized separately/locally at each relay, the probability that the network coding system matrix $\mathbf{A}$ is not of full rank, i.e. $|\mathbf{A}| = 0$, in which case the destination actually cannot decode the original messages efficiently. With the average of 10000 randomly generated channel realizations, it can be observed from Fig. 6 the severity of this issue. For example, when $L = 3$ and $P = 1$dB-8dB, the probability of rank failure with local optimized network coding vectors is always beyond 0.4. This further assures the importance and necessity of our proposed algorithms.

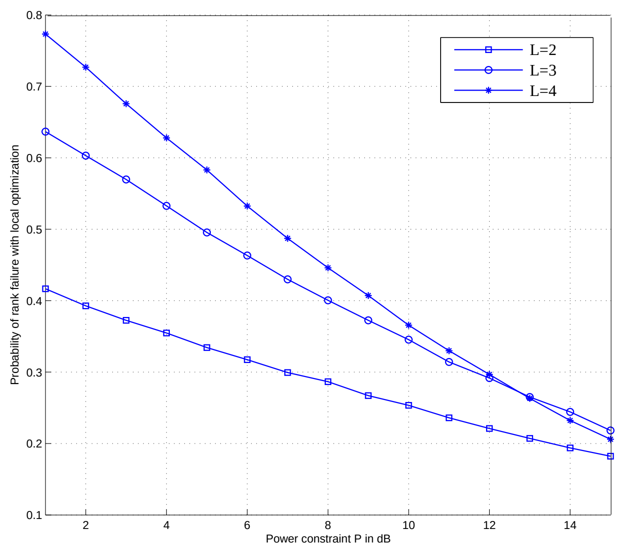

Fig. 6. Probability of rank failure with local optimization for $L = 2, 3, 4$

In Fig. 7, we compare the overall transmission rate $\mathscr{R}_D$ at destination, with the average of 10000 randomly generated channel realizations, of several different strategies in multi-source multi-relay channels with $L = 3$ and $T_{max} = 5$. (i) The "*DF with interference as noise*" is a strategy in which relay m is trying to decode one message from source m and treat other messages as noise. In this special case, the system matrix $\mathbf{A} = \mathbf{I}_L$. (ii) The "*CPF NC with Round-H*" is a strategy that each relay decodes a linear integer combination of transmitted messages, while the network coding coefficients are set by a simplified method, i.e. rounding the channel coefficients directly to the nearest integers. (iii) The "*CPF NC with local optimization*" is a strategy that each relay also decodes a linear integer combination of transmitted messages, while the network coding coefficients are optimized locally/separately. Due to the rank failure issue of network coding system matrix, in which case the destination cannot decode all messages, the rate is decreased. Finally, (iv) the "*CPF NC with proposed algorithms*" is the strategy that each relay decodes a linear integer combination of transmitted messages with our proposed FP based candidate set searching algorithm and network coding system matrix constructing algorithm.

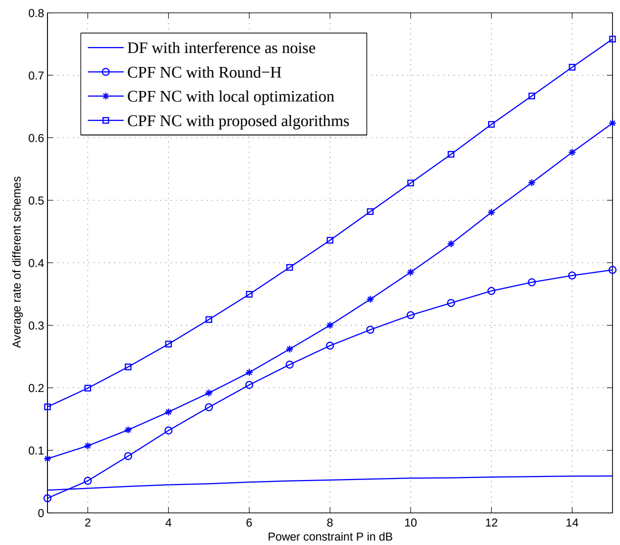

Fig. 7. Rate comparisons of different schemes for $L = 3$

As shown in Fig. 7, the performance differences are significant. "*DF with interference as noise*" gives very poor result. Furthermore, increasing power constraint has not much effect on this strategy since as the power increases for the interested message, the corresponding interference power is also raised. The "*CPF NC with Round-H*" strategy works a little better since it somehow takes advantage of network coding to improve the rate, but the coefficients are chosen in a simplified way and not optimal. The "*CPF NC with proposed algorithms*" strategy, in which case the network coding coefficients are optimized systematically, performs superior to all other strategies and has about 3dB gain compared with the "*CPF NC with local optimization*".

