# Asymptotic Performance Analysis of GSVD-NOMA Systems with a Large-Scale Antenna Array 

Zhuo Chen, Zhiguo Ding, Senior Member, IEEE, Xuchu Dai, and Robert<br>Schober, Fellow, IEEE


#### Abstract

This paper considers a multiple-input multiple-output (MIMO) downlink communication scenario with one base station and two users, where each user is equipped with $m$ antennas and the base station is equipped with $n$ antennas. To efficiently exploit the spectrum resources, we propose a transmission protocol which combines generalized singular value decomposition (GSVD) and non-orthogonal multiple access (NOMA). The average data rates achieved by the two users are adopted as performance metrics for evaluation of the proposed GSVD-NOMA scheme. In particular, we first characterize the limiting distribution of the squared generalized singular values of the two users' channel matrices for the asymptotic case where the numbers of transmit and receive antennas approach infinity. Then, we calculate the normalized average individual rates of the users in the considered asymptotic regime. Furthermore, we extend the proposed GSVD-NOMA scheme to the MIMO downlink communication scenario with more than two users by using a hybrid multiple access (MA) approach, where the base station first divides the users into different groups, then the proposed GSVD-NOMA scheme is implemented within each group, and different groups are allocated with orthogonal bandwidth resources. Finally, numerical results are provided to validate the effectiveness of the proposed GSVD-NOMA protocol, and the accuracy of the developed analytical results.


## Index Terms

[^0][^1]
## I. Introduction

Non-orthogonal multiple access (NOMA) is an effective approach to improve the spectral efficiency of wireless networks and has been recognized as a promising candidate for 5G multiple access (MA) [1]. The key idea of NOMA is to serve multiple users at the same frequency resource blocks and at the same time. In general, NOMA can be implemented in two ways: first, by using single-carrier NOMA [2], [3], where the principle of NOMA, spectrum sharing, is implemented on one resource block, such as one subcarrier, and, second, by using multi-carrier NOMA [4], where the principle of NOMA is jointly implemented across multiple orthogonal resource blocks. In the past two years, NOMA has been widely used and investigated due to its superior spectral efficiency and flexibility compared to conventional orthogonal MA (OMA) [5], [6]. In [7], the performance of a downlink NOMA network with randomly deployed users was investigated. An uplink NOMA transmission scheme was proposed in [8] and its performance was evaluated. Moreover, practical forms of multi-carrier NOMA, such as sparse code multiple access (SCMA) and low-density spreading (LDS) NOMA, were proposed in [9], [10], which introduce redundancy via coding/spreading among multiple subcarriers to facilitate interference cancelation.

On the other hand, multiple-input multiple-output (MIMO) techniques have also been identified as a key enabling technology for improving the 5 G system throughput. It is well known that if perfect channel state information at the transmitter (CSIT) is available, the capacity region of MIMO broadcast channels can be achieved by using dirty paper coding (DPC) [11]. However, due to its prohibitive computational complexity, it is difficult to implement DPC in practice [12]. Compared to DPC, MIMO-NOMA schemes have a relatively low computational complexity at the expense of a small loss in performance [12]. Therefore, MIMO-NOMA systems have attracted considerable research interest. In [13], the ergodic rate was maximized for MIMONOMA systems having statistical CSIT only. In [14], a layered transmission scheme was applied to MIMO-NOMA systems and an optimal power allocation policy was developed. In [15], a hybrid MIMO-NOMA scheme was proposed, where users were grouped into smallsize clusters. NOMA was implemented within each cluster, and MIMO detection was used
to cancel inter-cluster interference. In [16], a new MIMO-NOMA scheme based on QR decomposition was proposed and power allocation policies for this scheme were investigated. In [17], coordinated beamforming techniques were developed to enhance the performance of MIMO-NOMA communications in the presence of inter-cell interference. In [18], NOMA was applied to downlink multiuser MIMO cellular systems and a linear beamforming technique was proposed to cancel inter-cluster interference. A more detailed literature review on MIMO-NOMA can be found in [19].

The generalized singular value decomposition (GSVD) is an efficient tool to decompose the MIMO-NOMA channel into parallel single-input single-output (SISO) channels, such that the NOMA principle can be applied to each SISO channel individually]. In [20], the GSVD was applied to MIMO-NOMA uplink and downlink transmission. However, the authors in [20] only consider the special case where all nodes are equipped with the same number of antennas. Also, in [20], the performance evaluation of the proposed GSVD-NOMA scheme relied on computer simulations, and more insightful analytical results are missing. Motivated by this, in this paper, we apply GSVD-NOMA to a general MIMO downlink communication scenario with one base station and two users, where each user is equipped with $m$ antennas and the base station is equipped with $n$ antennas. Moreover, the average data rates achieved by the two users are adopted as performance metrics for evaluation of the proposed GSVD-NOMA scheme.

The main contributions of this paper are summarized as follows.

- We characterize the limiting distribution of the squared generalized singular values of the two users' channel matrices for the asymptotic case where the numbers of transmit and receive antennas approach infinity, i.e., $n, m \rightarrow \infty$ with $\frac{m}{n} \rightarrow \eta$, where $\eta$ is a constant denoting the ratio of the numbers of receive and transmit antennas. To the best of the authors' knowledge, the limiting distribution of the squared generalized singular values in the considered asymptotic regime, has not been characterized before. Compared to the eigenvalues and conventional singular values [21], it is more challenging to characterize the distribution of the squared generalized singular values, since they depend on the channel matrices of both users. Long-term power normalization is applied at the base station. In

[^2]order to investigate the impact of power normalization on the proposed GSVD-NOMA scheme, we study the properties of the GSVD decomposition matrix and characterize the long-term power normalization factor.

- Furthermore, in order to evaluate the performance of the proposed GSVD-NOMA scheme, we characterize the normalized average individual rates ${ }^{2}$ of the two users in the considered asymptotic regime. The developed analytical results are easy to evaluate the performance of the proposed GSVD-NOMA scheme and can help avoid extensive computer simulation in the considered asymptotic case where the numbers of transmit and receive antennas are large. Also, when the base station and the users have moderate numbers of antennas (e.g. $m=$ $2, n=5$ ), the derived analytical results still provide good approximations, as indicated by the presented simulation results. In addition, simulation results are provided to corroborate the improvement of the normalized average sum rate of the proposed GSVD-NOMA scheme compared to "conventional" OMA and QR-NOMA in [16]. Moreover, a hybrid NOMA scheme is proposed to extend the proposed GSVD-NOMA scheme to the MIMO downlink communication scenario with more than two users, where the base station first divides the users into different groups, then the proposed GSVD-NOMA scheme is implemented within each group, and different groups are allocated with orthogonal bandwidth resources. Also, numerical results are provided to demonstrate the performance of this hybrid NOMA scheme.

The rest of the paper is organized as follows. In Section III, we introduce the system model considered in this paper. In Section IIII, we present the proposed GSVD-NOMA design which efficiently exploits the spectral resources. In Section IV, we derive new analytical results for the squared generalized singular values and develop analytical expressions for the normalized average individual rates of the two users. Numerical results are provided in Section V and Section VI concludes this paper.

## II. System model

In this paper, we consider a general MIMO downlink communication scenario with one base station and two users, where each user is equipped with $m$ antennas and the base station is
${ }^{2}$ Consider a linear memoryless vector channel of the form $\mathbf{y}=\mathbf{H x}+\mathbf{n}$, where $\mathbf{H}$ is the $m \times n$ channel matrix, $\mathbf{x}$ is the $n$-dimensional input vector, $\mathbf{y}$ is the $m$-dimensional output vector, and the $m$-dimensional vector $\mathbf{n}$ models the additive circularly symmetric Gaussian noise. The normalized input-output mutual information of this channel is defined as $\frac{1}{m} I(\mathbf{x} ; \mathbf{y})$ [21]. Similarly, in this paper, we define the normalized rate of a user as $\frac{1}{m} R_{d}$, where $R_{d}$ is the overall data rate of the user.
equipped with $n$ antennas. We assume block fading, i.e., the user channels are constant for the transmission of one codeword and change independently from one codeword to the next. The $m \times n$ channel matrix from the base station to the $i$-th user, $i \in\{1,2\}$, is denoted by $\mathbf{G}_{i}$. The considered composite channels are modeled as $\mathbf{G}_{i}=\frac{1}{\sqrt{d_{i}^{\tau}}} \mathbf{H}_{i}$, where $\mathbf{H}_{i} \in \mathbb{C}^{m \times n}$ models the small-scale fading, $\frac{1}{\sqrt{d_{i}^{\tau}}}$ models large-scale path loss, $\tau$ is the path loss exponent, and $d_{i}$ denotes the distance between the base station and the $i$-th user, $i \in\{1,2\}$. The locations of the users affect the user channels via large-scale path loss $\frac{1}{\sqrt{d_{i}^{\tau}}}$. All small-scale fading coefficients are assumed to be independent and identically distributed Rayleigh with unit variance, i.e., $\mathbf{H}_{i}$ is an $m \times n$ matrix whose elements are mutually independent and identically distributed (i.i.d.) complex Gaussian random variables with zero mean and unit variance. Full CSIT is assumed to be available at the base station.

Denote the $n \times n$ precoding matrix at the base station by $\mathbf{P}_{b}$. Then, the base station transmits an $n \times 1$ signal vector $\mathbf{x}$ to the two users, where $\mathbf{x}=\mathbf{P}_{b} \mathbf{S}$ and the $n \times 1$ vector $\mathbf{s}=\left[s_{1}, s_{2}, \cdots s_{n}\right]^{T}$ is the information bearing vector. The elements of $\mathbf{s}, s_{i}, i \in\{1, \cdots, n\}$, are coded symbols representing the messages intended for the two users and are taken from Gaussian codebooks, see Section III-B for detailed description. The observations at the two users can be expressed as follows:

$$
\begin{equation*}
\mathbf{y}_{1}=\frac{1}{\sqrt{d_{1}^{\tau}}} \mathbf{H}_{1} \mathbf{x}+\mathbf{n}_{1} \quad \text { and } \quad \mathbf{y}_{2}=\frac{1}{\sqrt{d_{2}^{\tau}}} \mathbf{H}_{2} \mathbf{x}+\mathbf{n}_{2} \tag{1}
\end{equation*}
$$

respectively, where $\mathbf{n}_{i}, i \in\{1,2\}$, denotes the Gaussian additive noise vector of user $i$, whose elements are mutually independent and identically distributed (i.i.d.) complex Gaussian random variables with zero mean and unit variance. Define the detection matrices at user 1 and user 2 as $\mathbf{D}_{1} \in \mathbb{C}^{m \times m}$ and $\mathbf{D}_{2} \in \mathbb{C}^{m \times m}$, respectively. After applying the detection matrices, user 1 and user 2 observe the following signals:

$$
\begin{equation*}
\mathbf{D}_{1} \mathbf{y}_{1}=\frac{1}{\sqrt{d_{1}^{\tau}}} \mathbf{D}_{1} \mathbf{H}_{1} \mathbf{P}_{b} \mathbf{s}+\mathbf{n}_{1}^{\prime} \quad \text { and } \quad \mathbf{D}_{2} \mathbf{y}_{2}=\frac{1}{\sqrt{d_{2}^{\tau}}} \mathbf{D}_{2} \mathbf{H}_{2} \mathbf{P}_{b} \mathbf{s}+\mathbf{n}_{2}^{\prime} \tag{2}
\end{equation*}
$$

respectively, where $\mathbf{n}_{1}^{\prime}=\mathbf{D}_{1} \mathbf{n}_{1}$ and $\mathbf{n}_{2}^{\prime}=\mathbf{D}_{2} \mathbf{n}_{2}$.

## III. Description of the Proposed GSVD-NOMA Scheme

In the following, we propose a transmission protocol which combines GSVD and NOMA. To be more specific, we first introduce the GSVD of the two $m \times n$ channel matrices $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$.

Please note that in this paper, we assume that $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$ are full rank ${ }^{3}$. Then, we explain the designs of $\mathbf{P}_{b}, \mathbf{D}_{1}, \mathbf{D}_{2}$, and $\mathbf{s}$, which are based on GSVD and NOMA to efficiently exploit the spectrum resources.

## A. Definition of GSVD

We first introduce the GSVD of the two $m \times n$ channel matrices $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$ as follows [23]:

$$
\begin{equation*}
\mathrm{UH}_{1} \mathrm{Q}=\boldsymbol{\Sigma}_{1} \quad \text { and } \quad \mathrm{VH}_{2} \mathbf{Q}=\boldsymbol{\Sigma}_{2}, \tag{3}
\end{equation*}
$$

where $\mathbf{U} \in \mathbb{C}^{m \times m}$ and $\mathbf{V} \in \mathbb{C}^{m \times m}$ are two unitary matrices, $\mathbf{Q} \in \mathbb{C}^{n \times n}$ is a nonsingular matrix, and $\Sigma_{1} \in \mathbb{C}^{m \times n}$ and $\Sigma_{2} \in \mathbb{C}^{m \times n}$ are two nonnegative diagonal matrices. Moreover, $\boldsymbol{\Sigma}_{1}$ and $\boldsymbol{\Sigma}_{2}$ have the following forms depending on the choices of $m$ and $n$.

- When $m \geq n, \boldsymbol{\Sigma}_{1}$ and $\boldsymbol{\Sigma}_{2}$ can be expressed as follows:

$$
\begin{equation*}
\Sigma_{1}=\binom{\mathrm{S}_{1}}{\mathrm{O}_{(m-n) \times n}} \quad \text { and } \quad \Sigma_{2}=\binom{\mathrm{O}_{(m-n) \times n}}{\mathrm{~S}_{2}} \tag{4}
\end{equation*}
$$

where $\mathbf{O}_{(m-n) \times n}$ denotes the zero matrix of size $(m-n) \times n, \mathbf{S}_{1}=\operatorname{diag}\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ and $\mathbf{S}_{2}=\operatorname{diag}\left(\beta_{1}, \cdots, \beta_{n}\right)$ are two $n \times n$ nonnegative diagonal matrices, satisfying $1 \geq \alpha_{1} \geq$ $\alpha_{2} \cdots \geq \alpha_{n} \geq 0$ and $\mathbf{S}_{1}^{2}+\mathbf{S}_{2}^{2}=\mathbf{I}_{n}$, where $\mathbf{I}_{n}$ denotes the identity matrix of size $n$. Then, the squared generalized singular values are defined as $w_{i}^{2}=\alpha_{i}^{2} / \beta_{i}^{2}, i \in\{1, \cdots, n\}$.

- When $m<n<2 m$, let us define $r=n-m$ and $q=2 m-n$. Then, $\boldsymbol{\Sigma}_{1}$ and $\boldsymbol{\Sigma}_{2}$ can be expressed as follows:

$$
\Sigma_{1}=\left(\begin{array}{ccc}
\mathbf{I}_{r} & \mathrm{O}_{r \times q} & \mathrm{O}_{r \times r}  \tag{5}\\
\mathbf{O}_{q \times r} & \mathrm{~S}_{1} & \mathrm{O}_{q \times r}
\end{array}\right) \quad \text { and } \quad \Sigma_{2}=\left(\begin{array}{ccc}
\mathbf{O}_{q \times r} & \mathbf{S}_{2} & \mathrm{O}_{q \times r} \\
\mathbf{O}_{r \times r} & \mathrm{O}_{r \times q} & \mathbf{I}_{r}
\end{array}\right)
$$

where $\mathbf{S}_{1}=\operatorname{diag}\left(\alpha_{1}, \cdots, \alpha_{q}\right)$ and $\mathbf{S}_{2}=\operatorname{diag}\left(\beta_{1}, \cdots, \beta_{q}\right)$ are two $q \times q$ nonnegative diagonal matrices, satisfying $1 \geq \alpha_{1} \geq \alpha_{2} \cdots \geq \alpha_{q} \geq 0$ and $\mathbf{S}_{1}^{2}+\mathbf{S}_{2}^{2}=\mathbf{I}_{q}$. Similarly, we define the squared generalized singular values as $w_{i}^{2}=\alpha_{i}^{2} / \beta_{i}^{2}, i \in\{1, \cdots, q\}$.

- When $2 m \leq n, \boldsymbol{\Sigma}_{1}$ and $\boldsymbol{\Sigma}_{2}$ can be expressed as follows:

$$
\boldsymbol{\Sigma}_{1}=\left(\begin{array}{ll}
\mathbf{I}_{m} & \mathbf{O}_{m \times(n-m)}
\end{array}\right) \quad \text { and } \quad \boldsymbol{\Sigma}_{2}=\left(\begin{array}{cc}
\mathbf{O}_{m \times(n-m)} & \mathbf{I}_{m} \tag{6}
\end{array}\right) .
$$

Note that the generalized singular values are independent of $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$ in this case.

[^3]
## B. The proposed GSVD-NOMA design

As shown in (3), the GSVD of the two $m \times n$ matrices $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$ can be expressed as $\mathrm{UH}_{1} \mathbf{Q}=\boldsymbol{\Sigma}_{1} \quad$ and $\quad \mathrm{VH}_{2} \mathbf{Q}=\boldsymbol{\Sigma}_{2}$. Then, the precoding matrix $\mathbf{P}_{b}$ is given by $\mathbf{P}_{b}=\mathbf{Q} \sqrt{P} / t$, where $P$ denotes the total transmission power of the base station, and $t$ is a scalar for power normalization. In this paper, for the sake of analytical tractability, long-term power normalization is applied at the base station, i.e., $t^{2}=\mathcal{E}\left\{\operatorname{trace}\left(\mathbf{Q s s}{ }^{H} \mathbf{Q}^{H}\right)\right\}$, where $\mathbf{s}$ is the information bearing symbol vector, $\mathcal{E}\{\cdot\}$ denotes mathematical expectation, and $\operatorname{trace}\left(\mathbf{Q s s}{ }^{H} \mathbf{Q}^{H}\right)$ denotes the trace of Qss ${ }^{H} \mathbf{Q}^{H}$. The detection matrices $\mathbf{D}_{1}$ and $\mathbf{D}_{2}$ are chosen as $\mathbf{D}_{1}=\mathbf{U}$ and $\mathbf{D}_{2}=\mathbf{V}$, respectively. From (2), we conclude that, with these choices, user 1 and user 2 obtain the following signals:

$$
\begin{gather*}
\mathbf{U} \mathbf{y}_{1}=\frac{1}{\sqrt{d_{1}^{\tau}}} \mathbf{U H}_{1} \mathbf{P}_{b} \mathbf{s}+\mathbf{n}_{1}^{\prime}=\frac{\sqrt{P}}{t \sqrt{d_{1}^{\tau}}} \boldsymbol{\Sigma}_{1} \mathbf{s}+\mathbf{n}_{1}^{\prime}, \quad \text { and } \\
\mathbf{V} \mathbf{y}_{2}=\frac{1}{\sqrt{d_{2}^{\tau}}} \mathbf{V H}_{2} \mathbf{P}_{b} \mathbf{s}+\mathbf{n}_{2}^{\prime}=\frac{\sqrt{P}}{t \sqrt{d_{2}^{\tau}}} \boldsymbol{\Sigma}_{2} \mathbf{s}+\mathbf{n}_{2}^{\prime} \tag{7}
\end{gather*}
$$

where $\mathbf{n}_{1}^{\prime}=\mathbf{U n}_{1}$ and $\mathbf{n}_{2}^{\prime}=\mathbf{V n}_{2}$. Next, we explain the design of s which is based on NOMA in order to fully exploit the available spectrum resources.

1) The case when $m \geq n$ : As shown in Section III-A, when $m \geq n, \boldsymbol{\Sigma}_{1}$ and $\boldsymbol{\Sigma}_{2}$ can be expressed as in (4). So, when $m \geq n$, after applying detection matrices $\mathbf{U}$ and $\mathbf{V}$, we obtain the following:

$$
\mathbf{U} \mathbf{y}_{1}=\frac{\sqrt{P}}{t \sqrt{d_{1}^{\tau}}}\left[\begin{array}{c}
\alpha_{1} s_{1}  \tag{8}\\
\vdots \\
\alpha_{n} s_{n} \\
\mathbf{O}_{(m-n) \times 1}
\end{array}\right]+\mathbf{n}_{1}^{\prime} \quad \text { and } \quad \mathbf{V} \mathbf{y}_{2}=\frac{\sqrt{P}}{t \sqrt{d_{2}^{\tau}}}\left[\begin{array}{c}
\mathbf{O}_{(m-n) \times 1} \\
\beta_{1} s_{1} \\
\vdots \\
\beta_{n} s_{n}
\end{array}\right]+\mathbf{n}_{2}^{\prime}
$$

Therefore, the use of the GSVD converts the downlink MIMO channel into $n$ parallel SISO channels as follows:

$$
\begin{equation*}
y_{1 i}=\frac{\sqrt{P}}{t \sqrt{d_{1}^{\tau}}} \alpha_{i} s_{i}+n_{1 i} \quad \text { and } \quad y_{2 i}=\frac{\sqrt{P}}{t \sqrt{d_{2}^{\tau}}} \beta_{i} s_{i}+n_{2 i}, \quad i \in\{1, \cdots, n\} \tag{9}
\end{equation*}
$$

where $y_{1 i}$ and $y_{2 i}$ are the received signals at user 1 and user 2, respectively. As in [7], we consider fixed-NOMA (F-NOMA) in this paper. To be more specific, let us define two predefined power allocation coefficients $l_{1}$ and $l_{2}$ with $l_{1}>l_{2}$ and $l_{1}^{2}+l_{2}^{2}=1$. For the $i$-th SISO channel, when $\frac{\alpha_{i}}{\sqrt{d_{1}^{\tau}}}>\frac{\beta_{i}}{\sqrt{d_{2}^{\tau}}}$, i.e., user 1 has a stronger channel gain than user 2, $s_{i}$ is designed as $s_{i}=l_{2} s_{1 i}+l_{1} s_{2 i}$, where $s_{1 i}$ and $s_{2 i}$ are the information bearing messages for user 1 and user 2 , respectively. On
the other hand, for the $i$-th SISO channel, when $\frac{\alpha_{i}}{\sqrt{d_{1}^{\tau}}} \leq \frac{\beta_{i}}{\sqrt{d_{2}^{\tau}}}$, i.e., user 2 has a stronger channel gain than user $1, s_{i}$ is designed as $s_{i}=l_{1} s_{1 i}+l_{2} s_{2 i}$. Since Gaussian codebooks are employed, the input symbols $s_{1 i}$ and $s_{2 i}$ are independent zero mean complex Gaussian random variables. Moreover, as shown in (9), after applying the proposed GSVD-NOMA scheme, the base station decomposes the MIMO channels of the two users into parallel SISO channels. For simplicity, it is assumed that the base station allocates the same power to each parallel SISO channe 4 , i.e., $\mathcal{E}\left\{\left|s_{i}\right|^{2}\right\}=\mathcal{E}\left\{\left|s_{1 i}\right|^{2}\right\}=\mathcal{E}\left\{\left|s_{2 i}\right|^{2}\right\}=1, i \in\{1, \cdots, n\}$. Recall that $w_{i}^{2}=\alpha_{i}^{2} / \beta_{i}^{2}, i \in\{1, \cdots, n\}$, are the squared generalized singular values. Then, when $\frac{\alpha_{i}}{\sqrt{d_{1}^{\tau}}}>\frac{\beta_{i}}{\sqrt{d_{2}^{\tau}}}$, i.e., $w_{i}^{2}>\frac{d_{1}^{\tau}}{d_{2}^{\tau}}$, SIC is performed at user 1 to decode $s_{1 i}$ and the information rates of $s_{1 i}$ and $s_{2 i}$ can be expressed as

$$
\begin{equation*}
R_{1 i}=\log \left(1+\frac{P \alpha_{i}^{2} l_{2}^{2}}{t^{2} d_{1}^{\tau} N_{0}}\right) \quad \text { and } \quad R_{2 i}=\log \left(1+\frac{P \beta_{i}^{2} l_{1}^{2}}{P \beta_{i}^{2} l_{2}^{2}+t^{2} d_{2}^{\tau} N_{0}}\right) \tag{10}
\end{equation*}
$$

respectively, where $N_{0}$ denotes the noise power. On the other hand, when $\frac{\alpha_{i}}{\sqrt{d_{1}^{\tau}}} \leq \frac{\beta_{i}}{\sqrt{d_{2}^{\tau}}}$, i.e., $w_{i}^{2} \leq \frac{d_{1}^{\tau}}{d_{2}^{\tau}}$, SIC is performed at user 2 to decode $s_{2 i}$ and the information rates of $s_{1 i}$ and $s_{2 i}$ can be expressed as

$$
\begin{equation*}
R_{1 i}=\log \left(1+\frac{P \alpha_{i}^{2} l_{1}^{2}}{P \alpha_{i}^{2} l_{2}^{2}+t^{2} d_{1}^{\tau} N_{0}}\right) \quad \text { and } \quad R_{2 i}=\log \left(1+\frac{P \beta_{i}^{2} l_{2}^{2}}{t^{2} d_{2}^{\tau} N_{0}}\right) \tag{11}
\end{equation*}
$$

respectively. Note that (10) and (11) provide the achievable instantaneous rates of the users for given user channel matrices $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$.
2) The case when $m<n<2 m$ : As shown in Section 【II-A, when $m<n<2 m, \boldsymbol{\Sigma}_{1}$ and $\Sigma_{2}$ can be expressed as in (5). So, when $m<n<2 m$, after applying detection matrices $\mathbf{U}$ and $\mathbf{V}$, user 1 and user 2 can observe the following signals:

$$
\mathbf{U y}_{1}=\frac{\sqrt{P}}{t \sqrt{d_{1}^{\tau}}}\left[\begin{array}{c}
s_{1}  \tag{12}\\
\vdots \\
s_{r} \\
\alpha_{1} s_{r+1} \\
\vdots \\
\alpha_{s} s_{m}
\end{array}\right]+\mathbf{n}_{1}^{\prime} \quad \text { and } \quad \mathbf{V y}_{2}=\frac{\sqrt{P}}{t \sqrt{d_{2}^{\tau}}}\left[\begin{array}{c}
\beta_{1} s_{r+1} \\
\vdots \\
\beta_{s} s_{m} \\
s_{m+1} \\
\vdots \\
s_{n}
\end{array}\right]+\mathbf{n}_{2}^{\prime},
$$

[^4]respectively. Again, we assume that $\mathcal{E}\left\{\left|s_{i}\right|^{2}\right\}=1, i \in\{1, \cdots, n\}$. For $i \in\{1, \cdots, r\}, s_{i}$ is observed by user 1 only, and the observations of $s_{i}, i \in\{1, \cdots, r\}$, at the two users can be expressed as
\[

$$
\begin{equation*}
y_{1 i}=\frac{\sqrt{P}}{t \sqrt{d_{1}^{\tau}}} s_{i}+n_{1 i} \quad \text { and } \quad y_{2 i}=0, \tag{13}
\end{equation*}
$$

\]

respectively. So, when $i \in\{1, \cdots, r\}$, we adopt the OMA transmission strategy, and $s_{i}$ is designed as $s_{i}=s_{1 i}$, i.e., $s_{i}$ only contains the message for user 1 . Therefore, the corresponding information rate of $s_{1 i}, i \in\{1, \cdots, r\}$, can be expressed as $R_{1 i}=\log \left(1+\frac{P}{t^{2} d_{1}^{\tau} N_{0}}\right)$. When $i \in\{m+1, \cdots, n\}, s_{i}$ is observed by user 2 only and the observations of $s_{i}, i \in\{m+1, \cdots, n\}$, at the two users can be expressed as

$$
\begin{equation*}
y_{1 i}=0 \quad \text { and } \quad y_{2 i}=\frac{\sqrt{P}}{t \sqrt{d_{2}^{\tau}}} s_{i}+n_{2 i}, \tag{14}
\end{equation*}
$$

respectively. Hence, when $i \in\{m+1, \cdots, n\}$, we adopt the OMA transmission strategy, and $s_{i}$ is designed as $s_{i}=s_{2 i}$, i.e., $s_{i}$ only contains the message for user 2 . The corresponding information rate of $s_{2 i}, i \in\{m+1, \cdots, n\}$, can be expressed as $R_{2 i}=\log \left(1+\frac{P}{t^{2} d_{2}^{\tau} N_{0}}\right)$. When $i \in\{r+1, \cdots, m\}, s_{i}$ is observed by both users. The observations of $s_{i}, i \in\{r+1, \cdots, m\}$, at the two users can be expressed as

$$
\begin{equation*}
y_{1 i}=\frac{\sqrt{P}}{t \sqrt{d_{1}^{\tau}}} \alpha_{i-r} s_{i}+n_{1 i} \quad \text { and } \quad y_{2 i}=\frac{\sqrt{P}}{t \sqrt{d_{2}^{\tau}}} \beta_{i-r} s_{i}+n_{2 i}, \tag{15}
\end{equation*}
$$

respectively. Again, by applying F-NOMA, when $\frac{\alpha_{i-r}}{\sqrt{d_{1}^{\tau}}}>\frac{\beta_{i-r}}{\sqrt{d_{2}^{\tau}}}, i \in\{r+1, \cdots, m\}$, i.e., $w_{i-r}^{2}>$ $\frac{d_{1}^{\tau}}{d_{2}^{\tau}}, s_{i}, i \in\{r+1, \cdots, m\}$, is designed as $s_{i}=l_{2} s_{1 i}+l_{1} s_{2 i}$, where $s_{1 i}$ and $s_{2 i}$ are the information bearing messages for user 1 and user 2, respectively. When $\frac{\alpha_{i-r}}{\sqrt{d_{1}^{\tau}}} \leq \frac{\beta_{i-r}}{\sqrt{d_{2}^{\tau}}}, i \in\{r+1, \cdots, m\}$, i.e., $w_{i-r}^{2} \leq \frac{d_{1}^{\tau}}{d_{2}^{\tau}}, s_{i}, i \in\{r+1, \cdots, m\}$, is designed as $s_{i}=l_{1} s_{1 i}+l_{2} s_{2 i}$. Note that $\mathcal{E}\left\{\left|s_{i}\right|^{2}\right\}=$ $\mathcal{E}\left\{\left|s_{1 i}\right|^{2}\right\}=\mathcal{E}\left\{\left|s_{2 i}\right|^{2}\right\}=1, i \in\{r+1, \cdots, m\}$. Therefore, when $w_{i-r}^{2}>\frac{d_{1}^{\tau}}{d_{2}^{\tau}}, i \in\{r+1, \cdots, m\}$, SIC is carried out at user 1 and the information rates of $s_{1 i}$ and $s_{2 i}, i \in\{r+1, \cdots, m\}$, can be expressed as

$$
\begin{equation*}
R_{1 i}=\log \left(1+\frac{P \alpha_{i-r}^{2} l_{2}^{2}}{t^{2} d_{1}^{\tau} N_{0}}\right) \quad \text { and } \quad R_{2 i}=\log \left(1+\frac{P \beta_{i-r}^{2} l_{1}^{2}}{P \beta_{i-r}^{2} l_{2}^{2}+t^{2} d_{2}^{\tau} N_{0}}\right) \tag{16}
\end{equation*}
$$

respectively. When $w_{i-r}^{2} \leq \frac{d_{1}^{\tau}}{d_{2}^{\tau}}$, SIC is carried by user 2 and the information rates of $s_{1 i}$ and $s_{2 i}$, $i \in\{r+1, \cdots, m\}$, can be expressed as

$$
\begin{equation*}
R_{1 i}=\log \left(1+\frac{P \alpha_{i-r}^{2} l_{1}^{2}}{P \alpha_{i-r}^{2} l_{2}^{2}+t^{2} d_{1}^{\tau} N_{0}}\right) \quad \text { and } \quad R_{2 i}=\log \left(1+\frac{P \beta_{i-r}^{2} l_{2}^{2}}{t^{2} d_{2}^{\tau} N_{0}}\right) \tag{17}
\end{equation*}
$$

respectively. Note that, similar to (10) and (11), (16) and (17) provide the achievable instantaneous rates of the users for given user channel matrices $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$.
3) The case when $2 m \leq n$ : As shown in Section III-A, when $2 m \leq n, \boldsymbol{\Sigma}_{1}$ and $\boldsymbol{\Sigma}_{2}$ can be expressed as in (6). So, when $2 m \leq n$, after applying detection matrices $\mathbf{U}$ and $\mathbf{V}$, user 1 and user 2 observe

$$
\mathbf{U} \mathbf{y}_{1}=\frac{\sqrt{P}}{t \sqrt{d_{1}^{\tau}}}\left[\begin{array}{c}
s_{1}  \tag{18}\\
\vdots \\
s_{m}
\end{array}\right]+\mathbf{n}_{1}^{\prime} \quad \text { and } \quad \mathbf{V} \mathbf{y}_{2}=\frac{\sqrt{P}}{t \sqrt{d_{2}^{\tau}}}\left[\begin{array}{c}
s_{n-m+1} \\
\vdots \\
s_{n}
\end{array}\right]+\mathbf{n}_{2}^{\prime}
$$

respectively. Again, we assume that the base station allocates the same power to each parallel SISO channel, i.e., $\mathcal{E}\left\{\left|s_{i}\right|^{2}\right\}=1, i \in\{1, \cdots, m, n-m+1, \cdots, n\}$. For $i \in\{1, \cdots, m\}, s_{i}$ is observed by user 1 only, and the observations of $s_{i}, i \in\{1, \cdots, m\}$, at the two users can be expressed as

$$
\begin{equation*}
y_{1 i}=\frac{\sqrt{P}}{t \sqrt{d_{1}^{\tau}}} s_{i}+n_{1 i} \quad \text { and } \quad y_{2 i}=0 \tag{19}
\end{equation*}
$$

respectively. So, when $i \in\{1, \cdots, m\}$, we adopt the OMA transmission strategy, and $s_{i}$ is designed as $s_{i}=s_{1 i}$, i.e., $s_{i}$ only contains the information bearing message for user 1 . Therefore, the corresponding information rate of $s_{1 i}, i \in\{1, \cdots, m\}$, can be expressed as $R_{1 i}=\log \left(1+\frac{P}{t^{2} d_{1}^{\tau} N_{0}}\right)$. When $i \in\{n-m+1, \cdots, n\}, s_{i}$ is observed by user 2 only, and the observations of $s_{i}, i \in\{n-m+1, \cdots, n\}$, at the two users can be expressed as

$$
\begin{equation*}
y_{1 i}=0 \quad \text { and } \quad y_{2 i}=\frac{\sqrt{P}}{t \sqrt{d_{2}^{\tau}}} s_{i}+n_{2 i} \tag{20}
\end{equation*}
$$

respectively. Hence, when $i \in\{n-m+1, \cdots, n\}$, we also adopt the OMA transmission strategy, and $s_{i}$ is designed as $s_{i}=s_{2 i}$, i.e., $s_{i}$ only contains the information bearing message for user 2. The corresponding information rate of $s_{2 i}, i \in\{n-m+1, \cdots, n\}$, can be expressed as $R_{2 i}=\log \left(1+\frac{P}{t^{2} d_{2}^{\tau} N_{0}}\right)$. For $i \in\{m+1, \cdots, n-m\}, s_{i}$ is observed neither by user 1 nor by user 2. So, $s_{i}, i \in\{m+1, \cdots, n-m\}$, is set as $s_{i}=0$ in order to save transmit power.