We repeat our experiment with multi-source multi-relay channels of $L = 4$ and present the average rate comparisons of different schemes with respect to the power constraint. Similar results are shown as in Fig. 8. "*CPF NC with proposed algorithms*" strategy still gives the best performance and further demonstrates the effectiveness of our proposed algorithms.

V. CONCLUSION

In this work, we consider the problem of integer network coding coefficients design in a system level over a compute-and-forward multi-source multi-relay system. Instead of optimizing network coding vector of each relay separately, we propose the Fincke-Pohst based candidate set searching algorithm, to provide a network coding vector candidate set for each relay with corresponding computation rate in descending order. Then, with our proposed network coding system matrix constructing algorithm, we choose network coding vectors from candidate sets to construct network coding system matrix with full rank, while in the meantime the transmission rate of the overall

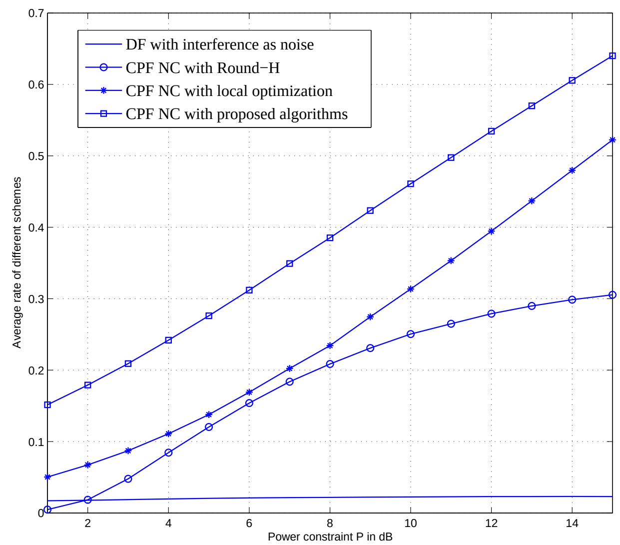

Fig. 8. Rate comparisons of different schemes for $L = 4$

system is maximized. Numerical results give the performance comparisons of our proposed compute-and-forward network coding algorithms and other strategies.

REFERENCES

[1] R. Ahlswede, N. Cai, S. R. Li and R. W. Yeung, “Network information flow”, *IEEE Trans. Info. Theory*, vol. 46, no. 4, pp. 1204-1216, July 2000.

[2] S. R. Li, R. Ahlswede and N. Cai, “Linear network coding”, *IEEE Trans. Info. Theory*, vol. 49, no. 2, pp. 371-381, Feb. 2003.

[3] R. Koetter and M. Medard, “An algebraic approach to network coding”, *IEEE/ACM Trans. Networking*, vol. 11, no. 5, pp.782-795, Oct. 2003.

[4] S. Jaggi, P. Sanders, P. A. Chou, M. Effros, S. Egner, K Jain and L. Tolhuizen, “Polynomial time algorithms for multicast network code construction”, *IEEE Trans. Info. Theory*, vol. 51, no.6, June. 2005.

[5] T. Ho, R. Koetter, M. Medard, D. R. Karger, M. Effros, J. Shi and B. Leong, “A random linear network coding approach to multicast”, *IEEE Trans. Info. Theory*, vol. 52, no. 10, pp. 4413-4430, Oct. 2006.

[6] S. Zhang, S. Liew and P. P. Lam, “Physical-layer network coding”, in *Proc. IEEE MobiComm*, pp. 358-365, Los Angeles, USA, Sept. 2006.

[7] T. Wang and G. B. Giannakis, “Complex field network coding for multiuser cooperative communications, *IEEE J. Sel. Areas Commun.*, vol. 26, no. 2, pp. 561-571, Apr. 2008.

[8] S. Katti, S. Gollakota, and D. Katabi, “Embracing wireless interference: Analog network coding”, in *Proc. ACM SIGCOMM*, Kyoto, Japan, pp. 397-408, Aug. 2007.

[9] S. Katti, H. Rahul, W. Hu, D. Katabi, M. Medard and J. Crowcroft, “XORs in the air: Practical wireless network coding”, in *Proc. ACM SIGCOMM*, Pisa, Italy, pp. 243-254, Sept. 2006.

[10] P. A. Chou, Y. Wu and K. Jain, “Practical network coding”, *Allerton Conf. Commun. Control, Computing*, Oct. 2003.

[11] Y. Wu, P. A. Chou and S. Kung, “Information exchange in wireless networks with network coding and physical-layer broadcast”, Microsoft Research, Redmond, WA, Tech. Rep. MSR-TR-2004-78, Aug. 2004.