## IV. Performance Analysis

In this section, we evaluate the performance of the proposed GSVD-NOMA protocol for the asymptotic case, where the numbers of transmit and receive antennas approach infinity while the
ratio of the numbers of receive and transmit antennas remains constant, i.e., $\eta=\frac{m}{n}$ is constan $\sqrt{5}$. To this end, we first characterize the limiting distribution of the squared generalized singular values $w_{i}^{2}=\alpha_{i}^{2} / \beta_{i}^{2}$ in the considered asymptotic regime. Then, we study the characteristics of power normalization factor $t$. Finally, we calculate the normalized average individual rates of the two users. The derived analytical results can be applied when the base station and the users have large numbers of antennas. For example, in heterogenous networks, a macro base station may communicate with two micro base stations by using GSVD-NOMA. In this case, it is reasonable to assume that both the transmitter and the receivers have large numbers of antennas. Furthermore, the numerical results in Section $\nabla$ reveal that when the base station and the users have moderate numbers of antennas (e.g. $m=2, n=5$ ), the derived analytical results provide still accurate approximations. Therefore, our asymptotic results provide insight into the performance achieved by the proposed GSVD-NOMA scheme for the realistic case where the base station and the users have moderate numbers of antennas.

## A. The limiting distribution of the squared generalized singular values

As shown in Section III-B, the rates of the users depend on the squared generalized singular values of the two users' channel matrices. Therefore, in order to calculate the normalized average individual rates of the two users, we first need to characterize the limiting distribution of the squared generalized singular values.

Assuming that the elements of $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$ are i.i.d. complex Gaussian random variables with zero mean and unit variance, the distribution of the squared generalized singular values $w_{i}^{2}$ can be characterized as follows. First, let us define the empirical distribution function (e.d.f.) of $k$ random variables $v_{i}, i \in\{1, \cdots, k\}$, as $\mathbf{F}_{v_{i}}^{k}(x)$, where

$$
\begin{equation*}
\mathbf{F}_{v_{i}}^{k}(x)=\frac{1}{k} \sum_{j=1}^{k} 1\left\{v_{j} \leq x\right\}, \tag{21}
\end{equation*}
$$

[^5]and $1\{\cdot\}$ is the indicator function. Next, we define function $f_{y, y^{\prime}}(x)$ as follows:
\[

f_{y, y^{\prime}}(x)= $$
\begin{cases}\frac{\left(1-y^{\prime}\right) \sqrt{\left(x-\left(\frac{1-g\left(y, y^{\prime}\right)}{1--y^{\prime}}\right)^{2}\right)\left(\left(\frac{1+g\left(y, y^{\prime}\right)}{1-y^{\prime}}\right)^{2}-x\right)}}{2 \pi x\left(x y^{\prime}+y\right)}, & \left(\frac{1-g\left(y, y^{\prime}\right)}{1-y^{\prime}}\right)^{2}<x<\left(\frac{1+g\left(y, y^{\prime}\right)}{1-y^{\prime}}\right)^{2}  \tag{22}\\ 0, & \text { otherwise }\end{cases}
$$
\]

where $x$ is the argument of the function, $y$ and $y^{\prime}$ are two parameters of the function, and $g\left(y, y^{\prime}\right)=\sqrt{1-(1-y)\left(1-y^{\prime}\right)}$.

Then, equipped with there definitions, we can characterize the distribution of the squared generalized singular values $w_{i}^{2}$ as in the following theorem.

Theorem 1: Suppose that $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$ are two $m \times n$ matrices whose elements are i.i.d. complex Gaussian random variables with zero mean and unit variance and their GSVD is defined as in (3).

- When $m \geq n$, almost surely, $\mathbf{F}_{w_{i}}^{n}(x)$, the e.d.f. of their squared generalized singular values $w_{i}^{2}, i \in\{1, \cdots, n\}$, converges, as $m, n \rightarrow \infty$ with $\frac{m}{n} \rightarrow \eta$, to a nonrandom cumulative distribution function (c.d.f.) $\mathbf{F}_{w_{i}}(x)$, whose probability density function (p.d.f.) is $f_{\frac{1}{\eta}, \frac{1}{\eta}}(x)$.
- When $m<n<2 m$, almost surely, $\mathbf{F}_{w_{i}}^{2 m-n}(x)$, the e.d.f. of their squared generalized singular values $w_{i}^{2}, i \in\{1, \cdots, 2 m-n\}$, converges, as $m, n \rightarrow \infty$ with $\frac{m}{n} \rightarrow \eta$, to a nonrandom c.d.f. $\mathbf{F}_{w_{i}}(x)$, whose p.d.f. is $\frac{\eta}{(2 \eta-1)^{2}} f_{\frac{\eta}{2 \eta-1}, \eta}\left(\frac{x}{2 \eta-1}\right)$.
- When $2 m \leq n, \boldsymbol{\Sigma}_{1}$ and $\boldsymbol{\Sigma}_{2}$ are deterministic and given by $\left[\begin{array}{lll}\mathbf{I}_{m} & \mathbf{O}_{m \times(n-m)}\end{array}\right]$ and $\left[\begin{array}{ll}\mathbf{O}_{m \times(n-m)} & \mathbf{I}_{m}\end{array}\right]$, respectively.

Proof: See Appendix A.

## B. The value of the power normalization factor $t$

As shown in Section 【II-B, the power normalization factor $t$ is a key parameter affecting the rates of the users. In this subsection, we characterize the value of this power normalization factor. Recall that $t^{2}$ can be expressed as $t^{2}=\mathcal{E}\left\{\operatorname{trace}\left(\mathbf{Q s s}^{H} \mathbf{Q}^{H}\right)\right\}$, where $\mathbf{Q}$ is the GSVD decomposition matrix defined as in (3), and $s$ is the information bearing symbol vector. Moreover, as shown in Section III-B, $\mathcal{E}\left\{\mathbf{S s}^{H}\right\}$ can be expressed as $\mathcal{E}\left\{\mathbf{s s}^{H}\right\}=\left\{\begin{array}{ll}\mathbf{I}_{n} & 2 m \geq n \\ \mathbf{B} & 2 m<n\end{array}\right.$, where $n \times n$ matrix $\mathbf{B}=\operatorname{diag}\left[\mathbf{I}_{m}, \mathbf{O}_{(n-2 m) \times(n-2 m)}, \mathbf{I}_{m}\right]$. Thus, $t^{2}$ can be further expressed as follows:

$$
t^{2}=\mathcal{E}\left\{\operatorname{trace}\left(\mathbf{Q} \mathbf{Q s}^{H} \mathbf{Q}^{H}\right)\right\}=\left\{\begin{array}{ll}
\mathcal{E}\left\{\operatorname{trace}\left(\mathbf{Q Q}^{H}\right)\right\} & 2 m \geq n  \tag{23}\\
\mathcal{E}\left\{\operatorname{trace}\left(\mathbf{\mathbf { Q B Q } ^ { H }}\right)\right\} & 2 m<n
\end{array} .\right.
$$

Moreover, let us define $\mathbf{H}=\binom{\mathbf{H}_{1}}{\mathbf{H}_{2}}$ and $\eta=\frac{m}{n}$. Then, the value of $t^{2}$ is discussed as follows.

- For the case of $2 m>n$, the value of $t^{2}$ is given by the following theorem.

Theorem 2: Assume that $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$ are two $m \times n$ matrices whose elements are i.i.d. complex Gaussian random variables with zero mean and unit variance and their GSVD is defined as in (3). Then, for $2 m>n, t^{2}=\mathcal{E}\left\{\operatorname{trace}\left(\mathbf{Q Q}^{H}\right)\right\}=\mathcal{E}\left\{\operatorname{trace}\left(\left(\mathbf{H}^{H} \mathbf{H}\right)^{-1}\right)\right\}=\frac{1}{2 \eta-1}$. Proof: See Appendix B.

- For the case of $2 m<n$, the value of $t^{2}$ is given by the following theorem.

Theorem 3: Assume that $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$ are two $m \times n$ matrices whose elements are i.i.d. complex Gaussian random variables with zero mean and unit variance and their GSVD is defined as in (3). Then, for $2 m<n, t^{2}=\mathcal{E}\left\{\operatorname{trace}\left(\mathbf{Q B Q}^{H}\right)\right\}=\mathcal{E}\left\{\operatorname{trace}\left(\left(\mathbf{H H}^{H}\right)^{-1}\right)\right\}=$ $\frac{1}{1 /(2 \eta)-1}$.
Proof: See Appendix C.

- For the case of $2 m=n$, from Appendix $\mathbf{B}$, it can be shown that $t^{2}=\mathcal{E}\left\{\operatorname{trace}\left(\mathbf{Q Q}^{H}\right)\right\}=$ $\mathcal{E}\left\{\operatorname{trace}\left(\left(\mathbf{H}^{H} \mathbf{H}\right)^{-1}\right)\right\}$. Note that the elements of $\mathbf{H}$ are i.i.d. complex Gaussian random variables with zero mean and unit variance. From [21], it is easy to show that when $2 m=n$, $\mathcal{E}\left\{\operatorname{trace}\left(\left(\mathbf{H}^{H} \mathbf{H}\right)^{-1}\right)\right\}$ approaches infinity. Therefore, when $n=2 m$, the average power of the GSVD precoding matrix, $t^{2}$, approaches infinity. Thus, when $n=2 m$, the long-term power constraint is not applicable, and in practice more sophisticated precoding schemes based on an instantaneous power constraint should be applied at the base station. Studying such precoders is beyond the scope of this paper.


## C. The normalized average individual rates of the two users

In this subsection, we focus on the normalized average individual rates of the two users.

1) The case of $m \geq n$ : As shown in Section III-B1, when $m \geq n, s_{1 i}$ and $s_{2 i}, i \in\{1, \cdots, n\}$, are broadcasted by the base station, where $s_{1 i}$ and $s_{2 i}$ are the information bearing messages for user 1 and user 2, respectively. The instantaneous information rates of $s_{1 i}$ and $s_{2 i}$ can be expressed as in (10) and (11). Recall that the squared generalized singular values are defined as $w_{i}^{2}=\alpha_{i}^{2} / \beta_{i}^{2}, i \in\{1, \cdots, n\}$. From the fact that $\alpha_{i}^{2}+\beta_{i}^{2}=1$, we can obtain that $\alpha_{i}^{2}=\frac{w_{i}^{2}}{1+w_{i}^{2}}$ and $\beta_{i}^{2}=\frac{1}{1+w_{i}^{2}}$. Therefore, when $m \geq n$, substituting $\alpha_{i}^{2}=\frac{w_{i}^{2}}{1+w_{i}^{2}}$ and $\beta_{i}^{2}=\frac{1}{1+w_{i}^{2}}$ into (10), we can
show that when $w_{i}^{2}>\frac{d_{1}^{\tau}}{d_{2}^{\tau}}$, the instantaneous information rates of $s_{1 i}$ and $s_{2 i}$ can be expressed as follows:

$$
\begin{equation*}
R_{1 i}=\log \left(1+\frac{P \alpha_{i}^{2} l_{2}^{2}}{t^{2} d_{1}^{\tau} N_{0}}\right)=\log \left(1+\frac{P w_{i}^{2} l_{2}^{2}}{t^{2} d_{1}^{\tau} N_{0}\left(1+w_{i}^{2}\right)}\right) \tag{24}
\end{equation*}
$$

and

$$
\begin{equation*}
R_{2 i}=\log \left(1+\frac{P \beta_{i}^{2} l_{1}^{2}}{P \beta_{i}^{2} l_{2}^{2}+t^{2} d_{2}^{\tau} N_{0}}\right)=\log \left(1+\frac{P l_{1}^{2}}{P l_{2}^{2}+t^{2} d_{2}^{\tau} N_{0}\left(1+w_{i}^{2}\right)}\right) . \tag{25}
\end{equation*}
$$

Furthermore, substituting $\alpha_{i}^{2}=\frac{w_{i}^{2}}{1+w_{i}^{2}}$ and $\beta_{i}^{2}=\frac{1}{1+w_{i}^{2}}$ into (11), we can show that when $w_{i}^{2} \leq \frac{d_{1}^{\tau}}{d_{2}^{\tau}}$, the instantaneous information rates of $s_{1 i}$ and $s_{2 i}$ can be expressed as as follows:

$$
\begin{equation*}
R_{1 i}=\log \left(1+\frac{P \alpha_{i}^{2} l_{1}^{2}}{P \alpha_{i}^{2} l_{2}^{2}+t^{2} d_{1}^{\tau} N_{0}}\right)=\log \left(1+\frac{P w_{i}^{2} l_{1}^{2}}{P w_{i}^{2} l_{2}^{2}+t^{2} d_{1}^{\tau} N_{0}\left(1+w_{i}^{2}\right)}\right) \tag{26}
\end{equation*}
$$

and

$$
\begin{equation*}
R_{2 i}=\log \left(1+\frac{P \beta_{i}^{2} l_{2}^{2}}{t^{2} d_{2}^{\tau} N_{0}}\right)=\log \left(1+\frac{P l_{2}^{2}}{t^{2} d_{2}^{\tau} N_{0}\left(1+w_{i}^{2}\right)}\right) . \tag{27}
\end{equation*}
$$

So when $m \geq n, m, n \rightarrow \infty$ with $\frac{m}{n} \rightarrow \eta$, the normalized average individual rates of the two users can be characterized as in the following corollary.

Corollary 1: When $m \geq n, m, n \rightarrow \infty$ with $\frac{m}{n} \rightarrow \eta$, the normalized average individual rates of user 1 and user 2 can be expressed as follows:

$$
\begin{align*}
R_{1}= & \frac{1}{\eta}\left(C\left(\frac{d_{1}^{\tau}}{d_{2}^{\tau}}, B, \frac{d_{1}^{\tau} N_{0}}{d_{1}^{\tau} N_{0}+P l_{2}^{2}(2 \eta-1)}\right)-C\left(\frac{d_{1}^{\tau}}{d_{2}^{\tau}}, B, 1\right)\right.  \tag{28}\\
& +\log \left(\frac{d_{1}^{\tau} N_{0}+P l_{2}^{2}(2 \eta-1)}{d_{1}^{\tau} N_{0}}\right) D\left(\frac{d_{1}^{\tau}}{d_{2}^{\tau}}, B\right)+C\left(A, \frac{d_{1}^{\tau}}{d_{2}^{\tau}}, \frac{d_{1}^{\tau} N_{0}}{d_{1}^{\tau} N_{0}+P(2 \eta-1)}\right) \\
& \left.-C\left(A, \frac{d_{1}^{\tau}}{d_{2}^{\tau}}, \frac{d_{1}^{\tau} N_{0}}{d_{1}^{\tau} N_{0}+P l_{2}^{2}(2 \eta-1)}\right)+\log \left(\frac{d_{1}^{\tau} N_{0}+P(2 \eta-1)}{d_{1}^{\tau} N_{0}+P l_{2}^{2}(2 \eta-1)}\right) D\left(A, \frac{d_{1}^{\tau}}{d_{2}^{\tau}}\right)\right)
\end{align*}
$$

and

$$
\begin{align*}
R_{2}= & \frac{1}{\eta}\left(C\left(\frac{d_{1}^{\tau}}{d_{2}^{\tau}}, B, 1+\frac{P(2 \eta-1)}{d_{2}^{\tau} N_{0}}\right)-C\left(\frac{d_{1}^{\tau}}{d_{2}^{\tau}}, B, 1+\frac{P l_{2}^{2}(2 \eta-1)}{d_{2}^{\tau} N_{0}}\right)\right. \\
& \left.+C\left(A, \frac{d_{1}^{\tau}}{d_{2}^{\tau}}, \frac{P l_{2}^{2}(2 \eta-1)+d_{2}^{\tau} N_{0}}{d_{2}^{\tau} N_{0}}\right)-C\left(A, \frac{d_{1}^{\tau}}{d_{2}^{\tau}}, 1\right)\right), \tag{29}
\end{align*}
$$

where $A=\left(\frac{1-\sqrt{1-(1-1 / \eta)(1-1 / \eta)}}{1-1 / \eta}\right)^{2}, B=\left(\frac{1+\sqrt{1-(1-1 / \eta)(1-1 / \eta)}}{1-1 / \eta}\right)^{2}, C\left(y_{1}, y_{2}, y_{3}\right)=\int_{y_{1}}^{y_{2}} \log (x+$ $\left.y_{3}\right) f_{\frac{1}{\eta}, \frac{1}{\eta}}(x) d x$, and $D\left(y_{1}, y_{2}\right)=\int_{y_{1}}^{y_{2}} f_{\frac{1}{\eta}, \frac{1}{\eta}}(x) d x$.

Proof: See Appendix D.
2) The case when $m<n<2 m$ : As shown in Section III-B2, when $m<n<2 m$, $s_{1 i}$, $i \in\{1, \cdots, m\}$, and $s_{2 i}, i \in\{r+1, \cdots, n\}$, are broadcasted by the base station, where $r=n-m$, and $s_{1 i}$ and $s_{2 i}$ are the information bearing messages for user 1 and user 2 , respectively. The instantaneous information rate of $s_{1 i}, i \in\{1, \cdots, r\}$, is given by $R_{1 i}=\log \left(1+\frac{P}{t^{2} d_{1}^{f} N_{0}}\right)$. The instantaneous information rate of $s_{2 i}, i \in\{m+1, \cdots, n\}$, is given by $R_{2 i}=\log \left(1+\frac{P}{t^{2} d_{2}^{\tau} N_{0}}\right)$. When $i \in\{r+1, \cdots, m\}, s_{1 i}$ and $s_{2 i}$ are observed by both users. The instantaneous information rates of $s_{1 i}$ and $s_{2 i}, i \in\{r+1, \cdots, m\}$, are given by (16) and (17). Recall that when $m<n<2 m$, the squared generalized singular values are defined as $w_{i}^{2}=\alpha_{i}^{2} / \beta_{i}^{2}, i \in\{1, \cdots, 2 m-n\}$. As $\alpha_{i}^{2}+\beta_{i}^{2}=1$, we obtain $\alpha_{i}^{2}=\frac{w_{i}^{2}}{1+w_{i}^{2}}$ and $\beta_{i}^{2}=\frac{1}{1+w_{i}^{2}}$. Therefore, when $m<n<2 m$, from (16) and (17), it can be shown that when $w_{i-r}^{2}>\frac{d_{1}^{\tau}}{d_{2}^{\tau}}, i \in\{r+1, \cdots, m\}$, the instantaneous information rates of $s_{1 i}$ and $s_{2 i}$ can be expressed as follows:

$$
\begin{equation*}
R_{1 i}=\log \left(1+\frac{P \alpha_{i-r}^{2} l_{2}^{2}}{t^{2} d_{1}^{\tau} N_{0}}\right)=\log \left(1+\frac{P w_{i-r}^{2} l_{2}^{2}}{t^{2} d_{1}^{\tau} N_{0}\left(1+w_{i-r}^{2}\right)}\right) \tag{30}
\end{equation*}
$$

and

$$
\begin{equation*}
R_{2 i}=\log \left(1+\frac{P \beta_{i-r}^{2} l_{1}^{2}}{P \beta_{i-r}^{2} l_{2}^{2}+t^{2} d_{2}^{\tau} N_{0}}\right)=\log \left(1+\frac{P l_{1}^{2}}{P l_{2}^{2}+t^{2} d_{2}^{\tau} N_{0}\left(1+w_{i-r}^{2}\right)}\right) \tag{31}
\end{equation*}
$$

On the other hand, when $w_{i-r}^{2} \leq \frac{d_{1}^{\tau}}{d_{2}^{\tau}}, i \in\{r+1, \cdots, m\}$, the instantaneous information rates of $s_{1 i}$ and $s_{2 i}$ can be expressed as follows:

$$
\begin{equation*}
R_{1 i}=\log \left(1+\frac{P \alpha_{i-r}^{2} l_{1}^{2}}{P \alpha_{i-r}^{2} l_{2}^{2}+t^{2} d_{1}^{\tau} N_{0}}\right)=\log \left(1+\frac{P w_{i-r}^{2} l_{1}^{2}}{P w_{i-r}^{2} l_{2}^{2}+t^{2} d_{1}^{\tau} N_{0}\left(1+w_{i-r}^{2}\right)}\right) \tag{32}
\end{equation*}
$$

and

$$
\begin{equation*}
R_{2 i}=\log \left(1+\frac{P \beta_{i-r}^{2} l_{2}^{2}}{t^{2} d_{2}^{\tau} N_{0}}\right)=\log \left(1+\frac{P l_{2}^{2}}{t^{2} d_{2}^{\tau} N_{0}\left(1+w_{i-r}^{2}\right)}\right) . \tag{33}
\end{equation*}
$$

Thus, when $m<n<2 m, m, n \rightarrow \infty$ with $\frac{m}{n} \rightarrow \eta$, the normalized average individual rates of the two users can be characterized as in the following corollary.

Corollary 2: When $m<n<2 m, m, n \rightarrow \infty$ with $\frac{m}{n} \rightarrow \eta$, the normalized average individual rates of user 1 and user 2 can be expressed as follows:

$$
\begin{align*}
R_{1}= & \left(2-\frac{1}{\eta}\right)\left(G\left(\frac{d_{1}^{\tau}}{d_{2}^{\tau}}, F, \frac{d_{1}^{\tau} N_{0}}{d_{1}^{\tau} N_{0}+P l_{2}^{2}(2 \eta-1)}\right)-G\left(\frac{d_{1}^{\tau}}{d_{2}^{\tau}}, F, 1\right)\right.  \tag{34}\\
& +\log \left(\frac{d_{1}^{\tau} N_{0}+P l_{2}^{2}(2 \eta-1)}{d_{1}^{\tau} N_{0}}\right) H\left(\frac{d_{1}^{\tau}}{d_{2}^{\tau}}, F\right)+G\left(E, \frac{d_{1}^{\tau}}{d_{2}^{\tau}}, \frac{d_{1}^{\tau} N_{0}}{d_{1}^{\tau} N_{0}+P(2 \eta-1)}\right) \\
& \left.-G\left(E, \frac{d_{1}^{\tau}}{d_{2}^{\tau}}, \frac{d_{1}^{\tau} N_{0}}{d_{1}^{\tau} N_{0}+P l_{2}^{2}(2 \eta-1)}\right)+\log \left(\frac{d_{1}^{\tau} N_{0}+P(2 \eta-1)}{d_{1}^{\tau} N_{0}+P l_{2}^{2}(2 \eta-1)}\right) H\left(E, \frac{d_{1}^{\tau}}{d_{2}^{\tau}}\right)\right) \\
& +\left(\frac{1}{\eta}-1\right) \log \left(1+\frac{P(2 \eta-1)}{d_{1}^{\tau} N_{0}}\right)
\end{align*}
$$

and

$$
\begin{align*}
R_{2}= & \left(2-\frac{1}{\eta}\right)\left(G\left(\frac{d_{1}^{\tau}}{d_{2}^{\tau}}, F, 1+\frac{P(2 \eta-1)}{d_{2}^{\tau} N_{0}}\right)-G\left(\frac{d_{1}^{\tau}}{d_{2}^{\tau}}, F, 1+\frac{P l_{2}^{2}(2 \eta-1)}{d_{2}^{\tau} N_{0}}\right)\right. \\
& \left.+G\left(E, \frac{d_{1}^{\tau}}{d_{2}^{\tau}}, \frac{P l_{2}^{2}(2 \eta-1)+d_{2}^{\tau} N_{0}}{d_{2}^{\tau} N_{0}}\right)-G\left(E, \frac{d_{1}^{\tau}}{d_{2}^{\tau}}, 1\right)\right) \\
& +\left(\frac{1}{\eta}-1\right) \log \left(1+\frac{P(2 \eta-1)}{d_{2}^{\tau} N_{0}}\right), \tag{35}
\end{align*}
$$

where $E=(2 \eta-1)\left(\frac{1-\sqrt{1-(1-\eta /(2 \eta-1))(1-\eta)}}{1-\eta}\right)^{2}, F=(2 \eta-1)\left(\frac{1+\sqrt{1-(1-\eta /(2 \eta-1))(1-\eta)}}{1-\eta}\right)^{2}$, $G\left(y_{1}, y_{2}, y_{3}\right)=\int_{y_{1}}^{y_{2}} \log \left(x+y_{3}\right) \frac{\eta}{(2 \eta-1)^{2}} f_{\frac{\eta}{2 \eta-1}, \eta}(x /(2 \eta-1)) d x, \quad$ and $\quad H\left(y_{1}, y_{2}\right) \quad=$ $\int_{y_{1}}^{y_{2}} \frac{\eta}{(2 \eta-1)^{2}} f_{\frac{\eta}{2 \eta-1}, \eta}(x /(2 \eta-1)) d x$.

Proof: Following steps similar to those in the proof of Corollary 1, the normalized average individual rates of the two users can be obtained.
3) The case when $2 m<n$ : As shown in Section 【II-B3, when $2 m<n, s_{1 i}$ and $s_{2 i}, i \in$ $\{1, \cdots, n\}$, are broadcasted by the base station. The information rates of $s_{1 i}$ and $s_{2 i}$ can be expressed as $R_{1 i}=\log \left(1+\frac{P}{t^{2} d_{1}^{f} N_{0}}\right)$ and $R_{2 i}=\log \left(1+\frac{P}{t^{2} d_{2}^{f} N_{0}}\right)$, respectively. Moreover, when $2 m<n, t^{2}=\operatorname{trace}\left(\mathbf{Q s s}^{H} \mathbf{Q}^{H}\right)=\operatorname{trace}\left(\mathbf{Q B Q}{ }^{H}\right)$. Theorem 3 obtains that when $2 m<n$, $t^{2}=\operatorname{trace}\left(\mathbf{Q B Q}^{H}\right)$ converges, as $m, n \rightarrow \infty$ with $\frac{m}{n} \rightarrow \eta$, to $\frac{1}{1 /(2 \eta)-1}$. Thus, when $2 m<n$, $m, n \rightarrow \infty$ with $\frac{m}{n} \rightarrow \eta$, the normalized average individual rates of the two users can be expressed as follows:

$$
\begin{equation*}
R_{1}=\log \left(1+\frac{P(1 /(2 \eta)-1)}{d_{1}^{\tau} N_{0}}\right) \quad \text { and } \quad R_{2}=\log \left(1+\frac{P(1 /(2 \eta)-1)}{d_{2}^{\tau} N_{0}}\right) \tag{36}
\end{equation*}
$$

## V. Numerical Results

In this section, we first provide computer simulation results by focusing on the MIMO downlink communication scenario with one base stations and two users, where the base station and the users have large but finite numbers of antennas, to demonstrate the performance of the proposed GSVD-NOMA scheme, and to verify the correctness of the developed analytical results. Then, we propose a hybrid NOMA scheme to extend the proposed GSVD-NOMA scheme to the MIMO downlink communication scenario with more than two users, and provide numerical results to demonstrate the performance of this hybrid NOMA scheme.

In Fig. 1, we compare the normalized average sum rates achieved by OMA and GSVD-NOMA when $m=28, d_{1}=10 \mathrm{~m}, d_{2}=100 \mathrm{~m}, \tau=2, N_{0}=-35 \mathrm{dBm}$, and $l_{2}^{2}=0.2$. Here, "BPCU" denotes bit per channel use. From this figure, we observe that the proposed GSVD-NOMA


Fig. 1. Comparing the normalized average sum rates achieved Fig. 2. The normalized average sum rates achieved by OMA by OMA and GSVD-NOMA. and GSVD-NOMA versus distance $d_{2}$.
scheme outperforms conventional OMA by a considerable margin. As the transmit power $P$ increases, the performance gap between the two protocols also increases, which confirms that the proposed GSVD-NOMA scheme can exploit the spatial degrees of freedom of the channel more efficiently than OMA in the high SNR regime. Moreover, when $\eta=\frac{m}{n}$ changes from $\frac{4}{5}$ to $\frac{2}{5}$, i.e., the number of the transmit antennas increases, the normalized average sum rate achieved by OMA stays practically constant while the normalized average sum rate achieved by the proposed GSVD-NOMA scheme increases considerably. This can be explained as follows, with OMA, the base station can perform SVD to convert the $n \times m$ MIMO channel of each user into $k_{1}$ parallel SISO channels, where $k_{1}=\min \{m, n\}$. Thus, in OMA systems, when $n$ exceeds $m$, the number of the parallel SISO channels, $k_{1}$, will stop growing, which results in the saturation of the normalized average sum rate. On the other hand, as shown in Section $\Pi I-B$, for the proposed GSVD-NOMA scheme, the base station decomposes the $n \times m$ MIMO channels of both users into $k_{2}$ parallel SISO channels, where $k_{2}=\min \{2 m, n\}$. Therefore, when $n$ increases from $m$ to $2 m$, while $k_{1}$ will stay constant, $k_{2}$ will grow. Hence, while the normalized average sum rate achieved by OMA saturates quickly, that of the proposed GSVD-NOMA scheme does not.

Fig. 2 compares the normalized average sum rates achieved by OMA and GSVD-NOMA as a function of $d_{2}$ when $m=28, d_{1}=50 \mathrm{~m}, \tau=2, P=30 \mathrm{dBm}, N_{0}=-35 \mathrm{dBm}$, and $l_{2}^{2}=0.2$. From this figure, we observe that the proposed GSVD-NOMA outperforms conventional OMA for all considered values of $d_{2}$. In fact, regardless of whether $d_{1}<d_{2}$ or $d_{1}>d_{2}$, the proposed GSVD-NOMA achieves a higher sum rate than conventional OMA. Moreover, when the number of the transmit antennas increases, the performance gap between the two protocols increases

[^6]

Fig. 3. The normalized average rates achieved by the QR- Fig. 4. The normalized average individual rates of the two users NOMA scheme in [16] and the proposed GSVD-NOMA scheme. versus $n$.
from 1.5 BPCU to 9.5 BPCU for $d_{2}=10 \mathrm{~m}$.
In Fig. 3, the normalized average rates achieved by the proposed GSVD-NOMA scheme are compared to those achieved by the QR-NOMA scheme in [16]. In this figure, it is assumed that $m=40, n=50, d_{1}=10 \mathrm{~m}, d_{2}=10 \mathrm{~m}, \tau=2, N_{0}=-35 \mathrm{dBm}$, and $l_{2}^{2}=0.2$. For QR-NOMA [16], the channel of one user will become extremely weak, which negatively affects the average sum rate. Therefore, as shown in Fig. 3, when increasing the transmit power, the average sum rate of the proposed GSVD-NOMA scheme increases more rapidly than that of the QR-NOMA scheme. Moreover, for the proposed GSVD-NOMA scheme, the normalized average individual rates of the two users are identical, i.e., $R_{1}=R_{2}$. In contrast, for the QR-NOMA scheme [16], $R_{1}$ is much greater than $R_{2}$. Therefore, the proposed GSVD-NOMA scheme ensures better fairness between the two users than QR-NOMA.

Fig. 4 shows the normalized average individual rates of the two users with $n$ increasing from 5 to 30 , when $d_{1}=10 \mathrm{~m}, d_{2}=40 \mathrm{~m}, \tau=2, P=15 \mathrm{dBm}, N_{0}=-35 \mathrm{dBm}$, and $l_{2}^{2}=0.2$. As can be seen from this figure, when $m$ and $n$ is small (e.g. $m=2, n=5$ ), the numerical results still coincide with the analytical results perfectly, which validates the analytical results developed in Section IV-C. In this figure, although the normalized average rates ( $R_{1}$ and $R_{2}$ ) decrease as $m$ increases, the overall average rates of the users ( $m R_{1}$ and $m R_{2}$ ) still increase with $m$.

Fig. [5 shows the normalized average individual rates of the two users when $m=2, d_{1}=10 \mathrm{~m}$, $d_{2}=100 \mathrm{~m}, \tau=2, N_{0}=-35 \mathrm{dBm}$, and $l_{2}^{2}=0.2$. From this figure, we observe that the numerical results match the analytical results perfectly and the normalized average individual rates of both users increase when the number of transmit antennas increases, i.e., increasing the number of transmit antennas can improve the performance of the proposed GSVD-NOMA scheme. Moreover, as expected, user 1 which is closer to the base station achieves a higher


Fig. 5. The normalized average individual rates of the two users Fig. 6. The normalized average rates of the two users for versus transmission power. different power allocation coefficients.
average rate than user 2.
Please note that in the considered asymptotic scenario, when both users have the same distance to the base station, i.e, $d_{1}=d_{2}$, for the proposed GSVD-NOMA scheme, the normalized average individual rates of both users are the same, i.e., $R_{1}=R_{2}$, as shown in Fig. 3. Hence, the proposed GSVD-NOMA scheme ensures user fairness. When $d_{1} \neq d_{2}$, there can be significant difference between the users' channel conditions. In this scenario, the use of the proposed GSVD-NOMA scheme can efficiently explore the dynamic range of channel difference to improve the overall system throughput. However, we note that, when $d_{1} \neq d_{2}$, the near user will achieve a higher information rate as shown in Fig. 4 and Fig. [5, i.e., the overall system throughput is improved at a price of reduced user fairness. This is reasonable, since a user with better channel conditions can support a higher achievable rate, and allocating more bandwidth resources to the weak user is not beneficial to the spectral efficiency.

Fig. 6 shows the impact of different power allocation coefficients on GSVD-NOMA when $m=7, n=5, d_{1}=10 \mathrm{~m}, d_{2}=100 \mathrm{~m}, \tau=2$, and $N_{0}=-35 \mathrm{dBm}$. From this figure, we observe that when the power allocation coefficient $l_{2}^{2}$ increases, the normalized average rate $R_{1}$ of user 1 increases and the normalized average rate $R_{2}$ of user 2 decreases, since more powers have been allocated to the near user. On the other hand, for large $P$, i.e., in the high SNR regime, the normalized average sum rate of the two users $R_{1}+R_{2}$ remains almost the same when the power allocation coefficient $l_{2}^{2}$ changes from 0.2 to 0.4 . Thus, in the high SNR regime, $l_{2}^{2}$ has an insignificant impact on the average sum rate.