[12] C. Fragouli, D. Katabi, A. Markopoulou, M. Medard and H. Rahul, “Wireless network coding: Opportunities & Challenges”, in *Proc. IEEE MILCOM*, Orlando, FL, Oct. 2007.

[13] R. Zamir, “Lattices are everywhere”, in *Proceedings of the 4th Annual Workshop on Info. Theory and its Applications*, La Jolla, CA, Feb. 2009.

[14] F. Oggier and E. Viterbo, “Algebraic number theory and code design for rayleigh fading channels”, in *Foundations and Trends in Commun. and Info. Theory*, vol. 1, pp. 333-415, 2004.

[15] R. Zamir, S. Shamai, and U. Erez, “Nested linear/lattice codes for structed multiterminal binning”, *IEEE Trans. Info. Theory*, vol. 48, no. 6, pp. 1250-1276, June 2002.

[16] U. Erez and R. Zamir, “Achieving (1/2)log(1+SNR) on the AWGN channel with lattice encoding and decoding”, *IEEE Trans. Info. Theory*, vol. 50, no. 10, pp. 2293-2314, Oct. 2004.

[17] U. Fincke and M. Pohst, “Improved methods for calculating vectors of short length in a lattice, including a complexity analysis,” *Math. Comput.*, vol. 44, pp. 463-471, Apr. 1985.

[18] E. Viterbo and J. Boutros, “A universal lattice code decoder for fading channels,” *IEEE Trans. Info. Theory*, vol. 45, no. 5, pp. 1635-1642, July 1999.

[19] O. Damen, A. Chkeif, and J. C. Belfiore, ”Lattice code decoder for space-time codes,” *IEEE Commun. Letters*, vol. 4, no. 5, pp. 161-163, May 2000.

[20] M. O. Damen, H. El Gamal and G. Carie, “On maximum-likelihood detection and the search for the closest lattice point,” *IEEE Trans. Info. Theory*, vol. 49, no. 10, pp. 2389-2402, Oct. 2003.

[21] M. Wilson, K. Narayanan, H. Pfister and A. Sprintson, “Joint physical layer coding and network coding for bidirectional relaying,” *IEEE Trans. Info. Theory*, vol. 56, pp. 5641-5654, Nov. 2010.

[22] B. Hassibi, A. Sezgin, M. Khajehnejad and A. Avestimehr, “Approximate capacity region of the two-pair bidirectional gaussian relay network,” in *IEEE Inter. Symp. Info. Theory*, pp. 2018-2022, Seoul, Korea, June, 2009.

[23] I. Baik and S. Y. Chung, “Network coding for two-way relay channels using lattices,” in *IEEE Inter. Conf. Commun.*, pp. 3898-3902, Beijing, China, May 2008.

[24] B. Nazer and M. Gastpar, “Compute-and-forward: harnessing interference through structured codes”, *IEEE Trans. Info. Theory*, vol. 57, no. 10, pp. 6463-6486, Oct. 2011.

[25] B. Nazer and M. Gastpar, “Reliable physical layer network coding”, *IEEE Proceedings*, vol. 99, no. 3, pp. 438-460, Mar. 2011.

[26] A. Osmane and J. C. Belfiore, “The compute-and-forward protocol: implementation and practical aspects”, *IEEE Commun. Lett.*, submitted. Available online: http://arxiv.org/abs/1107.0300.

[27] C. Feng, D. Silva, and F. R. Kschischang, “Design criteria for lattice network coding”, *IEEE 45th Annual Conf. Info. Sci.* &*, Sys.*, Baltimore, USA, Mar. 2011.

[28] M. Nokleby and B. Aazhang, “Lattice coding over the relay channel“, *IEEE Inter. Conf. Commun.*, Kyoto, Japan, June 2011.

[29] C. Feng, D. Silva and F. R. Kschischang, “An algebraic approach to physical-layer network coding”, *IEEE Inter. Sym. Info. Theory*, Austin, TX, USA, June. 2010.

[30] J. Zhan, U. Erez, M. Gastpar and B. Nazer, ”MIMO compute-and-forward,” in *IEEE Inter. Symp. Info. Theory*, pp. 2848-2852, Seoul, Korea, June 2009.

[31] D. Tse and P. Viswanath, *Fundamentals of wireless communications*. New York, NY, USA: Cambridge University Press, 2005.

[32] K. J. R. Liu, A. K. Sadek, W. Su and A. Kwasinski, *Cooperative Communications and Networking*. Cambridge Univ. Press, 2009.