Fig. 7 compares the average sum rates achieved by DPC and GSVD-NOMA in a downlink massive MIMO scenario with one base station and two users, where the base station is equipped with $n$ antennas and each user is equipped with single antenna. Moreover, it is assumed that



Fig. 7. The average sum rates achieved by DPC and GSVD- Fig. 8. The impact of user clustering on the hybrid GSVDNOMA. NOMA system
$d_{1}=10 \mathrm{~m}, d_{2}=10 \mathrm{~m}, \tau=2$, and $N_{0}=-35 \mathrm{dBm}$. From this figure, we observe that the numerical results match the analytical results perfectly and the average sum rate of GSVDNOMA approaches that of DPC, which demonstrates the performance of the proposed GSVDNOMA scheme, and verifies the correctness of the developed analytical results.

Although this paper focuses on the special case of two NOMA users, the proposed GSVDNOMA scheme can be easily extended to the MIMO downlink communication scenario with more than two users by using a hybrid MA approach. In particular, consider a downlink communication scenario with one base station and multiple users. The base station first divides all users into multiple groups, where each group contains two users. Then, the proposed GSVDNOMA scheme can be implemented within each group and different groups are allocated with orthogonal bandwidth resources. This hybrid NOMA scheme can be used to e.g. upgrade an existing OMA system without changing its fundamental resource blocks, while realizing the performance gains of GSVD-NOMA. The reason for including only two users in each group is to reduce the processing complexity of the SIC receiver, and the two-user pairing consideration is consistent with the specifications in 3GPP-LTE system [26].

In order to demonstrate the performance of hybrid GSVD-NOMA, we consider a multiuser scenario with one base station and four users. The distance between user 1 (user 2) and the base station is $d_{1}=15 \mathrm{~m}\left(d_{2}=10 \mathrm{~m}\right)$, and the distance between user 3 (user 4$)$ and the base station is $d_{3}=200 \mathrm{~m}\left(d_{4}=300 \mathrm{~m}\right)$. In Fig. 8, we compare the normalized average sum rates achieved by OMA and hybrid GSVD-NOMA for $m=40, n=50, \tau=2, N_{0}=-35 \mathrm{dBm}$, and $l_{2}^{2}=0.2$. From the figure, one can observe that no matter how the users are grouped, the proposed GSVD-NOMA scheme always achieves a larger normalized average sum rate than OMA. Moreover, pairing user 1 with user 2 results in the worst performance among the three
possible pairing strategies. This suggests that it is preferable to pair users which are close to the base station with users which are far from the base station 7 . In addition, the performance gap between NOMA and OMA grows as the transmit power $P$ increases.

## VI. Conclusion

In this paper, a MIMO downlink communication scenario with one base station and two users has been considered, where each user is equipped with $m$ antennas and the base station is equipped with $n$ antennas. To fully exploit the available spatial degrees of freedom, a transmission protocol which combines GSVD and NOMA has been proposed. The performance of the proposed protocols has been characterized in the asymptotic regime, where the numbers of transmit and receive antennas approach infinity. To be more specific, we have characterized the limiting distribution of the squared generalized singular values of the two users' channel matrices. Then, the normalized average individual rates of the two users have been analyzed. The proposed GSVD-NOMA scheme uses GSVD to decompose the two-user MIMO broadcast channel into broadcast two-user SISO channels, without interference among different transmit antennas. Moreover, the provided numerical results have shown that GSVD-NOMA achieves considerable improvements in terms of the normalized average sum rate compared to conventional OMA and QR-NOMA in [16], and when the base station and the users have moderate numbers of antennas (e.g. $m=2, n=5$ ), the derived analytical results still provide good approximations. Furthermore, we use a hybrid MA approach to extend the proposed GSVD-NOMA scheme to the MIMO downlink communication scenario with more than two users, where the base station first divides the users into different groups, then the proposed GSVD-NOMA scheme is implemented within each group, and different groups are allocated with orthogonal bandwidth resources.

## Appendix A: Proof of Theorem 1

Since the elements of $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$ are i.i.d. complex Gaussian random variables, they are full rank with probability one [22]. Thus, in this appendix, it is assumed that $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$ are full rank.

[^7]
## A. The case when $m \geq n$

First, we revisit Zha's method [23] to construct the GSVD of the two $m \times n$ channel matrices $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$. Then, we obtain the limiting distribution of the squared generalized singular values based on this GSVD construction method.

1) Steps of GSVD when $m \geq n$ : When $m \geq n$, we can construct the GSVD of the two $m \times n$ channel matrices $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$ with the following steps:

- Denote the SVD of matrix $\mathbf{H}_{2}$ by

$$
\begin{equation*}
\mathbf{U}_{2} \mathbf{H}_{2} \mathbf{V}_{2}=\binom{\mathbf{O}_{(m-n) \times n}}{\boldsymbol{\Sigma}_{H 2}} \tag{37}
\end{equation*}
$$

where $\boldsymbol{\Sigma}_{H 2}=\operatorname{diag}\left(z_{1}, z_{2}, \cdots z_{n}\right)$ and $z_{1} \geq z_{2} \cdots \geq z_{n}$. Moreover, define $\mathbf{Q}_{1}$ as follows:

$$
\begin{equation*}
\mathbf{Q}_{1}=\mathbf{V}_{2} \operatorname{diag}\left(\frac{1}{z_{1}}, \cdots, \frac{1}{z_{n}}\right) . \tag{38}
\end{equation*}
$$

Then, define $\mathbf{H}_{1}^{\prime}$ and $\mathbf{H}_{2}^{\prime}$ as follows:

$$
\begin{equation*}
\mathbf{H}_{1}^{\prime}=\mathbf{H}_{1} \mathbf{Q}_{1}=\mathbf{H}_{1} \mathbf{V}_{2} \operatorname{diag}\left(\frac{1}{z_{1}}, \cdots, \frac{1}{z_{n}}\right) \quad \text { and } \quad \mathbf{H}_{2}^{\prime}=\mathbf{U}_{2} \mathbf{H}_{2} \mathbf{Q}_{1}=\binom{\mathbf{O}_{(m-n) \times n}}{\mathbf{I}_{n}} . \tag{39}
\end{equation*}
$$

- Next, define the SVD of matrix $\mathbf{H}_{1}^{\prime}$ as

$$
\begin{equation*}
\mathbf{U}_{3} \mathbf{H}_{1}^{\prime} \mathbf{V}_{3}=\binom{\operatorname{diag}\left(w_{1}, \cdots, w_{n}\right)}{\mathbf{O}_{(m-n) \times n}} \tag{40}
\end{equation*}
$$

where $w_{1} \geq w_{2} \cdots \geq w_{n}$ are the $n$ ordered singular values of $\mathbf{H}_{1}^{\prime}$. Furthermore, define $\mathbf{Q}_{2}$ as follows:

$$
\begin{equation*}
\mathbf{Q}_{2}=\mathbf{V}_{3} \operatorname{diag}\left(\beta_{1}, \beta_{2}, \cdots \beta_{n}\right) \tag{41}
\end{equation*}
$$

where $\beta_{i}=\left(1+w_{i}^{2}\right)^{-1 / 2}$.

- Finally, it can be shown that

$$
\begin{equation*}
\mathbf{U}_{3} \mathbf{H}_{1}^{\prime} \mathbf{Q}_{2}=\mathbf{U}_{3} \mathbf{H}_{1} \mathbf{Q}_{1} \mathbf{Q}_{2}=\binom{\operatorname{diag}\left(\alpha_{1}, \cdots, \alpha_{n}\right)}{\mathbf{O}_{(m-n) \times n}} \tag{42}
\end{equation*}
$$

where $\alpha_{i}=w_{i} \beta_{i}=\frac{w_{i}}{\left(1+w_{i}^{2}\right)^{1 / 2}}$ and it is easy to see that $\alpha_{i}^{2}+\beta_{i}^{2}=1$. Moreover, it can be shown that

$$
\begin{align*}
\left(\begin{array}{cc}
\mathbf{I}_{m-n} & \mathbf{O}_{(m-n) \times n} \\
\mathbf{O}_{n \times(m-n)} & \mathbf{V}_{3}^{H}
\end{array}\right) \mathbf{H}_{2}^{\prime} \mathbf{Q}_{2} & =\left(\begin{array}{cc}
\mathbf{I}_{m-n} & \mathbf{O}_{(m-n) \times n} \\
\mathbf{O}_{n \times(m-n)} & \mathbf{V}_{3}^{H}
\end{array}\right) \mathbf{U}_{2} \mathbf{H}_{2} \mathbf{Q}_{1} \mathbf{Q}_{2} \\
& =\binom{\mathbf{O}_{(m-n) \times n}}{\operatorname{diag}\left(\beta_{1}, \cdots, \beta_{n}\right)} \tag{43}
\end{align*}
$$

Then, for the case of $m \geq n$, we have provided the steps needed for constructing the GSVD defined as in (3) with $\mathbf{Q}=\mathbf{Q}_{1} \mathbf{Q}_{2}, \mathbf{U}=\mathbf{U}_{3}$ and $\mathbf{V}=\left(\begin{array}{cc}\mathbf{I}_{m-n} & \mathbf{O}_{(m-n) \times n} \\ \mathbf{O}_{n \times(m-n)} & \mathbf{V}_{3}^{H}\end{array}\right) \mathbf{U}_{2}$.
2) The limiting distribution of the squared generalized singular values when $m \geq n$ : From (40), it can be shown that $\mathbf{U}_{3} \mathbf{H}_{1}^{\prime} \mathbf{V}_{3}=\binom{\operatorname{diag}\left(w_{1}, \cdots, w_{n}\right)}{\mathbf{O}_{(m-n) \times n}}$, and the squared generalized singular values $w_{i}^{2}=\alpha_{i}^{2} / \beta_{i}^{2}, i \in\{1, \cdots, n\}$, are the eigenvalues of $\mathbf{H}_{1}^{\prime H} \mathbf{H}_{1}^{\prime}$, where $\mathbf{H}_{1}^{\prime}=\mathbf{H}_{1} \mathbf{Q}_{1}=\mathbf{H}_{1} \mathbf{V}_{2} \operatorname{diag}\left(\frac{1}{z_{1}}, \cdots, \frac{1}{z_{n}}\right)$. Recall that the SVD of $\mathbf{H}_{2}$ is $\mathbf{U}_{2} \mathbf{H}_{2} \mathbf{V}_{2}=$ $\binom{\mathbf{O}_{(m-n) \times n}}{\operatorname{diag}\left(z_{1}, \cdots z_{n}\right)}$, and we have that $\mathbf{V}_{2}^{H} \mathbf{H}_{2}^{H} \mathbf{H}_{2} \mathbf{V}_{2}=\operatorname{diag}\left(z_{1}, \cdots, z_{n}\right)^{2}$. Also, since $\mathbf{H}_{2}^{H} \mathbf{H}_{2}$ is a central Wishart matrix which is unitarily invariant [21], it can be shown that $\mathbf{V}_{2}$ is a Haar matrix which is independent of $\operatorname{diag}\left(z_{1}, \cdots, z_{n}\right)^{2}$. So $\mathbf{H}_{1}^{\prime}=\mathbf{H}_{1} \mathbf{V}_{2} \operatorname{diag}\left(\frac{1}{z_{1}}, \cdots, \frac{1}{z_{n}}\right)$ is the product of three independent matrices $\mathbf{H}_{1}, \mathbf{V}_{2}$, and $\operatorname{diag}\left(\frac{1}{z_{1}}, \cdots, \frac{1}{z_{n}}\right)$.

Define $\mathbf{G}_{1}=\mathbf{H}_{1} \mathbf{V}_{2}$. Then, it can be shown that $\mathbf{G}_{1}$ is an $m \times n$ matrix whose elements are i.i.d. complex Gaussian random variables with zero mean and unit variance. Furthermore, we have that $\mathbf{H}_{1}^{\prime}=\mathbf{G}_{1} \operatorname{diag}\left(\frac{1}{z_{1}}, \cdots, \frac{1}{z_{n}}\right)$ and the squared generalized singular values $w_{i}^{2}, i \in\{1, \cdots, n\}$, can be expressed as the eigenvalues of $\mathbf{H}_{1}^{\prime H} \mathbf{H}_{1}^{\prime}=\operatorname{diag}\left(\frac{1}{z_{1}}, \cdots, \frac{1}{z_{n}}\right) \mathbf{G}_{1}^{H} \mathbf{G}_{1} \operatorname{diag}\left(\frac{1}{z_{1}}, \cdots, \frac{1}{z_{n}}\right)$. Moreover, the squared generalized singular values $w_{i}^{2}, i \in\{1, \cdots, n\}$, can be also expressed as the eigenvalues of $\mathbf{G}_{1}^{H} \mathbf{G}_{1} \operatorname{diag}\left(\frac{1}{z_{1}}, \cdots, \frac{1}{z_{n}}\right)^{2}$ or the nonzero eigenvalues of $\mathbf{G}_{1} \operatorname{diag}\left(\frac{1}{z_{1}}, \cdots, \frac{1}{z_{n}}\right)^{2} \mathbf{G}_{1}^{H}$. Note that $\mathbf{G}_{1}$ is independent of $z_{i}, i \in\{1, \cdots, n\}$, where $z_{i}$ is a nonzero singular value of $\mathbf{H}_{2}$. Thus, directly from [21, Theorem 2.39 and Theorem 2.40], the e.d.f. of the nonzero eigenvalues of $\mathbf{G}_{1} \operatorname{diag}\left(\frac{1}{z_{1}}, \cdots, \frac{1}{z_{n}}\right)^{2} \mathbf{G}_{1}^{H}$, which is identical to the e.d.f. of the generalized singular values $\mathbf{F}_{w_{i}}^{n}(x)$, converges, almost surely, as $m, n \rightarrow \infty$ with $\frac{m}{n} \rightarrow \eta$, to a nonrandom c.d.f. $\mathbf{F}_{w_{i}}(x)$, whose p.d.f. is $f_{\frac{1}{\eta}, \frac{1}{\eta}}(x)$. Then, when $m \geq n$, we have characterized the limiting distribution of the squared generalized singular values $w_{i}^{2}$.
B. The case when $m<n<2 m$

Again, we first use Zha's method to construct the GSVD of the two $m \times n$ channel matrices $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$.

1) Steps of GSVD when $m<n<2 m$ : When $m<n<2 m$, we can construct the GSVD of the two $m \times n$ channel matrices $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$ with the following steps:

- First, we define the SVD of matrix $\mathbf{H}_{2}$ as

$$
\mathbf{U}_{2} \mathbf{H}_{2} \mathbf{V}_{2}=\left(\begin{array}{ll}
\mathbf{O}_{m \times(n-m)} & \operatorname{diag}\left(z_{1}, z_{2}, \cdots z_{m}\right) \tag{44}
\end{array}\right)
$$

where $z_{1} \geq z_{2} \cdots \geq z_{m}$. Let us define $\mathbf{Q}_{1}$ as follows:

$$
\mathbf{Q}_{1}=\mathbf{V}_{2}\left(\begin{array}{cc}
\mathbf{I}_{n-m} & \mathbf{O}_{(n-m) \times m}  \tag{45}\\
\mathbf{O}_{m \times(n-m)} & \operatorname{diag}\left(z_{1}, \cdots z_{m}\right)^{-1}
\end{array}\right)
$$

Then, it can be shown that

$$
\mathbf{H}_{1}^{\prime}=\mathbf{H}_{1} \mathbf{Q}_{1}=\mathbf{H}_{1} \mathbf{V}_{2}\left(\begin{array}{cc}
\mathbf{I}_{n-m} & \mathbf{O}_{(n-m) \times m}  \tag{46}\\
\mathbf{O}_{m \times(n-m)} & \operatorname{diag}\left(z_{1}, \cdots z_{m}\right)^{-1}
\end{array}\right)=\left(\begin{array}{ll}
\mathbf{H}_{11}^{\prime} & \mathbf{H}_{12}^{\prime}
\end{array}\right),
$$

where $\mathbf{H}_{11}^{\prime}=\mathbf{H}_{1} \mathbf{V}_{2}\binom{\mathbf{I}_{n-m}}{\mathbf{O}_{m \times(n-m)}}$ and $\mathbf{H}_{12}^{\prime}=\mathbf{H}_{1} \mathbf{V}_{2}\binom{\mathbf{O}_{(n-m) \times m}}{\operatorname{diag}\left(z_{1}, \cdots z_{m}\right)^{-1}}$. Also, $\mathbf{H}_{2}^{\prime}$ can be defined as follows:

$$
\begin{align*}
\mathbf{H}_{2}^{\prime} & =\mathbf{U}_{2} \mathbf{H}_{2} \mathbf{Q}_{1}=\left(\begin{array}{ll}
\mathbf{O}_{m \times(n-m)} & \operatorname{diag}\left(z_{1}, \cdots z_{m}\right)
\end{array}\right)\left(\begin{array}{cc}
\mathbf{I}_{n-m} & \mathbf{O}_{(n-m) \times m} \\
\mathbf{O}_{m \times(n-m)} & \operatorname{diag}\left(z_{1}, \cdots z_{m}\right)^{-1}
\end{array}\right) \\
& =\left(\begin{array}{ll}
\mathbf{O}_{m \times(n-m)} & \mathbf{I}_{m}
\end{array}\right) . \tag{47}
\end{align*}
$$

- It is easy to see that the rank of $\mathbf{H}_{11}^{\prime}$ is $r$, where $r=n-m$. Then, we can define the SVD of matrix $\mathbf{H}_{11}^{\prime}$ as follows:

$$
\begin{equation*}
\mathbf{U}_{11} \mathbf{H}_{11}^{\prime} \mathbf{V}_{11}=\binom{\boldsymbol{\Sigma}_{A}}{\mathbf{O}_{(m-r) \times r}} \tag{48}
\end{equation*}
$$

where $\boldsymbol{\Sigma}_{A}=\operatorname{diag}\left(t_{1}, \cdots, t_{r}\right)$ with $t_{1} \geq t_{2} \cdots \geq t_{r}>0$. Furthermore, let us define $\mathbf{Q}_{2}$ as follows:

$$
\mathbf{Q}_{2}=\left(\begin{array}{cc}
\mathbf{V}_{11} & \mathbf{O}_{r \times m}  \tag{49}\\
\mathbf{O}_{m \times r} & \mathbf{I}_{m}
\end{array}\right)\left(\begin{array}{cc}
\Sigma_{A}^{-1} & \mathbf{O}_{r \times m} \\
\mathbf{O}_{m \times r} & \mathbf{I}_{m}
\end{array}\right)
$$

Then, it can be shown that

$$
\begin{align*}
& \mathbf{H}_{2}^{\prime \prime}=\mathbf{H}_{2}^{\prime} \mathbf{Q}_{2}=\left(\begin{array}{ll}
\mathbf{O}_{m \times(n-m)} & \mathbf{I}_{m}
\end{array}\right) \mathbf{Q}_{2}=\left(\begin{array}{ll}
\mathbf{O}_{m \times(n-m)} & \mathbf{I}_{m}
\end{array}\right), \quad \text { and }  \tag{50}\\
& \mathbf{H}_{1}^{\prime \prime}=\mathbf{U}_{11} \mathbf{H}_{1}^{\prime} \mathbf{Q}_{2}=\left(\begin{array}{ll}
\mathbf{U}_{11} \mathbf{H}_{11}^{\prime} & \mathbf{U}_{11} \mathbf{H}_{12}^{\prime}
\end{array}\right)\left(\begin{array}{cc}
\mathbf{V}_{11} & \mathbf{O}_{r \times m} \\
\mathbf{O}_{m \times r} & \mathbf{I}_{m}
\end{array}\right)\left(\begin{array}{cc}
\boldsymbol{\Sigma}_{A}^{-1} & \mathbf{O}_{r \times m} \\
\mathbf{O}_{m \times r} & \mathbf{I}_{m}
\end{array}\right) \\
& =\left(\binom{\boldsymbol{\Sigma}_{A}}{\mathbf{O}_{(m-r) \times r}} \quad \mathbf{U}_{11} \mathbf{H}_{12}^{\prime}\right)\left(\begin{array}{cc}
\boldsymbol{\Sigma}_{A}^{-1} & \mathbf{O}_{r \times m} \\
\mathbf{O}_{m \times r} & \mathbf{I}_{m}
\end{array}\right) \\
& =\left(\binom{\mathbf{I}_{r}}{\mathbf{O}_{(m-r) \times r}} \quad \mathbf{U}_{11} \mathbf{H}_{12}^{\prime}\right)=\left(\begin{array}{cc}
\mathbf{I}_{r} & \mathbf{A}_{13} \\
\mathbf{O}_{(m-r) \times r} & \mathbf{A}_{23}
\end{array}\right) \text {, } \tag{51}
\end{align*}
$$

where $\binom{\mathbf{A}_{13}}{\mathbf{A}_{23}}=\mathbf{U}_{11} \mathbf{H}_{12}^{\prime}, \mathbf{A}_{13} \in \mathbb{C}^{r \times(n-r)}$, and $\mathbf{A}_{23} \in \mathbb{C}^{(m-r) \times(n-r)}$.

- It is easy to see that the rank of $\mathbf{A}_{23}$ is $m-r$. Let us define $q=2 m-n=m-r$. Then, we can rewrite the SVD of matrix $\mathbf{A}_{23}$ as

$$
\mathbf{U}_{23} \mathbf{A}_{23} \mathbf{V}_{23}=\left(\begin{array}{cc}
\boldsymbol{\Sigma}_{A_{23}} & \mathbf{O}_{q \times r} \tag{52}
\end{array}\right)
$$

where $\boldsymbol{\Sigma}_{A_{23}}=\operatorname{diag}\left(w_{1}, \cdots, w_{q}\right)$ with $w_{1} \geq w_{2} \cdots \geq w_{q}>0$. Also, let us define $\mathbf{S}_{2}=$ $\operatorname{diag}\left(\beta_{1}, \cdots, \beta_{q}\right)$ with $\beta_{i}=\left(1+w_{i}^{2}\right)^{-1 / 2}$ and $\mathbf{Q}_{3}$ as follows:

$$
\mathbf{Q}_{3}=\left(\begin{array}{cc}
\mathbf{I}_{r} & -\mathbf{A}_{13}  \tag{53}\\
\mathbf{O}_{m \times r} & \mathbf{I}_{m}
\end{array}\right)\left(\begin{array}{cc}
\mathbf{I}_{r} & \mathbf{O}_{r \times m} \\
\mathbf{O}_{m \times r} & \mathbf{V}_{23}
\end{array}\right) \operatorname{diag}\left(\mathbf{I}_{r}, \mathbf{S}_{2}, \mathbf{I}_{r}\right) .
$$

- Finally, it can be shown that

$$
\begin{align*}
& \mathbf{V}_{23}^{H} \mathbf{H}_{2}^{\prime \prime} \mathbf{Q}_{3}=\mathbf{V}_{23}^{H} \mathbf{U}_{2} \mathbf{H}_{2} \mathbf{Q}_{1} \mathbf{Q}_{2} \mathbf{Q}_{3}=\mathbf{V}_{23}^{H}\left(\begin{array}{cc}
\mathbf{O}_{m \times r} & \mathbf{I}_{m}
\end{array}\right)\left(\begin{array}{cc}
\mathbf{I}_{r} & -\mathbf{A}_{13} \\
\mathbf{O}_{m \times r} & \mathbf{I}_{m}
\end{array}\right)  \tag{54}\\
& \times\left(\begin{array}{cc}
\mathbf{I}_{r} & \mathbf{O}_{r \times m} \\
\mathbf{O}_{m \times r} & \mathbf{V}_{23}
\end{array}\right) \operatorname{diag}\left(\mathbf{I}_{r}, \mathbf{S}_{2}, \mathbf{I}_{r}\right)=\left(\begin{array}{ccc}
\mathbf{O}_{q \times r} & \mathbf{S}_{2} & \mathbf{O}_{q \times r} \\
\mathbf{O}_{r \times r} & \mathbf{O}_{r \times q} & \mathbf{I}_{r}
\end{array}\right) .
\end{align*}
$$

Moreover, it can be shown that

$$
\begin{align*}
& \left(\begin{array}{cc}
\mathbf{I}_{r} & \mathbf{O}_{r \times q} \\
\mathbf{O}_{q \times r} & \mathbf{U}_{23}
\end{array}\right) \mathbf{H}_{1}^{\prime \prime} \mathbf{Q}_{3}=\left(\begin{array}{cc}
\mathbf{I}_{r} & \mathbf{O}_{r \times q} \\
\mathbf{O}_{q \times r} & \mathbf{U}_{23}
\end{array}\right) \mathbf{U}_{11} \mathbf{H}_{1} \mathbf{Q}_{1} \mathbf{Q}_{2} \mathbf{Q}_{3}=\left(\begin{array}{cc}
\mathbf{I}_{r} & \mathbf{O}_{r \times q} \\
\mathbf{O}_{q \times r} & \mathbf{U}_{23}
\end{array}\right) \\
& \times\left(\begin{array}{cc}
\mathbf{I}_{r} & \mathbf{A}_{13} \\
\mathbf{O}_{q \times r} & \mathbf{A}_{23}
\end{array}\right)\left(\begin{array}{cc}
\mathbf{I}_{r} & -\mathbf{A}_{13} \\
\mathbf{O}_{m \times r} & \mathbf{I}_{m}
\end{array}\right)\left(\begin{array}{cc}
\mathbf{I}_{r} & \mathbf{O}_{r \times m} \\
\mathbf{O}_{m \times r} & \mathbf{V}_{23}
\end{array}\right) \operatorname{diag}\left(\mathbf{I}_{r}, \mathbf{S}_{2}, \mathbf{I}_{r}\right)= \\
& \left(\begin{array}{ccc}
\mathbf{I}_{r} & \mathbf{O}_{r \times q} & \mathbf{O}_{r \times r} \\
\mathbf{O}_{q \times r} & \boldsymbol{\Sigma}_{A_{23}} \mathbf{S}_{2} & \mathbf{O}_{q \times r}
\end{array}\right) . \tag{55}
\end{align*}
$$

Furthermore, define $\mathbf{S}_{1}=\boldsymbol{\Sigma}_{A_{23}} \mathbf{S}_{2}=\operatorname{diag}\left(\alpha_{1}, \cdots, \alpha_{q}\right)$ with $\alpha_{i}=w_{i} \beta_{i}=\frac{w_{i}}{\left(1+w_{i}^{2}\right)^{1 / 2}}$. Then, for the case $m<n<2 m$, we have provided the steps for constructing the GSVD defined in (3) with $\mathbf{Q}=\mathbf{Q}_{1} \mathbf{Q}_{2} \mathbf{Q}_{3}, \mathbf{U}=\left(\begin{array}{cc}\mathbf{I}_{r} & \mathbf{O}_{r \times q} \\ \mathbf{O}_{q \times r} & \mathbf{U}_{23}\end{array}\right) \mathbf{U}_{11}$, and $\mathbf{V}=\mathbf{V}_{23}^{H} \mathbf{U}_{2}$.
2) The limiting distribution of the squared generalized singular values when $m<n<2 m$ :

From (52), we can see that $\mathbf{U}_{23} \mathbf{A}_{23} \mathbf{V}_{23}=\left(\operatorname{diag}\left(w_{1}, \cdots, w_{2 m-n}\right) \quad \mathbf{O}_{(2 m-n) \times(n-m)}\right)$, and the squared generalized singular values $w_{i}^{2}=\alpha_{i}^{2} / \beta_{i}^{2}, i \in\{1, \cdots, 2 m-n\}$, are the eigenvalues of $\mathbf{A}_{23} \mathbf{A}_{23}^{H}$. Recall that

$$
\binom{\mathbf{A}_{13}}{\mathbf{A}_{23}}=\mathbf{U}_{11} \mathbf{H}_{12}^{\prime}, \quad \mathbf{H}_{1} \mathbf{V}_{2}\left(\begin{array}{cc}
\mathbf{I}_{n-m} & \mathbf{O}_{(n-m) \times m}  \tag{56}\\
\mathbf{O}_{m \times(n-m)} & \operatorname{diag}\left(z_{1}, \cdots, z_{m}\right)^{-1}
\end{array}\right)=\left(\begin{array}{ll}
\mathbf{H}_{11}^{\prime} & \mathbf{H}_{12}^{\prime}
\end{array}\right),
$$

$\mathbf{U}_{2} \mathbf{H}_{2} \mathbf{V}_{2}=\left(\begin{array}{ll}\mathbf{O}_{m \times(n-m)} & \operatorname{diag}\left(z_{1}, \cdots, z_{m}\right)\end{array}\right)$, and $\mathbf{U}_{11} \mathbf{H}_{11}^{\prime} \mathbf{V}_{11}=\binom{\boldsymbol{\Sigma}_{A}}{\mathbf{O}_{(2 m-n) \times(n-m)}}$. Similarly, from [21], it can be shown that $\mathbf{V}_{2}$ is a Haar matrix which is independent of $\operatorname{diag}\left(z_{1}, \cdots, z_{m}\right)$. Let us define $\mathbf{P}=\mathbf{H}_{1} \mathbf{V}_{2}=\left(\begin{array}{ll}\mathbf{P}_{1} & \mathbf{P}_{2}\end{array}\right)$ with $\mathbf{P}_{1} \in \mathbb{C}^{m \times(n-m)}$ and $\mathbf{P}_{2} \in \mathbb{C}^{m \times m}$. Then, it can be shown that $\mathbf{P}$ is an $m \times n$ matrix whose elements are i.i.d. complex Gaussian random variables with zero mean and unit variance and independent of $\operatorname{diag}\left(z_{1}, \cdots, z_{m}\right)$. It is easy to see that $\mathbf{P}_{1}=\mathbf{H}_{11}^{\prime}$, so $\mathbf{H}_{11}^{\prime}$ is independent of $\mathbf{P}_{2}$. Thus, $\mathbf{U}_{11}$ is independent of $\mathbf{P}_{2}$. Then, $\mathbf{W}=\mathbf{U}_{11} \mathbf{P}_{2}$ is an $m \times m$ matrix whose elements are i.i.d. complex Gaussian random variables with zero mean and unit variance and independent of $\operatorname{diag}\left(z_{1}, \cdots, z_{m}\right)$.

Let $\mathbf{W}^{H}=\left(\begin{array}{ll}\mathbf{W}_{1}^{H} & \mathbf{W}_{2}^{H}\end{array}\right)$ with $\mathbf{W}_{1} \in \mathbb{C}^{(n-m) \times m}$ and $\mathbf{W}_{2} \in \mathbb{C}^{(2 m-n) \times m}$. Then, we can rewrite $\mathbf{A}_{23}$ as $\mathbf{A}_{23}=\mathbf{W}_{2} \operatorname{diag}\left(z_{1}, \cdots, z_{m}\right)^{-1}$. The squared generalized singular values $w_{i}^{2}$, $i \in\{1, \cdots, 2 m-n\}$, can be expressed as the eigenvalues of $\mathbf{W}_{2} \operatorname{diag}\left(z_{1}, \cdots, z_{m}\right)^{-2} \mathbf{W}_{2}^{H}$. Note that $\mathbf{W}_{2}$ is independent of $z_{i}, i \in\{1, \cdots, m\}$, where $z_{i}$ is a nonzero singular value of $\mathbf{H}_{2}$. Thus, directly from [21, Theorem 2.39 and Theorem 2.40], the e.d.f. of the nonzero eigenvalues of $\mathbf{W}_{2} \operatorname{diag}\left(z_{1}, \cdots, z_{m}\right)^{-2} \mathbf{W}_{2}^{H}$, which is identical to the e.d.f. of the generalized singular values $\mathbf{F}_{w_{i}}^{2 m-n}(x)$, converges, almost surely, as $m, n \rightarrow \infty$ with $\frac{m}{n} \rightarrow \eta$, to a nonrandom c.d.f. $\mathbf{F}_{w_{i}}(x)$, whose p.d.f. is $\frac{\eta}{(2 \eta-1)^{2}} \frac{\eta}{\frac{\eta}{2 \eta-1}, \eta}\left(\frac{x}{2 \eta-1}\right)$. Then, when $m<n<2 m$, we have characterized the limiting distribution of the squared generalized singular values $w_{i}^{2}$.

## C. The case when $2 m \leq n$

For the case of $2 m \leq n$, the construction of the GSVD is different from those described before. In the following, we again use Zha's method to construct the GSVD of the two $m \times n$ channel matrices $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$.

1) Steps of GSVD when $2 m<n$ : When $2 m<n$, we can construct the GSVD of the two $m \times n$ channel matrices $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$ with the following steps:

- First, we define the SVD of matrix $\mathbf{H}_{2}$ as

$$
\mathbf{U}_{2} \mathbf{H}_{2} \mathbf{V}_{2}=\left(\begin{array}{ll}
\mathbf{O}_{m \times(n-m)} & \operatorname{diag}\left(z_{1}, z_{2}, \cdots z_{m}\right) \tag{57}
\end{array}\right)
$$

where $z_{1} \geq z_{2} \cdots \geq z_{m}$. Let us define $\mathbf{Q}_{1}=\mathbf{V}_{2}\left(\begin{array}{cc}\mathbf{I}_{n-m} & \mathbf{O}_{(n-m) \times m} \\ \mathbf{O}_{m \times(n-m)} & \operatorname{diag}\left(z_{1}, \cdots z_{m}\right)^{-1}\end{array}\right)$. Then, it can be shown that

$$
\mathbf{H}_{1}^{\prime}=\mathbf{H}_{1} \mathbf{Q}_{1}=\mathbf{H}_{1} \mathbf{V}_{2}\left(\begin{array}{cc}
\mathbf{I}_{n-m} & \mathbf{O}_{(n-m) \times m}  \tag{58}\\
\mathbf{O}_{m \times(n-m)} & \operatorname{diag}\left(z_{1}, \cdots z_{m}\right)^{-1}
\end{array}\right)=\left(\begin{array}{ll}
\mathbf{H}_{11}^{\prime} & \mathbf{H}_{12}^{\prime}
\end{array}\right),
$$

where $\mathbf{H}_{11}^{\prime}=\mathbf{H}_{1} \mathbf{V}_{2}\binom{\mathbf{I}_{n-m}}{\mathbf{O}_{m \times(n-m)}}, \mathbf{H}_{12}^{\prime}=\mathbf{H}_{1} \mathbf{V}_{2}\binom{\mathbf{O}_{(n-m) \times m}}{\operatorname{diag}\left(z_{1}, \cdots z_{m}\right)^{-1}}, \mathbf{H}_{11}^{\prime} \in$ $\mathbb{C}^{m \times(n-m)}$, and $\mathbf{H}_{12}^{\prime} \in \mathbb{C}^{m \times m}$. Also, it can be shown that

$$
\begin{align*}
\mathbf{H}_{2}^{\prime}= & \mathbf{U}_{2} \mathbf{H}_{2} \mathbf{Q}_{1}=\left(\begin{array}{ll}
\mathbf{O}_{m \times(n-m)} & \operatorname{diag}\left(z_{1}, \cdots z_{m}\right)
\end{array}\right)  \tag{59}\\
& \times\left(\begin{array}{cc}
\mathbf{I}_{n-m} & \mathbf{O}_{(n-m) \times m} \\
\mathbf{O}_{m \times(n-m)} & \operatorname{diag}\left(z_{1}, \cdots z_{m}\right)^{-1}
\end{array}\right)=\left(\begin{array}{ll}
\mathbf{O}_{m \times(n-m)} & \mathbf{I}_{m}
\end{array}\right) .
\end{align*}
$$

- It is easy to see that the rank of $\mathbf{H}_{11}^{\prime}$ is $m$. Then, we can define the SVD of matrix $\mathbf{H}_{11}^{\prime}$ as

$$
\mathbf{U}_{11} \mathbf{H}_{11}^{\prime} \mathbf{V}_{11}=\left(\begin{array}{cc}
\boldsymbol{\Sigma}_{A} & \mathbf{O}_{m \times(n-2 m)} \tag{60}
\end{array}\right)
$$

where $\boldsymbol{\Sigma}_{A}=\operatorname{diag}\left(t_{1}, \cdots, t_{m}\right)$ with $t_{1} \geq t_{2} \cdots \geq t_{m}>0$. Furthermore, let us define $\mathbf{Q}_{2}=\left(\begin{array}{cc}\mathbf{V}_{11} & \mathbf{O}_{(n-m) \times m} \\ \mathbf{O}_{m \times(n-m)} & \mathbf{I}_{m}\end{array}\right)\left(\begin{array}{cc}\Sigma_{A}^{-1} & \mathbf{O}_{m \times(n-m)} \\ \mathbf{O}_{(n-m) \times m} & \mathbf{I}_{n-m}\end{array}\right)$. Then, it can be shown that

$$
\mathbf{H}_{2}^{\prime \prime}=\mathbf{H}_{2}^{\prime} \mathbf{Q}_{2}=\left(\begin{array}{ll}
\mathbf{O}_{m \times(n-m)} & \mathbf{I}_{m}
\end{array}\right) \mathbf{Q}_{2}=\left(\begin{array}{ll}
\mathbf{O}_{m \times(n-m)} & \mathbf{I}_{m} \tag{61}
\end{array}\right), \quad \text { and }
$$

$$
\mathbf{H}_{1}^{\prime \prime}=\mathbf{U}_{11} \mathbf{H}_{1}^{\prime} \mathbf{Q}_{2}=\left(\begin{array}{ll}
\mathbf{U}_{11} \mathbf{H}_{11}^{\prime} & \mathbf{U}_{11} \mathbf{H}_{12}^{\prime}
\end{array}\right)\left(\begin{array}{cc}
\mathbf{V}_{11} & \mathbf{O}_{(n-m) \times m}  \tag{62}\\
\mathbf{O}_{m \times(n-m)} & \mathbf{I}_{m}
\end{array}\right)
$$

$$
\times\left(\begin{array}{cc}
\boldsymbol{\Sigma}_{A}^{-1} & \mathbf{O}_{m \times(n-m)} \\
\mathbf{O}_{(n-m) \times m} & \mathbf{I}_{n-m}
\end{array}\right)=\left(\begin{array}{ccc}
\mathbf{I}_{m} & \mathbf{O}_{m \times(n-2 m)} & \mathbf{U}_{11} \mathbf{H}_{12}^{\prime}
\end{array}\right)
$$

- Finally, let us define $\mathbf{Q}_{3}=\left(\begin{array}{cc}\mathbf{I}_{n-m} \\ \mathbf{O}_{m \times(n-m)} & \binom{-\mathbf{U}_{11} \mathbf{H}_{12}^{\prime}}{\mathbf{O}_{(n-2 m) \times m}} \\ \mathbf{I}_{m}\end{array}\right)$. Then, it can be shown that $\mathbf{H}_{2}^{\prime \prime} \mathbf{Q}_{3}=\mathbf{U}_{2} \mathbf{H}_{2} \mathbf{Q}_{1} \mathbf{Q}_{2} \mathbf{Q}_{3}=\left(\begin{array}{ll}\mathbf{O}_{m \times(n-m)} & \mathbf{I}_{m}\end{array}\right) \mathbf{Q}_{3}=\left(\begin{array}{ll}\mathbf{O}_{m \times(n-m)} & \mathbf{I}_{m}\end{array}\right)$. Moreover, it
can be shown that $\mathbf{H}_{1}^{\prime \prime} \mathbf{Q}_{3}=\mathbf{U}_{11} \mathbf{H}_{1} \mathbf{Q}_{1} \mathbf{Q}_{2} \mathbf{Q}_{3}=\left(\begin{array}{ccc}\mathbf{I}_{m} & \mathbf{O}_{m \times(n-2 m)} & \mathbf{U}_{11} \mathbf{H}_{12}^{\prime}\end{array}\right) \mathbf{Q}_{3}=$ $\left(\begin{array}{ll}\mathbf{I}_{m} & \mathbf{O}_{m \times(n-m)}\end{array}\right)$. Thus, for $2 m<n$, we have provided the steps for constructing the GSVD defined in (3) with $\mathbf{Q}=\mathrm{Q}_{1} \mathrm{Q}_{2} \mathrm{Q}_{3}, \mathrm{U}=\mathrm{U}_{11}$, and $\mathrm{V}=\mathbf{U}_{2}$.

2) The squared generalized singular values when $2 m<n$ : When $2 m<n$, after using Zha's GSVD method, we have transformed the two $m \times n$ channel matrices $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$ into $\left(\begin{array}{ll}\mathbf{I}_{m} & \mathbf{O}_{m \times(n-m)}\end{array}\right)$ and $\left(\begin{array}{ll}\mathbf{O}_{m \times(n-m)} & \mathbf{I}_{m}\end{array}\right)$, respectively. Thus, the squared generalized singular values become constants in this case.

This concludes the proof of the theorem.

## Appendix B: Proof of Theorem 2

In this section, we characterize $t^{2}=\mathcal{E}\left\{\operatorname{trace}\left(\mathbf{Q Q}^{H}\right)\right\}$ when $2 m \geq n$. From (3), it can be shown that

$$
\left(\begin{array}{cc}
\mathbf{U} & \mathbf{O}_{m \times m}  \tag{63}\\
\mathbf{O}_{m \times m} & \mathbf{V}
\end{array}\right)\binom{\mathbf{H}_{1}}{\mathbf{H}_{2}} \mathbf{Q}=\binom{\boldsymbol{\Sigma}_{1}}{\boldsymbol{\Sigma}_{2}}
$$

Moreover, let us define $\Sigma=\binom{\boldsymbol{\Sigma}_{1}}{\boldsymbol{\Sigma}_{2}}$. Then, when $2 m \geq n$, from (4) and (5), it can be shown that $\boldsymbol{\Sigma}^{H} \boldsymbol{\Sigma}=\left(\begin{array}{cc}\boldsymbol{\Sigma}_{1}^{H} & \boldsymbol{\Sigma}_{2}^{H}\end{array}\right)\binom{\boldsymbol{\Sigma}_{1}}{\boldsymbol{\Sigma}_{2}}=\mathbf{I}_{n}$. Let us define $\mathbf{H}=\binom{\mathbf{H}_{1}}{\mathbf{H}_{2}}$. Then, we have that

$$
\mathbf{Q}^{H} \mathbf{H}^{H}\left(\begin{array}{cc}
\mathbf{U}^{H} & \mathbf{O}_{m \times m}  \tag{64}\\
\mathbf{O}_{m \times m} & \mathbf{V}^{H}
\end{array}\right)\left(\begin{array}{cc}
\mathbf{U} & \mathbf{O}_{m \times m} \\
\mathbf{O}_{m \times m} & \mathbf{V}
\end{array}\right) \mathbf{H Q}=\mathbf{Q}^{H} \mathbf{H}^{H} \mathbf{H Q}=\boldsymbol{\Sigma}^{H} \boldsymbol{\Sigma}=\mathbf{I}_{n} .
$$

Thus, it can be shown that $\mathbf{H}^{H} \mathbf{H}=\mathbf{Q}^{-H} \mathbf{I}_{n} \mathbf{Q}^{-1}=\mathbf{Q}^{-H} \mathbf{Q}^{-1}$ and $\mathbf{Q Q}^{H}=\left(\mathbf{H}^{H} \mathbf{H}\right)^{-1}$. Therefore, we have that when $2 m \geq n$, $\operatorname{trace}\left\{\mathbf{Q Q}^{H}\right\}=\operatorname{trace}\left\{\left(\mathbf{H}^{H} \mathbf{H}\right)^{-1}\right\}$. Then, it can be shown that when $2 m>n, t^{2}=\mathcal{E}\left\{\operatorname{trace}\left(\mathbf{Q Q}^{H}\right)\right\}=\mathcal{E}\left\{\operatorname{trace}\left(\left(\mathbf{H}^{H} \mathbf{H}\right)^{-1}\right)\right\}$. Assume that $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$ are two $m \times n$ matrices whose elements are i.i.d. complex Gaussian random variables with zero mean and unit variance. From [21, Lemma 2.10], it is easy to see that for $2 m>n$, $t^{2}=\mathcal{E}\left\{\operatorname{trace}\left(\mathbf{Q Q}^{H}\right)\right\}=\mathcal{E}\left\{\operatorname{trace}\left(\left(\mathbf{H}^{H} \mathbf{H}\right)^{-1}\right)\right\}=\frac{1}{2 \eta-1}$, where $\eta=\frac{m}{n}$.

This completes the proof of the theorem.

## Appendix C: Proof of Theorem 3

In this section, we characterize $t^{2}=\mathcal{E}\left\{\operatorname{trace}\left(\mathbf{Q B Q}{ }^{H}\right)\right\}$ when $2 m<n$, where the $n \times n$ matrix B can be expressed as $\operatorname{diag}\left[\mathbf{I}_{m}, \mathbf{O}_{(n-2 m) \times(n-2 m)}, \mathbf{I}_{m}\right]$. From (3) , it can be shown that

$$
\left(\begin{array}{cc}
\mathbf{U} & \mathbf{O}_{m \times m}  \tag{65}\\
\mathbf{O}_{m \times m} & \mathbf{V}
\end{array}\right)\binom{\mathbf{H}_{1}}{\mathbf{H}_{2}} \mathbf{Q}=\binom{\boldsymbol{\Sigma}_{1}}{\boldsymbol{\Sigma}_{2}}
$$

Moreover, let us define $\mathbf{H}=\binom{\mathbf{H}_{1}}{\mathbf{H}_{2}}, \boldsymbol{\Sigma}=\binom{\boldsymbol{\Sigma}_{1}}{\boldsymbol{\Sigma}_{2}}, \mathbf{M}=\left(\begin{array}{cc}\mathbf{U} & \mathbf{O}_{m \times m} \\ \mathbf{O}_{m \times m} & \mathbf{V}\end{array}\right)$. Then, it can be shown that $\mathbf{H}=\mathbf{M}^{H} \boldsymbol{\Sigma} \mathbf{Q}^{-1}$. Thus, we have that $\left(\mathbf{H H}^{H}\right)^{-1}=\left(\mathbf{M}^{H} \boldsymbol{\Sigma} \mathbf{Q}^{-1} \mathbf{Q}^{-H} \boldsymbol{\Sigma}^{H} \mathbf{M}\right)^{-1}$, and

$$
\begin{align*}
\operatorname{trace}\left\{\left(\mathbf{H H}^{H}\right)^{-1}\right\} & =\operatorname{trace}\left\{\left(\mathbf{M}^{H} \boldsymbol{\Sigma} \mathbf{Q}^{-1} \mathbf{Q}^{-H} \boldsymbol{\Sigma}^{H} \mathbf{M}\right)^{-1}\right\}  \tag{66}\\
& =\operatorname{trace}\left\{\mathbf{M}^{H}\left(\boldsymbol{\Sigma} \mathbf{Q}^{-1} \mathbf{Q}^{-H} \boldsymbol{\Sigma}^{H}\right)^{-1} \mathbf{M}\right\} \\
& =\operatorname{trace}\left\{\mathbf{M M}^{H}\left(\boldsymbol{\Sigma} \mathbf{Q}^{-1} \mathbf{Q}^{-H} \boldsymbol{\Sigma}^{H}\right)^{-1}\right\} \\
& =\operatorname{trace}\left\{\left(\boldsymbol{\Sigma} \mathbf{Q}^{-1} \mathbf{Q}^{-H} \boldsymbol{\Sigma}^{H}\right)^{-1}\right\}
\end{align*}
$$

From (6), it can be shown that

$$
\Sigma=\left(\begin{array}{ccc}
\mathbf{I}_{m} & \mathbf{O}_{m \times(n-2 m)} & \mathbf{O}_{m \times m}  \tag{67}\\
\mathbf{O}_{m \times m} & \mathbf{O}_{m \times(n-2 m)} & \mathbf{I}_{m}
\end{array}\right)
$$

and $\mathbf{B}=\operatorname{diag}\left[\mathbf{I}_{m}, \mathbf{O}_{(n-2 m) \times(n-2 m)}, \mathbf{I}_{m}\right]=\boldsymbol{\Sigma}^{H} \boldsymbol{\Sigma}$. Then, we have

$$
\begin{equation*}
\operatorname{trace}\left\{\mathbf{Q B} \mathbf{Q}^{H}\right\}=\operatorname{trace}\left\{\mathbf{Q} \boldsymbol{\Sigma}^{H} \boldsymbol{\Sigma} \mathbf{Q}^{H}\right\}=\operatorname{trace}\left\{\boldsymbol{\Sigma} \mathbf{Q}^{H} \mathbf{Q} \boldsymbol{\Sigma}^{H}\right\} \tag{68}
\end{equation*}
$$

Next, we show that $\boldsymbol{\Sigma} \mathbf{Q}^{H} \mathbf{Q} \boldsymbol{\Sigma}^{H}=\left(\boldsymbol{\Sigma} \mathbf{Q}^{-1} \mathbf{Q}^{-H} \boldsymbol{\Sigma}^{H}\right)^{-1}$.
Let us define $\mathbf{X}=\mathbf{U}_{11} \mathbf{H}_{12}^{\prime}$. From Appendix A, we know that when $2 m<n$, $\mathbf{Q}$ can be expressed as follows:

$$
\left.\begin{array}{rl}
\mathbf{Q}= & \mathbf{Q}_{1} \mathbf{Q}_{2} \mathbf{Q}_{3}  \tag{69}\\
= & \mathbf{V}_{2}\left(\begin{array}{cc}
\mathbf{I}_{n-m} & \mathbf{O}_{(n-m) \times m} \\
\mathbf{O}_{m \times(n-m)} & \operatorname{diag}\left(z_{1}, \cdots z_{m}\right)^{-1}
\end{array}\right)\left(\begin{array}{cc}
\mathbf{V}_{11} & \mathbf{O}_{(n-m) \times m} \\
\mathbf{O}_{m \times(n-m)} & \mathbf{I}_{m}
\end{array}\right) \\
& \times\left(\begin{array}{cc}
\operatorname{diag}\left(t_{1}, \cdots t_{m}\right)^{-1} & \mathbf{O}_{m \times(n-m)} \\
\mathbf{O}_{(n-m) \times m} & \mathbf{I}_{n-m}
\end{array}\right)\left(\begin{array}{c}
-\mathbf{X} \\
\mathbf{I}_{n-m} \\
\mathbf{O}_{(n-2 m) \times m}
\end{array}\right) \\
\mathbf{O}_{m \times(n-m)} & \mathbf{I}_{m}
\end{array}\right) . .
$$

Thus, using (69) and (67), it can be shown that

$$
\boldsymbol{\Sigma} \mathbf{Q}^{H} \mathbf{Q} \boldsymbol{\Sigma}^{H}=\left(\begin{array}{cc}
\operatorname{diag}\left(t_{1}, \cdots t_{m}\right)^{-2} & -\operatorname{diag}\left(t_{1}, \cdots t_{m}\right)^{-2} \mathbf{X}  \tag{70}\\
-\mathbf{X}^{H} \operatorname{diag}\left(t_{1}, \cdots t_{m}\right)^{-2} & \mathbf{X}^{H} \operatorname{diag}\left(t_{1}, \cdots t_{m}\right)^{-2} \mathbf{X}+\operatorname{diag}\left(z_{1}, \cdots z_{m}\right)^{-2}
\end{array}\right)
$$

Moreover, based on (69), we can show that

$$
\begin{align*}
\mathbf{Q}^{-H}= & \mathbf{V}_{2}\left(\begin{array}{cc}
\mathbf{I}_{n-m} & \mathbf{O}_{(n-m) \times m} \\
\mathbf{O}_{m \times(n-m)} & \operatorname{diag}\left(z_{1}, \cdots z_{m}\right)
\end{array}\right)\left(\begin{array}{cc}
\mathbf{V}_{11} & \mathbf{O}_{(n-m) \times m} \\
\mathbf{O}_{m \times(n-m)} & \mathbf{I}_{m}
\end{array}\right)  \tag{71}\\
& \left(\begin{array}{cc}
\operatorname{diag}\left(t_{1}, \cdots t_{m}\right) & \mathbf{O}_{m \times(n-m)} \\
\mathbf{O}_{(n-m) \times m} & \mathbf{I}_{n-m}
\end{array}\right)\left(\begin{array}{cc}
\mathbf{I}_{n-m} & \mathbf{O}_{(n-m) \times m} \\
\left(\begin{array}{lll}
\mathbf{X}^{H} & \mathbf{O}_{m \times(n-2 m)}
\end{array}\right) & \mathbf{I}_{m}
\end{array}\right) .
\end{align*}
$$

Thus, based on (71) and (67), it can be obtained that

$$
\boldsymbol{\Sigma} \mathbf{Q}^{-1} \mathbf{Q}^{-H} \boldsymbol{\Sigma}^{H}=\left(\begin{array}{cc}
\operatorname{diag}\left(t_{1}, \cdots t_{m}\right)^{2}+\mathbf{X} \operatorname{diag}\left(z_{1}, \cdots z_{m}\right)^{2} \mathbf{X}^{H} & \mathbf{X} \operatorname{diag}\left(z_{1}, \cdots z_{m}\right)^{2}  \tag{72}\\
\operatorname{diag}\left(z_{1}, \cdots z_{m}\right)^{2} \mathbf{X}^{H} & \operatorname{diag}\left(z_{1}, \cdots z_{m}\right)^{2}
\end{array}\right)
$$

Then, with (70) and (72), it can be shown that $\boldsymbol{\Sigma} \mathbf{Q}^{H} \mathbf{Q} \boldsymbol{\Sigma}^{H} \boldsymbol{\Sigma} \mathbf{Q}^{-1} \mathbf{Q}^{-H} \boldsymbol{\Sigma}^{H}=\mathbf{I}_{2 m}$ and

$$
\begin{equation*}
\boldsymbol{\Sigma} \mathbf{Q}^{H} \mathbf{Q} \boldsymbol{\Sigma}^{H}=\left(\boldsymbol{\Sigma} \mathbf{Q}^{-1} \mathbf{Q}^{-H} \boldsymbol{\Sigma}^{H}\right)^{-1} \tag{73}
\end{equation*}
$$

Finally, based on (66), (68), and (73), it can be shown that when $2 m<n$, trace $\left\{\mathrm{QBQ}^{H}\right\}=$ $\operatorname{trace}\left\{\boldsymbol{\Sigma} \mathbf{Q}^{H} \mathbf{Q} \boldsymbol{\Sigma}^{H}\right\}=\operatorname{trace}\left\{\left(\boldsymbol{\Sigma} \mathbf{Q}^{-1} \mathbf{Q}^{-H} \boldsymbol{\Sigma}^{H}\right)^{-1}\right\}=\operatorname{trace}\left\{\left(\mathbf{H H}^{H}\right)^{-1}\right\}$, where $\mathbf{H}=\binom{\mathbf{H}_{1}}{\mathbf{H}_{2}}$.
Assume that $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$ are two $m \times n$ matrices whose elements are i.i.d. complex Gaussian random variables with zero mean and unit variance. Using [21, Lemma 2.10], it is easy to see that for $2 m<n, t^{2}=\mathcal{E}\left\{\operatorname{trace}\left(\mathbf{Q B Q}^{H}\right)\right\}=\mathcal{E}\left\{\operatorname{trace}\left(\left(\mathbf{H H}^{H}\right)^{-1}\right)\right\}=\frac{1}{1 /(2 \eta)-1}$, where $\eta=\frac{m}{n}$.

This completes the proof of the theorem.

## Appendix D: Proof of Corollary 1

When $m \geq n$, from (24) and (26), the information rate of user 1 can be expressed as follows:

$$
R_{1 i}= \begin{cases}\log \left(1+\frac{P w_{i}^{2} l_{2}^{2}}{t^{2} d_{1}^{\tau} N_{0}\left(1+w_{i}^{2}\right)}\right) & w_{i}^{2}>\frac{d_{1}^{\tau}}{d_{2}^{\tau}}  \tag{74}\\ \log \left(1+\frac{P w_{i}^{2}}{P w_{i}^{2} l_{2}^{2}+t^{2} d_{1}^{2} N_{0}\left(1+w_{i}^{2}\right)}\right) & w_{i}^{2} \leq \frac{d_{1}^{\tau}}{d_{2}^{\tau}}\end{cases}
$$

where $w_{i}^{2}, i \in\{1, \cdots, n\}$, is the squared generalized singular value. From Theorem 2, it can be shown that $t^{2}=\frac{1}{2 \eta-1}$, where $\eta=\frac{m}{n}$. Then, user 1's average information rate, $\bar{R}_{1 i}^{n}$, can be expressed as follows:

$$
\begin{align*}
\bar{R}_{1 i}^{n}=\frac{1}{n} \sum_{i=1}^{n} R_{1 i}= & \int_{0}^{\infty} R_{1 i} d \mathbf{F}_{w_{i}}^{n} \\
= & \int_{\frac{d_{1}^{\tau}}{d_{2}}}^{\infty} \log \left(1+\frac{P x l_{2}^{2}}{\frac{1}{2 \eta-1} d_{1}^{\tau} N_{0}(1+x)}\right) d \mathbf{F}_{w_{i}}^{n}(x)  \tag{75}\\
& +\int_{0}^{\frac{d_{1}^{\tau}}{d_{2}^{2}}} \log \left(1+\frac{P x l_{1}^{2}}{P x l_{2}^{2}+\frac{1}{2 \eta-1} d_{1}^{\tau} N_{0}(1+x)}\right) d \mathbf{F}_{w_{i}}^{n}(x)
\end{align*}
$$

where $\mathbf{F}_{w_{i}}^{n}(x)$ is the e.d.f. of the squared generalized singular values $w_{i}^{2}, i \in\{1, \cdots, n\}$, of the two users. Moreover, Theorem 1 shows that when $m \geq n$, almost surely, $\mathbf{F}_{w_{i}}^{n}(x)$, converges, as $m, n \rightarrow \infty$ with $\frac{m}{n} \rightarrow \eta$, to a nonrandom c.d.f. $\mathbf{F}_{w_{i}}(x)$, whose p.d.f. is $f_{\frac{1}{\eta}, \frac{1}{\eta}}(x)$. Furthermore, it is easy to see that both $\log \left(1+\frac{P x l_{2}^{2}}{\frac{1}{2 \eta-1} d_{1}^{\tau} N_{0}(1+x)}\right)$ and $\log \left(1+\frac{P x l_{1}^{2}}{P x l_{2}^{2}+\frac{1}{2 \eta-1} d_{1}^{\tau} N_{0}(1+x)}\right)$ are continuous bounded functions in the domain $[0, \infty]$. Then, from the bounded convergence theorem [29, Theorem 6.3], it can be shown that as $m, n \rightarrow \infty$ with $\frac{m}{n} \rightarrow \eta, \bar{R}_{1 i}^{n}$ converges to $\bar{R}_{1 i}$, where

$$
\begin{align*}
\bar{R}_{1 i}= & \int_{\frac{d_{1}^{\tau}}{d_{2}^{\tau}}}^{\infty} \log \left(1+\frac{P x l_{2}^{2}}{\frac{1}{2 \eta-1} d_{1}^{\tau} N_{0}(1+x)}\right) d \mathbf{F}_{w_{i}}(x) \\
& +\int_{0}^{\frac{d_{1}^{\tau}}{d_{2}^{2}}} \log \left(1+\frac{P x l_{1}^{2}}{P x l_{2}^{2}+\frac{1}{2 \eta-1} d_{1}^{\tau} N_{0}(1+x)}\right) d \mathbf{F}_{w_{i}}(x) \tag{76}
\end{align*}
$$

Finally, when $m \geq n, m, n \rightarrow \infty$ with $\frac{m}{n} \rightarrow \eta$, the normalized average rate of user $1, R_{1}$, can be expressed as $\frac{1}{\eta} \bar{R}_{1 i}$ and the expression for $\frac{1}{\eta} \bar{R}_{1 i}$ is shown in Corollary 11. The normalized average individual rate of user 2 can be obtained in a similar way. Thus, the corollary is proved.

This completes the proof of the corollary.

## References

[1] Y. Saito, Y. Kishiyama, A. Benjebbour, T. Nakamura, A. Li, and K. Higuchi, "Non-orthogonal multiple access (NOMA) for cellular future radio access," in IEEE 77th Vehicular Technology Conference (VTC Spring). IEEE, Jun. 2013, pp. 1-5.
[2] Z. Ding, Y. Liu, J. Choi, Q. Sun, M. Elkashlan, I. Chih-Lin, and H. V. Poor, "Application of non-orthogonal multiple access in LTE and 5G networks," IEEE Commun. Mag., vol. 55, no. 2, pp. 185-191, Feb. 2017.
[3] P. Xu and K. Cumanan, "Optimal power allocation scheme for non-orthogonal multiple access with $\alpha$-fairness," IEEE. J. Sel. Areas Commun, vol. 35, no. 10, pp. 2357-2369, Oct. 2017.
[4] Y. Sun, D. W. K. Ng, Z. Ding, and R. Schober, "Optimal joint power and subcarrier allocation for full-duplex multicarrier non-orthogonal multiple access systems," IEEE Trans. Commun., vol. 65, no. 3, pp. 1077-1091, Mar. 2017.
[5] K. Higuchi and Y. Kishiyama, "Non-orthogonal access with random beamforming and intra-beam SIC for cellular MIMO downlink," in IEEE 78th Vehicular Technology Conference (VTC Fall). IEEE, Sept. 2013, pp. 1-5.
[6] H. Haci and H. Zhu, "Performance of non-orthogonal multiple access with a novel interference cancellation method," in Communications (ICC), 2015 IEEE International Conference on. IEEE, Jun. 2015, pp. 2912-2917.
[7] Z. Ding, Z. Yang, P. Fan, and H. V. Poor, "On the performance of non-orthogonal multiple access in 5G systems with randomly deployed users," IEEE Signal Process. Lett., vol. 21, no. 12, pp. 1501-1505, Dec. 2014.
[8] M. Al-Imari, P. Xiao, M. A. Imran, and R. Tafazolli, "Uplink non-orthogonal multiple access for 5G wireless networks," in Wireless Communications Systems (ISWCS), 2014 11th International Symposium on. IEEE, Aug. 2014, pp. 781-785.
[9] H. Nikopour and H. Baligh, "Sparse code multiple access," in Personal Indoor and Mobile Radio Communications (PIMRC), 2013 IEEE 24th International Symposium on. IEEE, 2013, pp. 332-336.
[10] R. Razavi, A.-I. Mohammed, M. A. Imran, R. Hoshyar, and D. Chen, "On receiver design for uplink low density signature OFDM (LDS-OFDM)," IEEE Trans. Commun., vol. 60, no. 11, pp. 3499-3508, Sep. 2012.
[11] H. Weingarten, Y. Steinberg, and S. Shamai, "The capacity region of the Gaussian multiple-input multiple-output broadcast channel," IEEE Trans. Inf. Theory., vol. 52, no. 9, pp. 3936-3964, Sept. 2006.
[12] Z. Chen and X. Dai, "MED precoding for multi-user MIMO-NOMA downlink transmission," IEEE Trans. Veh. Technol. to be published, 2016.
[13] Q. Sun, S. Han, I. Chin-Lin, and Z. Pan, "On the ergodic capacity of MIMO NOMA systems," IEEE Wireless Commun. Lett., vol. 4, no. 4, pp. 405-408, Aug. 2015.
[14] J. Choi, "On the power allocation for MIMO-NOMA systems with layered transmissions," IEEE Trans. Wireless Commun., vol. 15, no. 5, pp. 3226-3237, May. 2016.
[15] Z. Ding, F. Adachi, and H. V. Poor, "The application of MIMO to non-orthogonal multiple access," IEEE Trans. Wireless Commun., vol. 15, no. 1, pp. 537-552, Jan. 2016.
[16] Z. Ding, L. Dai, and H. V. Poor, "MIMO-NOMA design for small packet transmission in the Internet of Things," IEEE Access, vol. 4, pp. 1393-1405, Apr. 2016.
[17] W. Shin, M. Vaezi, B. Lee, D. J. Love, J. Lee, and H. V. Poor, "Coordinated beamforming for multi-cell MIMO-NOMA," IEEE Commun. Lett., vol. 21, no. 1, pp. 84-87, Jan. 2017.
[18] S. Ali, E. Hossain, and D. I. Kim, "Non-orthogonal multiple access (NOMA) for downlink multiuser MIMO systems: User clustering, beamforming, and power allocation," IEEE Access, vol. 5, pp. 565-577, Dec. 2017.
[19] Z. Ding, X. Lei, G. K. Karagiannidis, R. Schober, J. Yuan, and V. Bhargava, "A survey on non-orthogonal multiple access for 5G networks: Research challenges and future trends," arXiv preprint arXiv:1706.05347, 2017.
[20] Z. Ma, Z. Ding, P. Fan, and S. Tang, "A general framework for MIMO uplink and downlink transmissions in 5G multiple access," in IEEE 83rd Vehicular Technology Conference (VTC Spring). IEEE, Sept. 2016, pp. 1-4.
[21] A. Tulino and S. Verdú, Foundations and Trends in Commun. and Inform. Theory: Random Matrix Theory and Wireless Communications. now Publishers Inc., 2004.
[22] R. J. Muirhead, Aspects of multivariate statistical theory. John Wiley \& Sons, 2009.
[23] H. Zha, "The restricted singular value decomposition of matrix triplets," SIAM Journal on Matrix Analysis and Applications, vol. 12, no. 1, pp. 172-194, Jan. 1991.
[24] M. Peng, C. Wang, J. Li, H. Xiang, and V. Lau, "Recent advances in underlay heterogeneous networks: Interference control, resource allocation, and self-organization," IEEE Commun. Surv. Tuts., vol. 17, no. 2, pp. 700-729, Mar. 2015.
[25] C. Mobile, "C-RAN: the road towards green RAN," White Paper, ver 2.5, Oct. 2011.
[26] 3rd Generation Partnership Project, Study on Downlink Multiuser Superposition Transmission for LTE, Mar. 2015.
[27] J. Cui, Z. Ding, and P. Fan, "The Application of Machine Learning in mmWave-NOMA Systems." Submitted to Communications (ICC), 2018 IEEE International Conference on.
[28] K. Wang, Y. Liu, Z. Ding, and A. Nallanathan, "User association in Non-Orthogonal multiple access networks." Submitted to Communications (ICC), 2018 IEEE International Conference on.
[29] R. Couillet and M. Debbah, Random matrix methods for wireless communications. Cambridge University Press, 2011.


[^0]:    Zhuo Chen and Xuchu Dai are with Key Lab of Wireless-Optical Commun., Chinese Acad. of Sciences, Sch. Info Science \& Tech., Univ. Science \& Tech. China, Hefei, Anhui, 230027, P.R.China (E-mail: cz1992@mail.ustc.edu.cn, daixc@ustc.edu.cn). Zhiguo Ding is with the School of Computer and Communications, Lancaster University, LA1 4YW, UK (E-mail: z.ding@lancaster.ac.uk).

    Robert Schober is with the Institute for Digital Communications, University Erlangen-Nürnberg, Cauerstrasse 7, D-91058 Erlangen, Germany (Email: robert.schober@fau.de).

[^1]:    Non-orthogonal multiple access (NOMA), generalized singular value decomposition (GSVD), multiple-input multiple-output (MIMO), empirical probability density function, asymptotic analysis.

[^2]:    ${ }^{1}$ The proposed GSVD-NOMA scheme can be viewed as an extension of conventional MIMO processing for a single-user point-to-point link where the singular value decomposition (SVD) is performed to convert the MIMO channel into parallel SISO channels.

[^3]:    ${ }^{3}$ Since the elements of $\mathbf{H}_{1}$ and $\mathbf{H}_{2}$ are i.i.d. complex Gaussian random variables, they are full rank with probability one [22].

[^4]:    ${ }^{4}$ More sophisticated, optimized power allocation strategies can further increase the sum rate of the two users, but make implementation and performance analysis more complicated. Nevertheless, the development of optimal power allocation strategies, which maximize the sum rate and ensure user fairness at the same time, is an important direction of future research.

[^5]:    ${ }^{5}$ Please note that, in this section, the performance analysis of the GSVD-NOMA system encompasses all possible cases of $m$ and $n$, i.e., $2 m \leq n, m<n<2 m$, and $m \geq n$. The case $m \leq n$ applies when the users have fewer antennas than the base station, which is an expected scenario in the Internet of Things and cellular networks. Furthermore, the case $m \geq n$ is applicable to e.g. small cells [24] and cloud radio access networks (C-RANs) [25], where low-cost base stations are deployed with high density and these base stations are expected to have capabilities similar to those of advanced smart phones and tablets.

[^6]:    ${ }^{6}$ Here, the time division multiple access (TDMA) protocol is used as the benchmark.

[^7]:    ${ }^{7}$ Obviously the performance of this hybrid NOMA scheme depends on how the users are grouped, where sophisticated algorithms such as monotonic optimization, machine learning, and game theory can be used for performance optimization [4], [27], [28]. We also suggest this as a topic for future work.

